Quantized coinvariants at transcendental q

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Abstract

A general method is developed for deriving Quantum First and Second Fundamental Theorems of Coinvariant Theory from classical analogs in Invariant Theory, in the case that the quantization parameter q is transcendental over a base field. Several examples are given illustrating the utility of the method; these recover earlier results of various researchers including Domokos, Fioresi, Hacon, Rigal, Strickland, and the present authors.

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Introduction and background

In the classic terminology of Hermann Weyl [**18**], a full solution to any invariant theory problem should incorporate a *First Fundamental Theorem*, giving a set of generators (finite, where possible) for the ring of invariants, and a *Second Fundamental Theorem*, giving generators for the ideal of relations among the generators of the ring of invariants. Many of the classical settings of Invariant Theory have quantized analogs, and one seeks corresponding analogs of the classical First and Second Fundamental Theorems. However, the setting must be dualized before potential quantized analogs can be framed, since there are no quantum analogs of the original objects, only quantum analogs of their coordinate rings. Hence, one rephrases the classical results in terms of rings of <u>co</u>invariants (see below), and then seeks quantized versions of these. A first stumbling block is that in general these coactions are not algebra homomorphisms, and so at the outset it is not even obvious that the coinvariants form a subalgebra. However, this can often be established; cf [**9**], Proposition 1.1; [**4**], Proposition 1.3.

Typically, a classical invariant theoretical setting is quantized uniformly with respect to a parameter q, that is, there is a family of rings of coinvariants to be determined, parametrized by nonzero scalars in a base field, such that the case $q = 1$ is the classical one. (Many authors, however, restrict attention to special values of q, such as those

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transcendental over the rational number field, and do not address the general case.) As in the classical setting, one usually has identified natural candidates to generate the ring of coinvariants. Effectively, then, one has a parametrized inclusion of algebras (candidate subalgebras inside the algebras of coinvariants), which is an equality at $q = 1$, and one seeks equality for other values of q. In the best of all worlds, the equality at $q = 1$ could be "lifted" by some general process to equality at arbitrary q. Lifting to transcendental q has been done succesfully in some cases, by ad hoc methods – see, for example, [**6**], [**17**]. Also, an early version of [**9**] for the transcendental case was obtained in this way. Quantum Second Fundamental Theorems can be approached in a similar manner. We develop here a general method for lifting equalities of inclusions from $q = 1$ to transcendental q, which applies to many analyses of quantized coinvariants.

In order to apply classical results as indicated above, we must be able to transform invariants to coinvariants. For morphic actions of algebraic groups, the setting of interest to us, invariants and coinvariants are related as follows. Suppose that $\gamma : G \times V \to V$ is a morphic action of an algebraic group G on a variety V. This action induces an action of G on $\mathcal{O}(V)$, where $(x.f)(v) = f(x^{-1}.v)$ for $x \in G$, $f \in \mathcal{O}(V)$, and $v \in V$. The invariants for this action are, of course, those functions in $\mathcal{O}(V)$ which are constant on G-orbits. The comorphism of γ is an algebra homomorphism $\gamma^* : \mathcal{O}(V) \to \mathcal{O}(G) \otimes \mathcal{O}(V)$, with respect to which $\mathcal{O}(V)$ becomes a left $\mathcal{O}(G)$-comodule. Now a function $f \in \mathcal{O}(V)$ is a coinvariant in this comodule when $\gamma^*(f) = 1 \otimes f$. Since $1 \otimes f$ corresponds to the function $(x, v) \mapsto f(v)$ on $G \times V$, we see that $\gamma^*(f) = 1 \otimes f$ if and only if $f(x.v) = f(v)$ for all $x \in G$ and $v \in V$, that is, if and only if f is an invariant function. To summarize:

$$\mathcal{O}(V)^{\mathrm{co}\,\mathcal{O}(G)} = \mathcal{O}(V)^G.$$

Quantized coordinate rings have been constructed for all complex semisimple algebraic groups G. These quantized coordinate rings are Hopf algebras, which we denote $\mathcal{O}_q(G)$, since we are concentrating on the single parameter versions. In those cases where a morphic action of G on a variety V has been quantized, we have a quantized coordinate ring $\mathcal{O}_q(V)$ which supports an $\mathcal{O}_q(G)$-coaction. The coaction is often not an algebra homomorphism, but nonetheless – as mentioned above – the set of $\mathcal{O}_q(G)$-coinvariants in $\mathcal{O}_q(V)$ is typically a subalgebra. The goal of Quantum First and Second Fundamental Theorems for this setting is to give generators and relations for the algebra $\mathcal{O}_q(V)^{\mathrm{co}\,\mathcal{O}_q(G)}$ of $\mathcal{O}_q(G)$-coinvariants in $\mathcal{O}_q(V)$. We discuss several standard settings in later sections of the paper, and outline how our general method applies. We recover known Quantum First and Second Fundamental Theorems at transcendental q in these settings, with some simplifications to the original proofs, and in some cases extending the range of the theorems.

Throughout, k will denote a field, which may be of arbitrary characteristic and need not be algebraically closed.

