SMSTC Geometry and Topology 2012-2013

Lecture 7

The fundamental group and covering spaces

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The method of algebraic topology

- Algebraic topology uses algebra to classify topological spaces.
- A functor on topological spaces is a function

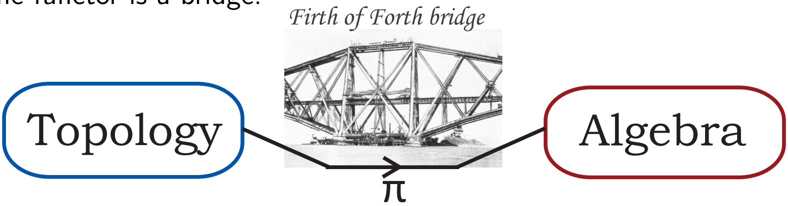
$$\pi \ : \ \{\text{topological spaces}\} \to \{\text{sets}\}$$

which sends a topological space X to a set $\pi(X)$, and a continuous function $f : X \to Y$ to a function $f_* : \pi(X) \to \pi(Y)$, satisfying the relations

$$(1 : X \to X)_* \ = \ 1 \ : \ \pi(X) \to \pi(X) \ ,$$

$$(gf)_* \ = \ g_* f_* \ : \pi(X) \to \pi(Z) \text{ for } f : X \to Y,\ g : Y \to Z \ .$$

- The functor is a bridge:



What is a functor on topological spaces good for?

- **Consequence 1** If $f : X \to Y$ is a homeomorphism of spaces then $f_* : \pi(X) \to \pi(Y)$ is a bijection.
- **Consequence 2** If X, Y are such that there does not exist a bijection $\pi(X) \cong \pi(Y)$ then X, Y are not homeomorphic.
- There are also functors

$$\pi \ : \ \{\text{topological spaces}\} \to \{\text{abelian groups}\}\ ,$$

$$\pi \ : \ \{\text{topological spaces}\} \to \{\text{groups}\}$$

  with similar properties, requiring f_* to be group morphisms.
- A functor π is homotopy invariant if

$$(f_0)_* \ = \ (f_1)_* \ : \ \pi(X) \to \pi(Y)$$

  for homotopic $f_0, f_1 : X \to Y$. If $f : X \to Y$ is a homotopy equivalence then $f_* : \pi(X) \to \pi(Y)$ is a bijection or an isomorphism.
- It is much easier to decide if f_* is a bijection than to decide if f is a homeomorphism or a homotopy equivalence.

Our functors $\pi_0(X)$, $H_*(X)$ and $\pi_1(X)$

- The set of path components (Lecture 2) is a homotopy invariant functor

$$\pi_0 \ : \ \{\text{topological spaces}\} \to \{\text{sets}\} \ ; \ X \mapsto \pi_0(X) \ .$$

- The homology groups (Lectures 4/5/6) are homotopy invariant functors

$$H_n \ : \ \{\text{topological spaces}\} \to \{\text{abelian groups}\} \ ; \ X \mapsto H_n(X)$$

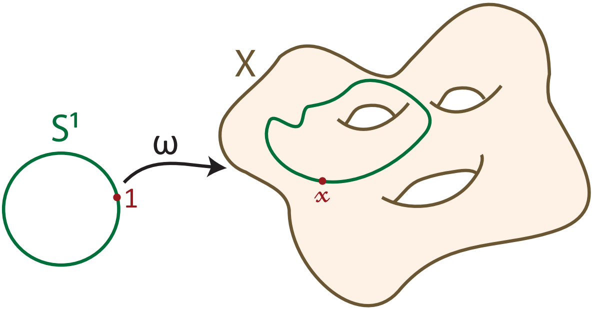
  for $n = 0, 1, 2, \ldots$
- The fundamental group is a homotopy invariant functor

$$\pi_1 \ : \ \{\text{topological spaces}\} \to \{\text{groups}\} \ ; \ X \mapsto \pi_1(X) \ .$$

  Strictly speaking, $\pi_1(X)$ is defined for a space X with a choice of base point $x \in X$.
- Can also define higher homotopy groups $\pi_n(X)$, with morphisms $\pi_n(X) \to H_n(X)$ for $n \geqslant 0$. Abelian for $n \geqslant 2$.
- We shall concentrate on $\pi_1(X)$.

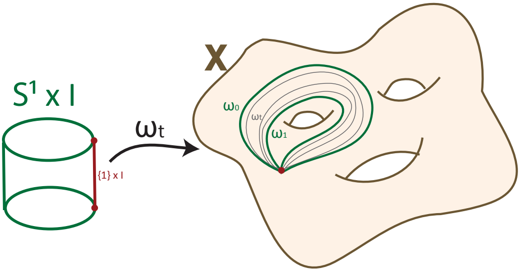
Loops

- Let X be a topological space.
- Fix a point $x \in X$.
- A loop in X at $x \in X$ is a continuous map $\omega : S^1 \to X$ such that $\omega(1) = x \in X$.



The unofficial definition of the fundamental group $\pi_1(X, x)$. Part I.

- The fundamental group of a space X at a point $x \in X$ is the geometrically defined group of homotopy classes $[\omega]$ of loops $\omega : S^1 \to X$ which are tethered at x, that is $\omega(1) = x \in X$.
- The homotopies are also to be tethered at $x \in X$.



The unofficial definition of the fundamental group $\pi_1(X, x)$. Part II.

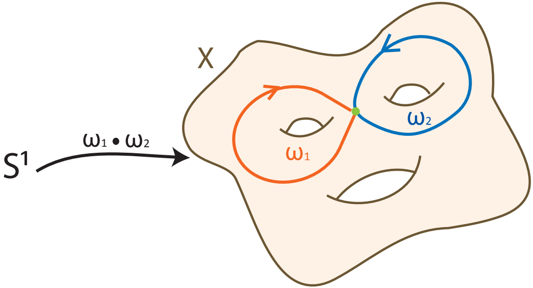
- The identity element $1 = [e_x] \in \pi_1(X, x)$ is the homotopy class of the constant loop

$$e_x \ : \ S^1 \to X \ ; \ s \mapsto x \ .$$

- Group law

$$\pi_1(X, x) \times \pi_1(X, x) \to \pi_1(X, x) \ ; \ ([\omega_1], [\omega_2]) \mapsto \omega_1 \bullet \omega_2$$

defined by the 'concatenation' of loops.



