THE NUMBER EIGHT IN TOPOLOGY

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http://www.maths.ed.ac.uk/∼aar/eight.htm

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Sociology and topology

- *It is a fact of sociology that topologists are interested in quadratic forms* (Serge Lang)
- The 8 in the title refers to the applications in topology of the mod 8 properties of the signatures of integral symmetric matrices, such as the celebrated 8×8 matrix E_8 with

$$\text{signature}(E_8) \; = \; 8 \in \mathbb{Z} \; .$$

- A compact oriented $4k$-manifold with boundary has an integral symmetric matrix of intersection numbers. The signature of the manifold is defined by

$$\text{signature(manifold)} \; = \; \text{signature(matrix)} \in \mathbb{Z} \; .$$

- Manifolds with intersection matrix E_8 have been used to distinguish the categories of differentiable, PL and topological manifolds, and so are of particular interest to topologists!

Quadratic forms and manifolds

- The algebraic properties of quadratic forms were already studied in the 19th century: Sylvester, H.J.S. Smith, ...
- Similarly, the study of the topological properties of manifolds reaches back to the 19th century: Riemann, Poincaré, ...
- The combination of algebra and topology is very much a 20th century story. But in 1923 when Weyl first proposed the definition of the signature of a manifold, topology was so dangerous that he thought it wiser to write the paper in Spanish and publish it in Spain. And this is his signature :

Symmetric matrices

- $R =$ commutative ring. Main examples today: $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{Z}_4$, $\mathbb{Z}_2$.
- The **transpose** of an $m \times n$ matrix $\Phi = (\Phi_{ij})$ with $\Phi_{ij} \in R$ is the $n \times m$ matrix Φ^T with

$$(\Phi^T)_{ji} \;=\; \Phi_{ij}\ (1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n)\ .$$

- Let $\mathrm{Sym}_n(R)$ be the set of $n \times n$ matrices Φ which are **symmetric** $\Phi^T = \Phi$.
- $\Phi, \Phi' \in \mathrm{Sym}_n(R)$ are **conjugate** if $\Phi' = A^T \Phi A$ for an invertible $n \times n$ matrix $A \in GL_n(R)$.
- Can also view Φ as a symmetric bilinear pairing on the n-dimensional f.g. free R-module R^n

$$\Phi\ :\ R^n \times R^n \to R\ ;\ ((x_1,\dots,x_n),(y_1,\dots,y_n)) \mapsto \sum_{i=1}^{n}\sum_{j=1}^{n} \Phi_{ij} x_i y_j\ .$$

- $\Phi \in \mathrm{Sym}_n(R)$ is **unimodular** if it is invertible, or equivalently if $\det(\Phi) \in R$ is a unit.

The signature

- The **signature** of $\Phi \in \mathrm{Sym}_n(\mathbb{R})$ is
$$\sigma(\Phi) \ = \ p_+ - p_- \in \mathbb{Z}$$
with p_+ the number of eigenvalues > 0 and p_- the number of eigenvalues < 0.
- **Law of Inertia** (Sylvester 1853)
Symmetric matrices $\Phi, \Phi' \in \mathrm{Sym}_n(\mathbb{R})$ are conjugate if and only if
$$p_+ \ = \ p'_+ \ , \ p_- \ = \ p'_- \ .$$
- The signature of $\Phi \in \mathrm{Sym}_n(\mathbb{Z})$
$$\sigma(\Phi) \ = \ \sigma(\mathbb{R} \otimes_{\mathbb{Z}} \Phi) \in \mathbb{Z}.$$
is an integral conjugacy invariant.
- The conjugacy classification of symmetric matrices is much harder for $\mathbb{Z}$ than $\mathbb{R}$. For example, can diagonalize over $\mathbb{R}$ but not over $\mathbb{Z}$.

Type I and type II

- $\Phi \in \mathrm{Sym}_n(\mathbb{Z})$ is of **type I** if at least one of the diagonal entries $\Phi_{ii} \in \mathbb{Z}$ is odd.
- Φ is of **type II** if each $\Phi_{ii} \in \mathbb{Z}$ is even.
- Type I cannot be conjugate to type II. So unimodular type II cannot be diagonalized, i.e. not conjugate to $\bigoplus_n \pm 1$.
- Φ is **positive definite** if $n = p_+$, or equivalently if $\sigma(\Phi) = n$. Choosing an orthonormal basis for $\mathbb{R} \otimes_{\mathbb{Z}} (\mathbb{Z}^n, \Phi)$ defines an embedding as a lattice $(\mathbb{Z}^n, \Phi) \subset (\mathbb{R}^n, \text{dot product})$. Lattices (including E_8) much used in coding theory.
- **Examples**
  (i) $\Phi = (1) \in \mathrm{Sym}_1(\mathbb{Z})$ is unimodular, positive definite, type I, signature 1.
  (ii) $\Phi = (2) \in \mathrm{Sym}_1(\mathbb{Z})$ is positive definite, type II, signature 1.
  (iii) $\Phi = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in \mathrm{Sym}_2(\mathbb{Z})$ is unimodular, type I, signature 0.
  (iv) $\Phi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{Sym}_2(\mathbb{Z})$ is unimodular, type II, signature 0.