1 Reduction modulo $q - 1$

Throughout this section, we work with a field extension $k^\circ \subset k$ and a scalar $q \in k \setminus k^\circ$ which is transcendental over k°. Thus, the k°-subalgebra $R = k^\circ[q, q^{-1}] \subset k$ is a Laurent polynomial ring. Let us denote reduction modulo $q-1$ by overbars, that is, given any R-module homomorphism $\phi : A \to B$, we write $\overline{\phi} : \overline{A} \to \overline{B}$ for the induced map $A/(q-1)A \to B/(q-1)B$.

Proposition 1.1 *Let $A \xrightarrow{\phi} B \xrightarrow{\psi} C$ be a complex of R-modules, such that C is torsion-free. Suppose that there are R-module decompositions*

$$A = \bigoplus_{j\in J} A_j, \qquad B = \bigoplus_{j\in J} B_j, \qquad C = \bigoplus_{j\in J} C_j$$

such that B_j is finitely generated, $\phi(A_j) \subseteq B_j$, and $\psi(B_j) \subseteq C_j$ for all $j \in J$.

If the reduced complex $\overline{A} \xrightarrow{\overline{\phi}} \overline{B} \xrightarrow{\overline{\psi}} \overline{C}$ is exact, then so is

$$k \otimes_R A \xrightarrow{\mathrm{id}\otimes\phi} k \otimes_R B \xrightarrow{\mathrm{id}\otimes\psi} k \otimes_R C.$$

Proof. The hypotheses and the conclusions all reduce to the direct sum components of the given decompositions, so it is enough to work in one component. Hence, there is no loss of generality in assuming that B is finitely generated.

Let S denote the localization of R at the maximal ideal $(q-1)R$. Set

$$\widetilde{A} = S \otimes_R A, \qquad \widetilde{B} = S \otimes_R B, \qquad \widetilde{C} = S \otimes_R C,$$

and let $\widetilde{A} \xrightarrow{\widetilde{\phi}} \widetilde{B} \xrightarrow{\widetilde{\psi}} \widetilde{C}$ denote the induced complex of S-modules. Since $\widetilde{A}/(q-1)\widetilde{A}$ is naturally isomorphic to $A/(q-1)A = \overline{A}$, and similarly for B and C, there is a commutative diagram

$$\begin{tikzcd}
\widetilde{A} \arrow[r, "\widetilde{\phi}"] \arrow[d, "\alpha"'] & \widetilde{B} \arrow[r, "\widetilde{\psi}"] \arrow[d, "\beta"'] & \widetilde{C} \arrow[d, "\gamma"] \\
\overline{A} \arrow[r, "\overline{\phi}"] & \overline{B} \arrow[r, "\overline{\psi}"] & \overline{C}
\end{tikzcd}$$

where α, β, γ are epimorphisms with kernels $(q-1)\widetilde{A}$ etc.

The bottom row of the diagram is exact by hypothesis, and we claim that the top row is exact. Consider an element $x \in \ker\widetilde{\psi}$. Chasing x around the diagram, we see that $x - \widetilde{\phi}(y) = (q-1)z$ for some $y \in \widetilde{A}$ and $z \in \widetilde{B}$. Note that $(q-1)z \in \ker\widetilde{\psi}$. Since $\widetilde{C}$ is torsionfree, it follows that $z \in \ker\widetilde{\psi}$. Thus $\ker\widetilde{\psi} \subseteq \widetilde{\phi}(\widetilde{A}) + (q-1)\ker\widetilde{\psi}$, whence

$$(\ker\widetilde{\psi})/\widetilde{\phi}(\widetilde{A}) = (q-1)\big[(\ker\widetilde{\psi})/\widetilde{\phi}(\widetilde{A})\big].$$

Since $\ker\widetilde{\psi}$ is a finitely generated S-module, it follows from Nakayama's Lemma that $(\ker\widetilde{\psi})/\widetilde{\phi}(\widetilde{A}) = 0$, establishing the claim.

Since k is a flat S-module, the sequence $k \otimes_S \widetilde{A} \xrightarrow{\mathrm{id}\otimes\phi} k \otimes_S \widetilde{B} \xrightarrow{\mathrm{id}\otimes\psi} k \otimes_S \widetilde{C}$ is exact. This is isomorphic to the sequence

$$k \otimes_R A \xrightarrow{\mathrm{id}\otimes\phi} k \otimes_R B \xrightarrow{\mathrm{id}\otimes\psi} k \otimes_R C$$

and therefore the latter is exact. $\square$

Proposition 1.1 is useful in obtaining quantized versions of both First and Second Fundamental Theorems. For ease of application, we write out general versions of both situations with appropriate notation, as follows.

Theorem 1.2 *Let H be an R-bialgebra and B a left H-comodule with structure map $\lambda : B \to H \otimes_R B$, and let $\phi : A \to B$ be an R-module homomorphism whose image is contained in $B^{\mathrm{co}\, H}$. Assume that B and H are torsionfree R-modules, and suppose that there exist R-module decompositions $A = \bigoplus_{j \in J} A_j$ and $B = \bigoplus_{j \in J} B_j$ such that B_j is finitely generated, $\phi(A_j) \subseteq B_j$, and $\lambda(B_j) \subseteq H \otimes_R B_j$ for all $j \in J$.*

Now $\overline{B}$ is a left comodule over the k°-bialgebra $\overline{H}$, and $k \otimes_R B$ is a left comodule over the k-bialgebra $k \otimes_R H$. If $\overline{B}^{\mathrm{co}\, \overline{H}}$ equals the image of the induced map $\overline{\phi} : \overline{A} \to \overline{B}$, then $[k \otimes_R B]^{\mathrm{co}\, k \otimes_R H}$ equals the image of the induced map $\mathrm{id} \otimes \phi : k \otimes_R A \to k \otimes_R B$.