Some properties of the fundamental group $\pi_1(X, x)$

- $\pi_1(\{x\}, x) = \{1\}$, i.e. the fundamental group of a one-point space is the one-element group.
- A continuous map $f : X \to Y$ induces a morphism of groups

$$f_* \ : \ \pi_1(X, x) \to \pi_1(Y, f(x)) \ ; \ [\omega] \mapsto [f\omega] \ .$$

  This is an isomorphism if f is a homotopy equivalence.
- **Consequence** $\pi_1(X) = \{1\}$ for contractible X.
- Later in lecture will show that a path $\alpha : I \to X$ induces an isomorphism

$$\alpha_\# \ : \ \pi_1(X, \alpha(0)) \to \pi_1(X, \alpha(1)) \ .$$

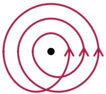
  So for path-connected space X can write

$$\pi_1(X) \equiv \pi_1(X, x) \text{ for any } x \in X \ .$$

- Every group G is the fundamental group $G = \pi_1(X)$ of a path-connected space X. (Hatcher, p.89)

The fundamental group of the circle, $\pi_1(S^1) = \mathbb{Z}$

- View S^1 as the unit circle in the complex plane
$$S^1 \ = \ \{z \in \mathbb{C} \,|\, |z| = 1\} \ .$$
- For any $d \in \mathbb{Z}$ use complex multiplication to define the standard loop of degree d
$$\omega_d \ : \ S^1 \to S^1 \ ; \ z \mapsto z^d$$
winding round the circle d times.
- **Main Theorem** The function
$$\mathbb{Z} \to \pi_1(S^1) \ ; \ d \mapsto [\omega_d]$$
is an isomorphism of groups.
- Idea of proof: define an inverse $\pi_1(S^1) \to \mathbb{Z}$ by counting the number of times a loop $\omega : S^1 \to S^1$ goes around S^1.
- **Example** ω_3



The fundamental group of the torus, $\pi_1(S^1 \times S^1) = \mathbb{Z} \oplus \mathbb{Z}$

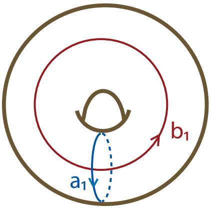
- The fundamental group of the torus is the free abelian group on 2 generators a_1, b_1

$$\pi_1(S^1 \times S^1) \ = \ \mathbb{Z} \oplus \mathbb{Z} \ = \ \mathbb{Z}^2$$

with $(c, d) \in \mathbb{Z} \oplus \mathbb{Z}$ the homotopy class of the loop

$$\omega_{c,d} \ : \ S^1 \to S^1 \times S^1 \ ; \ z \mapsto (z^c, z^d) \ .$$

- The generators $(1, 0)$,$(0, 1) \in \pi_1(S^1 \times S^1)$ are represented by the meridian and longitude loops $a_1, b_1 : S^1 \to S^1 \times S^1$

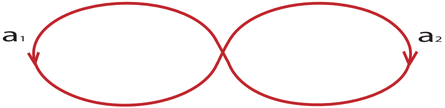


The fundamental group of the figure eight, $\pi_1(S^1 \vee S^1)$

- The fundamental group of the figure 8 is the free nonabelian group on 2 generators a_1, a_2

$$\pi_1(S^1 \vee S^1) \ = \ F_2 \ = \ \{a_1, a_2\}$$

- The evident inclusions $a_1, a_2 : S^1 \to S^1 \vee S^1$ are loops representing the two generators

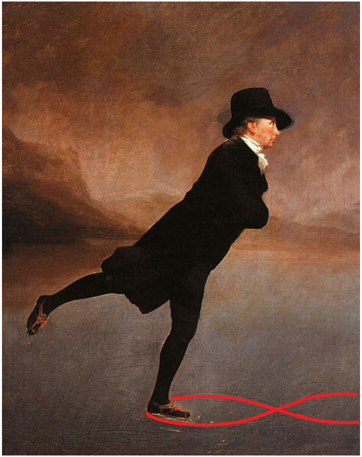


- The homotopy class of the commutator

$$[a_1, a_2] \ = \ a_1 a_2 (a_1)^{-1} (a_2)^{-1} \in \pi_1(S^1 \vee S^1)$$

is represented by the loop which goes round a_1 counterclockwise, then round a_2 clockwise, then round a_1 clockwise, and finally round a_2 counterclockwise.

The loop traced out by a figure eight skater
$\in \pi_1(S^1 \vee S^1) = F_2$



The Reverend Robert Walker Skating on Duddingston Loch (Raeburn)

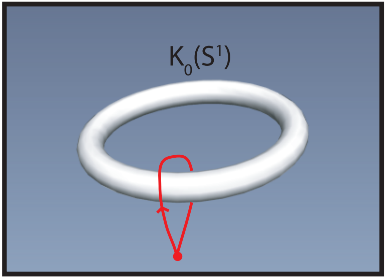
The knot group

- If $K : S^1 \subset S^3$ is a knot the fundamental group of the complement
$$X_K \ = \ S^3 \backslash K(S^1) \subset S^3$$
is a topological invariant of the knot.
- **Definition** Two knots $K, K' : S^1 \subset S^3$ are equivalent if there exists a homeomorphism $h : S^3 \to S^3$ such that $K' = hK$.
- Equivalent knots have isomorphic groups, since
$$(h|)_* \ : \ \pi_1(X_K) \to \pi_1(X_{K'})$$
is an isomorphism of groups.
- So knots with non-isomorphic groups cannot be equivalent!

