NOTES ON ENDS OF COMPLEXES

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(Talks at Princeton, Notre Dame and Aberdeen, March-May, 1996)

These are notes for talks based on material in the book Ends of Complexes *by Bruce Hughes and A.R., published by Cambridge University Press as Cambridge Tract in Mathematics 123. The introduction is available on WWW from*
http://www.maths.ed.ac.uk/~aar
The paper of A.R. Finite domination and Novikov rings *(Topology 34, 619–632 (1995)) is also relevant. (Offprint available from same source).*

At the Princeton 250th anniversary conference in March of this year Milnor recalled his 1950's and 1960's prejudice in favour of compact manifolds and finite CW complexes, and his reasons for overcoming this prejudice. Milnor mentioned only Chapman's 1974 proof of the topological invariance of Whitehead torsion, and dynamical systems, but there are others, most notably Novikov's 1965 proof of the topological invariance of the rational Pontrjagin classes. I inherited Milnor's prejudice from his papers and books, and I have also become a convert to the wide open spaces of non-compact manifolds. In my case this was largely due to the development of *controlled topology*: this is a combination of algebra and geometry which is particularly suited to the asymptotic topology of non-compact manifolds.*

The algebraic topology of finite CW complexes suffices for compact PL manifolds. However, compact topological manifolds require the algebraic topology of infinite CW complexes. The non-compact manifolds usually arise as universal covers of compact manifolds (especially aspherical ones) with infinite fundamental group. The traditional algebraic topology method of studying non-compact spaces uses inverse limits of systems of neighbourhoods. Geometric topology deals with ends of non-compact spaces using methods of differential geometry, surgery theory and controlled algebra. The homotopy theory of inverse systems is not well adapted to geometry, so had to invent suitable chain homotopy theory, halfway between the geometry and algebra.

The *complexes* of the title of the book refer to CW (or simplicial) complexes in topology and chain complexes in algebra. An *end* of a complex is a subcomplex with a particular type of infinite behaviour, involving non-compactness in topology and infinite generation in algebra. These are to be regarded as tools in the study of manifolds, both compact and non-compact. *Tameness* is a topological growth condition on ends: tame ends arise fairly naturally in geometry, and are relatively easy to handle in algebra. Tame ends of manifolds just non-compact enough for finite domination and for K- and L-theory to decide if they are *collared* (= look like $K \times \mathbb{R}^+$). (The main result of the Topology paper is

* Actually, I have not entirely shed my prejudice in favour of compactness/finiteness, but non-compact spaces/infinite complexes happen to be the most powerful tools for studying the *topological* properties of compact manifolds!

the homological characterization of tameness: an end of an infinite cyclic cover of compact space is tame if and only if Novikov homology is 0). In section 1. there is a brief survey of various results from the last 50 years which motivated me. (I cannot vouch for Bruce Hughes' motivations, but not for nothing was he a student of Chapman.)

