SMSTC Geometry and Topology 2012–2013

Lecture 9

The Seifert – van Kampen Theorem

The classification of surfaces

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Introduction

- Topology and groups are closely related via the fundamental group construction
$$\pi_1 \ : \ \{\text{spaces}\} \to \{\text{groups}\} \ ; \ X \mapsto \pi_1(X) \ .$$
- The Seifert - van Kampen Theorem expresses the fundamental group of a union $X = X_1 \cup_Y X_2$ of path-connected spaces in terms of the fundamental groups of X_1, X_2, Y.
- The Theorem is used to compute the fundamental group of a space built up using spaces whose fundamental groups are known already.
- The Theorem is used to prove that every group G is the fundamental group $G = \pi_1(X)$ of a space X, and to compute the fundamental groups of surfaces.
- Treatment of Seifert-van Kampen will follow Section I.1.2 of Hatcher's Algebraic Topology, but not slavishly so.

Three ways of computing the fundamental group

I. By geometry

- For an infinite space X there are far too many loops $\omega : S^1 \to X$ in order to compute $\pi_1(X)$ from the definition.
- A space X is **simply-connected** if X is path connected and the fundamental group is trivial

$$\pi_1(X) \;=\; \{e\} \;.$$

- Sometimes it is possible to prove that X is simply-connected by geometry.
- **Example**: If X is contractible then X is simply-connected.
- **Example**: If $X = S^n$ and $n \geqslant 2$ then X is simply-connected.
- **Example**: Suppose that (X, d) is a metric space such that for any $x, y \in X$ there is unique geodesic (= shortest path) $\alpha_{x,y} : I \to X$ from $\alpha_{x,y}(0) = x$ to $\alpha_{x,y}(1) = y$. If $\alpha_{x,y}$ varies continuously with x, y then X is contractible. Trees. Many examples of such spaces in differentiable geometry.

Three ways of computing the fundamental group

II. From above

- If

$$\begin{tikzcd} \widetilde{X} \arrow[d, "p"] \\ X \end{tikzcd}$$

  is a covering projection and $\widetilde{X}$ is simply-connected then $\pi_1(X)$ is isomorphic to the group of covering translations

  $\text{Homeo}_p(\widetilde{X}) = \{\text{homeomorphisms } h : \widetilde{X} \to \widetilde{X} \text{ such that } ph = p\}$

- **Example** If

$$p : \widetilde{X} = \mathbb{R} \to X = S^1 \; ; \; x \mapsto e^{2\pi i x}$$

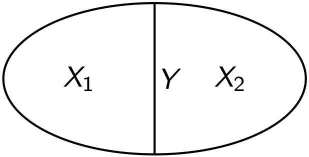
  then

$$\pi_1(S^1) = \text{Homeo}_p(\mathbb{R}) = \mathbb{Z} \; .$$

Three ways of computing the fundamental group

III. From below

- **Seifert-van Kampen Theorem** (preliminary version)



If a path-connected space X is a union $X = X_1 \cup_Y X_2$ with X_1, X_2 and $Y = X_1 \cap X_2$ path-connected then the fundamental group of X is the **free product with amalgamation**

$$\pi_1(X) \ = \ \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2) \ .$$

- $G_1 *_H G_2$ defined for group morphisms $H \to G_1$, $H \to G_2$.
- First proved by van Kampen (1933) in the special case when Y is simply-connected, and then by Seifert (1934) in general.

Seifert and van Kampen

Herbert Seifert
(1907-1996)

Egbert van Kampen
(1908-1942)

The free product of groups

- Let G_1, G_2 be groups with units $e_1 \in G_1$, $e_2 \in G_2$.
- A **reduced word** $g_1 g_2 \dots g_m$ is a finite sequence of length $m \geqslant 1$ with
  - $g_i \in G_1 \setminus \{e_1\}$ or $g_i \in G_2 \setminus \{e_2\}$,
  - g_i, g_{i+1} not in the same G_j.
- The **free product** of G_1 and G_2 is the group

$$G_1 * G_2 \;=\; \{e\} \cup \{\text{reduced words}\}$$

  with multiplication by concatenation and reduction.
- The unit $e =$ empty word of length 0.
- See p.42 of Hatcher for detailed proof that $G_1 * G_2$ is a group.
- **Exercise** Prove that

$$\{e\} * G \cong G \;,\; G_1 * G_2 \cong G_2 * G_1 \;,\; (G_1 * G_2) * G_3 \cong G_1 * (G_2 * G_3)$$

The free group F_g

- For a set S let

$$\langle S \rangle \;=\; \text{free group generated by } S \;=\; \mathop{\star}_{s \in S} \mathbb{Z} \;.$$

- Let $g \geqslant 1$. The **free group on g generators** is the free product of g copies of $\mathbb{Z}$

$$F_g \;=\; \langle a_1, a_2, \dots, a_g \rangle \;=\; \mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z} \;.$$

- Every element $x \in F_g$ has an expression as a word

$$x \;=\; (a_1)^{m_{11}}(a_2)^{m_{12}} \dots (a_g)^{m_{1g}} a_1^{m_{21}} \dots a_g^{m_{Ng}}$$

  with (m_{ij}) an $N \times g$ matrix (N large), $m_{ij} \in \mathbb{Z}$.
- $F_1 = \mathbb{Z}$.
- For $g \geqslant 2$ F_g is nonabelian.
- $F_g * F_h = F_{g+h}$.

The subgroups generated by a subset

- Needed for statement and proof the Seifert - van Kampen Theorem.
- Let G be a group. The **subgroup generated** by a subset $S \subseteq G$
$$\langle S \rangle \subseteq G$$
is the smallest subgroup of G containing S.
- $\langle G \rangle$ consists of finite length words in elements of S and their inverses.
- Let S^G be the subset of G consisting of the conjugates of S
$$S^G \ = \ \{gsg^{-1} \,|\, s \in S,\, g \in G\}$$
- The **normal subgroup generated** by a subset $S \subseteq G$ $\langle S^G \rangle \subseteq G$ is the smallest normal subgroup of G containing S.

Group presentations

- Given a set S and a subset $R \subseteq G = \langle S \rangle$ define the group

$$\begin{aligned}\langle S|R\rangle &= G/\langle R^G\rangle \\ &= G/\text{normal subgroup generated by } R\ .\end{aligned}$$

  with **generating set** S and **relations** R.
- **Example** Let $m \geqslant 1$. The function

$$\langle a \,|\, a^m\rangle \;=\; \{e, a, a^2, \ldots, a^{m-1}\} \to \mathbb{Z}_m\ ;\ a^n \mapsto n$$

  is an isomorphism of groups, with $\mathbb{Z}_m$ the finite cyclic group of order m.
- R can be empty, with

$$\langle S|\emptyset\rangle \;=\; \langle S\rangle \;=\; \mathop{\star}_{S}\mathbb{Z}$$

  the free group generated by S.
- The free product of $G_1 = \langle S_1|\, R_1\rangle$ and $G_2 = \langle S_2 \,|\, R_2\rangle$ is

$$G_1 * G_2 = \langle S_1 \cup S_2 \,|\, R_1 \cup R_2\rangle\ .$$

Every group has a presentation

- Every group G has a group presentation, i.e. is isomorphic to $\langle S|R\rangle$ for some sets S, R.
- **Proof** Let $S = \langle G\rangle = \underset{G}{\star}\mathbb{Z}$ and let $R = \ker(\Phi)$ be the kernel of the surjection of groups

$$\Phi \ : \ S \to G \ ; \ (g_1^{n_1}, g_2^{n_2}, \dots) \mapsto (g_1)^{n_1}(g_2)^{n_2}\dots \ .$$

  Then

$$\langle S|R\rangle \to G \ ; \ [x] \mapsto \Phi(x)$$

  is an isomorphism of groups.
- It is a nontrivial theorem that R is a free subgroup of the free group S. But we are only interested in S and R as sets here. This presentation is too large to be of use in practice! But the principle has been established.
- While presentations are good for specifying groups, it is not always easy to work out what the group actually is. Word problem: when is $\langle S|R\rangle \cong \langle S'|R'\rangle$? Undecidable in general.