Characteristic elements and the signature mod 8

- An element $u \in R^n$ is **characteristic** for $\Phi \in \mathrm{Sym}_n(R)$ if

$$\Phi(x,u) - \Phi(x,x) \in 2R \subseteq R \text{ for all } x \in R^n \ .$$

- Every unimodular Φ admits characteristic elements $u \in R^n$ which constitute a coset of $2R^n \subseteq R^n$.
- **Theorem** (van der Blij, 1958) The mod 8 signature of a unimodular $\Phi \in \mathrm{Sym}_n(\mathbb{Z})$ is such that

$$\sigma(\Phi) \ \equiv \ \Phi(u,u) \bmod 8$$

  for any characteristic element $u \in \mathbb{Z}^n$.
- **Corollary** A unimodular $\Phi \in \mathrm{Sym}_n(\mathbb{Z})$ is of type II if and only if $u = 0 \in \mathbb{Z}^n$ is characteristic, in which case

$$\sigma(\Phi) \ \equiv \ 0 \bmod 8 \ .$$

The E_8-form I.

- **Theorem** (H.J.S. Smith 1867, Korkine and Zolotareff 1873) There exists an 8-dimensional unimodular positive definite type II symmetric matrix

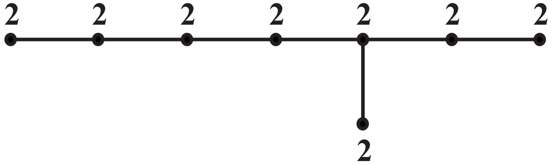
$$E_8 = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 \end{pmatrix} \in \mathrm{Sym}_8(\mathbb{Z}) \ .$$

- E_8 has signature

$$\sigma(E_8) = 8 \in \mathbb{Z} \ .$$

The E_8-form II.

- $E_8 \in \mathrm{Sym}_8(\mathbb{Z})$ is determined by the Dynkin diagram of the simple Lie algebra E_8



weighted by $\chi(S^2) = 2$ at each vertex, with

$$\Phi_{ij} = \begin{cases} 1 & \text{if } i\text{th vertex is adjacent to } j\text{th vertex} \\ 2 & \text{if } i = j \\ 0 & \text{otherwise .} \end{cases}$$

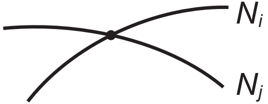
- **Theorem** (Mordell, 1938) Any unimodular positive definite type II symmetric matrix $\Phi \in \mathrm{Sym}_8(\mathbb{Z})$ is conjugate to E_8.

The intersection matrix of a $4k$-manifold

- The **intersection matrix** of a $4k$-manifold with boundary $(M, \partial M)$ with respect to a basis $(b_1, b_2, \ldots, b_n)$ for $H_{2k}(M)/\text{torsion} \cong \mathbb{Z}^n$ is the symmetric matrix

$$\Phi(M) = (b_i \cap b_j)_{1 \leqslant i,j \leqslant n} \in \mathrm{Sym}_n(\mathbb{Z})$$

  with $b_i \cap b_j \in \mathbb{Z}$ the homological intersection number.
- If b_i, b_j are represented by disjoint closed $2k$-submanifolds $N_i, N_j \subset M$ which intersect transversely then $b_i \cap b_j \in \mathbb{Z}$ is the number of points in the actual intersection $N_i \cap N_j \subset M$, counted algebraically.



- A different basis gives a conjugate intersection matrix.

$(2k-1)$-connected $4k$-manifolds

- A space M is **0-connected** if it is connected.
- For $j \geqslant 1$ a space M is **j-connected** if it is connected, and $\pi_i(M) = \{1\}$ for $1 \leqslant i \leqslant j$ or equivalently if M is simply-connected ($\pi_1(M) = \{1\}$) and $H_i(M) = 0$ for $1 \leqslant i \leqslant j$.
- An m-manifold with boundary $(M, \partial M)$ is **j-connected** if M is j-connected and ∂M is $(j-1)$-connected.
- **Proposition** If $(M, \partial M)$ is a $(2k-1)$-connected $4k$-manifold with boundary then
  - $H_{2k}(M)$ is f.g. free,
  - there is an exact sequence

$$0 \to H_{2k}(\partial M) \to H_{2k}(M) \xrightarrow{\Phi(M)} H_{2k}(M)^* \to H_{2k-1}(\partial M) \to 0$$

with $H_{2k}(M)^* = \mathrm{Hom}_{\mathbb{Z}}(H_{2k}(M), \mathbb{Z})$.