The unknot

- The unknot $K_0 : S^1 \subset S^3$ has complement $X_{K_0} = S^1 \times \mathbb{R}^2$, with group

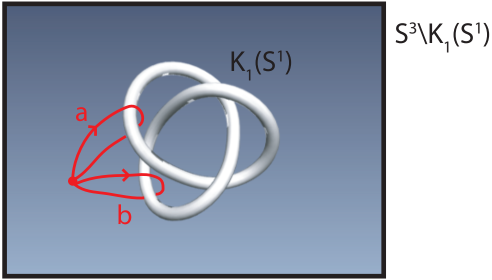
$$\pi_1(X_{K_0}) \ = \ \mathbb{Z} \ .$$



The trefoil knot

- The trefoil knot $K_1 : S^1 \subset S^3$ has group

$$\pi_1(X_{K_1}) = \{a, b \mid aba = bab\} .$$



- **Conclusion** The groups of the knots K_0, K_1 are not isomorphic: $\pi_1(X_{K_0})$ is abelian while $\pi_1(X_{K_1})$ is not abelian. It follows that the knots K_0, K_1 are not equivalent: the algebra shows that the trefoil knot cannot be unknotted.
- $H_1(X_{K_0}) = H_1(X_{K_1}) = \mathbb{Z}$, so have to use π_1 rather than H_1.

The Hurewicz map $\pi_1(X) \to H_1(X)$

- The Hurewicz map is defined by

$$\pi_1(X) \to H_1(X) \; ; \; [\omega : S^1 \to X] \mapsto \omega_*(1)$$

  with $\omega_* : H_1(S^1) \to H_1(X)$ the morphism induced in H_1, and $1 \in H_1(S^1) = \mathbb{Z}$ the generator.
- The Hurewicz theorem states that $\pi_1(X) \to H_1(X)$ is surjective with kernel the normal subgroup generated by the commutators

$$[a, b] \;=\; aba^{-1}b^{-1} \in \pi_1(X) \; (a, b \in \pi_1(X)) \; .$$

- In general, $\pi_1(X)$ is not abelian. If $\pi_1(X)$ is abelian then it is isomorphic to $H_1(X)$.
- Reference: Theorem 4.32 of Hatcher.

The fundamental group $\pi_1(X)$ and the homology group $H_1(X)$ for some spaces X

X	$\pi_1(X)$	$H_1(X)$
$\mathbb{R}^n$	0	0
D^n	0	0
S^n $(n \geqslant 2)$	0	0
$\mathbb{CP}^n$ $(n \geqslant 1)$	0	0
S^1	$\{a\} = \mathbb{Z}$	$\mathbb{Z}$
$S^1 \vee S^1$	$\{a_1, a_2\} = F_2$	$\mathbb{Z}^2$
$\bigvee_g S^1$	$\{a_1, a_2, \ldots, a_g\} = F_g$	$\mathbb{Z}^g$
$M(1) = S^1 \times S^1$	$\{a, b \mid [a, b]\} = \mathbb{Z}^2$	$\mathbb{Z}^2$
$M(g) = \Sigma_g$	$\{a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \ldots [a_g, b_g]\}$	$\mathbb{Z}^{2g}$
$\mathbb{RP}^n$ $(n \geqslant 2)$	$\{a \mid a^2\} = \mathbb{Z}_2$	$\mathbb{Z}_2$

Methods of computing $\pi_1(X)$

- By 'lassooing': if every loop $\omega : S^1 \to X$ can be extended to a continuous map $\delta\omega : D^2 \to X$ then $\pi_1(X) = \{1\}$.
- By covering space theory: for a 'universal covering projection' $p : \widetilde{X} \to X$ the fundamental group $\pi_1(X)$ is isomorphic to the group of homeomorphisms $h : \widetilde{X} \to \widetilde{X}$ such that $ph = p$.
- If X is a simplicial complex or a CW complex can compute $\pi_1(X)$ by an algorithm, keeping track of π_1 as X is built up. Need only go up to 2-dimensional simplices or cells. Adding n-dimensional ones for $n \geqslant 3$ does not change π_1.
- The inductive procedure requires the 'Seifert-van Kampen theorem' for the fundamental group of a union

$$\pi_1(X_1 \cup_Y X_2) \ = \ \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2) \ .$$

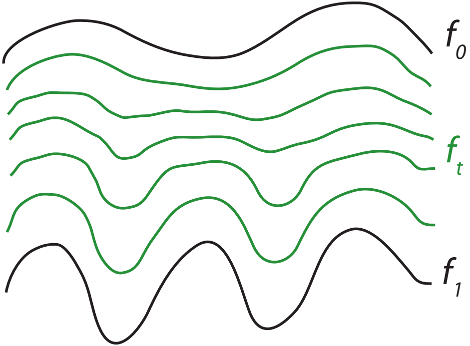
  This is the π_1-analogue of the Mayer-Vietoris exact sequence for $H_*(X_1 \cup_Y X_2)$ (Lecture 9).

Review of homotopy

- **Definition** A homotopy of continuous maps $f_0 : X \to Y$, $f_1 : X \to Y$ is a continuous map $h : X \times I \to Y$ such that for all $x \in X$

$$h(x, 0) \ = \ f_0(x) \ , \ h(x, 1) \ = \ f_1(x) \in Y \ .$$

Starts at f_0 and ends at f_1, like the first and last shot of a take in a film, with $h(x, t) = f_t(x)$. The world's most boring film:



Joined up thinking I.

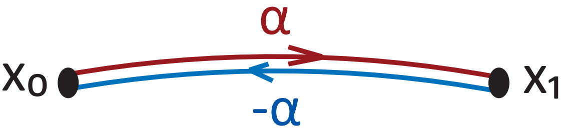
- **Proposition** The relation on X defined by $x_0 \sim x_1$ if there exists a path $\alpha : I \to X$ with $\alpha(0) = x_0$, $\alpha(1) = x_1$ is an equivalence relation.
- **Proof** (i) Every point $x \in X$ is related to itself by the constant path

$$e_x \ : I \to X \ ; \ t \mapsto x$$

which always stays at $x \in X$.
- (ii) The reverse of a path $\alpha : I \to X$ from $\alpha(0) = x_0$ to $\alpha(1) = x_1$ is the path

$$-\alpha \ : \ I \to X \ ; \ t \mapsto \alpha(1 - t)$$

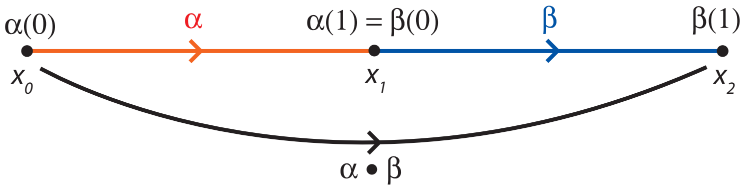
going backwards, from $-\alpha(0) = x_1$ to $-\alpha(1) = x_0 \in X$.