How to make a group abelian

- The **commutator** of $g, h \in G$ is
$$[g,h] \ = \ ghg^{-1}h^{-1} \in G \ .$$
Let $F = \{[g,h]\} \subseteq G$. G is abelian if and only if $F = \{e\}$.
- The **abelianization** of a group G is the abelian group
$$G^{ab} \ = \ G/\langle F^G \rangle \ ,$$
with $\langle F^G \rangle \subseteq G$ the normal subgroup generated by F.
- If $G = \langle S|R\rangle$ then $G^{ab} = \langle S|R \cup F\rangle$.
- **Universal property** G^{ab} is the largest abelian quotient group of G, in the sense that for any group morphism $f : G \to A$ to an abelian group A there is a unique group morphism $f^{ab} : G^{ab} \to A$ such that
$$f \ : \ G \longrightarrow G^{ab} \xrightarrow{f^{ab}} A \ .$$
- $\pi_1(X)^{ab} = H_1(X)$ is the first homology group of a space X.

Free abelian at last

- **Example** Isomorphism of groups

$$(F_2)^{ab} \;=\; \langle a, b \,|\, aba^{-1}b^{-1}\rangle \to \mathbb{Z} \oplus \mathbb{Z} \;;$$
$$a^{m_1} b^{n_1} a^{m_2} b^{n_2} \ldots \mapsto (m_1 + m_2 + \ldots, n_1 + n_2 + \ldots)$$

  with $\mathbb{Z} \oplus \mathbb{Z}$ the free abelian group on 2 generators.
- More generally, the abelianization of the free group on g generators is the free abelian group on g generators

$$(F_g)^{ab} \;=\; \bigoplus_g \mathbb{Z} \text{ for any } g \geqslant 1 \,.$$

- It is clear from linear algebra (Gaussian elimination) that

$$\bigoplus_g \mathbb{Z} \text{ is isomorphic to } \bigoplus_h \mathbb{Z} \text{ if and only if } g = h \,.$$

- It follows that

$$F_g \text{ is isomorphic to } F_h \text{ if and only if } g = h \,.$$

Amalgamated free products

- The **amalgamated free product** of group morphisms

$$i_1 \;:\; H \to G_1 \;,\; i_2 \;:\; H \to G_2$$

  is the group

$$G_1 *_H G_2 \;=\; (G_1 * G_2)/N$$

  with $N \subseteq G_1 * G_2$ the normal subgroup generated by the elements

$$i_1(h)i_2(h)^{-1} \;(h \in H) \;.$$

- For any $h \in H$

$$i_1(h) \;=\; i_2(h) \in G_1 *_H G_2 \;.$$

- In general, the natural morphisms of groups

$$j_1 : G_1 \to G_1 *_H G_2 \;,\; j_2 : G_2 \to G_1 *_H G_2 \;,\; j_1 i_1 = j_2 i_2 : H \to G_1 *_H G_2$$

  are not injective.

Some examples of amalgamated free products

- **Example** $G *_G G = G$.
- **Example** $\{e\} *_H \{e\} = \{e\}$.
- **Example** For $H = \{e\}$ the amalgamated free product is just the free product

$$G_1 *_{\{e\}} G_2 \ = \ G_1 * G_2 \ .$$

- **Example** For any group morphism $i : H \to G$

$$G *_H \{e\} \ = \ G/N$$

with $N = \langle i(H)^G \rangle \subseteq G$ the normal subgroup generated by the subgroup $i(H) \subseteq G$.

The Seifert - van Kampen Theorem

- Let $X = X_1 \cup_Y X_2$ with X_1, X_2 and $Y = X_1 \cap X_2$ open in X and path-connected. Let
$$i_1 : \pi_1(Y) \to \pi_1(X_1) \ , \ i_2 : \pi_1(Y) \to \pi_1(X_2)$$
be the group morphisms induced by $Y \subseteq X_1$, $Y \subseteq X_2$, and let
$$j_1 : \pi_1(X_1) \to \pi_1(X) \ , \ j_2 : \pi_1(X_2) \to \pi_1(X)$$
be the group morphisms induced by $X_1 \subseteq X$, $X_2 \subseteq X$. Then
$$\Phi \ : \ \pi_1(X_1) * \pi_1(X_2) \to \pi_1(X) \ ; \ x_k \mapsto j_k(x_k)$$
$(x_k \in \pi_1(X_k), k = 1$ or $2)$ is a surjective group morphism with
$$\begin{aligned} \ker \Phi \ = \ N \ = \ & \text{the normal subgroup of } \pi_1(X_1) * \pi_1(X_2) \\ & \text{generated by } i_1(y) i_2(y)^{-1} \ (y \in \pi_1(Y)) \ . \end{aligned}$$
- **Theorem** Φ induces an isomorphism of groups
$$\Phi \ : \ \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2) \ \cong \ \pi_1(X) \ .$$

Φ is surjective I.

- Will only prove the easy part, that Φ is surjective.
- For the hard part, that Φ is injective, see pp.45-46 of Hatcher.
- Choose the base point $x \in Y \subseteq X$. Regard a loop $\omega : (S^1, 1) \to (X, x)$ as a closed path

$$f \ : \ I = [0,1] \to X = X_1 \cup_X X_2$$

such that $f(0) = f(1) = x \in X$, with $\omega(e^{2\pi is}) = f(s) \in X$.
- By the compactness of I there exist

$$0 = s_0 < s_1 < s_2 < \cdots < s_m = 1$$

such that $f[s_i, s_{i+1}] \subseteq X_1$ or X_2, written $f[s_i, s_{i+1}] \subseteq X_i$.
- Then

$$f = f_1 \bullet f_2 \bullet \cdots \bullet f_m : I \to X$$

is the concatenation of paths $f_i : I \to X_i$ with

$$f_i(1) \ = \ f_{i+1}(0) \ = \ f(s_i) \in Y \ (1 \leqslant i \leqslant m) \ .$$

Φ is surjective II.

- Since Y is path-connected there exists paths $g_i : I \to Y$ from $g_i(0) = x$ to $g_i(1) = f(s_i) \in X$. The loop

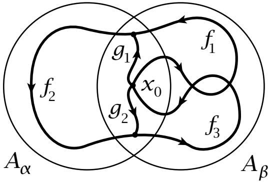
  $(f_1 \bullet \overline{g}_1) \bullet (g_1 \bullet f_2 \bullet \overline{g}_2) \bullet \cdots \bullet (g_{m-2} \bullet f_{m-1} \bullet \overline{g}_{m-1}) \bullet (g_{m-1} \bullet f_m) : I \to X$

  is homotopic to f rel $\{0,1\}$, with

  $$[g_i \bullet f_i \bullet \overline{g}_{i+1}] \in \mathrm{im}(\pi_1(X_i)) \subseteq \pi_1(X) \ ,$$

  so that

  $$[f] = [f_1 \bullet \overline{g}_1][g_1 \bullet f_2 \bullet \overline{g}_2] \dots [g_{m-2} \bullet f_{m-1} \bullet \overline{g}_{m-1}][g_{m-1} \bullet f_m] \in \mathrm{im}(\Phi) \subseteq \pi_1(X) \ .$$



Hatcher diagram: $A_\alpha = X_1$, $A_\beta = X_2$

The universal property I.

- An amalgamated free product $G_1 *_H G_2$ defines a commutative square of groups and morphisms

$$\begin{tikzcd} H \arrow[r, "i_1"] \arrow[d, "i_2"'] & G_1 \arrow[d, "j_1"] \\ G_2 \arrow[r, "j_2"] & G_1 *_H G_2 \end{tikzcd}$$

  with the universal property that for any commutative square

$$\begin{tikzcd} H \arrow[r, "i_1"] \arrow[d, "i_2"'] & G_1 \arrow[d, "k_1"] \\ G_2 \arrow[r, "k_2"] & G \end{tikzcd}$$

  there is a unique group morphism $\Phi : G_1 *_H G_2 \to G$ such that $k_1 = \Phi j_1 : G_1 \to G$ and $k_2 = \Phi j_2 : G_2 \to G$.

The universal property II.