Homology spheres

- A **homology ℓ-sphere** Σ is a closed ℓ-manifold such that

$$H_*(\Sigma) = H_*(S^\ell) .$$

- An m-manifold with boundary $(M, \partial M)$ is **almost closed** if

  either M is closed, i.e. $\partial M = \emptyset$,

  or ∂M is a homology $(m-1)$-sphere

$$H_*(\partial M) = H_*(S^{m-1}) .$$

- **Proposition** The intersection matrix $\Phi(M) \in \mathrm{Sym}_n(\mathbb{Z})$ of a $(2k-1)$-connected $4k$-dimensional manifold with boundary $(M, \partial M)$ with $H_{2k}(M) = \mathbb{Z}^n$ is unimodular if and only if $(M, \partial M)$ is almost closed.

The $2k^{th}$ Wu class of an almost closed $(M^{4k}, \partial M)$

- **Proposition** For an almost closed $(2k-1)$-connected $4k$-manifold with boundary $(M^{4k}, \partial M)$ and intersection matrix $\Phi(M) \in \mathrm{Sym}_n(\mathbb{Z})$ the Poincaré dual of the $2k^{th}$ Wu characteristic class of the normal bundle ν_M
$$v_{2k}(\nu_M) \in H^{2k}(M;\mathbb{Z}_2) \;\cong\; H_{2k}(M;\mathbb{Z}_2)$$
is characteristic for $1 \otimes \Phi(M) \in \mathrm{Sym}_n(\mathbb{Z}_2)$. An element $u \in H_{2k}(M)$ is characteristic for $\Phi(M)$ if and only if
$$[u] \;=\; v_{2k}(\nu_M) \in H_{2k}(M)/2H_{2k}(M) \;=\; H_{2k}(M;\mathbb{Z}_2)\ .$$
.
- $\Phi(M)$ is of type II if and only if $v_{2k}(\nu_M) = 0$.
- By van der Blij's theorem, for any lift $u \in H_{2k}(M)$ of $v_{2k}(\nu_M)$.
$$\sigma(M) \;\equiv\; \Phi(u,u) \ \bmod 8\ .$$
- If $(M^{4k}, \partial M)$ is framed, i.e. ν_M is trivial, then
$$v_{2k}(\nu_M) \;=\; 0\ ,\ \ u \;= 0 \text{ and } \sigma(M) \;\equiv\; 0\,(\mathrm{mod} 8)\ .$$

The Poincaré homology 3-sphere and E_8

- Poincaré (1904) constructed a differentiable homology 3-sphere

$$\Sigma^3 \;=\; \text{dodecahedron}/\text{opposite faces}$$

  with $\pi_1(\Sigma^3) = \text{binary icosahedral group of order } 120 \neq \{1\}$. This disproved the naive **Poincaré conjecture** that every homology 3-sphere is homeomorphic to S^3.

- Modern construction: $\Sigma^3 = \partial M$ is the boundary of the 1-connected framed differentiable 4-manifold with boundary $(M^4, \partial M)$ with intersection matrix $\Phi(M) = E_8$ obtained by the "geometric plumbing" of 8 copies of τ_{S^2} using the E_8 tree.

Exotic spheres and E_8

- An **exotic ℓ-sphere** Σ^ℓ is a differentiable ℓ-manifold which is homeomorphic but not diffeomorphic to S^ℓ.
- Milnor (1956) constructed the first exotic spheres, Σ^7, using the Hirzebruch signature theorem (1953) to detect non-standard differentiable structure.
- Kervaire and Milnor (1963) classified exotic ℓ-spheres Σ^ℓ for all $\ell \geqslant 7$, involving the finite abelian groups Θ_ℓ of differentiable structures on S^ℓ.
- The subgroup $bP_{4k} \subseteq \Theta_{4k-1}$ consists of the exotic $(4k-1)$-spheres $\Sigma^{4k-1} = \partial M$ which are the boundary of a framed $(2k-1)$-connected $4k$-manifold $(M^{4k}, \partial M)$ obtained by geometric plumbing, with $\Phi(M) = \bigoplus E_8$.
- In particular, the Brieskorn (1965) exotic spheres arising in algebraic geometry are such boundaries, including the Poincaré homology 3-sphere Σ^3 as a special case.

bP_{4k}

- The subgroup $bP_{4k} \subseteq \Theta_{4k-1}$ of diffeomorphism classes of the bounding exotic spheres $\Sigma^{4k-1} = \partial M$ is a finite cyclic group $\mathbb{Z}_{bp_{4k}}$, with an isomorphism

$$bP_{4k} \xrightarrow{\cong} \mathbb{Z}_{bp_{4k}} \ ; \ \Sigma^{4k-1} = \partial M \mapsto \sigma(M)/8 \ .$$

- The order $|bP_{4k}| = bp_{4k}$ is related to the numerators of the Bernoulli numbers.
- The group

$$bP_8 \ = \ \Theta_7 \ = \ \mathbb{Z}_{28}$$

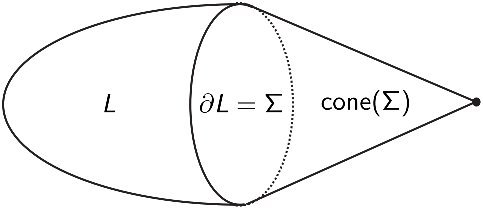
  of 28 differentiable structures on S^7 is generated by $\Sigma^7 = \partial M$ with $\Phi(M) = E_8$.