Proof. Since R is a principal ideal domain, $H \otimes_R B$ is a torsionfree R-module. We may identify $\overline{H \otimes_R B}$ with $\overline{H} \otimes_{k^\circ} \overline{B}$ and $k \otimes_R (H \otimes_R B)$ with $(k \otimes_R H) \otimes_k (k \otimes_R B)$. Now apply Proposition 1.1 to the complex $A \xrightarrow{\phi} B \xrightarrow{\lambda - \gamma} H \otimes_R B$, where $\gamma : B \to H \otimes_R B$ is the map $b \mapsto 1 \otimes b$. $\square$

Theorem 1.3 *Let $\phi : A \to B$ be a homomorphism of R-algebras, and I an ideal of A contained in $\ker \phi$. Assume that B is a torsionfree R-module, and suppose that there exist R-module decompositions*

$$I = \bigoplus_{j \in J} I_j, \qquad A = \bigoplus_{j \in J} A_j, \qquad B = \bigoplus_{j \in J} B_j$$

such that A_j is finitely generated, $I_j \subseteq A_j$, and $\phi(A_j) \subseteq B_j$ for all $j \in J$.

If $\ker \overline{\phi}$ equals the image of $\overline{I}$, then the kernel of the k-algebra homomorphism $\mathrm{id} \otimes \phi : k \otimes_R A \to k \otimes_R B$ equals the image of $k \otimes_R I$.

Proof. Apply Proposition 1.1 to the complex $I \xrightarrow{\eta} A \xrightarrow{\phi} B$, where η is the inclusion map. $\square$

Proposition 1.1 can easily be adapted to yield other exactness conclusions. In particular, suppose that, in addition to the given hypotheses, the cokernels of ϕ and ψ are torsionfree R-modules. Then one can show that for all nonzero scalars $\lambda \in k^\circ$, the induced complex

$$A/(q - \lambda)A \longrightarrow B/(q - \lambda)B \longrightarrow C/(q - \lambda)C$$

is exact. Unfortunately, in the applications to quantized coinvariants, it appears to be a difficult task to verify the above additional hypotheses.

2 The interior action of GL_t on $M_{m,t} \times M_{t,n}$

Let m, n, t be positive integers with $t \leq \min\{m, n\}$. The group $GL_t(k)$ acts on the variety $V := M_{m,t}(k) \times M_{n,t}(k)$ via the rule

$$g \cdot (A, B) = (Ag^{-1}, gB),$$

and consequently on the coordinate ring $\mathcal{O}(V) \cong \mathcal{O}(M_{m,t}(k)) \otimes \mathcal{O}(M_{t,n}(k))$. The invariant theory of this action is closely related to the matrix multiplication map

$$\mu : V \longrightarrow M_{m,n}(k)$$

and its comorphism μ^*. The classical Fundamental Theorems for this situation ([**2**], Theorems 3.1, 3.4) are

(1) The ring of invariants $\mathcal{O}(V)^{GL_t(k)}$ equals the image of μ^*.

(2) The kernel of μ^* is the ideal of $\mathcal{O}(M_{m,n}(k))$ generated by all $(t+1) \times (t+1)$ minors.

As discussed in the introduction, the comorphism of the action map $\gamma : GL_t(k) \times V \to V$ is an algebra homomorphism $\gamma^* : \mathcal{O}(V) \to \mathcal{O}(GL_t(k)) \otimes \mathcal{O}(V)$ which makes $\mathcal{O}(V)$ into a left comodule over $\mathcal{O}(GL_t(k))$, and the coinvariants for this coaction equal the invariants for the action above: $\mathcal{O}(V)^{GL_t(k)} = \mathcal{O}(V)^{\mathrm{co}\,\mathcal{O}(GL_t(k))}$.

We may consider quantum versions of this situation, relative to a parameter $q \in k^\times$, using the standard quantized coordinate rings $\mathcal{O}_q(M_{m,t}(k))$, $\mathcal{O}_q(M_{t,n}(k))$, and $\mathcal{O}_q(GL_t(k))$. We write X_{ij} for the standard generators in each of these algebras; see, e.g., [**8**], (1.1), for the standard relations.

To quantize the classical coaction above, one combines quantized versions of the actions of $GL_t(k)$ on $M_{m,t}(k)$ and $M_{n,t}(k)$ by right and left multiplication, respectively. The latter quantizations are algebra morphisms

$$\rho_q^* : \mathcal{O}_q(M_{m,t}(k)) \longrightarrow \mathcal{O}_q(M_{m,t}(k)) \otimes \mathcal{O}_q(GL_t(k))$$
$$\lambda_q^* : \mathcal{O}_q(M_{t,n}(k)) \longrightarrow \mathcal{O}_q(GL_t(k)) \otimes \mathcal{O}_q(M_{t,n}(k))$$