- By the Seifert - van Kampen Theorem for $X = X_1 \cup_Y X_2$ the commutative square

$$\begin{tikzcd} \pi_1(Y) \arrow[r, "i_1"] \arrow[d, "i_2"'] & \pi_1(X_1) \arrow[d, "j_1"] \\ \pi_1(X_2) \arrow[r, "j_2"] & \pi_1(X) \end{tikzcd}$$

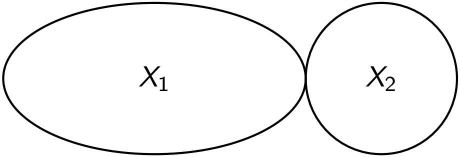
has the universal property of an amalgamated free product, with an isomorphism

$$\Phi \ : \ \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2) \ \cong \ \pi_1(X) \ .$$

The one-point union

- Let X_1, X_2 be spaces with base points $x_1 \in X_1$, $x_2 \in X_2$. The **one-point union** is the quotient space of the disjoint union $X_1 \sqcup X_2$

$$X_1 \vee X_2 \;=\; (X_1 \sqcup X_2)/\{x_1 \sim x_2\}\ .$$



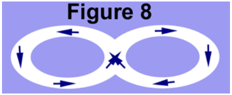
- **The Seifert - van Kampen Theorem for** $X_1 \vee X_2$
If X_1, X_2 are path connected then so is $X_1 \vee X_2$, with fundamental group the free product

$$\pi_1(X_1 \vee X_2) \;=\; \pi_1(X_1) * \pi_1(X_2)\ .$$

The fundamental group of the figure eight

- The figure eight is the one-point union of two circles

$$X = S^1 \vee S^1$$



- The fundamental group is the free **nonabelian** group on two generators:

$$\pi_1(X) = \pi_1(S^1) * \pi_1(S^1) = \langle a, b \rangle = \mathbb{Z} * \mathbb{Z} .$$

- An element

$$a^{m_1} b^{n_1} a^{m_2} b^{n_2} \cdots \in \pi_1(X)$$

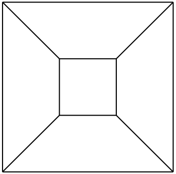
  can be regarded as the loop traced out by an iceskater who traces out a figure 8, going round the first circle m_1 times, then round the second circles n_1 times, then round the first circle m_2 times, then round the second circle n_2 times, ...

The fundamental group of a graph

- Let X be the path-connected space defined by a connected finite graph with V vertices and E edges, and let

$$g \;=\; 1 - V + E \;.$$

- **Exercise** Prove that X is homotopy equivalent to the one-point union $S^1 \vee S^1 \vee \cdots \vee S^1$ of g circles, and hence that $\pi_1(X) = F_g$, the free group on g generators. Prove that X is a tree if and only if it is contractible, if and only if $g = 0$.
- Example I.1.22 of Hatcher is a special case with $V = 8$, $E = 12$, $g = 5$.



Cell attachment

- Let $n \geqslant 0$. Given a space W and a map $f : S^{n-1} \to W$ let
$$X = W \cup_f D^n$$
be the space obtained from W by **attaching an n-cell**.
- X is the quotient of the disjoint union $W \cup D^n$ by the equivalence relation generated by
$$(x \in S^{n-1}) \sim (f(x) \in W) .$$
- An **n-dimensional cell complex** is a space obtained from $\emptyset$ by successively attaching k-cells, with $k = 0, 1, 2, \ldots, n$
- **Example** A graph is a 1-dimensional cell complex.
- **Example** S^n is the n-dimensional cell complex obtained from $\emptyset$ by attaching a 0-cell and an n-cell
$$S^n = D^0 \cup_f D^n$$
with $f : S^{n-1} \to D^0$ the unique map.

The Euler characteristic

- **Definition** The **Euler characteristic** of a finite cell complex

$$X = \bigcup_{c_0} D^0 \cup \bigcup_{c_1} D^1 \cup \bigcup_{c_2} D^2 \cup \cdots \cup \bigcup_{c_n} D^n$$

with c_k k-cells is

$$\chi(X) = \sum_{k=0}^{n} (-1)^k c_k \in \mathbb{Z} .$$

- $\chi(D^n) = 1$, $\chi(S^n) = 1 + (-1)^n$
- If X is homotopy equivalent to Y then $\chi(X) = \chi(Y)$
- $\chi(X \cup Y) = \chi(X) + \chi(Y) - \chi(X \cap Y) \in \mathbb{Z}$.
- If $F \to \widetilde{X} \to X$ is a regular cover with finite fibre F then $\chi(\widetilde{X}) = \chi(F)\chi(X)$, with $\chi(F) = |F|$.

The effect on π_1 of a cell attachment I.

- Let $n \geqslant 1$. If W is path-connected then so is $X = W \cup_f D^n$.
- What is $\pi_1(X)$?
- If $n = 1$ then X is homotopy equivalent to $W \vee S^1$, so that

$$\pi_1(X) \ = \ \pi_1(W \vee S^1) \ = \ \pi_1(W) * \mathbb{Z} \ .$$

- For $n \geqslant 2$ apply the Seifert - van Kampen Theorem to the decomposition

$$X \ = \ X_1 \cup_Y X_2$$

with

$$\begin{aligned} X_1 \ &= \ W \cup_f \{x \in D^n \,|\, \|x\| \geqslant 1/2\} \ , \\ X_2 \ &= \ \{x \in D^n \,|\, \|x\| \leqslant 1/2\} \ , \\ Y \ &= \ X_1 \cap X_2 \ = \ \{x \in D^n \,|\, \|x\| = 1/2\} \ = \ S^{n-1} \end{aligned}$$

The effect on π_1 of a cell attachment II.

- The inclusion $W \subset X_1$ is a homotopy equivalence, and $X_2 \cong D^n$ is simply-connected, so that

$$\begin{aligned}\pi_1(X) &= \pi_1(X_1) *_{\pi_1(Y)} \pi_1(X_2) \\ &= \pi_1(W) *_{\pi_1(S^{n-1})} \pi_1(D^n) = \pi_1(W) *_{\pi_1(S^{n-1})} \{e\} .\end{aligned}$$

- If $n \geqslant 3$ then $\pi_1(S^{n-1}) = \{e\}$, so that

$$\pi_1(X) = \pi_1(W) *_{\{e\}} \{e\} = \pi_1(W) .$$

- If $n = 2$ then

$$\pi_1(X) = \pi_1(W) *_{\mathbb{Z}} \{e\} = \pi_1(W)/N$$

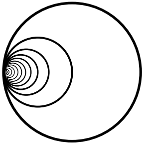
  the quotient of $\pi_1(W)$ by the normal subgroup $N \subseteq \pi_1(W)$ generated by the homotopy class $[f] \in \pi_1(W)$ of $f : S^1 \to W$.
- See Hatcher's Proposition I.1.26 for detailed exposition.
- If $X = \bigvee_S S^1 \cup \bigcup_R D^2 \cup \bigcup_{n \geqslant 3} \bigcup D^n$ is a cell complex with a single 0-cell, S 1-cells and R 2-cells then $\pi_1(X) = \langle S | R \rangle$.

Every group is a fundamental group

- Let $G = \langle S|R\rangle$ be a group with a presentation.
- Realize the generators S by the 1-dimensional cell complex

$$W = \bigvee_S S^1$$

with $\pi_1(W) = \langle S\rangle$ the free group generated by S.



- Realize each relation $r \in R \subseteq \pi_1(W)$ by a map $r : S^1 \to W$.
- Attach a 2-cell to W for each relation, to obtain a 2-dimensional cell complex $X = W \cup \bigcup_R D^2$ such that

$$\pi_1(X) = \langle S|R\rangle = G .$$

Realizing the cyclic groups topologically

- **Example** Let $m \geqslant 1$. The cyclic group $\mathbb{Z}_m = \langle a \,|\, a^m \rangle$ of order m is the fundamental group

$$\pi_1(X_m) \;=\; \mathbb{Z}_m$$

of the 2-dimensional cell complex

$$X_m \;=\; S^1 \cup_m D^2 \;,$$

with the 2-cell attached to $S^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$ by

$$m \;:\; S^1 \to S^1 \;;\; z \mapsto z^m \;.$$

- $X_1 = D^2$ is contractible, with $\pi_1(X_1) = \{e\}$
- X_2 is homeomorphic to the real projective plane

$$\begin{aligned} \mathbb{RP}^2 \;&=\; S^2/\{v \sim -v \,|\, v \in S^2\} \\ &=\; D^2/\{w \sim -w \,|\, w \in S^1\} \end{aligned}$$

with $\pi_1(X_2) = \pi_1(\mathbb{RP}^2) = \mathbb{Z}_2$.