PL manifolds without differentiable structure I.

- Cairns (1935) proved that a differentiable manifold has a canonical PL structure.
- If $(L^m, \partial L)$ is a differentiable m-manifold with boundary $\partial L = \Sigma^{m-1}$ an exotic $(m-1)$-sphere then

$$K^m \;=\; L^m \cup_{\Sigma} \text{cone}(\Sigma)$$

is a closed PL m-manifold without a differentiable structure.



PL manifolds without differentiable structure II.

- The first PL manifold without a differentiable structure was the closed 4-connected PL 10-manifold constructed by Kervaire (1960)
$$K^{10} \;=\; L^{10} \cup_{\partial L} c\partial L$$
using a framed differentiable 4-connected 10-manifold $(L^{10}, \partial L)$ with boundary an exotic 9-sphere ∂L, obtained by plumbing two τ_{S^5}'s. The corresponding $\mathbb{Z}_2$-valued quadratic form on $H^5(K;\mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ has Arf invariant $1 \in \mathbb{Z}_2$.
- The E_8-plumbing $(M^8, \partial M)$ gives a closed 3-connected PL 8-manifold $M^8 \cup_{\partial M} c\partial M$ without a differentiable structure.
- In fact, there is a close connection between the $\mathbb{Z}_8$-valued signature mod 8 and the $\mathbb{Z}_2$-valued Arf invariant, which is best understood using symmetric matrices in $\mathbb{Z}_4$.

The classification of 1-connected 4-manifolds

- Milnor (1958) proved that $M^4 \mapsto \Phi(M)$ defines a bijection

  $$\{\text{homotopy equivalence classes of closed } 1\text{-connected differentiable 4-manifolds } M^4\} \xrightarrow{\cong} \{\text{conjugacy classes of unimodular integral symmetric matrices } \Phi\} .$$

- **Diagonalisation Theorem** (Donaldson 1982) If M^4 is a closed 1-connected differentiable 4-manifold and $\Phi(M)$ is positive definite then $\Phi(M)$ is diagonalizable over $\mathbb{Z}$.
- **Non-diagonalisation Theorem** (Freedman 1982) Every unimodular matrix $\Phi \in \mathrm{Sym}_n(\mathbb{Z})$ is realized as $\Phi = \Phi(M)$ for a closed 1-connected topological 4-manifold M^4. If Φ is of type II and M has a PL structure then $\sigma(M) \equiv 0 (\bmod 16)$ (Rochlin 1952).
- **Nontriangulable manifolds** Casson (1990) : M^4 with $\Phi(M) = E_8$ is nontriangulable. Manolescu (2013) : there are nontriangulable topological m-manifolds M^m for all $m \geqslant 4$.

Which integral symmetric matrices are realized as intersection matrices of manifolds? I.

- Adams (1962) proved that there exists a map $S^{4k-1} \to S^{2k}$ of Hopf invariant 1 if and only if $k = 1, 2, 4$. It followed that there exists a closed differentiable $(2k-1)$-connected $4k$-manifold M^{4k} with intersection matrix $\Phi(M)$ of type I if and only if $k = 1, 2, 4$.
- The standard examples of $(2k-1)$-connected M^{4k} with

$$(H_{2k}(M), \Phi(M)) \ = \ (\mathbb{Z}, 1)$$

of type I :
(i) $k = 1$: the complex projective plane $\mathbb{C}\,\mathbb{P}^2$,
(ii) $k = 2$: the quaternionic projective plane $\mathbb{H}\,\mathbb{P}$ (Hamilton),
(iii) $k = 4$: the octonionic projective plane $\mathbb{O}\,\mathbb{P}$ (Cayley).

Which integral symmetric matrices are realized as intersection matrices of manifolds? II.