which are both determined by $X_{ij} \mapsto \sum_{l=1}^t X_{il} \otimes X_{lj}$ for all i, j. The right comodule structure on $\mathcal{O}_q(M_{m,t}(k))$ determines a left comodule structure in the standard way, via the map $\tau \circ (\mathrm{id} \otimes S) \circ \rho_q^*$, where τ is the flip and S is the antipode in $\mathcal{O}_q(GL_t(k))$. We set $\mathcal{O}_q(V) = \mathcal{O}_q(M_{m,t}(k)) \otimes \mathcal{O}_q(M_{t,n}(k))$, which is a left comodule over $\mathcal{O}_q(GL_t(k))$ in the standard way. The resulting structure map $\gamma_q^* : \mathcal{O}_q(V) \to \mathcal{O}_q(GL_t(k)) \otimes \mathcal{O}_q(V)$ is determined by the rule

$$\gamma_q^*(a \otimes b) = \sum_{(a),(b)} S(a_1)b_{-1} \otimes a_0 \otimes b_0$$

for $a \in \mathcal{O}_q(M_{m,t}(k))$ and $b \in \mathcal{O}_q(M_{t,n}(k))$, where $\rho_q^*(a) = \sum_{(a)} a_0 \otimes a_1$ and $\lambda_q^*(b) = \sum_{(b)} b_{-1} \otimes b_0$. It is, unfortunately, more involved to work with γ_q^* than with its classical predecessor γ^* because γ_q^* is not an algebra homomorphism.

The analog of μ^* is the algebra homomorphism $\mu_q^* : \mathcal{O}_q(M_{m,n}(k)) \to \mathcal{O}_q(V)$ such that $X_{ij} \mapsto \sum_{l=1}^t X_{il} \otimes X_{lj}$ for all i, j. We can now state the Fundamental Theorems for the quantized coinvariants in the current situation:

(1_q) The set of coinvariants $\mathcal{O}_q(V)^{\mathrm{co}\,\mathcal{O}_q(GL_t(k))}$ equals the image of μ_q^* (and so is a subalgebra of $\mathcal{O}_q(V)$).

(2_q) The kernel of μ_q^* is the ideal of $\mathcal{O}_q(M_{m,n}(k))$ generated by all $(t+1)\times(t+1)$ quantum minors.

These theorems were proved for arbitrary q in [**9**], Theorem 4.5, and [**8**], Proposition 2.4, respectively.

To illustrate the use of Theorems 1.2 and 1.3, we specialize to the case that q is transcendental over a subfield $k^\circ \subset k$. Set $R = k^\circ[q, q^{-1}]$ as in the previous section. Since the construction of quantum matrix algebras requires only a commutative base ring and an invertible element in that ring, we can form $\mathcal{O}_q(M_{m,t}(R))$ and $\mathcal{O}_q(M_{t,n}(R))$. These R-algebras are iterated skew polynomial extensions of R, and thus are free R-modules, as is the algebra $\mathcal{O}_q(V_R) := \mathcal{O}_q(M_{m,t}(R)) \otimes_R \mathcal{O}_q(M_{t,n}(R))$. Similarly, $\mathcal{O}_q(M_t(R))$ is a free R-module and an integral domain, and so the localization $\mathcal{O}_q(GL_t(R))$, obtained by inverting the quantum determinant, is a torsionfree R-module, as well as a Hopf R-algebra. Since the restrictions of ρ_q^* and λ_q^* make $\mathcal{O}_q(M_{m,t}(R))$ and $\mathcal{O}_q(M_{t,n}(R))$ into right and left comodules over $\mathcal{O}_q(GL_t(R))$, respectively, the restriction γ_R^* of γ_q^* makes $\mathcal{O}_q(V_R)$ into a left $\mathcal{O}_q(GL_t(R))$-comodule. Finally, μ_q^* restricts to an R-algebra homomorphism $\mu_R^* : \mathcal{O}_q(M_{m,n}(R)) \to \mathcal{O}_q(V_R)$. It is easily checked that the image of μ_R^* is contained in $\mathcal{O}_q(V_R)^{\mathrm{co}\,\mathcal{O}_q(GL_t(R))}$, and that the kernel of μ_R^* contains the ideal I generated by all $(t+1)\times(t+1)$ quantum minors (cf [**9**], Proposition 2.3 and [**8**], (2.1)).

All of the quantum matrix algebras $\mathcal{O}_q(M_{\bullet,\bullet}(R))$ are positively graded R-algebras, with the generators X_{ij} having degree 1, and $\mathcal{O}_q(V_R)$ inherits a positive grading from its two factors. In each of these algebras, the homogeneous components are finitely generated free R-modules. It is easily checked that

$$\mu_R^*\big(\mathcal{O}_q(M_{m,n}(R))_j\big) \subseteq \mathcal{O}_q(V_R)_{2j} \qquad \text{and} \qquad \gamma_R^*\big(\mathcal{O}_q(V_R)_j\big) \subseteq \mathcal{O}_q(GL_t(R)) \otimes_R \mathcal{O}_q(V_R)_j$$

for all $j \geq 0$. Moreover, the ideal I of $\mathcal{O}_q(M_{m,n}(R))$ is homogeneous with respect to this grading. Note that when we come to apply Theorem 1.3, we should replace the grading on $\mathcal{O}_q(V_R)$ by the vector space decomposition $\bigoplus_{j=0}^\infty \big(\mathcal{O}_q(V_R)_{2j} \oplus \mathcal{O}_q(V_R)_{2j+1}\big)$, for instance.