Manifolds

- An n-**dimensional manifold** M is a topological space such that each $x \in M$ has an open neighbourhood $U \subset M$ homeomorphic to n-dimensional Euclidean space $\mathbb{R}^n$
$$U \cong \mathbb{R}^n .$$
- Strictly speaking, need to include the condition that M be Hausdorff and paracompact = every open cover has a locally finite refinement.
- Called n-**manifold** for short.
- Manifolds are the topological spaces of greatest interest, e.g. $M = \mathbb{R}^n$. Appear in algebraic geometry, analysis as well as topology.
- Study of manifolds initiated by Riemann (1854).
- A **surface** is a 2-dimensional manifold.
- Will be mainly concerned with manifolds which are compact = every open cover has a finite refinement.

Why are manifolds interesting?

- Topology.
- Differential equations.
- Differential geometry.
- Hyperbolic geometry.
- Algebraic geometry. Uniformization theorem.
- Complex analysis. Riemann surfaces.
- Dynamical systems,
- Mathematical physics.
- Combinatorics.
- Topological quantum field theory.
- Computational topology.
- Pattern recognition: body and brain scans.
- . . .

Examples of n-manifolds

- The n-**dimensional Euclidean space** $\mathbb{R}^n$
- The n-**sphere** S^n.
- The n-**dimensional projective space**

$$\mathbb{RP}^n \;=\; S^n/\{z \sim -z\} \;.$$

- **Rank theorem in linear algebra**. If $J : \mathbb{R}^{n+k} \to \mathbb{R}^k$ is a linear map of rank k (i.e. onto) then $J^{-1}(0) = \ker(J) \subseteq \mathbb{R}^{n+k}$ is an n-dimensional vector subspace.
- **Implicit function theorem**. The solutions of differential equations are generically manifolds. If $f : \mathbb{R}^{n+k} \to \mathbb{R}^k$ is a differentiable function such that for every $x \in f^{-1}(0)$ the Jacobian $k \times (n+k)$ matrix $J = (\partial f_i/\partial x_j)$ has rank k, then

$$M \;=\; f^{-1}(0) \subseteq \mathbb{R}^{n+k}$$

is an n-manifold.
- In fact, every n-manifold M admits an embedding $M \subseteq \mathbb{R}^{n+k}$ for some large k.

Manifolds with boundary

- An n-**dimensional manifold with boundary** $(M, \partial M \subset M)$ is a pair of topological spaces such that
  1. $M \backslash \partial M$ is an n-manifold called the **interior**,
  2. ∂M is an $(n-1)$-manifold called the **boundary**,
  3. Each $x \in \partial M$ has an open neighbourhood $U \subset M$ such that

$$(U, \partial M \cap U) \cong \mathbb{R}^{n-1} \times ([0, \infty), \{0\}) .$$

- A manifold M is **closed** if $\partial M = \emptyset$.
- The boundary ∂M of a manifold with boundary $(M, \partial M)$ is closed, $\partial\partial M = \emptyset$.
- **Example** (D^n, S^{n-1}) is an n-manifold with boundary.
- **Example** The product of an m-manifold with boundary $(M, \partial M)$ and an n-manifold with boundary $(N, \partial N)$ is an $(m+n)$-manifold with boundary

$$(M \times N, M \times \partial N \cup_{\partial M \times \partial N} \partial M \times N) .$$

The classification of n-manifolds I.

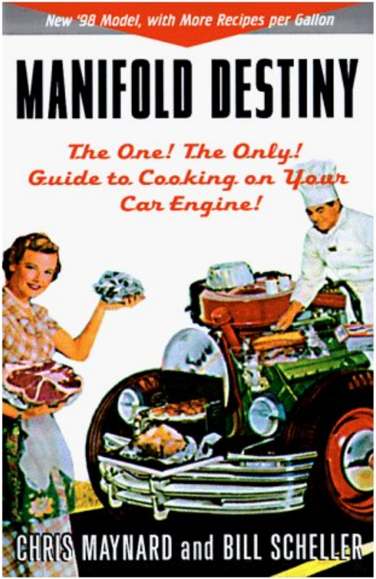
- Will only consider compact manifolds from now on.
- A function

$$i \ : \ \text{a class of manifolds} \to \text{a set} \ ; \ M \mapsto i(M)$$

  is a **topological invariant** if $i(M) = i(M')$ for homeomorphic M, M'. Want the set to be finite, or at least countable.
- **Example 1** The dimension $n \geqslant 0$ of an n-manifold M is a topological invariant (Brouwer, 1910).
- **Example 2** The number of components $\pi_0(M)$ of a manifold M is a topological invariant.
- **Example 3** The orientability $w(M) \in \{-1, +1\}$ of a connected manifold M is a topological invariant.
- **Example 4** The Euler characteristic $\chi(M) \in \mathbb{Z}$ of a manifold M is a topological invariant.
- A **classification** of n-manifolds is a topological invariant i such that $i(M) = i(M')$ if and only if M, M' are homeomorphic.

The classification of n-manifolds II. $n = 0, 1, 2, \ldots$

- **Classification of 0-manifolds** A 0-manifold M is a finite set of points. Classified by $\pi_0(M) =$ no. of points $\geqslant 1$.
- **Classification of 1-manifolds** A 1-manifold M is a finite set of circles S^1. Classified by $\pi_0(M) =$ no. of circles $\geqslant 1$.
- **Classification of 2-manifolds** Classified by $\pi_0(M)$, and for connected M by the fundamental group $\pi_1(M)$. Details to follow!
- For $n \leqslant 2$ homeomorphism $\iff$ homotopy equivalence.
- ***n*-dimensional Poincaré conjecture** A connected n-manifold M is homeomorphic to S^n if and only if $\pi_1(M) = \{1\}$ and $H_*(M) = H_*(S^n)$. Proved by Smale (1960) for $n \geqslant 5$, Freedman (1982) for $n = 4$ and Perelman (2003) for $n = 3$.
- It is not possible to classify n-manifolds for $n \geqslant 4$. Every finitely presented group $G = \langle S | R \rangle$ is realized as the fundamental group $G = \pi_1(M)$ for a 4-manifold M. The word problem is undecidable, so cannot classify $\pi_1(M)$, let alone M.


New '98 Model, with More Recipes per Gallon
MANIFOLD DESTINY
The One! The Only! Guide to Cooking on Your Car Engine!
CHRIS MAYNARD and BILL SCHELLER

How does one classify surfaces?

- (1) Every surface M can be **triangulated**, in the weak sense of being homotopy equivalent to a finite 2-dimensional cell complex
$$M \cong \bigcup_{c_0} D^0 \cup \bigcup_{c_1} D^1 \cup \bigcup_{c_2} D^2 .$$
- (2) Every connected M is homeomorphic to a normal form
$$M(g) \text{ orientable, genus } g \geqslant 0 ,$$
$$N(g) \text{ nonorientable, genus } g \geqslant 1$$
- (3) No two normal forms are homeomorphic.
- Similarly for surfaces with boundary, with $M(g,h)$, $N(g,h)$.
- History: (2)+(3) already in 1860-1920 (Möbius, Clifford, von Dyck, Dehn and Heegaard, Brahana). (1) only in the 1920's (Rado, Kerékjártó).
- Today will only do (3), using $\pi_1(M(g))$ of normal forms. Could use genus g or Euler characteristic $\chi(M(g))$ instead!

A page from Dehn and Heegaard's Analysis Situs (1907)

Jede geschlossene Fläche kann stets mit drei Elementarflächenstücken bedeckt werden. Jede nicht geschlossene Fläche und jede Kugelfläche kann mit zwei Elementarflächen bedeckt werden[95a]).

d) *Normalformen für geschlossene Flächen*[96]).

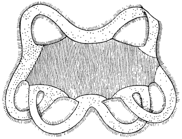


Fig. 8.

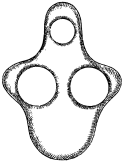


Fig. 9.

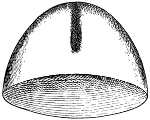


Fig. 10.

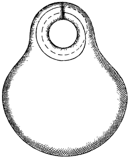


Fig. 11.

α) *Zweiseitige Flächen.* Eine Fläche, deren Restfläche p Doppelbänder hat, ist homöomorph mit einer Kugel mit p „*Henkeln*" (Fig. 9);

95) *Möbius*, Leipzig Ber. 15 (1863) = Werke 2, p. 450.