1. Historical background.

1950 Whitehead torsion.
1952 Serre uses path space fibrations for homotopy group computations.
1955 Nash uses a path space model PM for the tangent bundle TM of a compact manifold M to prove Thom's result on the topological invariance of the Stiefel-Whitney classes. (PM = the space of paths $\omega : [0,1] \longrightarrow M$ such that $\omega(t) \neq \omega(0)$ for $t \neq 0$).
1960 Smale h-cobordism theorem.
1961 Milnor's disproves the Hauptvermutung for compact polyhedra: combinatorial type is not topologically invariant.
1962 Stallings proves that for $n \geq 5$ a contractible open n-dimensional PL manifold which is simply connected at infinity is PL homeomorphic to $\mathbb{R}^n$.
1964 Barden-Mazur-Stallings s-cobordism theorem.
1964 Stallings' open h-cobordism theorem.
1965 Faddell develops path space model for normal bundle of submanifold.
1965 Browder proves that a simply-connected open n-dimensional PL manifold W with 2 ends which are simply connected at infinity is PL homeomorphic to $M \times \mathbb{R}$ for a compact $(n-1)$-dimensional manifold M.
1965 Wall finiteness obstruction: a finitely dominated CW complex X is homotopy equivalent to a finite CW complex if and only if an algebraic K-theory invariant $[X] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$ is 0.
1965 Browder, Levine and Livesay use surgery to prove that for $n \geq 6$ a simply-connected open n-dimensional PL manifold which is simply connected at infinity is interior of compact n-dimensional PL manifold.
1965 Siebenmann end obstruction: for $n \geq 6$ an open n-dimensional manifold with tame end ϵ is interior of compact n-dimensional manifold if and only if Wall finiteness obstruction $[\epsilon] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(\epsilon)])$ is 0.
1966 Novikov proves topological invariance of rational Pontrjagin classes.
1969 Casson and Sullivan disprove manifold Hauptvermutung. (Hauptvermutung book to be published by Kluwer in 1996).
1969 Kirby and Siebenmann structure theory for high-dimensional topological manifolds.
1970 Browder-Novikov-Sullivan-Wall surgery theory for classifying high-dimensional manifolds is complete.
1970 Siebenmann's infinite Whitehead torsion, and open s-cobordism theorem.
1974 Chapman uses Hilbert cube manifolds to prove the topological invariance of Whitehead torsion
1975 Browder and Quinn propose surgery theory for stratified sets.
1976 West uses Hilbert cube manifolds to prove Borsuk conjecture: every compact ANR is homotopy equivalent to a finite CW complex.
1978 Chapman, Ferry and Quinn develop controlled topology, the version of B-N-S-W

compact manifold surgery theory appropriate for non-compact manifolds. (Actually, controlled surgery theory is still under construction).

1979 Farrell and Hsiang apply controlled topology to prove Novikov conjectures for fundamental groups of manifolds with strong curvature conditions.

1985 Quinn develops surgery theory of stratified sets, using controlled surgery and path space model for the link of one stratum inside another.

1992 Cappell, Shaneson and Weinberger apply controlled surgery to stratified sets.

1993 Bryant, Ferry, Mio and Weinberger apply controlled surgery to construct compact ANR homology manifolds which cannot be resolved by topological manifolds.

1994 Carlsson and Pedersen apply controlled surgery and equivariant topology to prove Novikov conjectures for fundamental groups of manifolds with weak curvature conditions.

2. Homology at infinity.

The homology at ∞ is the classical algebraic topology invariant which depends only on the behaviour at ∞, i.e. away from compact subsets, being the difference between ordinary and locally finite homology.

Let W be a CW complex, and let I_r be an indexing set for the r-cells. The ordinary homology of W is the homology of the cellular chain complex

$$H_*(W) \;=\; H_*(C(W)) \;\;,\;\; C(W)_r \;=\; \sum_{I_r} \mathbb{Z}\;.$$

Milnor showed that *locally finite homology* of a finite-dimensional locally finite W is the homology of the locally finite cellular chain complex

$$H^{lf}_*(W) \;=\; H_*(C^{lf}(W)) \;\;,\;\; C^{lf}(W)_r \;=\; \prod_{I_r} \mathbb{Z}\;.$$

In fact, can use this to define H^{lf}_*. Define the homology at ∞ of W

$$H^{\infty}_*(W) \;=\; H_{*+1}(C(W)\longrightarrow C^{lf}(W))$$

to fit into the exact sequence

$$\dots \longrightarrow H^{\infty}_r(W) \longrightarrow H_r(W) \longrightarrow H^{lf}_r(W) \longrightarrow H^{\infty}_{r-1}(W) \longrightarrow \dots\;.$$

The homology at ∞ is a proper homotopy invariant of W which ignores all finite (= compact) subcomplexes $V \subseteq W$. (Proper means that inverse image of compact is compact). Here is the algebraic analogue:

Definition 1. Let A be any ring. Given a chain complex of based free A-modules C with $C_r = \sum_{I_r} A$ and such that the differentials $d : C_r \longrightarrow C_{r-1}$ are proper (i.e. the matrices have finite rows and finite columns) define the <u>end complex</u>

$$\begin{aligned} e(C) &= \mathcal{C}(i : C \longrightarrow C^{lf})_{*+1} \ , \\ i &= \text{inclusion} : C_r = \sum_{I_r} A \longrightarrow C^{lf}_r = \prod_{I_r} A \ , \\ e(C)_r &= C_r \oplus C^{lf}_{r+1} = \sum_{I_r} A \oplus \prod_{I_{r+1}} A \ , \ d_{e(C)} = \begin{pmatrix} d_C & 0 \\ (-)^r i & d_{C^{lf}} \end{pmatrix} . \end{aligned}$$

Key property: if $D \subseteq C$ is a f.g. free subcomplex then the natural projection $C \longrightarrow C/D$ induces a homology equivalence $e(C) \simeq e(C/D)$. Thus the homology $H_*(e(C))$ depends only on the *algebra at* ∞ of C, ignoring the f.g. free subcomplexes.

The *topology at* ∞ of an infinite CW complex W is given on the chain level by the end complex of the cellular chain complex $C(W)$

$$e(C(W)) = \mathcal{C}(C(W) \longrightarrow C^{lf}(W))_{*+1} \ .$$

<u>Example</u> If $A = \mathbb{Z}$, $C = C(W)$ then

$$H_*(e(C)) = H^\infty_*(W) \ .$$

<u>Example</u> If $W = K \times \mathbb{R}$ for a connected finite CW complex K then take $A = \mathbb{Z}$ and note that the standard CW structure on $\mathbb{R}$ gives

$$\begin{aligned} C(W) &= C(K) \otimes \mathcal{C}(1 - z : \mathbb{Z}[z, z^{-1}] \longrightarrow \mathbb{Z}[z, z^{-1}]) \simeq C(K) \ , \\ C^{lf}(W) &= C(K) \otimes \mathcal{C}(1 - z : \mathbb{Z}[[z, z^{-1}]] \longrightarrow \mathbb{Z}[[z, z^{-1}]]) \simeq C(K)_{*-1} \ , \\ e(C(W)) &\simeq C(K) \otimes \mathcal{C}(1 - z : \mathbb{Z}[[z, z^{-1}]]/\mathbb{Z}[z, z^{-1}] \longrightarrow \mathbb{Z}[[z, z^{-1}]]/\mathbb{Z}[z, z^{-1}]) \\ &\simeq C(K) \oplus C(K) \ \text{ corresponding to the two ends of } W, \end{aligned}$$

so that

$$H_*(W) = H_*(K) \ , \ H^{lf}_*(W) = H_{*-1}(K) \ , \ H^\infty_*(W) = H_*(K) \oplus H_*(K) \ .$$

In dealing with the universal cover $\widetilde{W}$ of a connected CW complex W with $\pi_1(W) = \pi$ take $A = \mathbb{Z}[\pi]$, $C(\widetilde{W})_r = \sum_{I_r} A$ and write

$$C^{lf,\pi}(\widetilde{W}) = C(\widetilde{W})^{lf}$$

with $C^{lf,\pi}(\widetilde{W})_r = \prod_{I_r} A$.

3. <u>Ends</u>

The ends of non-compact spaces were first studied in 3-dimensional manifolds, and also play an important role in the cohomology of infinite groups. Here, mainly interested in the structure of an individual tame end.

