Mathematics 4 Topology **Semester 2, 2009/2010**

Tutorial Sheet 4 (with solutions)

Problems 1,2,3 to be handed in at the lecture on Monday, 8 February, 2010

1. (Handin) Prove that a contractible space X is path-connected. [5 Marks]

Solution By definition, X is contractible if it is homotopy equivalent to a space $Y = \{y_0\}$ with one point. Let $f : X \to Y$, $g : Y \to X$ be inverse homotopy equivalences, so that there exists a homotopy $h : gf \simeq 1 : X \to X$. Let $g(y_0) = x_0 \in X$. By definition, $h : X \times I \to Y$ is a map such that

$$h(x,0) \;=\; gf(x) \;=\; g(y_0) \;=\; x_0 \;,\; h(x,1) \;=\; x \in X \quad (x \in X) \;.$$

For any $x \in X$ the map defined by

$$\omega_x \;:\; I \to X \;;\; t \mapsto h(x,t)$$

is a path from $\omega_x(0) = h(x,0) = x_0$ to $\omega_x(1) = h(x,1) = x$. So there is a path between any two points of X, and X is path connected.

2. (Handin) (i) Prove that every map $e : X \to \mathbb{R}^n$ is homotopic to a constant map. [2 Marks]
(ii) If $f : X \to S^n$ is a map that is not onto, show that f is homotopic to a constant map. [3 Marks]

Solution (i) Let $e_0 : X \to \mathbb{R}^n$ be the constant map defined by $e_0(x) = 0 \in \mathbb{R}^n$ for all $x \in X$. The map

$$d \;:\; X \times I \to \mathbb{R}^n \;;\; (x,t) \mapsto (1-t)e(x)$$

defines a homotopy $d : e \simeq e_0$.
(ii) Let $y \in S^n$ be such that $y \notin f(X)$, and define the maps

$$\begin{aligned} g \;&:\; X \to S^n \backslash \{y\} \;;\; x \mapsto f(x) \;, \\ i \;&=\; \text{inclusion} \;:\; S^n \backslash \{y\} \to S^n \end{aligned}$$

such that $f = ig$. Stereographic projection defines a homeomorphism $h : S^n \backslash \{x\} \to \mathbb{R}^n$. By (i) the composite $e = hg : X \to \mathbb{R}^n$ is homotopic to a constant map $e_0 : X \to \mathbb{R}^n$, with a homotopy $d : e \simeq e_0 : X \to \mathbb{R}^n$. The composite

$$ih^{-1}d \;:\; X \times I \xrightarrow{d} \mathbb{R}^n \xrightarrow{h^{-1}} S^n \backslash \{y\} \xrightarrow{i} S^n$$

is a homotopy from $ih^{-1}e = ig = f : X \to S^n$ to a constant map $ih^{-1}e_0 : X \to S^n$.

3. (Handin) Define the *mapping torus* of a homeomorphism $\phi : X \to X$ to be the identification space

$$T(\phi) \;=\; X \times I/\{(x,0) \sim (\phi(x),1)\,|\,x \in X\}\ .$$

Identify $T(\phi)$ with a standard space and prove that it is homotopy equivalent to S^1 by constructing explicit maps $f : S^1 \to T(\phi)$, $g : T(\phi) \to S^1$ and explicit homotopies $gf \simeq 1 : S^1 \to S^1$, $fg \simeq 1 : T(\phi) \to T(\phi)$, in the following two cases:

(i) $\phi(x) = x$ for $x \in X = I$. [2 Marks]

(ii) $\phi(x) = 1 - x$ for $x \in X = I$. [3 Marks]