96) Diese Formen für geschlossene Flächen sind, soweit zweiseitige Flächen in Betracht kommen, als betrachtet worden von *Riemann* (cf. *Klein*, Über Riemanns Theorie . . . (1882), p. IV), *Möbius*, a. a. O. § 16, *Tonelli* (Rom Linc. Atti (2) 2 (1875), p. 594, vgl. Rom Linc. Rend. (5) 4¹ (1895), p 300; *W. K. Clifford* London Proc. Math. Soc. 8 (1877), p. 292). Normalformen für einseitige Flächen sind von *Dyck* a. a. O. aufgestellt.

The connected sum I.

- Given an n-manifold with boundary $(M, \partial M)$ with M connected use any embedding $D^n \subset M \backslash \partial M$ to define the **punctured n-manifold with boundary**

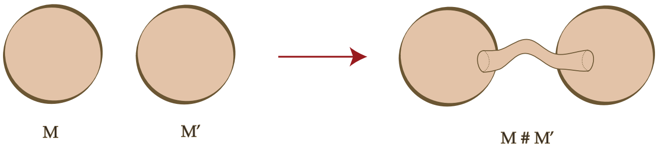
$$(M_0, \partial M_0) \ = \ (\mathrm{cl.}(M \backslash D^n), \partial M \cup S^{n-1}) \ .$$

- The **connected sum** of connected n-manifolds with boundary $(M, \partial M)$, $(M', \partial M')$ is the connected n-manifold with boundary

$$(M \# M', \partial(M \# M')) \ = \ (M_0 \cup_{S^{n-1}} M'_0, \partial M \cup \partial M') \ .$$

  Independent of choices of $D^n \subset M \backslash \partial M$, $D^n \subset M' \backslash \partial M'$.
- If M and M' are closed then so is $M \# M'$.

The connected sum II.

- 

- The connected sum $\#$ has a neutral element, is commutative and associative:

$$
\begin{array}{lrcl}
\text{(i)} & M\#S^n & \cong & M\ , \\
\text{(ii)} & M\#M' & \cong & M'\#M\ , \\
\text{(iii)} & (M\#M')\#M'' & \cong & M\#(M'\#M'')\ .
\end{array}
$$

- A punctured n-manifold has $\chi(M_0) = \chi(M) - (-1)^n$
- The connected sum of n-manifolds has Euler characteristic

$$\chi(M\#M') = \chi(M) + \chi(M') - \chi(S^n)\ .$$

The fundamental group of a connected sum

- If $(M, \partial M)$ is an n-manifold with boundary and M is connected then M_0 is also connected. Can apply the Seifert-van Kampen Theorem to
$$M \;=\; M_0 \cup_{S^{n-1}} D^n$$
to obtain
$$\pi_1(M) \;=\; \pi_1(M_0) *_{\pi_1(S^{n-1})} \{1\} \;=\; \begin{cases} \pi_1(M_0) & \text{for } n \geqslant 3 \\ \pi_1(M_0)/\langle\partial\rangle & \text{for } n = 2 \end{cases}$$
with $\langle\partial\rangle \triangleleft \pi_1(M_0)$ the normal subgroup generated by the boundary circle $\partial : S^1 \subset M_0$.
- Another application of the Seifert-van Kampen Theorem gives
$$\begin{aligned} \pi_1(M\#M') \;&=\; \pi_1(M_0) *_{\pi_1(S^{n-1})} \pi_1(M_0') \\ &=\; \begin{cases} \pi_1(M) * \pi_1(M') & \text{for } n \geqslant 3 \\ \pi_1(M_0) *_{\mathbb{Z}} \pi_1(M_0') & \text{for } n = 2 \;. \end{cases} \end{aligned}$$

Orientability for surfaces

- Let M be a connected surface.
- **Definition** An injective loop $\alpha : S^1 \to M$ is **orientable** if there exists an injective map $\overline{\alpha} : S^1 \times [-1, 1] \to M$ with $\overline{\alpha}(x, 0) = \alpha(x) \in M$ for all $x \in S^1$.
- **Definition** M is **orientable** if every α is orientable.
- **Example** The Euclidean 2-space $\mathbb{R}^2$, the 2-sphere S^2 and the torus $S^1 \times S^1$ are orientable.
- **Definition** M is **nonorientable** if there exists $\alpha : S^1 \to M$ which is not orientable, or equivalently if Möbius band $\subset M$.
- **Example** The Möbius band, the projective plane $\mathbb{RP}^2$ and the Klein bottle K are nonorientable.
- **Remark** Can similarly define orientability for connected n-manifolds M, using $\alpha : S^{n-1} \to M$.

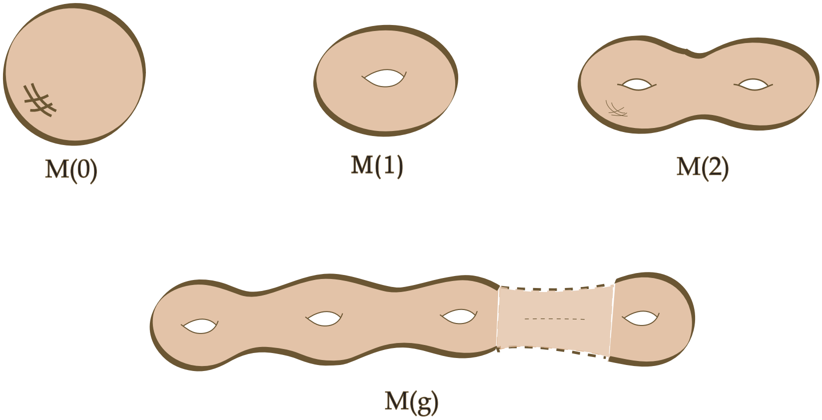
The orientable closed surfaces $M(g)$ I.

- **Definition** Let $g \geqslant 0$. The **orientable connected surface with genus** g is the connected sum of g copies of $S^1 \times S^1$

$$M(g) \;=\; \underset{g}{\#}(S^1 \times S^1)$$

- **Example** $M(0) = S^2$, the 2-sphere.
- **Example** $M(1) = S^1 \times S^1$, the torus.
- **Example** $M(2) =$ the 2-holed torus, by Henry Moore.

The orientable closed surfaces $M(g)$ II.



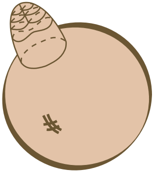
The nonorientable surfaces $N(g)$ I.

- Let $g \geqslant 1$. The **nonorientable connected surface with genus** g is the connected sum of g copies of $\mathbb{RP}^2$

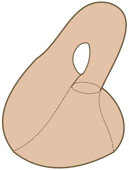
$$N(g) = \#_g \mathbb{RP}^2$$

- **Example** $N(1) = \mathbb{RP}^2$, the projective plane.
- Boy's immersion of $\mathbb{RP}^2$ in $\mathbb{R}^3$ (in Oberwolfach)

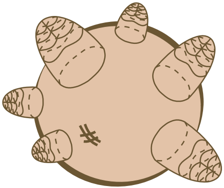
The nonorientable closed surfaces $N(g)$ II.



Projective plane = N(1)



Klein bottle = N(2)



N(g)

The Klein bottle

- **Example** $N(2) = K$ is the Klein bottle.
- The Klein bottle company

The classification theorem for closed surfaces

- **Theorem** Every connected closed surface M is homeomorphic to exactly one of

$$M(0)\ ,\ M(1)\ ,\ \ldots\ ,\ M(g) = \mathop{\#}_{g} S^1 \times S^1\ ,\ \ldots\ \text{(orientable)}$$
$$N(1)\ ,\ N(2)\ ,\ \ldots\ ,\ N(g) = \mathop{\#}_{g} \mathbb{RP}^2\ ,\ \ldots\ \text{(nonorientable)}$$

- Connected surfaces are classified by the genus g and orientability.
- Connected surfaces are classified by the fundamental group :

$$\pi_1(M(g)) \ = \ \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g \,|\, [a_1, b_1] \ldots [a_g, b_g] \rangle$$
$$\pi_1(N(g)) \ = \ \langle c_1, c_2, \ldots, c_g \,|\, (c_1)^2 (c_2)^2 \ldots (c_g)^2 \rangle$$

- Connected surfaces are classified by the Euler characteristic and orientability

$$\chi(M(g)) \ = \ 2 - 2g\ ,\ \chi(N(g)) \ = \ 2 - g\ .$$

The punctured torus I.