- **Theorem** (Milnor, Hirzebruch 1962) For every symmetric matrix $\Phi \in \mathrm{Sym}_n(\mathbb{Z})$ of type II and every $k \geqslant 1$ there exists a differentiable $(2k-1)$-connected $4k$-manifold $(M, \partial M)$ with intersection matrix $\Phi(M) = \Phi$.
- $(M, \partial M)$ is constructed by the "geometric plumbing" of a sequence of n oriented $2k$-plane bundles over S^{2k}

$$\mathbb{R}^{2k} \to E(w_i) \to S^{2k} \ (1 \leqslant i \leqslant n)$$

  classified by $w_i \in \pi_{2k}(BSO(2k))$, with Euler numbers $\chi(w_i) = \Phi_{ii} \in 2\mathbb{Z} \subset \mathbb{Z}$.
- The geometry reflects the way in which Φ is built up from 0 by the "algebraic plumbing" of its n principal minors

$$(\Phi_{11}) \ , \ \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} \ , \ \begin{pmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} \\ \Phi_{21} & \Phi_{22} & \Phi_{23} \\ \Phi_{31} & \Phi_{32} & \Phi_{33} \end{pmatrix} \ , \ \dots \ , \ \Phi$$

Algebraic plumbing

- **Definition** The **algebraic plumbing** of a symmetric $n \times n$ matrix $\Phi \in \mathrm{Sym}_n(\mathbb{Z})$ with respect to $v \in \mathbb{Z}^n$, $w \in \mathbb{Z}$ is the symmetric $(n+1) \times (n+1)$ matrix

$$\Phi' \;=\; \begin{pmatrix} \Phi & v^T \\ v & w \end{pmatrix} \in \mathrm{Sym}_{n+1}(\mathbb{Z}) \;.$$

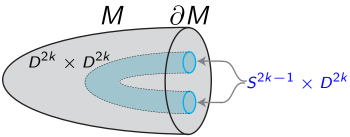
- Let $\Phi = \Phi(M) \in \mathrm{Sym}_n(\mathbb{Z})$ is the intersection matrix of a $(2k-1)$-connected $4k$-manifold with boundary $(M, \partial M)$, taken to be (D^{4k}, S^{4k-1}) if $n = 0$. It is frequently possible to realize the algebraic plumbing $\Phi \mapsto \Phi'$ by a geometric plumbing

$$(M, \partial M) \mapsto (M', \partial M') \;,\; \Phi(M') \;=\; \Phi' \in \mathrm{Sym}_{n+1}(\mathbb{Z})$$

  and $(M', \partial M')$ also $(2k-1)$-connected.
- Need $k = 1, 2, 4$ for type I. All $k \geqslant 1$ possible for type II. For $k = 1$ have to distinguish differentiable and topological categories.

Geometric plumbing I.

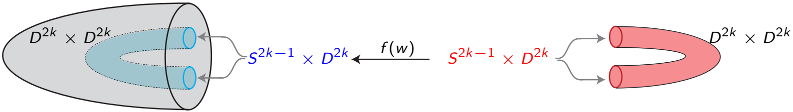
- **Input** (i) A $4k$-manifold with boundary $(M, \partial M)$,
  (ii) an embedding $v : (D^{2k} \times D^{2k}, S^{2k-1} \times D^{2k}) \subseteq (M, \partial M)$



  (iii) a map $w : S^{2k-1} \to SO(2k)$, the clutching map of the oriented $2k$-plane bundle over $S^{2k} = D^{2k} \cup_{S^{2k-1}} D^{2k}$ classified by $w \in \pi_{2k-1}(SO(2k)) = \pi_{2k}(BSO(2k))$

$$\mathbb{R}^{2k} \to E(w) \;=\; D^{2k} \times \mathbb{R}^{2k} \cup_{f(w)} D^{2k} \times \mathbb{R}^{2k} \to S^{2k}$$

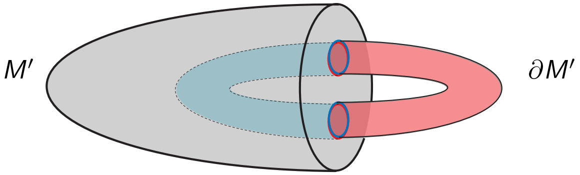
$$f(w) \;:\; S^{2k-1} \times \mathbb{R}^{2k} \to S^{2k-1} \times \mathbb{R}^{2k} \;;\; (x, y) \mapsto (x, w(x)(y)) \;.$$



Geometric plumbing II.

- **Output** The **plumbed** $4k$-manifold with boundary

$$(M', \partial M') = (M \cup_{f(w)} D^{2k} \times D^{2k}, \mathrm{cl.}(\partial M \backslash S^{2k-1} \times D^{2k}) \cup D^{2k} \times S^{2k-1}) \ .$$



- M' is obtained from M by attaching a $2k$-handle $D^{2k} \times D^{2k}$ at $S^{2k-1} \times D^{2k} \subset \partial M$.
- $\partial M'$ is obtained from ∂M by surgery on $S^{2k-1} \times D^{2k} \subset \partial M$.