The classical First and Second Fundamental Theorems (1) and (2) say that $\overline{\mu_R^*}$ maps $\overline{\mathcal{O}_q(M_{m,n}(R))}$ onto the coinvariants of $\overline{\mathcal{O}_q(V_R)}$, and that the kernel of $\overline{\mu_R^*}$ equals the image of $\overline{I}$. Therefore Theorems 1.2 and 1.3 yield the quantized Fundamental Theorems (1_q) and (2_q) with no further work in the transcendental case.

3 The right action of SL_r on $M_{n,r}$

Fix positive integers $r < n$. In this section, we consider the right action of $SL_r(k)$ on $M_{n,r}(k)$ by multiplication: $A.g = Ag$ for $A \in M_{n,r}(k)$ and $g \in SL_r(k)$. The First Fundamental Theorem for this case (cf [**7**], Proposition 2, p 138) says that

(1) The ring of invariants $\mathcal{O}(M_{n,r}(k))^{SL_r(k)}$ equals the subalgebra of $\mathcal{O}(M_{n,r}(k))$ generated by all $r \times r$ minors.

To state the Second Fundamental Theorem for this case, let $k[X]$ be the polynomial ring in a set of variables X_I, where I runs over all r-element subsets of $\{1, \ldots, n\}$. In view of the theorem above, there is a natural homomorphism of $\phi : k[X] \to \mathcal{O}(M_{n,r}(k))^{SL_r(k)}$.

(2) The kernel of ϕ is generated by the Plücker relations ([**7**], Proposition 2, p 138).

To quantize this situation, we make $\mathcal{O}_q(M_{n,r}(k))$ into a right comodule over $\mathcal{O}_q(SL_r(k))$ via the algebra homomorphism $\rho : \mathcal{O}_q(M_{n,r}(k)) \to \mathcal{O}_q(M_{n,r}(k)) \otimes \mathcal{O}_q(SL_r(k))$ such that $\rho(X_{ij}) = \sum_{l=1}^{r} X_{il} \otimes X_{lr}$ for all i, j. The First Fundamental Theorem for the quantized coinvariants is the statement

(1_q) The set of coinvariants $\mathcal{O}_q(M_{n,r}(k))^{\mathrm{co}\,\mathcal{O}_q(SL_r(k))}$ equals the subalgebra of $\mathcal{O}_q(M_{n,r}(k))$ generated by all $r \times r$ quantum minors.

This follows from work of Fioresi and Hacon under the assumptions that k is algebraically closed of characteristic 0 and q is transcendental over some subfield of k ([**6**], Theorem 3.12) – in fact, they prove (in our notation) that (1_q) holds with k replaced by a Laurent polynomial ring $k^\circ[q, q^{-1}]$. The result is obtained for any nonzero q in an arbitrary field k in [**11**], Theorem 5.2. The Fioresi-Hacon version of the Second Fundamental Theorem yields a presentation of $\mathcal{O}_q(M_{n,r}(k))^{\mathrm{co}\,\mathcal{O}_q(SL_r(k))}$ in terms of generators $\lambda_{i_1\cdots i_r}$ where $i_1 \cdots i_r$ runs over all unordered sequences of r distinct integers from $\{1, \dots, n\}$. Each $\lambda_{i_1\cdots i_r}$ is mapped to $(-q)^{\ell(\pi)} D_{\{i_1,\dots,i_r\}}$ where $\ell(\pi)$ denotes the length of the permutation π sending $l \mapsto i_l$ for $l = 1, \dots, r$, and $D_{\{i_1,\dots,i_r\}}$ denotes the $r \times r$ quantum minor with row index set $\{i_1, \dots, i_r\}$.

(2_q) The algebra $\mathcal{O}_q(M_{n,r}(k))^{\mathrm{co}\,\mathcal{O}_q(SL_r(k))}$ is isomorphic to the quotient of the free algebra $k\langle\lambda_{i_1\cdots i_r}\rangle$ modulo the ideal generated by the relations (c) and (y) given in [**6**], Theorem 3.14.

The hypotheses on k and q are as above.

Now let k°, k, q, R be as in Section 2, and define everything over R. In particular, $\mathcal{O}_q(M_{n,r}(R))$ is a right comodule algebra over $\mathcal{O}_q(SL_r(R))$. As already mentioned, Fioresi and Hacon have proved the R-algebra versions of (1_q) and (2_q), assuming that k is algebraically closed of characteristic zero. The easier parts of their work – proving inclusions rather than equalities in these R-algebra forms – lead directly to the k-algebra forms of these theorems, as follows.