- The computation of $\pi_1(M(g))$ for $g \geqslant 0$ will be by induction, using the connected sum

$$M(g+1) \;=\; M(g)\#M(1)$$

- So need to understand the fundamental group of the torus $M(1) = T = S^1 \times S^1$ and the punctured torus (T_0, S^1).
- Clear from $T = S^1 \times S^1$ that $\pi_1(T) = \mathbb{Z} \oplus \mathbb{Z}$.
- Can also get this by applying the Seifert-van Kampen theorem to $M(1) = M(1)\#M(0)$, i.e. $T = T_0 \cup_{S^1} D^2$.
- The punctured torus

$$(T_0, \partial T_0) \;=\; (\mathrm{cl.}(S^1 \times S^1 \backslash D^2), S^1)$$

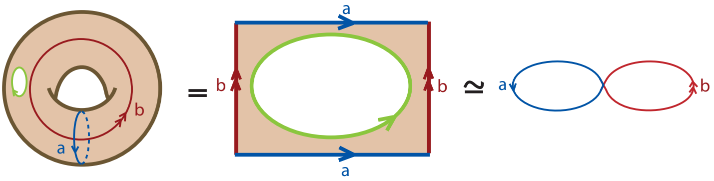
is such that $S^1 \vee S^1 \subset T_0$ is a homotopy equivalence.

The punctured torus II.

- The inclusion $\partial T_0 = S^1 \subset T_0$ induces

$$\pi_1(S^1) \;=\; \mathbb{Z} \to \pi_1(T_0) \;=\; \pi_1(S^1 \vee S^1) \;=\; \mathbb{Z} * \mathbb{Z} \;=\; \langle a, b \rangle \;;$$

$$1 \mapsto [a, b] \;=\; aba^{-1}b^{-1} \;.$$

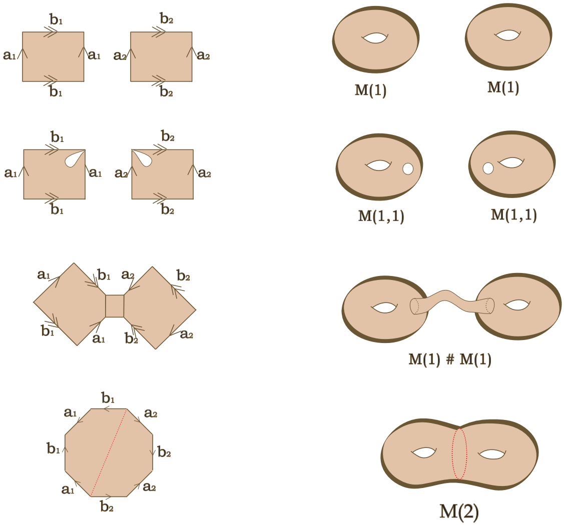


- The Seifert-van Kampen Theorem gives

$$\pi_1(T) \;=\; \pi_1(T_0) *_{\mathbb{Z}} \{1\} \;=\; \langle a, b \,|\, [a, b] \rangle \;=\; \mathbb{Z} \oplus \mathbb{Z} \;.$$

The calculation of $\pi_1(M(g))$ I.

- The initial case $g = 2$, using $M(2) = M(1)\#M(1)$



The calculation of $\pi_1(M(g))$ II. General case

- Assume inductively that
  - $\pi_1(M(g)) \;=\; \langle a_1, b_1, \dots, a_g, b_g \,|\, [a_1, b_1] \dots [a_g, b_g] \rangle$,
  - the punctured surface

    $$(M(g)_0, \partial M(g)_0) \;=\; (\mathrm{cl.}(M(g)\backslash D^2), S^1)$$

    is such that $\bigvee\limits_{2g} S^1 \subset M(g)_0$ is a homotopy equivalence,
  - the inclusion $\partial M(g)_0 = S^1 \subset M(g)_0$ induces

    $$\begin{aligned}\pi_1(S^1) \;=\; \mathbb{Z} \to \pi_1(M(g)_0) \;=\; \mathop{*}_{2g} \mathbb{Z} \;=\; \langle a_1, b_1, \dots, a_g, b_g \rangle \;;\\ 1 \mapsto [a_1, b_1][a_2, b_2] \dots [a_g, b_g]\;.\end{aligned}$$

- Apply the Seifert-van Kampen Theorem to

  $$M(g+1) \;=\; M(g)\# M(1)$$

  to obtain

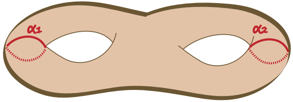
  $$\begin{aligned}\pi_1(M(g+1)) \;&=\; \pi_1(M(g)_0) *_{\mathbb{Z}} \pi_1(M(1)_0)\\ &=\; \langle a_1, b_1, \dots, a_{g+1}, b_{g+1} \,|\, [a_1, b_1] \dots [a_{g+1}, b_{g+1}] \rangle\end{aligned}$$

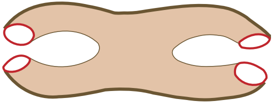
The genus measures connectivity. The orientable case

- The genus g of an orientable surface M is the maximum number of disjoint loops $\alpha_1, \alpha_2, \ldots, \alpha_g : S^1 \to M$ such that the complement $M \backslash \bigcup_{i=1}^{g} \alpha_i(S^1)$ is connected. The complement is homeomorphic to $M(0, 2g) \backslash \partial M(0, 2g)$.
- **Example** For $M = M(2)$ let $\alpha_1, \alpha_2 : S^1 \to M$ be disjoint loops which go round as in the diagram.
  The complement

$$M \backslash (\alpha_1(S^1) \cup \alpha_2(S^1)) \; = \; M(0, 4) \backslash \partial M(0, 4)$$

  is the sphere $M(0) = S^2$ with 4 holes punched out.





Morse theory

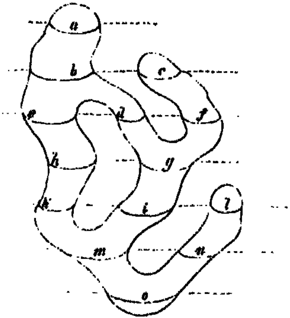
- For an orientable surface $M \subset \mathbb{R}^3$ in general position the height function

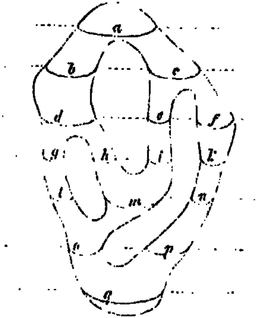
$$f \; : \; M \to \mathbb{R} \; ; \; (x, y, z) \mapsto z$$

  has the property that the inverse image $f^{-1}(c) \subset M$ is a 1-dimensional submanifold for all except a finite number $c \in \mathbb{R}$ called the critical values of f.
- Can recover the genus g of M by looking at the jumps in the number of circles in $f^{-1}(a)$ and $f^{-1}(b)$ for $a < b < c$.
- Morse theory developed (since 1926) is the key tool for studying n-manifolds for all $n \geqslant 0$.

An early exponent of Morse theory on a surface

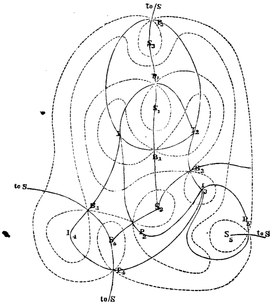
- August Ferdinand Möbius
  Theorie der elementaren Verwandschaften (1863)
- Fill a surface shaped bathtub with water, and recover the genus of the surface from a film of the cross-sections.





Another early exponent of Morse theory on a surface

- James Clerk Maxwell (1870) On hills and dales
- Reconstruct surface of the earth ($= S^2$) from contour lines.