Definition 2. An end ϵ of a non-compact space W is an equivalence class of sequences $W \supset U_1 \supset U_2 \supset \ldots$ of connected open subsets such that each U_i is a component of $W \backslash K_i$, for a sequence of compact subsets $K_1 \subset K_2 \subset \ldots \subset W$, such that $\bigcap\limits_{i=1}^{\infty} \mathrm{cl}(U_i) = \emptyset$, subject to

$$(W \supset U_1 \supset U_2 \supset \ldots) \sim (W \supset V_1 \supset V_2 \supset \ldots)$$

if for each U_i there exists j with $U_i \subseteq V_j$, and for each V_j there exists i with $V_j \subseteq U_i$. $\square$

Example Let K be a connected compact space.
(i) $K \times \mathbb{R}^+$ has one end ϵ, with neighbourhoods $U_i = K \times (i, \infty) \subset K \times \mathbb{R}^+$.
(ii) $K \times \mathbb{R}$ has two ends ϵ^+, ϵ^-, with neighbourhoods $U_i^{\pm} = K \times (\pm i, \pm\infty) \subset K \times \mathbb{R}$.

Example If W is a non-compact space with a proper map $f : W \longrightarrow [0, \infty)$ such that the inverse images $W_t = f^{-1}[t, \infty)$ $(t \geq 0)$ are connected then W has one end.

4. Tame ends

Definition 3. A space X is finitely dominated if there exist a finite CW complex K and maps $f : X \longrightarrow K$, $g : K \longrightarrow X$ such that $gf \simeq 1 : X \longrightarrow X$. $\square$

Wall proved that a connected infinite CW complex X is finitely dominated if and only if $\pi_1(X)$ is finitely presented and the cellular $\mathbb{Z}[\pi_1(X)]$-module chain complex $C(\widetilde{X})$ of the universal cover $\widetilde{X}$ is chain equivalent to a finite f.g. projective $\mathbb{Z}[\pi_1(X)]$-module chain complex P. The reduced projective class

$$[X] = [P] = \sum_{r=0}^{\infty} (-)^r [P_r] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$$

is the Wall finiteness obstruction, such that $[X] = 0$ if and only if X is homotopy equivalent to a finite CW complex.

Definition 4. (Siebenmann) (i) An end ϵ of an open manifold W is tame if it admits a sequence $W \supset U_1 \supset U_2 \supset \ldots$ of finitely dominated connected open sets with

$$\bigcap_{i=1}^{\infty} \mathrm{cl}(U_i) = \emptyset \ , \ \pi_1(U_1) = \pi_1(U_2) = \ldots = \pi_1(\epsilon) \ .$$

(ii) An end ϵ of an open n-dimensional manifold W is collared if it has an open neighbourhood of the type $U = M \times \mathbb{R} \subset W$ for a closed $(n-1)$-dimensional manifold M, i.e. if ϵ is the interior of a closed manifold. $\square$

5. Wrapping up

The infinite cyclic covers of compact manifolds are the most interesting examples of manifold ends:

Theorem 1. (Siebenmann, 1965, Hughes-Ranicki, 1995)
Let W be a connected open n-dimensional manifold with compact boundary ∂W and one end ϵ.
(i) *The end ϵ is tame if (and for $n \geq 6$ only if) it has an open neighbourhood $U \subset W$ which is a finitely dominated infinite cyclic cover $U = \overline{V}$ of a compact n-dimensional manifold V (the wrapping up of ϵ) with $\pi_1(\epsilon) = \pi_1(\overline{V})$ and with a homeomorphism $U \times S^1 \cong V \times \mathbb{R}$.*
(ii) *For a tame ϵ the Wall finiteness obstruction of $\overline{V}^+$*

$$[\epsilon] \;=\; [\overline{V}^+] \in \widetilde{K}_0(\mathbb{Z}[\pi_1(\epsilon)])$$

is such that $[\epsilon] = 0$ if (and for $n \geq 6$ only if) ϵ can be collared, in which case $U = M \times \mathbb{R}$, $V = M \times S^1$. The wrapping up is such that $\epsilon \times S^1$ is collared, with $W \times S^1 = \overline{W \times S^1} \backslash V$ for a compact $(n+1)$-dimensional cobordism $(\overline{W \times S^1}; \partial W \times S^1, V)$. □

6. Forward and reverse tameness.

In the 1980's Quinn developed a version of the Siebenmann end invariant appropriate to stratified sets, distinguishing two kinds of tameness for a locally compact Hausdorff space W:

Definition 5. (i) The space W is reverse tame if for every cocompact subspace $U \subseteq W$ there exist a cocompact subspace $V \subseteq W$ with $V \subseteq U$ such that U is dominated by $U\backslash V$, by a homotopy $h : W \times I \longrightarrow W$ such that

(a) $h_0 = \mathrm{id}_W$,
(b) $h_t|_{(W\backslash U)} \;=\;$ inclusion $:\; W\backslash U \longrightarrow W$ for every $t \in I$,
(c) $h(U \times I) \subseteq U$,
(d) $h_1(W) \subseteq W\backslash V$.

(cocompact = closure of complement is compact).
(ii) The space W is reverse collared if for every cocompact subspace $U \subseteq W$ there exists a cocompact subspace $V \subseteq U$ such that $U\backslash V$ is is a strong deformation retract of U, in which case there exists a homotopy $h : W \times I \longrightarrow W$ as in (i). □

A *reverse* tame space admits a push in *from* infinity.

Definition 6. (i) The space W is forward tame if there exists a closed cocompact subspace $V \subseteq W$ such that the inclusion $V \times \{1\} \longrightarrow W$ extends to a proper map $q : V \times (0,1] \longrightarrow W$.
(ii) The space W is forward collared if there exists a closed cocompact ANR subspace $V \subseteq W$ with an extension of the identity $V \times \{1\} \longrightarrow V$ to a proper map $q : V \times (0,1] \longrightarrow V$. □

A *forward* tame space admits a push out *to* infinity.

Quinn (and H–R) use geometric Poincaré duality to prove a manifold end ϵ is forward tame if and only if it is reverse tame, if and only if it is tame in sense of Siebenmann.

7. End space

The end space $e(W)$ of a non-compact space W is a space whose homotopy type describes the algebraic topology of W at infinity, a kind of homotopy theoretic ideal boundary. Here is a typical application. If W is an open n-dimensional manifold then the one-point compactification $W^\infty = W \cup \{\infty\}$ is not in general a manifold. Is W the interior of compact n-dimensional manifold with boundary, i.e. can the singularity at ∞ in W^∞ be *resolved*? If W is (forward or reverse) tame then $(W, e(W))$ is an n-dimensional Poincaré pair. End spaces of tame manifold ends are thus a geometric source of Poincaré duality spaces. Algebraic K- and L-theory can be used to decide if $e(W)$ is homotopy equivalent to a closed $(n-1)$-dimensional manifold. The book (and talk) deals with the following questions: what is the end space $e(W)$?, what does it mean for W to be tame?, when can tame manifold W be closed? In principle, such questions were already answered by Siebenmann in 1965, but the answers need to be retooled for modern applications, e.g. to controlled topology and stratified sets.

Definition 7. (Quinn) The end space of a non-compact space W is the space of paths $\omega : ([0,\infty], \{\infty\}) \longrightarrow (W^\infty, \{\infty\})$ such that $\omega^{-1}(\infty) = \{\infty\}$, with the compact-open topology. □

The end space $e(W)$ is the *homotopy link* of $\{\infty\}$ in W^∞, with $\pi_0(e(W))$ the number of ends of W. An element $\omega \in e(W)$ can also be viewed as a path $\omega : [0,\infty) \longrightarrow W$ such that $\omega(t)$ *diverges to* ∞ as $t \to \infty$, meaning that for every compact subspace $K \subset W$ there exists $N > 0$ with $\omega([N,\infty)) \subset W \backslash K$.