Let $R\langle\lambda_\bullet\rangle$ denote the free R-algebra on generators $\lambda_{i_1\cdots i_r}$, and let $\phi_q : R\langle\lambda_\bullet\rangle \to \mathcal{O}_q(M_{n,r}(R))$ be the R-algebra homomorphism sending each $\lambda_{i_1\cdots i_r}$ to $(-q)^{\ell(\pi)} D_{\{i_1,\dots,i_r\}}$. The image of ϕ_q is the R-subalgebra generated by the $D_{\{i_1,\dots,i_r\}}$, which is contained in $\mathcal{O}_q(M_{n,r}(R))^{\mathrm{co}\,\mathcal{O}_q(SL_r(R))}$ by [**6**], Lemma 3.5. Let I_q be the ideal of $R\langle\lambda_\bullet\rangle$ generated by the relations (c) and (y) of [**6**], Theorem 3.14; that $I_q \subseteq \ker\phi_q$ was checked by Fioresi ([**5**], Proposition 2.21 and Theorem 3.6). Now $\mathcal{O}_q(M_{n,r}(R))$ and $R\langle\lambda_\bullet\rangle$ are positively graded R-algebras, with the X_{ij} having degree 1 and the $\lambda_\bullet$ having degree r. The homogeneous components with respect to these gradings are finitely generated free R-modules, I_q is a homogeneous ideal of $R\langle\lambda_\bullet\rangle$, and the map ϕ_q is homogeneous of degree 0.

The classical Fundamental Theorems (1) and (2) say that $\overline{\phi_q}$ maps $\overline{R\langle\lambda_\bullet\rangle}$ onto the coinvariants of $\overline{\mathcal{O}_q(M_{n,r}(R))}$, and the kernel of $\overline{\phi_q}$ equals $\overline{I_q}$. (The latter statement requires a change of relations using the results of [**7**], Chapter 8, as observed in [**6**], p 435.) Therefore Theorems 1.2 and 1.3 yield the quantized Fundamental Theorems (1_q) and (2_q). Note that the hypotheses of algebraic closure and characteristic zero on k are not needed, although q is still assumed to be transcendental over a subfield of k.

4 The right action of Sp_{2n} on $M_{m,2n}$

Fix positive integers m and n, and consider the right action of $Sp_{2n}(k)$ on $M_{m,2n}(k)$ by multiplication. Here we take $Sp_{2n}(k)$ to preserve the standard alternating bilinear form on k^{2n}, which we denote $\langle -,- \rangle$, and we view $M_{m,2n}(k)$ as the variety of m vectors of length $2n$. Thus, each row $x_i = (X_{i1}, \dots, X_{i,2n})$ of generators in $\mathcal{O}(M_{m,2n}(k))$ corresponds to the i-th of m generic $2n$-vectors. We shall need the functions describing the values of $\langle -,- \rangle$ on two of these generic vectors:

$$z_{ij} := \langle x_i, x_j \rangle = \sum_{l=1}^{n} (X_{i,2l-1}X_{j,2l} - X_{i,2l}X_{j,2l-1})$$

for $i, j = 1, \dots, m$.

Next, consider the variety $\mathrm{Alt}_m(k)$ of alternating $m \times m$ matrices over k; its coordinate ring is the algebra

$$\mathcal{O}(\mathrm{Alt}_m(k)) = \mathcal{O}(M_m(k))/\langle X_{ii},\ X_{ij} + X_{ji} \mid i, j = 1, \dots, m \rangle.$$

Let us write Y_{ij} for the coset of X_{ij} in $\mathcal{O}(\mathrm{Alt}_m(k))$. There is a morphism $\nu : M_{m,2n}(k) \to \mathrm{Alt}_m(k)$ given by the rule $\nu(A) = ABA^{\mathrm{tr}}$, where B is the matrix of the symplectic form $\langle -,- \rangle$, and the comorphism $\nu^* : \mathcal{O}(\mathrm{Alt}_m(k)) \to \mathcal{O}(M_m(k))$ sends $Y_{ij} \mapsto z_{ij}$ for all i, j. For any even number h of distinct indices $i_1, \dots, i_h$ from $\{1, \dots, m\}$, let $[i_1, \dots, i_h] \in \mathcal{O}(\mathrm{Alt}_m(k))$ be the Pfaffian of the submatrix of (Y_{ij}) obtained by taking the rows and columns with indices $i_1, \dots, i_h$.

The First and Second Fundamental Theorems for the present situation ([**2**], Theorems 6.6, 6.7) state that

(1) The ring of invariants $\mathcal{O}(M_{m,2n}(k))^{Sp_{2n}(k)}$ equals the subalgebra of $\mathcal{O}(M_{m,2n}(k))$ generated by the z_{ij} for $i < j$, that is, the image of ν^*.

(2) If $m \leq 2n + 1$, the kernel of ν^* is zero, while if $m \geq 2n + 2$, the kernel of ν^* is the ideal of $\mathcal{O}(\mathrm{Alt}_m(k))$ generated by all the Pfaffians $[i_1, \dots, i_{2n+2}]$.

A quantized version of this situation was studied by Strickland [**17**], who quantized the $U(\mathfrak{sp}_{2n}(k))$-module action on $\mathcal{O}(M_{2n,m}(k))$ rather than its dual, the $\mathcal{O}(Sp_{2n}(k))$ coaction. We transpose the matrices and indices from her paper in order to match the notation for the classical situation used above. Since the approaches to quantized enveloping algebras for $\mathfrak{sp}_{2n}(k)$ and quantized coordinate rings for $Sp_n(k)$ become more complicated at roots of unity, let us restrict our discussion to a quantum parameter $q \in k^\times$ which is not a root of unity.