- Mountaineer's equation for surface of Earth

$$\text{no. of peaks} - \text{no. of pits} + \text{no. of passes} \ = \ \chi(S^2) \ = \ 2 \ .$$

Modern account in Chapter 8 of Surfaces (CUP, 1976) by H.B.Griffiths

Cross-capMetric

- If M is a surface the connected sum

$$M' = M\#\mathbb{RP}^2$$

  is the surface obtained from M by forming a **crosscap** (*Kreuzhaube* in German).
- M' is homeomorphic to the identification space obtained from the punctured surface (M_0, S^1) by identifying $z \sim -z$ for $z \in S^1$

$$M' = M_0/\{z \sim -z\} .$$

- Equivalently, M' is obtained from M by punching out $D^2 \subset M$ and replacing it by a Möbius band.
- M' is nonorientable.
- **Example** If $M = S^2$ then $M' = \mathbb{RP}^2$.

The punctured projective plane I.

- The computation of $\pi_1(N(g))$ for $g \geqslant 1$ will be by induction, using the connected sum

$$N(g+1) \;=\; N(g)\# N(1)$$

with $N(1) = \mathbb{RP}^2$. Abbreviate $\mathbb{RP}^2 = P$.

- Need to understand the fundamental group of P and the punctured projective plane (P_0, S^1), i.e. the Möbius band.
- Clear from the universal double cover $p : S^2 \to P$ that

$$\pi_1(P) \;=\; \mathrm{Homeo}_p(P) \;=\; \mathbb{Z}_2\ .$$

- Can also get this by applying the Seifert-van Kampen Theorem to $N(1) = N(1)\# M(0)$, i.e. $P = P_0 \cup_{S^1} D^2$.

The punctured projective plane II.

- The punctured projective plane

$$(P_0, \partial P_0) \; = \; (\mathrm{cl.}(P \backslash D^2), S^1)$$

  is a Möbius band, such that $S^1 \subset P_0 \backslash \partial P_0$ is a homotopy equivalence.

- The inclusion $\partial P_0 = S^1 \subset P_0$ induces

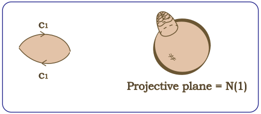
$$\pi_1(S^1) \; = \; \mathbb{Z} \to \pi_1(P_0) \; = \; \pi_1(S^1) \; = \; \mathbb{Z} \; ; \; 1 \mapsto 2 \; .$$

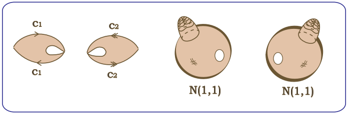
- The Seifert-van Kampen Theorem gives

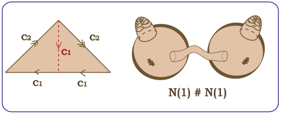
$$\pi_1(P) \; = \; \pi_1(P_0) *_{\mathbb{Z}} \{1\} \; = \; \langle c \, | \, c^2 \rangle \; = \; \mathbb{Z}_2 \; .$$

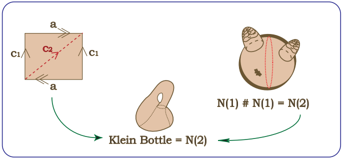
The calculation of $\pi_1(N(g))$ I.

- The initial case $g = 2$, using $N(2) = N(1)\#N(1)$ and $(N(1)_0, S^1) = $ (Möbius band,boundary circle).









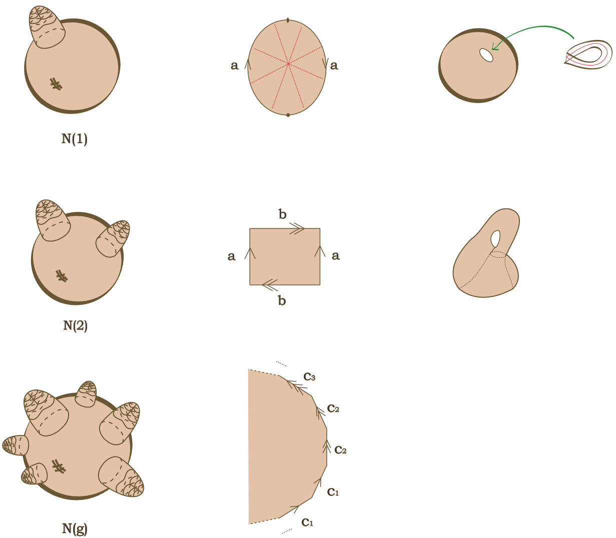
- By the Seifert-van Kampen Theorem, with $c_2 = (c_1')^{-1}$,

$$\begin{aligned}\pi_1(N(2)) &= \pi_1(N(1)\#N(1)) \\ &= \langle c_1, c_1' \,|\, (c_1)^2 = (c_1')^2 \rangle \;=\; \langle c_1, c_2 \,|\, (c_1)^2 (c_2)^2 \rangle \ .\end{aligned}$$

The calculation of $\pi_1(N(g))$ II.

- Assume inductively that
  - $\pi_1(N(g)) \;=\; \langle c_1, c_2, \dots, c_g \,|\, (c_1)^2(c_2)^2 \dots (c_g)^2 \rangle$,
  - the punctured surface
$$(N(g)_0, \partial N(g)_0) \;=\; (\text{cl.}(N(g)\backslash D^2), S^1)$$
is such that $\bigvee_g S^1 \subset N(g)_0$ is a homotopy equivalence,
  - the inclusion $\partial N(g)_0 = S^1 \subset N(g)_0$ induces
$$\begin{aligned} \pi_1(S^1) \;=\; \mathbb{Z} \to \pi_1(N(g)_0) \;=\; \ast_g \mathbb{Z} \;=\; \langle c_1, c_2, \dots, c_g \rangle \;; \\ 1 \mapsto (c_1)^2 \dots (c_g)^2 \;. \end{aligned}$$
- Apply the Seifert-van Kampen Theorem to
$$N(g+1) \;=\; N(g) \# N(1)$$
to obtain
$$\begin{aligned} \pi_1(N(g+1)) &\;=\; \pi_1(N(g)_0) \ast_{\mathbb{Z}} \pi_1(N(1)_0) \\ &\;=\; \langle c_1, \dots, c_{g+1} \,|\, (c_1)^2 \dots (c_{g+1})^2 \rangle \;. \end{aligned}$$

The calculation of $\pi_1(N(g))$ III.



The Euler characteristic of $M(g)$

- The fundamental group of $M(g)$ determines the genus g.
- The first homology group of $M(g)$ is the free abelian group of rank $2g$
$$H_1(M(g)) \ = \ \pi_1(M(g))^{ab} \ = \ \bigoplus_{2g} \mathbb{Z}$$
- $M(g)$ is homotopy equivalent to the 2-dimensional cell complex
$$(\bigvee_{2g} S^1) \cup_{[a_1,b_1]\dots[a_g,b_g]} D^2 \ = \ D^0 \cup \bigcup_{2g} D^1 \cup_{[a_1,b_1]\dots[a_g,b_g]} D^2 \ .$$
- The Euler characteristic of $M(g)$ is
$$\chi(M(g)) \ = \ 2 - 2g \ .$$
- A closed surface M is homeomorphic to S^2 if and only if $\chi(M) = 2$.

The Euler characteristic of $N(g)$

- The fundamental group determines the genus g.
- The first homology group of $N(g)$ is direct sum of the free abelian group of rank $g-1$ and the cyclic group of order 2

$$H_1(N(g)) = \pi_1(N(g))^{ab} = (\bigoplus_{g} \mathbb{Z})/(2,2,\ldots,2) = (\bigoplus_{g-1} \mathbb{Z}) \oplus \mathbb{Z}_2$$

- $N(g)$ is homotopy equivalent to the 2-dimensional cell complex

$$(\bigvee_{g} S^1) \cup_{(c_1)^2(c_2)^2\ldots(c_g)^2} D^2 = D^0 \cup \bigcup_{g} D^1 \cup_{(c_1)^2\ldots(c_g)^2} D^2 .$$

- $N(g)$ has Euler characteristic

$$\chi(N(g)) = 2-g .$$

The orientable surfaces with boundary $M(g,h)$

- Let $g \geqslant 0$, $h \geqslant 1$.
- **Definition** The **orientable surface of genus** g **and** h **boundary components** is

$$(M(g,h),\partial) \ = \ (\mathrm{cl.}(M(g)\backslash\bigcup_h D^2),\bigcup_h S^1)\ .$$

- Cell structure $M(g,h) \ \simeq \ \bigvee_{2g+h-1} S^1 = D^0 \cup \bigcup_{2g+h-1} D^1$
- Fundamental group $\pi_1(M(g,h)) = \mathop{*}_{2g+h-1} \mathbb{Z}$
- Euler characteristic $\chi(M(g,h)) = 2-2g-h$
- **Classification Theorem** Every connected orientable surface with non-empty boundary is homeomorphic to exactly one of $(M(g,h),\partial M(g,h))$.
- Set $M(g,0) = M(g)$.