Joined up thinking II.

- (iii) The concatenation of a path $\alpha : I \to X$ from $\alpha(0) = x_0$ to $\alpha(1) = x_1$ and of a path $\beta : I \to X$ from $\beta(0) = x_1$ to $\beta(1) = x_2$ is the path from x_0 to x_2 given by

$$\alpha \bullet \beta \ : \ I \to X \ ; \ t \mapsto \begin{cases} \alpha(2t) & \text{if } 0 \leqslant t \leqslant 1/2 \\ \beta(2t-1) & \text{if } 1/2 \leqslant t \leqslant 1 \ . \end{cases}$$



- **Warning** The triple concatenations of paths $\alpha, \beta, \gamma : I \to X$ with $\alpha(1) = \beta(0)$, $\beta(1) = \gamma(0)$ are paths from $\alpha(0)$ to $\gamma(1)$

$$(\alpha \bullet \beta) \bullet \gamma \ , \ \alpha \bullet (\beta \bullet \gamma) \ : \ I \to X \ .$$

Not the same, but homotopic, keeping end points fixed.

Based spaces

- **Definition** A based space (X, x) is a space with a base point $x \in X$.
- **Example** For $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ choose the base point $1 \in S^1$.
- **Definition** A based continuous map $f : (X, x) \to (Y, y)$ is a continuous map $f : X \to Y$ such that $f(x) = y \in Y$.
- **Definition** A based homotopy $h : f \simeq g : (X, x) \to (Y, y)$ is a homotopy $h : f \simeq g : X \to Y$ such that

$$h(x, t) = y \in Y \quad (t \in I) \ .$$

- For any based spaces (X, x), (Y, y) based homotopy is an equivalence relation on the set of based continuous maps $f : (X, x) \to (Y, y)$.

Loops = closed paths

- A path $\alpha : I \to X$ is closed if $\alpha(0) = \alpha(1) \in X$.
- A based loop is a based continuous map $\omega : (S^1, 1) \to (X, x)$.
- In view of the homeomorphism

$$I/\{0 \sim 1\} \to S^1 \; ; \; [t] \mapsto e^{2\pi i t} = \cos 2\pi t + i \sin 2\pi t$$

there is essentially no difference between based loops $\omega : (S^1, 1) \to (X, x)$ and closed paths $\alpha : I \to X$ at $x \in X$, with

$$\alpha(t) \;=\; \omega(e^{2\pi i t}) \in X \quad (t \in I)$$

such that

$$\alpha(0) \;=\; \omega(1) \;=\; \alpha(1) \in X \; .$$

Homotopy relative to a subspace

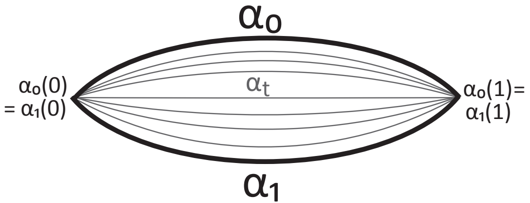
- Let X be a space, $A \subseteq X$ a subspace. If $f_0, f_1 : X \to Y$ are continuous maps such that $f_0(a) = f_1(a) \in Y$ for all $a \in A$ then a homotopy rel A (or relative to A) is a homotopy

$$X \times I \to Y \ ; \ (x,t) \mapsto f_t(x)$$

which is fixed on $A \subseteq X$, that is

$$f_0(a) \ = \ f_t(a) \ = \ f_1(a) \in Y \ \ (a \in A, t \in I) \ .$$

- A picture of a homotopy $\alpha_t : I \to X$ rel $\{0,1\}$ of paths $\alpha_0, \alpha_1 : I \to X$ with the same start point $\alpha_0(0) = \alpha_1(0) \in X$ and end point $\alpha_0(1) = \alpha_1(1) \in X$



Homotopy of paths

- **Exercise** If a space X is path-connected prove that any two paths $\alpha, \beta : I \to X$ are homotopic.
- **Exercise** (Hard) Let

  $$\alpha\ ,\ \beta\ :\ I \to X\ =\ \mathbb{C} - \{0\}$$

  be the paths defined by

  $$\alpha(t)\ =\ e^{\pi it}\ ,\ \beta(t)\ =\ e^{-\pi it}\ ,$$

  such that $\alpha(0) = \beta(0) = 1$, $\alpha(1) = \beta(1) = -1$. Prove that α, β are homotopic, but are not homotopic rel $\{0, 1\}$. Although hard to prove, it is easy to see why this is true!

The official definition of the fundamental group $\pi_1(X, x)$

- The fundamental group $\pi_1(X, x)$ is the set of based homotopy classes of loops $\omega : (S^1, 1) \to (X, x)$, or equivalently the rel $\{0, 1\}$ homotopy classes $[\alpha]$ of closed paths $\alpha : I \to X$ such that $\alpha(0) = \alpha(1) = x \in X$.
- The group law is by the concatenation of closed paths

$$\pi_1(X, x) \times \pi_1(X, x) \to \pi_1(X, x) \; ; \; ([\alpha], [\beta]) \mapsto [\alpha \bullet \beta]$$

- Inverses are by the reversal of paths

$$\pi_1(X, x) \to \pi_1(X, x) \; ; \; [\alpha] \mapsto [\alpha]^{-1} = [-\alpha] \; .$$

- The constant closed path e_x is the identity element, such that for any $[\alpha] \in \pi_1(X, x)$

$$[\alpha \bullet e_x] = [e_x \bullet \alpha] = [\alpha] \, , \; [\alpha \bullet -\alpha] = [-\alpha \bullet \alpha] = [e_x] \in \pi_1(X, x) \; .$$

- See Theorem 4.2.15 of the notes for a detailed proof that $\pi_1(X, x)$ is a group.