Proposition (Quinn) W is forward tame if and only if the homotopy commutative square

$$\begin{array}{ccc} e(W) & \longrightarrow & \{\infty\} \\ \downarrow & & \downarrow \\ W & \longrightarrow & W^\infty \end{array}$$

is a homotopy pushout. □

Thus for a forward tame space W there is an exact sequence

$$\dots \longrightarrow H_r(e(W)) \longrightarrow H_r(W) \longrightarrow \widetilde{H}_r(W^\infty) \longrightarrow H_{r-1}(e(W)) \longrightarrow \dots$$

and

$$\widetilde{H}_*(W^\infty) = H_*^{lf}(W) \ , \ H_*(e(W)) = H_*^\infty(W) \ .$$

Thus if W is a forward tame CW complex

$$H_*(e(W)) = H_{*+1}(C(W) \longrightarrow C^{lf}(W)) = H_*(e(C(W))) = H_*^\infty(W) \ .$$

8. Algebraic forward and reverse tameness

Our results are most complete for the tameness and collaring properties of infinite cyclic covers of finite CW complexes – in view of the wrapping up construction this is not too serious a restriction.

First a purely algebraic result. Given a ring A and an automorphism $\alpha : A \longrightarrow A$ let $A_\alpha[z]$, $A_\alpha[z,z^{-1}]$ be the α-twisted polynomial extensions, so that

$$A_\alpha[z,z^{-1}] = \{ \sum_{j=-\infty}^{\infty} a_j z^j \,|\, \{j \in \mathbb{Z} | a_j \neq 0 \in A\} \text{ finite}\}$$

with $az = z\alpha(a)$ $(a \in A)$. Define also the power series (Novikov) rings

$$A_\alpha[[z]] \;=\; \text{the } (z)\text{-adic completion of } A_\alpha[z] \;=\; \{\sum_{j=0}^{\infty} a_j z^j \,|\, a_j \in A\}\ ,$$

$$A_\alpha((z)) \;=\; \{ \sum_{j=-\infty}^{\infty} a_j z^j \,|\, \{j \in \mathbb{Z}^- | a_j \neq 0 \in A\} \text{ finite}\}$$

and similarly for $A_\alpha[z^{-1}]$, $A_\alpha[[z^{-1}]]$, $A_\alpha((z^{-1}))$. If C^+ is a based f.g. free $A_\alpha[z]$-module chain complex then

$$(C^+)^{lf} \;=\; A_\alpha[[z]] \otimes_{A_\alpha[z]} C^+\ .$$

The localization and completion fit into a cartesian square of rings

$$\begin{array}{ccc} A_\alpha[z] & \longrightarrow & A_\alpha[[z]] \\ \downarrow & & \downarrow \\ A_\alpha[z,z^{-1}] & \longrightarrow & A_\alpha((z)) \end{array}$$

Theorem 2. (Ranicki, 1995)
(i) *A finite f.g. free $A_\alpha[z]$-module chain complex C^+ is A-finitely dominated if and only if $H_*(A_\alpha((z^{-1})) \otimes_{A_\alpha[z]} C^+) = 0$.*
(ii) *A finite f.g. free $A_\alpha[z,z^{-1}]$-module chain complex C is A-finitely dominated if and only if $H_*(A_\alpha((z)) \otimes_{A_\alpha[z,z^{-1}]} C) = 0$ and $H_*(A_\alpha((z^{-1})) \otimes_{A_\alpha[z,z^{-1}]} C) = 0$. It is possible to express C as*

$$C \;=\; A_\alpha[z,z^{-1}] \otimes_{A_\alpha[z]} C^+ \;=\; A_\alpha[z,z^{-1}] \otimes_{A_\alpha[z^{-1}]} C^-$$

for $A_\alpha[z]$-module subcomplex $C^+ \subset C$ and $A_\alpha[z^{-1}]$-module subcomplex $C^- \subset C$. Let

$$C^{lf} \;=\; A_\alpha[[z,z^{-1}]] \otimes_{A_\alpha[z,z^{-1}]} C\ .$$

Then C is A-finitely dominated if and only if the A-module chain maps

$$e(C^+) \longrightarrow C \quad , \quad e(C^-) \longrightarrow C$$

are chain equivalences, if and only if the connecting maps in the Mayer-Vietoris exact sequence