Solution. (i) $T(\phi) = I \times S^1$ is a cylinder. The maps

$$\begin{aligned} f \ &:\ S^1 \to T(\phi)\ ;\ [t] \mapsto [1/2, t]\ ,\\ g \ &:\ T(\phi) \to S^1\ ;\ [x,y] \mapsto [y] \end{aligned}$$

are such that $gf = 1 : S^1 \to S^1$ (so $gf \simeq 1$ by the constant homotopy) with

$$fg \ :\ T(\phi) \to T(\phi)\ ;\ [x,y] \mapsto [1/2, y]\ .$$

The map

$$h \ :\ I \times I \times I \to I \times I\ ;\ (x,y,t) \mapsto (tx + (1-t)/2, y)$$

sends the relation $(x,0,t) \sim (x,1,t)$ on $I \times I \times I$ to the relation $(s,0) \sim (s,1)$ on $I \times I$, with $s = tx + (1-t)/2$. Passing to the identification spaces there is defined a map

$$\begin{aligned} h \ :\ (I \times I \times I)/\!\sim \ &=\ T(\phi) \times I \to (I \times I)/\!\sim \ =\ T(\phi)\ ;\\ &([x,y],t) \mapsto [tx + (1-t)/2, y] \end{aligned}$$

which is a homotopy $h : fg \simeq 1 : T(\phi) \to T(\phi)$.
(ii) $T(\phi)$ is a Möbius band. The maps

$$\begin{aligned} f \ &:\ S^1 \to T(\phi)\ ;\ [t] \mapsto [1/2, t]\ ,\\ g \ &:\ T(\phi) \to S^1\ ;\ [x,y] \mapsto [x] \end{aligned}$$

are such that $gf = 1 : S^1 \to S^1$ (so $gf \simeq 1$ by the constant homotopy) with

$$fg \ : \ T(\phi) \to T(\phi) \ ; \ [x,y] \mapsto [1/2, y] \ .$$

The map

$$h \ : \ I \times I \times I \to I \times I \ ; \ (x,y,t) \mapsto (tx + (1-t)/2, y)$$

sends the relation $(x,0,t) \sim (1-x,1,t)$ on $I \times I \times I$ to the relation $(s,0) \sim (1-s,1)$ on $I \times I$, with $s = tx + (1-t)/2$. Passing to the identification spaces there is defined a map

$$\begin{aligned} h \ : \ (I \times I \times I)/{\sim} \ &= \ T(\phi) \times I \to (I \times I)/{\sim} \ = \ T(\phi) \ ; \\ &([x,y],t) \mapsto [tx + (1-t)/2, y] \end{aligned}$$

which is a homotopy $h : fg \simeq 1 : T(\phi) \to T(\phi)$.

4. Let Δ be the triangle in $\mathbb{R}^2$ with vertices $a = (0,0)$, $b = (1,1)$, $c = (2,0)$, and let $f : \Delta \to X$ a continuous map with $f(a) = f(b) = f(c) \in X$. The closed paths defined by the restrictions of f to the line segments $[a,b]$, $[b,c]$, $[a,c]$ are respectively denoted $\alpha, \beta, \gamma : I \to X$. Construct a homotopy

$$h : \gamma \ \simeq \ \alpha \bullet \beta \ : \ I \to X \text{ rel } \{0,1\} \ .$$

Solution From the description given

$$\alpha(t) \ = \ f(t,t) \ , \ \beta(t) \ = \ f(t+1, 1-t) \ , \ \gamma(t) \ = \ f(2t, 0) \in X \quad (t \in I) \ .$$

For each $t \in I$ define a path in Δ by joining $(0,0)$ to $(1,t)$ and then $(1,t)$ to $(2,0)$ by straight lines

$$\gamma_t \ : \ I \to \Delta \ ; \ s \mapsto \begin{cases} (2s, 2st) & \text{if } 0 \leqslant s \leqslant 1/2 \\ (2s, (2-2s)t) & \text{if } 1/2 \leqslant s \leqslant 1 \ . \end{cases}$$