Homotopy equivalence

- **Definition** Two spaces X, Y are homotopy equivalent if there exist continuous maps $f : X \to Y$, $g : Y \to X$ and homotopies
$$h : gf \simeq 1_X : X \to X \ , \ k : fg \simeq 1_Y : Y \to Y \ .$$
- A continuous map $f : X \to Y$ is a homotopy equivalence if there exist such g, h, k. The continuous maps f, g are inverse homotopy equivalences.
- **Example** The inclusion $f : S^n \to \mathbb{R}^{n+1} \backslash \{0\}$ is a homotopy equivalence, with homotopy inverse
$$g : \mathbb{R}^{n+1} \backslash \{0\} \to S^n \ ; \ x \mapsto \frac{x}{\|x\|} \ .$$
- **Exercise** Prove that a homotopy equivalence $f : X \to Y$ induces a bijection $f_* : \pi_0(X) \to \pi_0(Y)$. Thus X is path-connected if and only if Y is path-connected.

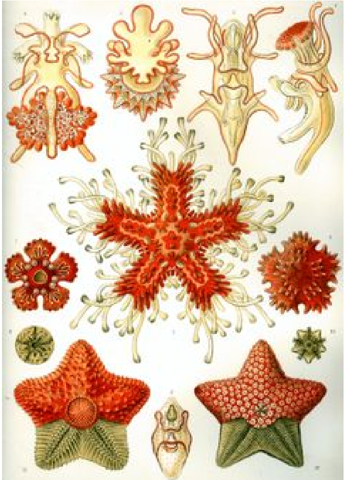
Contractible spaces

- A space X is contractible if it is homotopy equivalent to the space $\{\text{pt.}\}$ consisting of a single point.
- **Exercise** A subset $X \subseteq \mathbb{R}^n$ is star-shaped at $x \in X$ if for every $y \in X$ the line segment joining x to y

$$[x, y] \ = \ \{(1-t)x + ty \,|\, 0 \leqslant t \leqslant 1\}$$

  is contained in X. Prove that X is contractible.
- **Example** The n-dimensional Euclidean space $\mathbb{R}^n$ is contractible.
- **Example** The unit n-ball $D^n = \{x \in \mathbb{R}^n \,|\, \|x\| \leqslant 1\}$ is contractible.
- For any $n \geqslant 1$ the n-dimensional sphere S^n is not contractible: this follows from $H_n(S^n) = \mathbb{Z} \neq 0$.

Every starfish is contractible



"Asteroidea" from Ernst Haeckel's *Kunstformen der Natur* (1904)

Fundamental group morphisms

- **Proposition** A continuous map $f : X \to Y$ induces a group morphism

$$f_* \;:\; \pi_1(X,x) \to \pi_1(Y,f(x)) \;;\; [\omega] \mapsto [f\omega] \;.$$

with the following properties:

(i) The identity $1 : X \to X$ induces the identity, $1_* = 1 : \pi_1(X,x) \to \pi_1(X,x)$.

(ii) The composite of $f : X \to Y$ and $g : Y \to Z$ induces the composite, $(gf)_* = g_* f_* : \pi_1(X,x) \to \pi_1(Z,gf(x))$.

(iii) If $f,g : X \to Y$ are homotopic rel $\{x\}$ then $f_* = g_* : \pi_1(X,x) \to \pi_1(Y,f(x))$.

(iv) If $f : X \to Y$ is a homotopy equivalence then $f_* : \pi_1(X,x) \to \pi_1(Y,f(x))$ is an isomorphism.

(v) A path $\alpha : I \to X$ induces an isomorphism

$$\alpha_\# \;:\; \pi_1(X,\alpha(0)) \to \pi_1(X,\alpha(1)) \;;\; \omega \mapsto (-\alpha) \bullet \omega \bullet \alpha \;.$$

- Proofs in the notes, and also in Hatcher.
- (v) justifies $\pi_1(X) \equiv \pi_1(X,x)$ for path-connected X.

Simply-connected spaces I.

- **Definition** A space X is simply-connected if it is path-connected and $\pi_1(X) = \{1\}$. In words: every loop in X can be lassooed down to a point!
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Simply-connected spaces II.

- **Example** A contractible space is simply-connected.
- **Exercise** A space X is simply-connected if and only if for any points $x_0, x_1 \in X$ there is a unique rel $\{0,1\}$ homotopy class of paths $\alpha : I \to X$ from $\alpha(0) = x_0$ to $\alpha(1) = x_1$.
- **Exercise** If $n \geqslant 2$ then the n-sphere S^n is simply-connected: easy to prove if it can be assumed that every loop $\omega : S^1 \to S^n$ is homotopic to one which is not onto! (This is true, but hard to prove).
- **Remark** The circle S^1 is path-connected, but not simply-connected.

The fundamental group of the circle, $\pi_1(S^1) = \mathbb{Z}$

- **Theorem** Every loop $\omega : S^1 \to S^1$ is homotopic to exactly one of the standard loops

$$\omega_d \ : \ S^1 \to S^1 \ ; \ z \mapsto z^d \ (d \in \mathbb{Z}) \ .$$

  d is the degree of ω.
- This is the key step in the proof that the function

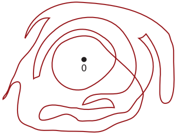
$$\mathbb{Z} \to \pi_1(S^1) \ ; \ d \mapsto [\omega_d]$$

  is an isomorphism of groups.
- Proved in lecture using covering $\mathbb{R} \to S^1$. Details on page 29 of Hatcher.
- How does one compute degree$(\omega) \in \mathbb{Z}$ for an arbitrary $\omega : S^1 \to S^1$?

The fundamental group of punctured plane, $\pi_1(\mathbb{C} - \{0\}) = \mathbb{Z}$

- The winding number of a loop $\sigma : S^1 \to \mathbb{C} - \{0\}$ is

$$W(\sigma) \ = \ \text{degree}(\omega) \in \mathbb{Z} \text{ with } \omega(z) = \sigma(z)/|\sigma(z)|$$



- The inclusion $S^1 \to \mathbb{C} - \{0\}$ is a homotopy equivalence, so isomorphism $\pi_1(S^1) \cong \pi_1(\mathbb{C} - \{0\})$; W defines isomorphism

$$W \ : \ \pi_1(\mathbb{C} - \{0\}) \to \mathbb{Z} \ ; \ [\sigma] \mapsto W(\sigma) \ .$$