$$\ldots \longrightarrow H_r(A_\alpha((z)) \otimes_{A[z,z^{-1}]} C) \oplus H_r(A_\alpha((z^{-1})) \otimes_{A[z,z^{-1}]} C) \longrightarrow H_r(C^{lf}) \xrightarrow{\partial} H_{r-1}(C) \longrightarrow \ldots$$

are isomorphisms

$$\partial \;:\; H_*(C^{lf}) \;\cong\; H_{*-1}(C)\;. \qquad \square$$

(In particular, it is no accident that $H^{lf}_*(\mathbb{R}) \cong H_{*-1}(\mathbb{R})$!).

Now the topology. Let X be a connected finite CW complex with

$$\pi_1(X) \;=\; \pi \times_\alpha \mathbb{Z}$$

for some automorphism $\alpha : \pi \longrightarrow \pi$ of a group π. The projection $p : \pi_1(X) \longrightarrow \mathbb{Z}$ is realized by a map $p : X \longrightarrow S^1$ classifying a connected infinite cyclic cover $\overline{X} = p^*\mathbb{R}$ of X with two ends ϵ^+, ϵ^-. By analogy with manifold transversality make p CW transverse at $* \in S^1$, so that

$$\overline{X} \;=\; \overline{X}^+ \cup \overline{X}^-$$

with $\overline{X}^+ \cap \overline{X}^- = p^{-1}(*)$ a connected finite subcomplex such that

$$\pi_1(\overline{X}^+ \cap \overline{X}^-) \;=\; \pi_1(\overline{X}^-) \;=\; \pi_1(\overline{X}^+) \;=\; \pi_1(\overline{X}) \;=\; \pi$$

and $\overline{X}^\pm$ a closed neighbourhood of $\epsilon^\pm$.

The following result is purely geometric:

Theorem 3. (i) $\overline{X}^+$ *is forward tame if and only if* $\overline{X}^-$ *is reverse tame.*
(ii) $\overline{X}^+$ *is forward collared if and only if* $\overline{X}^-$ *is reverse collared.* $\square$

The property of being forward/reverse tame is detected by homology, with forward/reverse collaring detected by an algebraic K-theory invariant, as follows.

Identify $\mathbb{Z}[\pi_1(X)]$ with the α-twisted Laurent polynomial extension of $\mathbb{Z}[\pi]$

$$\mathbb{Z}[\pi_1(X)] \;=\; \mathbb{Z}[\pi]_\alpha[z, z^{-1}]$$

Let $\widetilde{X}$ be the universal cover of X, so that

$$\widetilde{X} \;=\; \widetilde{X}^+ \cup \widetilde{X}^-$$

with $\widetilde{X}^+$, $\widetilde{X}^-$ the universal covers of X^+, X^-. The cellular chain complex $C(\widetilde{X})$ is a finite f.g. free $\mathbb{Z}[\pi]_\alpha[z, z^{-1}]$-module chain complex, and $C(\widetilde{X}^+)$ is a finite f.g. free $\mathbb{Z}[\pi]_\alpha[z]$-module chain complex. Define the locally finite cellular $\mathbb{Z}[\pi]_\alpha[[z]]$-module chain complex of $\widetilde{X}^+$

$$C^{lf,\pi}(\widetilde{X}^+) \;=\; C(\widetilde{X}^+)^{lf} \;=\; \mathbb{Z}[\pi]_\alpha[[z]] \otimes_{\mathbb{Z}[\pi]_\alpha[z]} C(\widetilde{X}^+)\;.$$

Theorem 4. *The following conditions on a finite CW complex X with* $\pi_1(X) = \pi \times_\alpha \mathbb{Z}$ *and* π *finitely presented are equivalent:*

(i) $\overline{X}^-$ *is reverse tame,*

(ii) $\overline{X}^-$ *is finitely dominated,*

(iii) $C(\widetilde{X}^-)$ *is* $\mathbb{Z}[\pi]$*-module chain equivalent to a finite f.g. projective* $\mathbb{Z}[\pi]$*-module chain complex,*