The algebraic effect of geometric plumbing

- **Proposition** If $(M^{4k}, \partial M)$ has symmetric intersection matrix $\Phi(M) \in \mathrm{Sym}_n(\mathbb{Z})$ the geometric plumbing $(M', \partial M')$ has the symmetric intersection matrix given by algebraic plumbing

$$\Phi(M') = \begin{pmatrix} \Phi(M) & v^T \\ v & \chi(w) \end{pmatrix} \in \mathrm{Sym}_{n+1}(\mathbb{Z})$$

with

$$v = v[D^{2k} \times D^{2k}] \in H_{2k}(M, \partial M) = H_{2k}(M)^* = \mathbb{Z}^n ,$$

$$\chi(w) = \mathrm{degree}(S^{2k-1} \to^w SO(2k) \to S^{2k-1}) \in \mathbb{Z} ,$$

$$SO(2k) \to S^{2k-1} \; ; \; A \mapsto A(0, \dots, 0, 1) .$$

Graph manifolds

- A **graph manifold** is a differentiable $4k$-manifold with boundary constructed from (D^{4k}, S^{4k-1}) by the geometric plumbing of n oriented $2k$-plane bundles $w_i \in \pi_{2k}(BSO(2k))$ over S^{2k}, using a graph with vertices $i = 1, 2, \ldots, n$ and weights $\chi_i = \chi(w_i) \in \mathbb{Z}$.
- **Theorem** (Milnor 1959, Hirzebruch 1961) Let $\Phi \in \mathrm{Sym}_n(\mathbb{Z})$.
  If Φ is of type I assume $k = 1, 2$ or 4.
  If Φ is of type II take any $k \geqslant 1$.
  Then Φ is the intersection matrix of a graph $4k$-manifold with boundary $(M, \partial M)$ such that

$$(H_{2k}(M), \Phi(M)) \;=\; (\mathbb{Z}^n, \Phi)\ .$$

- If the graph is a tree then $(M, \partial M)$ is $(2k-1)$-connected, and if Φ is unimodular then $(M, \partial M)$ is almost closed.

The A_2 graph manifold

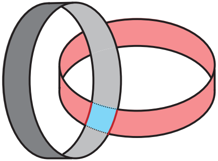
- The Dynkin diagram of the simple Lie algebra A_2 is the tree

which is here weighted by $\chi(S^2) = 2$ at each vertex.

- The corresponding symmetric matrix of type II

$$A_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \in \mathrm{Sym}_2(\mathbb{Z})$$

is the intersection matrix $\Phi(M)$ of the graph 1-connected 4-manifold with boundary $(M, \partial M)$ obtained by plumbing two copies of τ_{S^2}, with $\partial M = S^3/\mathbb{Z}_3 = L(3,2)$ a lens space.

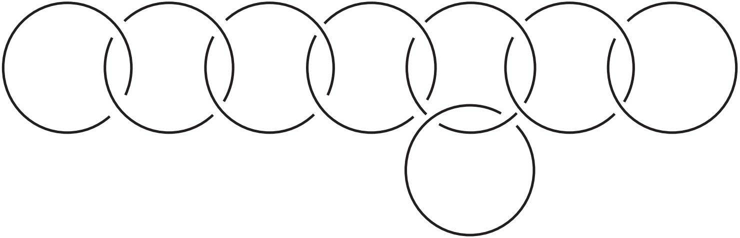


The E_8 graph manifold

- Geometric plumbing using $\Phi = E_8 \in \mathrm{Sym}_8(\mathbb{Z})$ and the Dynkin diagram of E_8 gives for each $k \geqslant 1$ a $(2k-1)$-connected graph $4k$-manifold $(M, \partial M)$ with

$$(H_{2k}(M), \Phi(M)) \ = \ (\mathbb{Z}^8, E_8) \ .$$

- The boundary $\partial M = \Sigma^{4k-1}$ is one of the interesting homology $(4k-1)$-spheres discussed already!



A doughnut of genus 2

The multiplicativity mod 8 signature of fibre bundles

- Convention: $\sigma(M) = 0 \in \mathbb{Z}$ for a $(4j+2)$-manifold M.
- What is the relationship between the signatures $\sigma(E), \sigma(B), \sigma(F) \in \mathbb{Z}$ of the manifolds in a fibre bundle
$$F^{2m} \to E^{4k} \to B^{2n} \text{ ?}$$
- **Theorem** (Chern, Hirzebruch, Serre 1956)
If $\pi_1(B)$ acts trivially on $H_*(F;\mathbb{R})$ then
$$\sigma(E) \ = \ \sigma(B)\sigma(F) \in \mathbb{Z} \ .$$
- Kodaira, Atiyah and Hirzebruch (1970) constructed examples with $\sigma(E) \neq \sigma(B)\sigma(F) \in \mathbb{Z}$.
- **Theorem** (Meyer 1972 for $k = 1$ using the **first Chern class**, Hambleton, Korzeniewski, Ranicki 2004 for all $k \geqslant 1$)
$$\sigma(E) \ \equiv \ \sigma(B)\sigma(F) \bmod 4 \ .$$
- What about mod 8? What is $(\sigma(E) - \sigma(B)\sigma(F))/4 \bmod 2$?