- The winding number of an analytic loop $\sigma : S^1 \to \mathbb{C} - \{0\}$ can be computed by Cauchy's theorem

$$W(\sigma) \ = \ \frac{1}{2\pi i} \oint_\sigma \frac{dz}{z} \in \mathbb{Z}$$

The motivation for covering spaces

- Roughly speaking, a 'covering' of a space X is a surjective map $p:\widetilde{X}\to X$ such that the inverse images $p^{-1}(x)\subseteq\widetilde{X}$ are discrete and homeomorphic to each other, for all $x\in X$.
- A 'universal cover' with X path-connected and $\widetilde{X}$ simply-connected gives a geometric method for computing the fundamental group : $\pi_1(X)$ is isomorphic to the group of covering translations

$$\text{Homeo}_p(\widetilde{X}) \;=\; \{h:\widetilde{X}\to\widetilde{X}\ \text{homeomorphism}\,|\,ph=p:\widetilde{X}\to\widetilde{X}\}$$

with group law by composition

$$\text{Homeo}_p(\widetilde{X})\times\text{Homeo}_p(\widetilde{X})\to\text{Homeo}_p(\widetilde{X})\ ;(h_1,h_2)\mapsto h_1\circ h_2$$

and inverses by inverses.

The official definition of a covering space

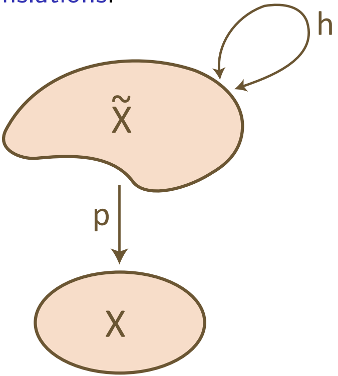
- **Definition** A covering space of a space X with fibre the discrete space F is a space $\widetilde{X}$ with a covering projection continuous map $p : \widetilde{X} \to X$ such that for each $x \in X$ there exists an open subset $U \subseteq X$ with $x \in U$, and with a homeomorphism $\phi : F \times U \to p^{-1}(U)$ such that

$$p\phi(a, u) \ = \ u \in U \subseteq X \ \ (a \in F, u \in U) \ .$$

- For each $x \in X$ $p^{-1}(x)$ is homeomorphic to F.
- The covering projection $p : \widetilde{X} \to X$ is a 'local homeomorphism': for each $\widetilde{x} \in \widetilde{X}$ there exists an open subset $U \subseteq \widetilde{X}$ such that $\widetilde{x} \in U$ and $U \to p(U); u \mapsto p(u)$ is a homeomorphism, with $p(U) \subseteq X$ an open subset.

The group of covering translations

- For any space X let $\mathrm{Homeo}(X)$ be the group of all homeomorphisms $h : X \to X$, with composition as group law.
- **Definition** Given a covering projection $p : \widetilde{X} \to X$ let $\mathrm{Homeo}_p(\widetilde{X})$ be the subgroup of $\mathrm{Homeo}(\widetilde{X})$ consisting of the homeomorphisms $h : \widetilde{X} \to \widetilde{X}$ such that $ph = p : \widetilde{X} \to X$, called covering translations.



The trivial covering

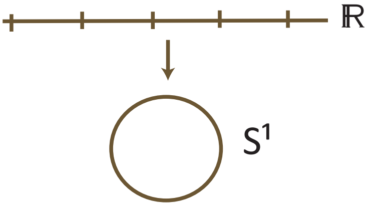
- **Definition** A covering projection $p : \widetilde{X} \to X$ with fibre F is trivial if there exists a homeomorphism $\phi : F \times X \to \widetilde{X}$ such that
$$p\phi(a,x) \ = \ x \in X \ \ (a \in F,\ x \in X)\ .$$
A particular choice of ϕ is a trivialisation of p.
- **Example** For any space X and discrete space F the covering projection
$$p \ : \ \widetilde{X} \ = \ F \times X \to X \ ; \ (a,x) \mapsto x$$
is trivial, with the identity trivialization $\phi = 1 : F \times X \to \widetilde{X}$. For path-connected X $\text{Homeo}_p(\widetilde{X})$ is isomorphic to the group of permutations of F, i.e. all the bijections $F \to F$.

The non-trivial covering $\mathbb{R} \to S^1$

- The projection

$$p \ : \ \mathbb{R} \to S^1 \ ; \ x \mapsto e^{2\pi i x}$$

is a covering.



- The fibre is $p^{-1}(1) = \mathbb{Z} \subset \mathbb{R}$, and the group of covering translations is

$$\mathrm{Homeo}_p(\mathbb{R}) \ = \ \{h^n \,|\, n \in \mathbb{Z}\}$$

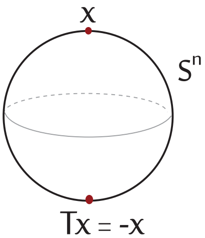
the infinite cyclic group generated by $h : \mathbb{R} \to \mathbb{R}; x \mapsto x + 1$.

The non-trivial covering $S^n \to \mathbb{RP}^n$

- Let $n \geqslant 1$. Recall that the *n*-dimensional real projective space is the quotient space of S^n by the antipodal map

$$T \ : \ S^n \to S^n \ ; \ x \mapsto -x$$

that is $\mathbb{RP}^n = S^n/\{x \sim -x\}$.

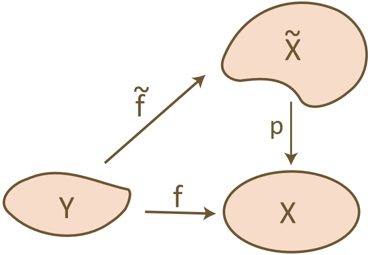


- The projection $p : S^n \to \mathbb{RP}^n$ is a non-trivial covering with fibre $F = \{1, 2\}$. The group of covering translations is

$$\text{Homeo}_p(S^n) \ = \ \{1, T\} \ = \ \mathbb{Z}_2 \ = \ \text{the cyclic group of two elements}\,.$$

Lifts

- **Definition** Let $p : \widetilde{X} \to X$ be a covering projection. A lift of a continuous map $f : Y \to X$ is a continuous map $\widetilde{f} : Y \to \widetilde{X}$ with $p(\widetilde{f}(y)) = f(y) \in X$ $(y \in Y)$.