(iv) $\overline{X}^+$ *is forward tame,*

(v) $C^{lf}(\widetilde{X}^+)$ *is* $\mathbb{Z}[\pi]$*-module chain equivalent to a finite f.g. projective* $\mathbb{Z}[\pi]$*-module chain complex,*

(vi) $H_*(X;\mathbb{Z}[\pi]_\alpha((z))) = 0$. □

Theorem 5. *If X is a finite CW complex such that the conditions of Theorem 4 are satisfied then the reduced projective class* $[\overline{X}^-] = [C(\widetilde{X}^-)] \in \widetilde{K}_0(\mathbb{Z}[\pi])$ *and the reduced locally finite projective class* $[\overline{X}^+]^{lf} = [C^{lf}(\widetilde{X}^+)] \in \widetilde{K}_0(\mathbb{Z}[\pi])$ *are such that*

$$[\overline{X}^+]^{lf} \ = \ -[\overline{X}^-] \in \widetilde{K}_0(\mathbb{Z}[\pi])$$

and the following conditions are equivalent:

(i) $\overline{X}^-$ *is reverse collared,*

(ii) $\overline{X}^-$ *is homotopy equivalent to a finite CW complex,*

(iii) $[\overline{X}^-] = 0 \in \widetilde{K}_0(\mathbb{Z}[\pi])$,

(iv) $\overline{X}^+$ *is forward collared,*

(v) $[\overline{X}^+]^{lf} = 0 \in \widetilde{K}_0(\mathbb{Z}[\pi])$. □

Example 3. For any integer $s \geq 2$ let $X = T(s : S^1 \longrightarrow S^1)$, so that $\pi_1(X) = \pi \times_\alpha \mathbb{Z}$ with $\pi = \mathbb{Z}[1/s]$, $\alpha = s$. Then $\overline{X}^+$ is forward collared but not reverse tame ($\pi_1(\overline{X}^+) = \pi$ is not finitely presented!), and $\overline{X}^-$ is reverse collared but not forward tame, with $H_*(T(s);\mathbb{Z}[\pi]_\alpha((z))) = 0$ and $H_*(T(s);\mathbb{Z}[\pi]_\alpha((z^{-1}))) = \mathbb{Z}[\pi]_\alpha((z^{-1})) \neq 0$. (In fact, $\overline{X}^- \simeq S^1$.) □

Theorem 6. (i) $\overline{X}^+$ *is forward tame if and only if the natural map* $e(\overline{X}^+) \longrightarrow \overline{X}$ *is a homotopy equivalence, if and only if the* $\mathbb{Z}[\pi]$*-module chain map* $e(C(\widetilde{X}^+)) \longrightarrow C(\widetilde{X})$ *is a chain equivalence.*

(ii) $\overline{X}$ *is finitely dominated if and only if* π *is finitely presented and the natural maps*

$$e(\overline{X}^+) \longrightarrow \overline{X} \quad , \quad e(\overline{X}^-) \longrightarrow \overline{X}$$

are homotopy equivalences, if and only if the $\mathbb{Z}[\pi]$*-module chain maps*

$$e(C(\widetilde{X}^+)) \longrightarrow C(\widetilde{X}) \quad , \quad e(C(\widetilde{X}^-)) \longrightarrow C(\widetilde{X})$$

are chain equivalences, if and only if the connecting maps in the Mayer-Vietoris exact sequence

$$\dots \longrightarrow H_r(X;\mathbb{Z}[\pi]_\alpha((z))) \oplus H_r(X;\mathbb{Z}[\pi]_\alpha((z^{-1}))) \longrightarrow H_r^{lf,\pi}(\widetilde{X}) \xrightarrow{\partial} H_{r-1}(\widetilde{X}) \longrightarrow \dots$$

are isomorphisms $\partial : H_*^{lf,\pi}(\widetilde{X}) \cong H_{*-1}(\widetilde{X})$. □