Recall that the quantized coordinate rings of the different classical groups coact on different quantized coordinate rings of affine spaces. In particular, $\mathcal{O}_q(Sp_{2n}(k))$ coacts on an algebra that we denote $\mathcal{O}_q(\mathfrak{sp}\, k^{2n})$, the *quantized coordinate ring of symplectic* $2n$*-space* ([**16**], Definition 14; see [**15**], §1.1, for a simpler set of relations). Thus, $\mathcal{O}(M_{m,2n}(k))$ needs to be quantized by a suitable algebra with $m \cdot 2n$ generators, each row of which generates a copy of $\mathcal{O}_q(\mathfrak{sp}\, k^{2n})$. Let us call this algebra $\mathcal{O}_q((\mathfrak{sp}\, k^{2n})^m)$ and take it to be the k-algebra with generators $X_{ij} = x_{j,i}$ (for $i = 1, \dots, m$ and $j = 1, \dots, 2n$) satisfying the relations given by Strickland in [**17**], Equations (2.1), (2.2), (2.3). (The $x_{j,i}$ are Strickland's generators for the algebra she denotes B; we transpose the indices for the reasons indicated above.) Similarly, let us write $\mathcal{O}_q(\mathrm{Alt}_m(k))$ for the k-algebra with generators $Y_{ij} = a_{j,i}$ for $1 \le j < i \le m$ satisfying the relations given in [**17**], Equations (1.1). As in [**17**], Theorem 2.5(2), there is a k-algebra homomorphism $\phi : \mathcal{O}_q(\mathrm{Alt}_m(k)) \to \mathcal{O}_q((\mathfrak{sp}\, k^{2n})^m)$ such that

$$\phi(Y_{ts}) = \sum_{l=1}^{n} q^{l-1-n} X_{sl} X_{t,2n-l+1} - \sum_{l=1}^{n} q^{n-l+1} X_{s,2n-l+1} X_{tl}$$

for $s < t$.

Set $B_q = \mathcal{O}_q((\mathfrak{sp}\, k^{2n})^m)$ for the time being. Strickland defines an action of $U_q(\mathfrak{sp}_{2n}(k))$ on B_q ([**17**], p 87). It is clear from the definition of this action that B_q is a locally finite dimensional left $U_q(\mathfrak{sp}_{2n}(k))$-module. Hence, B_q becomes a right comodule over the Hopf dual $U_q(\mathfrak{sp}_{2n}(k))^\circ$ in the standard way ([**14**], Lemma 1.6.4(2)), and the $U_q(\mathfrak{sp}_{2n}(k))$-invariants coincide with the $U_q(\mathfrak{sp}_{2n}(k))^\circ$-coinvariants ([**14**], Lemma 1.7.2(2)). The standard quantized coordinate ring of $Sp_{2n}(k)$, which we denote $\mathcal{O}_q(Sp_{2n}(k))$, is a sub-Hopf-algebra of $U_q(\mathfrak{sp}_{2n}(k))^\circ$, and one can check that the coaction $B_q \to B_q \otimes U_q(\mathfrak{sp}_{2n}(k))^\circ$ actually maps B_q into $B_q \otimes \mathcal{O}_q(Sp_{2n}(k))$. Consequently, B_q is a right comodule over $\mathcal{O}_q(Sp_{2n}(k))$, and its $\mathcal{O}_q(Sp_{2n}(k))$-coinvariants coincide with its $U_q(\mathfrak{sp}_{2n}(k))^\circ$-coinvariants.

Statements of First and Second Fundamental Theorems for quantized coinvariants in the symplectic situation above can be given as follows, where the *q-Pfaffians* $[i_1, \ldots, i_h]$ are defined as in [**17**], p 82:

(1_q) The set of coinvariants $\mathcal{O}_q((\mathfrak{sp}\, k^{2n})^m)^{\mathrm{co}\, \mathcal{O}_q(Sp_{2n}(k))}$ equals the image of ϕ.

(2_q) If $m \le 2n+1$, the kernel of ϕ is zero, while if $m \ge 2n+2$, the kernel of ϕ is the ideal of $\mathcal{O}_q(\mathrm{Alt}_m(k))$ generated by all the q-Pfaffians $[i_1, \ldots, i_{2n+2}]$.

Both statements have been proved by Strickland (modulo the changes of notation discussed above) under the assumptions that $\mathrm{char}(k) = 0$ and q is transcendental over a subfield of k ([**17**], Theorem 2.5). Via Theorems 1.2 and 1.3, we obtain the transcendental cases of (1_q) and (2_q) from the classical results (1) and (2) in arbitrary characteristic. (We note that part of Strickland's development also involves reduction modulo $q-1$. See the proof of [**17**], Theorem 1.5.) As far as we are aware, it is an open question whether (1_q) and (2_q) hold when q is algebraic over the prime subfield of k.

5 The conjugation action of GL_n on M_n

Fix a positive integer n, and assume that k has characteristic zero. In this section, we consider the quantum analogue of the classical conjugation action of $GL_n(k)$ on $M_n(k)$. We shall need the trace functions tr_i for $i = 1, \ldots, n$, where tr_i is the sum of the $i \times i$ principal minors. Note that tr_1 is the usual trace function, and that tr_n is the determinant function. The First and Second Fundamental Theorems for this situation ([**13**], Satz 3.1) can be stated as follows:

(1) The ring of invariants $\mathcal{O}(M_n(k))^{GL_n(k)}$ equals the subalgebra of $\mathcal{O}(M_n(k))$ generated by $\mathrm{tr}_1, \ldots, \mathrm{tr}_n$.