- **Example** For the trivial covering projection $p : \widetilde{X} = F \times X \to X$ define a lift of any continuous map $f : Y \to X$ by choosing a point $a \in F$ and setting

$$\widetilde{f}_a \ : \ Y \to \widetilde{X} \ = \ F \times X \ ; \ y \mapsto (a, f(y)) \ .$$

The path lifting property

- Let $p : \widetilde{X} \to X$ be a covering projection. Every path $\beta : I \to X$ lifts to a path $\alpha : I \to \widetilde{X}$. If β is closed, $\beta(0) = \beta(1) \in X$, there is a unique covering translation $h : \widetilde{X} \to \widetilde{X}$ such that $h(\alpha(0)) = \alpha(1) \in \widetilde{X}$.

$$\begin{array}{ccc} & & \widetilde{X} \\ & \overset{\alpha}{\nearrow} & \big\downarrow p \\ I & \xrightarrow[\beta]{} & X \end{array}$$

- Will need the path lifting property to relate a loop $\omega : S^1 \to X$ to a path $\alpha : I \to \widetilde{X}$ such that $p\alpha(t) = \omega(e^{2\pi i t}) \in X$.

$$\begin{array}{ccc} I & \xrightarrow{\alpha} & \widetilde{X} \\ \big\downarrow & & \big\downarrow p \\ S^1 & \xrightarrow{\omega} & X \end{array}$$

For ‘universal’ $p : \widetilde{X} \to X$ get isomorphism

$$\pi_1(X) \to \mathrm{Homeo}_p(\widetilde{X}) \; ; \; [\omega] \mapsto h \; .$$

Regular covers

- Recall: a subgroup $H \subseteq G$ is normal if $gH = Hg$ for all $g \in G$, in which case the quotient group G/H is defined.
- A covering projection $p : Y \to X$ of path-connected spaces induces an injective group morphism $p_* : \pi_1(Y) \to \pi_1(X)$: if $\omega : S^1 \to Y$ is a loop at $y \in Y$ such that there exists a homotopy $h : p\omega \simeq e_{p(y)} : S^1 \to X$ rel 1, then h can be lifted to a homotopy $\widetilde{h} : \omega \simeq e_y : S^1 \to Y$ rel 1.
- **Definition** A covering p is regular if $p_*(\pi_1(Y)) \subseteq \pi_1(X)$ is a normal subgroup.
- **Example** A covering $p : Y \to X$ with X path-connected and Y simply-connected is regular, since $\pi_1(Y) = \{1\} \subseteq \pi_1(X)$ is a normal subgroup.
- **Example** $p : \mathbb{R} \to S^1$ is regular.

A general construction of regular coverings

- Given a space Y and a subgroup $G \subseteq \text{Homeo}(Y)$ define an equivalence relation $\sim$ on Y by

$$y_1 \sim y_2 \text{ if there exists } g \in G \text{ such that } y_2 = g(y_1) \ .$$

Write

$$p \ : \ Y \to X \ = \ Y/\!\sim \ = \ Y/G \ ;$$

$$y \mapsto p(y) \ = \ \text{equivalence class of } y \ .$$

- Suppose that for each $y \in Y$ there exists an open subset $U \subseteq Y$ such that $y \in U$ and

$$g(U) \cap U = \emptyset \text{ for } g \neq 1 \in G \ .$$

(Such an action of a group G on a space Y is called free and properly discontinuous).

- **Theorem** $p : Y \to X$ is a regular covering projection with fibre G. If Y is path-connected then so is X, and the group of covering translations of p is $\text{Homeo}_p(Y) = G \subset \text{Homeo}(Y)$.

Regular covers and normal subgroups

- **Theorem** For a regular covering projection $p : Y \to X$ the induced morphism $p_* : \pi_1(Y) \to \pi_1(X)$ is the inclusion of a normal subgroup, and there is defined a group isomorphism

$$\pi_1(X)/p_*(\pi_1(Y)) \;\cong\; \mathrm{Homeo}_p(Y)\ .$$

- **Sketch proof** Let $x_0 \in X$, $y_0 \in Y$ be such that $p(y_0) = x_0$. Every closed path $\alpha : I \to X$ with $\alpha(0) = \alpha(1) = x_0$ has a unique lift to a path $\widetilde{\alpha} : I \to Y$ such that $\widetilde{\alpha}(0) = y_0$. Then

$$\pi_1(X, x_0)/p_*\pi_1(Y, y_0) \to p^{-1}(x_0)\ ;\ \alpha \mapsto \widetilde{\alpha}(1)$$

  is a bijection. For each $y \in p^{-1}(x_0)$ there is a unique covering translation $h_y \in \mathrm{Homeo}_p(Y)$ such that $h_y(y_0) = y \in Y$.
- The function $p^{-1}(x_0) \to \mathrm{Homeo}_p(Y); y \mapsto h_y$ is a bijection, with inverse $h \mapsto h(\widetilde{x}_0)$. The composite bijection

$$\pi_1(X, x_0)/p_*(\pi_1(Y)) \to p^{-1}(x_0) \to \mathrm{Homeo}_p(Y)$$

  is an isomorphism of groups.

Universal covers

- **Definition** A regular cover $p : Y = \widetilde{X} \to X$ of path-connected space X is universal if Y is simply-connected.
- **Theorem** (i) For a universal cover

$$\pi_1(X) \; = \; p^{-1}(x) \; = \; \text{Homeo}_p(Y)$$

for any $x \in X$.
- (ii) Any two universal covers are isomorphic.
- (iii) The regular covers $q : Y \to X$ of a path-connected space X with regular cover $p : \widetilde{X} \to X$ are quotients $Y = \widetilde{X}/G$ for normal subgroups $G \triangleleft \pi_1(X)$.
- (iv) A reasonable path-connected space X, e.g. a simplicial complex or a CW complex, has a universal covering projection $p : Y \to X$. The path-connected covers of X are the quotients Y/G by the subgroups $G \subseteq \pi_1(X)$