Examples of orientable surfaces with boundary

- $(M(0,1),\partial) = (D^2, S^1)$, 2-disk
- $(M(0,2),\partial) = (S^1 \times [0,1], S^1 \times \{0,1\})$, cylinder
- $(M(1,1),\partial) = ((S^1 \times S^1)_0, S^1)$, punctured torus.
- $(M(0,3),\partial) = (\text{pair of pants}, S^1 \cup S^1 \cup S^1)$.
- The pair of pants is an essential feature of topological quantum field theory, and so appeared in Ida's birthday cake for the 80th birthday of Michael Atiyah (29 April, 2009)

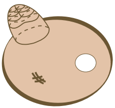
The nonorientable surfaces with boundary $N(g,h)$ I.

- Let $g \geqslant 1$, $h \geqslant 1$.
- **Definition** The **nonorientable surface with boundary with genus** g **with** h **boundary components** is

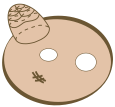
$$(N(g,h), \partial N(g,h)) \;=\; (\mathrm{cl.}(N(g)\backslash \bigcup_h D^2), \bigcup_h S^1)\ .$$

- Cell structure $N(g,h) \;\simeq\; \bigvee_{g+h-1} S^1 = D^0 \cup \bigcup_{g+h-1} D^1$.
- Fundamental group $\pi_1(N(g,h)) = \mathop{*}_{g+h-1} \mathbb{Z}$
- Euler characteristic $\chi(N(g,h)) = 2 - g - h$
- **Classification Theorem** Every connected nonorientable surface with non-empty boundary is homeomorphic to exactly one of $(N(g,h), \partial N(g,h))$.
- Set $N(g,0) = N(g)$.

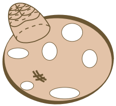
The nonorientable surfaces with boundary $N(g,h)$ II.



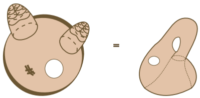
N(1,1)



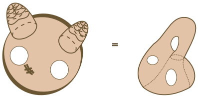
N(1,2)



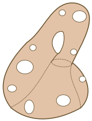
N(1, h)



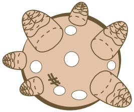
N(2,1)



N(2,1)



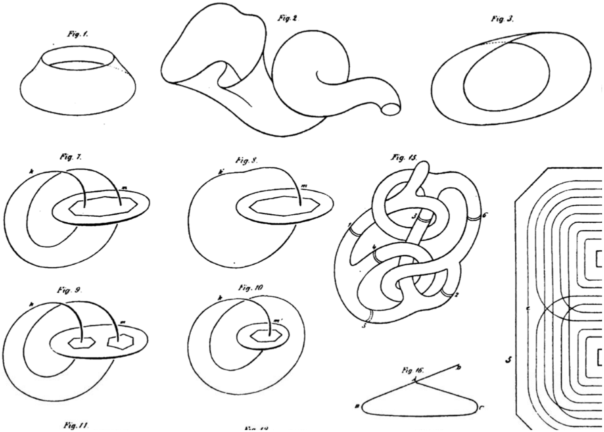
N(2, h)



N(g, h)

The Möbius band

- The Möbius band $(N(1,1), \partial N(1,1)) = ((\mathbb{RP}^2)_0, S^1)$.
- The first drawing of a Möbius band, from Listing' s 1862 Census der Räumlichen Complexe



The orientation double cover

- A **double cover** of a space N is a regular cover $\widetilde{N} \to N$ with fibre $F = \{0, 1\}$. Connected double covers of connected N are classified by index 2 subgroups $\pi_1(\widetilde{N}) \triangleleft \pi_1(N)$.
- A surface N has an **orientation double cover** $p : \widetilde{N} \to N$, with $\widetilde{N}$ an orientable surface. For connected N classified by the kernel of the orientation character group morphism

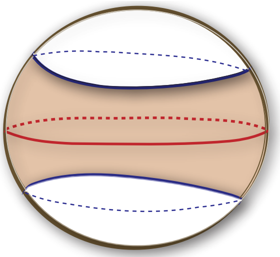
$$w \ : \ \pi_1(N) \to \mathbb{Z}_2 = \{+1, -1\}$$

  sending orientable (resp. nonorientable) α to $+1$ (resp. -1).
- If N is orientable $\widetilde{N} = N \cup N$ is the trivial double cover of N.
- If N is nonorientable w is onto, $\pi_1(\widetilde{N}) = \ker w$. Pullback along nonorientable $\alpha : S^1 \to N$ is the nontrivial double cover

$$q \ = \ \alpha^* p \ : \ S^1 \to S^1 \ ; \ z \mapsto z^2$$

$$\begin{tikzcd} S^1 \arrow[r, "\widetilde{\alpha}"] \arrow[d, "q"'] & \widetilde{N} \arrow[d, "p"] \\ S^1 \arrow[r, "\alpha"] & N \end{tikzcd}$$

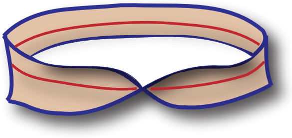
The orientation double cover of a Möbius band is a cylinder



$$S^1 \text{ x } I = M(0,2)$$

$$\downarrow$$

$$M = N(1,1)$$



$M(g-1,2h)$ is the orientation double cover of $N(g,h)$

- **Proposition** The orientation double cover of $N(g,h)$ is

$$\widetilde{N(g,h)} \ = \ M(g-1,2h) \ (g \geqslant 1, h \geqslant 0)$$

- **Proof** Let N be a connected nonorientable surface with orientation double cover $\widetilde{N}$. The boundary circle of $N_0 = \mathrm{cl.}(N \backslash D^2)$ is orientable. The orientation double cover of N_0 is the twice-punctured $\widetilde{N}$, $\widetilde{N}_{00} = \mathrm{cl.}(\widetilde{N} \backslash D^2 \cup D^2)$. The orientation double cover of $N' = N \# \mathbb{RP}^2$ is

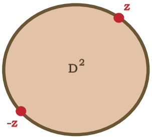
$$\widetilde{N}' \ = \ \widetilde{N}_{00} \cup_{S^1 \cup S^1} S^1 \times I \ .$$

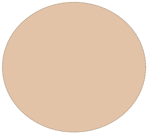
with $\chi(\widetilde{N}') = \chi(\widetilde{N}_{00}) = \chi(\widetilde{N}) - 2$. This gives the inductive step in checking that $\widetilde{N(g,h)} = M(g-1,2h)$.

- **Example** For $h = 0$, $g \geqslant 1$ have $\widetilde{N(g)} = M(g-1)$. Simply-connected for $g = 1$. For $g \geqslant 2$ universal cover $\mathbb{R}^2$.

The genus measures connectivity.
The nonorientable case

- The genus g of a nonorientable surface N is the maximum number of disjoint injective loops $\beta_1, \beta_2, \ldots, \beta_g : S^1 \to N$ such that the complement $N \backslash \bigcup_{i=1}^{g} \beta_i(S^1)$ is connected.
  The complement is homeomorphic to $M(0,g) \backslash \partial M(0,g)$.
- **Example** Let $N = \mathbb{RP}^2 = D^2/\{z \sim -z \mid z \in S^1\}$ and
  $$\beta \;:\; S^1 \;=\; \mathbb{RP}^1 \to \mathbb{RP}^2 \;;\; z \mapsto [\sqrt{z}] \;.$$
  The complement is
  $$\mathbb{RP}^2 \backslash \beta(S^1) \;=\; M(0,1) \backslash \partial M(0,1) \;=\; D^2 \backslash S^1 \;=\; \mathbb{R}^2 \;.$$





Further reading

- Google for "Classification of Surfaces" (147,000 hits)
- An Introduction to Topology. The classification theorem for surfaces by E.C. Zeeman (1966)
- A Guide to the Classification Theorem for Compact Surfaces by Jean Gallier and Dianna Xu (2011)
- Home Page for the Classification of Surfaces and the Jordan Curve Theorem Online resources, including many of the original papers.