(2) There are no relations among the tr_i, that is, $\mathrm{tr}_1, \ldots, \mathrm{tr}_n$ are algebraically independent over k.

The coinvariants of a quantum analogue of the conjugation action have been studied in [**3**]. The *right conjugation coaction* of $\mathcal{O}_q(GL_n(k))$ on $\mathcal{O}_q(M_n(k))$ is the right coaction $\beta : \mathcal{O}_q(M_n(k)) \longrightarrow \mathcal{O}_q(M_n(k)) \otimes \mathcal{O}_q(GL_n(k))$ given by $\beta(u) := \sum_{(u)} u_2 \otimes S(u_1)u_3$, where we are using the Sweedler notation. In particular, $\beta(X_{ij}) = \sum_{l,m} X_{lm} \otimes S(X_{il})X_{mj}$ for all i, j. However, as with the interior coaction studied in Section 2, the map β is not an algebra homomorphism.

Recall that if I and J are subsets of $\{1, \ldots, n\}$ of the same size, then the quantum determinant of the quantum matrix subalgebra generated by the X_{ij} with $i \in I$ and $j \in J$ is denoted by $[I|J]$ and called a *quantum minor* of the relevant quantum matrix algebra. For each $i = 1, \ldots, n$, define the weighted sums of principal quantum minors by $\tau_i := \sum_I q^{-2w(I)}[I|I]$, where I runs through all i element subsets of $\{1, \ldots, n\}$ and $w(I)$ is the sum of the entries in the index set I. These weighted sums of principal quantum minors provide quantized coinvariants for β by [**3**], Proposition 7.2. One can state First and Second Fundamental Theorems for quantized coinvariants in this situation as follows:

(1_q) The set of coinvariants $\mathcal{O}_q(M_n(k))^{\mathrm{co}\,\mathcal{O}_q(GL_n(k))}$ is equal to the subalgebra of $\mathcal{O}_q(M_n(k))$ generated by $\tau_1, \ldots, \tau_n$.

(2_q) The subalgebra $k[\tau_1, \ldots, \tau_n]$ is a commutative polynomial algebra of degree n.

These statements were proved for $k = \mathbb{C}$ and q not a root of unity in [**3**], Theorem 7.3, by using the corepresentation theory of the cosemisimple Hopf algebra $\mathcal{O}_q(GL_n(\mathbb{C}))$. We give a partial extension below.

Now let $k^\circ = \mathbb{Q}$. Suppose that $q \in k^\times$ is transcendental over $\mathbb{Q}$, set $R = \mathbb{Q}[q, q^{-1}]$ and define everything over R. Then $\mathcal{O}_q(M_n(R))$ is a right comodule over $\mathcal{O}_q(GL_n(R))$ by using β as above. A straightforward calculation shows that the R-subalgebra generated by $\tau_1, \ldots, \tau_n$ is contained in $\mathcal{O}_q(M_n(R))^{\mathrm{co}\,\mathcal{O}_q(GL_n(R))}$; see, for example, [**3**], Proposition 7.2. Also, note that when $q = 1$ the coinvariants $\tau_1, \ldots, \tau_n$ coincide with the classical traces $\mathrm{tr}_1, \ldots, \mathrm{tr}_n$. Let $F(R)$ denote the free R-algebra on generators $\gamma_1, \ldots, \gamma_n$, and consider $F(R)$ to be graded by setting $\deg(\gamma_i) = i$. Let ϕ_q be the R-algebra homomorphism sending γ_i to τ_i. Then ϕ_q is homogeneous of degree 0 and the image of ϕ_q is contained in $\mathcal{O}_q(M_n(R))^{\mathrm{co}\,\mathcal{O}_q(GL_n(R))}$. The homogeneous components of $\mathcal{O}_q(M_n(R))$ and $F(R)$ are finitely generated free R-modules with $\phi_q(F(R)_j) \subseteq \mathcal{O}_q(M_n(R))_j$ and $\beta(\mathcal{O}_q(M_n(R))_j) \subseteq \mathcal{O}_q(M_n(R))_j \otimes \mathcal{O}_q(GL_n(R))$. The classical First Fundamental Theorem (1) says that $\overline{\phi_q}$ maps $\overline{F(R)}$ onto the coinvariants of $\overline{\mathcal{O}_q(M_n(R))}$. Theorem 1.2 now yields the quantized First Fundamental Theorem (1_q) for the case that char(k) = 0 and q is transcendental over $\mathbb{Q}$.

In [**1**], Corollary 2.3, Cohen and Westreich show that the τ_i commute, by exploiting the coquasitriangular structure of $\mathcal{O}_q(GL_n(k))$ (see, e.g., [**10**], Theorem 3.1 and Proposition 4.1; a more detailed proof for the case $k = \mathbb{C}$ is given in [**12**], Theorem 10.9). Thus, if we set I to be the ideal of $F(R)$ generated by the commutators $\gamma_i\gamma_j - \gamma_j\gamma_i$, we see that $I \subseteq \ker(\phi_q)$. It is obvious that I is a homogeneous ideal. The classical theory shows that $\ker(\overline{\phi_q}) = \overline{I}$. Thus, Theorem 1.3 yields the Second Fundamental Theorem (2_q) in the case under discussion.

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