The map

$$h \ : \ I \times I \to X \ ; \ (s,t) \mapsto f(\gamma_t(s)) \ = \ \begin{cases} f(2s, 2st) & \text{if } 0 \leqslant s \leqslant 1/2 \\ f(2s, (2-2s)t) & \text{if } 1/2 \leqslant s \leqslant 1 \end{cases}$$

defines a homotopy from

$$\begin{aligned} h_0 \ : \ I \to X \ ; \ s \mapsto h(s,0) \ &= \ \begin{cases} f(2s,0) & \text{if } 0 \leqslant s \leqslant 1/2 \\ f(2s,0) & \text{if } 1/2 \leqslant s \leqslant 1 \end{cases} \\ &= \ \gamma(s) \end{aligned}$$

to

$$\begin{aligned} h_1 \ : \ I \to X \ ; \ s \mapsto h(s,1) \ &= \ \begin{cases} f(2s,2s) = \alpha(s) & \text{if } 0 \leqslant s \leqslant 1/2 \\ f(2s,2-2s) = \beta(2s-1) & \text{if } 1/2 \leqslant s \leqslant 1 \end{cases} \\ &= \ (\alpha \bullet \beta)(s) \ . \end{aligned}$$

Note that

$$h(0,t) \ = \ f(0,0) \ = \ f(a) \ , \ h(1,t) \ = \ f(2,0) \ = \ f(c) \in X \ (t \in I)$$

so h is a homotopy rel $\{0,1\}$.

5. (i) Prove that any two maps $f, g : \mathbb{R} \to X$ are homotopic, for any path-connected space X.
(ii) Prove that for a contractible space X any two maps $f, g : W \to X$ are homotopic.
Solution (i) Given maps $f, g : \mathbb{R} \to X$ let $\omega : I \to X$ be a path from $\omega(0) = f(0)$ to $\omega(1) = g(0)$, and define a homotopy $h : f \simeq g : \mathbb{R} \to X$ by

$$h(s,t) \ = \ \begin{cases} f(s(1-3t)) & \text{if } 0 \leqslant t \leqslant 1/3 \\ \omega(3t-1) & \text{if } 1/3 \leqslant t \leqslant 2/3 \\ g(s(3t-2)) & \text{if } 2/3 \leqslant t \leqslant 1. \end{cases}$$

(ii) Let $i : \{0\} \to X$, $j : X \to \{0\}$ be inverse homotopy equivalences, so that there exists a homotopy $h : ij \simeq 1 : X \to X$. The maps

$$\begin{aligned} h(f \times 1_I) \ &: \ W \times I \to X \ ; \ (w,t) \mapsto h(f(w),t) \\ h(g \times 1_I) \ &: \ W \times I \to X \ ; \ (w,t) \mapsto h(g(w),t) \end{aligned}$$

define homotopies

$$h(f \times 1_I) \ : \ ijf \simeq f \ : \ W \to X \ , \ h(g \times 1_I) \ : \ ijg \simeq g \ : \ W \to X$$

with

$$ijf \ = \ ijg \ : \ W \to X \ ; \ w \mapsto i(0) \ .$$

The concatenation

$$H \ = \ -h(f \times 1_I) \bullet h(g \times 1_I) \ : \ W \times I \to X$$

defines a homotopy $H : f \simeq g : W \to X$.

6. Construct examples of homotopy equivalent path-connected spaces X, Y such that $X \backslash \{x\}$ is not homotopy equivalent to $Y \backslash \{y\}$ for some $x \in X, y \in Y$.

Solution For any $x \in X = \mathbb{R}$, $y \in Y = \mathbb{R}^n$ $(n \geqslant 2)$ the complement $X \backslash \{x\}$ is disconnected and the complement $Y \backslash \{y\}$ is connected, so that $X \backslash \{x\}$ is not homotopy equivalent to $Y \backslash \{y\}$.