Symmetric forms over $\mathbb{Z}_2$

- A **symmetric form over** $\mathbb{Z}_2$ (V,λ) is a finite-dimensional vector space V over $\mathbb{Z}_2$ together with bilinear pairing

$$\lambda \ : \ V \times V \to \mathbb{Z}_2 \ ; \ (x,y) \mapsto \lambda(x,y) \ .$$

- The form is **nonsingular** if the adjoint $\mathbb{Z}_2$-linear map

$$\lambda \ : \ V \to V^* \ = \ \mathrm{Hom}_{\mathbb{Z}_2}(V,\mathbb{Z}_2)$$

  is an isomorphism.
- A nonsingular (V,λ) has a unique **characteristic element** $v \in V$ such that

$$\lambda(x,x) \ = \ \lambda(x,v) \in \mathbb{Z}_2 \ (x \in V) \ .$$

- (V,λ) is **isotropic** if $v = 0$, and **anisotropic** if $v \neq 0$.

$\mathbb{Z}_4$-quadratic enhancements

- Let (V, λ) be a nonsingular symmetric form over $\mathbb{Z}_2$.
- A $\mathbb{Z}_4$-**quadratic enhancement** of (V, λ) is a function $q : V \to \mathbb{Z}_4$ such that for all $x, y \in V$

$$\begin{aligned} q(x+y) - q(x) - q(y) &= 2\lambda(x,y) \in \mathbb{Z}_4 \ , \\ [q(x)] &= \lambda(x,x) \in \mathbb{Z}_2 \ . \end{aligned}$$

- Every (V, λ) admits $\mathbb{Z}_4$-quadratic enhancements q.
- **Example** $(V, \lambda) = (\mathbb{Z}_2, 1)$ has two $\mathbb{Z}_4$-quadratic enhancements

$$q_+(1) \ = \ 1 \in \mathbb{Z}_4 \ \text{ and } \ q_-(1) \ = \ -1 \in \mathbb{Z}_4 \ .$$

The Brown-Kervaire invariant

- The **Brown-Kervaire invariant** (1972) of a nonsingular symmetric form (V, λ) over $\mathbb{Z}_2$ with a $\mathbb{Z}_4$-quadratic enhancement q is the Gauss sum

$$\begin{aligned} \mathrm{BK}(V, \lambda, q) \ &= \ \frac{1}{\sqrt{|V|}} \sum_{x \in V} e^{\pi i q(x)/2} \\ &\in \mathbb{Z}_8 \ = \ \{\text{eighth roots of unity}\} \subset \mathbb{C} \ . \end{aligned}$$

- The Brown-Kervaire invariant has mod 4 reduction

$$[\mathrm{BK}(V, \lambda, q)] \ = \ q(v) \in \mathbb{Z}_4$$

where $v \in V$ is the characteristic element for (V, λ).
- The exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \xrightarrow{\ 4\ } \mathbb{Z}_8 \longrightarrow \mathbb{Z}_4 \longrightarrow 0$$

identifies a Brown-Kervaire invariant which has mod 4 reduction $0 \in \mathbb{Z}_4$ with a $\mathbb{Z}_2$-valued Arf invariant.

The Brown-Kervaire invariant of a symmetric matrix over $\mathbb{Z}$

- A unimodular symmetric matrix $\Phi \in \mathrm{Sym}_n(\mathbb{Z})$ determines

$$(V,\lambda,q) \;=\; ((\mathbb{Z}_2)^n, [\Phi], [x] \mapsto [\Phi(x,x)]) \;.$$

- Any lift of the characteristic element $v \in (\mathbb{Z}_2)^n$ for $[\Phi] \in \mathrm{Sym}_n(\mathbb{Z}_2)$ is a characteristic element $u \in \mathbb{Z}^n$ for Φ.
- The Brown-Kervaire invariant is the mod 8 reduction of the signature

$$\mathrm{BK}(V,\lambda,q) \;=\; [\sigma(\Phi)] \;=\; [\Phi(u,u)] \in \mathbb{Z}_8 \;.$$

- **Example** The unimodular symmetric matrix $\Phi = 1 \in \mathrm{Sym}_1(\mathbb{Z})$ determines

$$(V,\lambda,q) \;=\; (\mathbb{Z}_2,1,1) \;,\;\; u \;=\; 1 \in \mathbb{Z} \;,$$

$$\mathrm{BK}(V,\lambda,q) \;=\; 1 \in \mathbb{Z}_8 \;.$$

The Brown-Kervaire invariant of a symmetric matrix over $\mathbb{Z}_4$

- A unimodular symmetric matrix $\Phi \in \mathrm{Sym}_n(\mathbb{Z}_4)$ with mod 2 reduction $[\Phi] \in \mathrm{Sym}_n(\mathbb{Z}_2)$ determines