Examples of universal covers

- **Example** $p : S^n \to \mathbb{RP}^n$ is universal for $n \geqslant 2$, so

$$\pi_1(\mathbb{RP}^n) = \text{Homeo}_p(S^n) = \mathbb{Z}_2 .$$

- **Example** $p : \mathbb{R} \to S^1$ is universal, so

$$\pi_1(S^1) = \text{Homeo}_p(\mathbb{R}) = \mathbb{Z} .$$

- **Example** $p \times p : \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ is universal, so the fundamental group of the torus is the free abelian group on two generators

$$\pi_1(S^1 \times S^1) = \text{Homeo}_{p \times p}(\mathbb{R} \times \mathbb{R}) = \mathbb{Z} \oplus \mathbb{Z} .$$

The fundamental group of the circle

- $\mathrm{Homeo}_p(\mathbb{R})$ is the group of the homeomorphisms $h : \mathbb{R} \to \mathbb{R}$ such that $ph = p : \mathbb{R} \to S^1$. The group is infinite cyclic, with an isomorphism of groups

$$\mathbb{Z} \to \mathrm{Homeo}_p(\mathbb{R})\ ;\ n \mapsto (h_n : x \mapsto x + n)\ .$$

- Every loop $\omega : S^1 \to S^1$ lifts to a path $\alpha : I \to \mathbb{R}$ with

$$\omega(e^{2\pi it}) \ = \ e^{2\pi i\alpha(t)} \in S^1 \ \ (t \in I)\ .$$

  There is a unique $h \in \mathrm{Homeo}_p(\mathbb{R})$ with $h(\alpha(0)) = \alpha(1) \in \mathbb{R}$.
- The functions

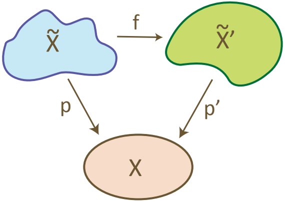
$$\mathrm{degree}\ :\ \pi_1(S^1) \to \mathrm{Homeo}_p(\mathbb{R}) = \mathbb{Z}\ ;\ \omega \mapsto \alpha(1) - \alpha(0)\ ,$$

$$\mathbb{Z} \to \pi_1(S^1)\ ;\ d \mapsto (\omega_d : S^1 \to S^1 ; z \mapsto z^d)$$

  are inverse isomorphisms of groups.

The classification of regular covers

- An isomorphism of coverings $p : \widetilde{X} \to X$, $p' : \widetilde{X}' \to X$ is a homeomorphism $f : \widetilde{X} \to \widetilde{X}'$ such that $p' \circ f = p$.



- **Example** A covering translation $h : \widetilde{X} \to \widetilde{X}$ is an isomorphism from a covering $p : \widetilde{X} \to X$ to itself.
- **Theorem** Let X be a path-connected space with a universal cover $p : \widetilde{X} \to X$. The isomorphism classes of regular covers $q : Y \to X$ are in one-one correspondence with the normal subgroups $G \triangleleft \pi_1(X)$, with $Y = \widetilde{X}/G$ and

$$\mathrm{Homeo}_q(Y) = \pi_1(X)/G .$$

The regular covers of S^1

- **Example** The isomorphism classes of regular covers of S^1 are in one-one correspondence with the subgroups

$$G \subseteq \pi_1(S^1) \ = \ \mathbb{Z} \ .$$

- (i) $G = \{0\} \subset \mathbb{Z}$ corresponds to the universal cover

$$p_\infty \ : \ \widetilde{S}^1 = \mathbb{R} \to S^1 \ ; \ x \mapsto e^{2\pi ix} \ .$$

- (ii) $G = n\mathbb{Z} \subset \mathbb{Z}$ corresponds to

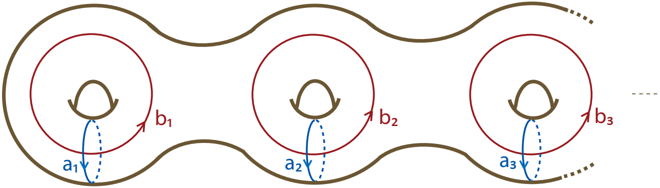
$$p_n \ : \ \widetilde{S}^1 = S^1 \to S^1 \ ; \ z \mapsto z^n \ .$$

- (iii) $G = \mathbb{Z}$ corresponds to

$$p_0 \ = \ 1 \ : \ \widetilde{S}^1 = S^1 \to S^1 \ .$$

The classification of surfaces I.

- Surface = 2-dimensional manifold.
- For $g \geqslant 0$ the closed orientable surface $M(g)$ is the surface obtained from S^2 by attaching g handles.



- **Example** $M(0) = S^2$ is the sphere, with $\pi_1(M(0)) = \{1\}$.
- **Example** $M(1) = S^1 \times S^1$, with $\pi_1(M(1)) = \mathbb{Z} \oplus \mathbb{Z}$.

The classification of surfaces II.

- **Theorem** The fundamental group of the orientable genus g surface $M(g)$ has $2g$ generators and 1 relation

$$\pi_1(M(g)) \ = \ \{a_1, b_1, \ldots, a_g, b_g \,|\, [a_1, b_1] \ldots [a_g, b_g]\}$$

  with $[a, b] = a^{-1}b^{-1}ab$ the commutator of a, b.
- $\pi_1(M(g))$ can be computed by the Seifert-van Kampen theorem for the fundamental group of unions, or by the universal cover $\mathbb{H} \to M(g)$ with $\mathbb{H}$ the hyperbolic plane.
- **Classification theorem** Every closed orientable surface M is diffeomorphic to $M(g)$ for a unique g.
- **Proof** A combination of algebra and topology is required to prove that M is diffeomorphic to some $M(g)$. Since the groups $\pi_1(M(g))$ $(g \geqslant 0)$ are all non-isomorphic, M is diffeomorphic to a unique $M(g)$. This can also be seen using $H_1(M(g)) = \mathbb{Z}^{2g}$.

What next?

- Lecture 8, 29 November. The Edinburgh algebraic geometer Vanya Cheltsov will describe some of the many ways in which the topology of surfaces features in algebraic geometry.
- Lecture 9, 6 December. I shall describe the Seifert-van Kampen theorem for the fundamental group of a union, and its application to the classification of surfaces. (Could also use H_1).
- Lecture 10, 13 December. John O'Connor and Edmund Robertson of the St. Andrews MacTutor History of Mathematics website

  http://www-history.mcs.st-and.ac.uk

  will talk on some of the rich history of geometry and topology.

A train delivering SMSTC Geometry and Topology around Scotland