7. Consider the letters of the alphabet

A B C D E F G H I J K L M N O P Q R S T U V W X Y Z

as topological spaces by regarding them as subspaces of $\mathbb{R}^2$.
(i) Classify the alphabet up to homeomorphism, i.e. group together the letters which are homeomorphism.
(ii) Classify the alphabet up to homotopy equivalence, i.e. group together the letters which are homotopy equivalent. (You may use the result that S^1 is neither homotopy equivalent to a point nor to the figure eight).
Detailed proofs are not expected!

Solution (i) The letter groups

$$\{\text{A,R}\} \ , \ \{\text{B}\} \ , \ \{\text{C,G,I,J,L,M,N,S,U,V,W,Z}\} \ ,$$
$$\{\text{D,O}\} \ , \ \{\text{E,F,T,Y}\} \ , \ \{\text{H,K}\} \ , \ \{\text{P}\} \ , \ \{\text{Q}\} \ , \ \{\text{X}\} \ .$$

are grouped according to homeomorphism class.
(ii) The letters

A D O P Q R

are homotopy equivalent to S^1. The letters

C E F G H I J K L M N S T U V W X Y Z

are contractible. The letter B is neither contractible, nor homotopy equivalent to S^1.

8. Show that the torus with a point removed is homotopy equivalent to a figure eight. (Hint: regard the torus $T^2 = S^1 \times S^1$ as the identification space $T^2 = A/\sim$ of the square $A = [-1,1] \times [-1,1]$ with $(x,-1) \sim (x,1)$ and $(-1,y) \sim (1,y)$, and remove $(0,0) \in A$.)

Solution The boundary of the square

$$B = \partial A = \{-1,1\} \times [-1,1] \cup \{-1,1\} \times [-1,1]$$

is such that $B/\sim$ is a figure eight. Let

$$g : T^2 \backslash \{(0,0)\} = C/\sim \to B/\sim$$

be the map which sends $(x,y) \in C$ to the unique point $g(x,y) \in B$ where the half-line $\{(\lambda x, \lambda y) \mid \lambda > 0\}$ meets B. (Draw a picture!). Let $C = A \backslash \{(0,0)\}$, and let

$$g : B/\sim \to T^2 \backslash \{(0,0)\} = C/\sim$$

be the inclusion. Then $fg = 1$ and the map

$$h : C/\sim \times I \to C/\sim \ ; \ (x,y,t) \mapsto t(x,y) + (1-t)g(x,y)$$

defines a homotopy $h : gf \simeq 1$.

9. Show that the group G_+ of upper triangular, non-singular, real $n \times n$ matrices (with positive entries on the diagonal) is homeomorphic to a Euclidean space and so is contractible. Let G be the (larger) group of all non-singular upper triangular matrices; determine the number of path components of G and hence describe a more familiar space to which it is homeomorphic.
(In the first instance, you may consider the special case $n = 2$.)

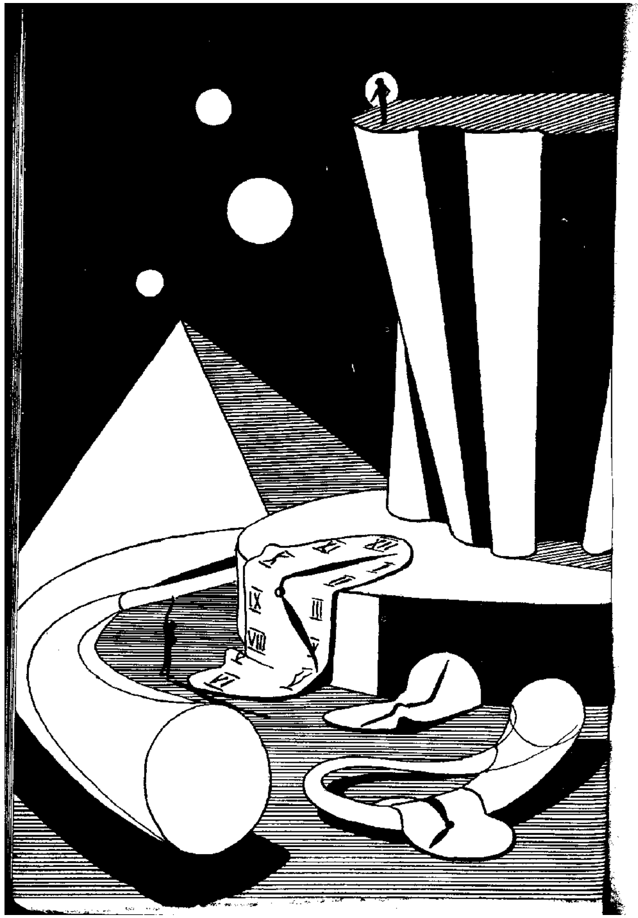
Solution We use the fact that $(0,\infty)$ is homeomorphic to $\mathbb{R}$. By counting the entries in a matrix, $G_+ \cong \mathbb{R}^N \times (0,\infty)^n$ where $N = 1+2+....+(n-1) = \binom{n}{2}$ is the total number of entries above the diagonal. So G_+ is homeomorphic to $\mathbb{R}^{\binom{n}{2}} \times \mathbb{R}^n \cong \mathbb{R}^{\binom{n+1}{2}}$. Any space homomorphic to a Euclidean space is contractible.

Each diagonal entry of an element of G is either positive or negative and we cannot change this sign in a path within G because the path would have to pass through an upper triangular matrix with 0 on the diagonal (which is a singular matrix). So the number of path components is 2^n (there are two independent possibilities for each diagonal entry of the matrix). Hence G is homeomorphic to $X_{2^n} \times \mathbb{R}^{\binom{n+1}{2}}$ where X_{2^n} is a discrete space with 2^n points.

10. The *mapping cylinder* M_f of a map $f : X \to Y$ is the identification space

$$M_f \;=\; X \times [0,1] \cup Y / \{(x,1) \sim f(x) \,|\, x \in X\} \ .$$

Prove the inclusion $i : Y \to M_f$ is a homotopy equivalence. On the next page is an illustration of the mapping cylinder with a geometric solution of the problem, a parody of *The persistence of memory* by Salvador Dali, from the original Russian edition of 'Homotopic Topology' by Fomenko et. al., JCMB Library 51.55 FOM.

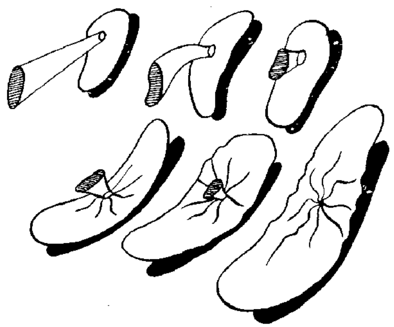


В самом деле $f: X_1 \to X_2$ – вложение, $g = f^{-1} \circ h : X_2 \to X_1$ (h – сжатие по радиусам),

Очевидно, что гомеоморфные пространства гомотопически эквивалентны. Из примера видно, что обратное неверно.

Задача. Доказать, что стягиваемое пространство гомотопически эквивалентно точке.

Задача. Покажите, что цилиндр отображения $f: X \to Y$ гомотопически эквивалентен пространству Y.



Задача. Постройте два пространства X_1, X_2, так, чтобы существовало взаимно однозначное непрерывное отображение $f: X_1 \to X_2$ и взаимно однозначное непрерывное отображение $g: X_2 \to X_1$, а пространства не были бы

Solution The map defined by

$$j \ : \ M_f \to Y \ ; \ \begin{cases} (x,t) \mapsto f(x) \\ y \mapsto y \end{cases}$$

is such that $ji = \text{identity} : Y \to Y$. Define a homotopy

$$h \ : \ ij \ \simeq \ \text{identity} \ : \ M_f \to M_f$$

by

$$h \ : \ M_f \times [0,1] \to M_f \ ; \ \begin{cases} ((x,s),t) \mapsto (x, st+1-t) \\ (y,t) \mapsto y \ . \end{cases}$$