$$(V, \lambda, q) \;=\; ((\mathbb{Z}_2)^n, [\Phi], [x] \mapsto \Phi(x, x)) \;.$$

- Any lift of the characteristic element $v \in V$ for $[\Phi] \in \mathrm{Sym}_n(\mathbb{Z}_2)$ is a characteristic element $u \in (\mathbb{Z}_4)^n$ for Φ.
- The mod 4 reduction of the Brown-Kervaire invariant is

$$[\mathrm{BK}(V, \lambda, q)] \;=\; q(v) \;=\; \Phi(u, u) \in \mathbb{Z}_4$$

  for any characteristic element $u \in (\mathbb{Z}_4)^n$ for Φ.
- **Example** The unimodular symmetric matrix $\Phi = 1 \in \mathrm{Sym}_1(\mathbb{Z}_4)$ has

$$(V, \lambda, q) \;=\; (\mathbb{Z}_2, 1, 1) \;,\;\; u \;=\; 1 \;,$$

$$\mathrm{BK}(V, \lambda, q) \;=\; 1 \in \mathbb{Z}_8 \;.$$

The Brown-Kervaire invariant of A_2

- The unimodular symmetric matrix over $\mathbb{Z}_4$

$$A_2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \in \mathrm{Sym}_2(\mathbb{Z}_4)$$

  has characteristic element $u = 0 \in (\mathbb{Z}_4)^2$.

- A_2 determines

$$\begin{aligned}
&(V, \lambda, q) \\
&= \ (\mathbb{Z}_2 \oplus \mathbb{Z}_2, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, (x, y) \mapsto 2(x^2 + xy + y^2)) \ , \\
&v \ = \ 0 \in V \ , \\
&\mathrm{BK}(V, \lambda, q) \ = \ 4 \in \mathrm{im}(4 : \mathbb{Z}_2 \to \mathbb{Z}_8) \ = \ \ker(\mathbb{Z}_8 \to \mathbb{Z}_4) \ .
\end{aligned}$$

Brown-Kervaire = signature mod 8

- **Theorem** (Morita 1974) A closed oriented $4k$-manifold M determines a nonsingular symmetric form $(H^{2k}(M;\mathbb{Z}_2),\lambda_M)$ over $\mathbb{Z}_2$, with

$$\lambda_M(x,y) \;=\; \langle x\cup y,[M]\rangle\in\mathbb{Z}_2$$

and characteristic element $v=v_{2k}(\nu_M)\in H^{2k}(M;\mathbb{Z}_2)$. The Pontrjagin square is a $\mathbb{Z}_4$-quadratic refinement

$$q_M \;=\; \mathcal{P}_{2k}\;:\; H^{2k}(M;\mathbb{Z}_2)\to H^{4k}(M;\mathbb{Z}_4)\;=\;\mathbb{Z}_4$$

with Brown-Kervaire invariant = the mod 8 reduction of the signature

$$\mathrm{BK}(H^{2k}(M;\mathbb{Z}_2),\lambda_M,q_M)\;=\;[\sigma(M)]\in\mathbb{Z}_8$$

and mod 4 reduction

$$q_M(v)\;=\;[[\sigma(M)]]\in\mathbb{Z}_4\ .$$

The Arf invariant I.

- Let (V, λ) be a nonsingular symmetric form over $\mathbb{Z}_2$.
- A $\mathbb{Z}_2$-**quadratic enhancement** of (V, λ) is a function $h : V \to \mathbb{Z}_2$ such that

$$h(x+y) - h(x) - h(y) \;=\; \lambda(x, y) \in \mathbb{Z}_2 \; (x, y \in V) \; .$$

- (V, λ) admits an h if and only if λ is isotropic, in which case there exists a basis $(b_1, b_2, \dots, b_n)$ for V with n even, such that

$$\lambda(b_i, b_j) \;=\; \begin{cases} 1 & \text{if } (i,j) = (1,2) \text{ or } (2,1) \text{ or } (3,4) \text{ or } (4,3) \; \dots \\ 0 & \text{otherwise.} \end{cases}$$

- The **Arf invariant** of (V, λ, h) is defined using any such basis

$$\mathrm{Arf}(V, \lambda, h) \;=\; \sum_{i=1}^{n/2} h(b_{2i-1}) h(b_{2i}) \in \mathbb{Z}_2 \; .$$

The Arf invariant II.

- Let (V,λ) be a nonsingular symmetric form over $\mathbb{Z}_2$.
- A $\mathbb{Z}_2$-quadratic enhancement $h : V \to \mathbb{Z}_2$ determines a $\mathbb{Z}_4$-quadratic enhancement
$$q ~=~ 2h ~:~ V \to \mathbb{Z}_4 ~;~ x \mapsto q(x) ~=~ 2h(x)$$
such that
$$\mathrm{BK}(V,\lambda,q) ~=~ 4\,\mathrm{Arf}(V,\lambda,h) \in 4\mathbb{Z}_2 \subset \mathbb{Z}_8 ~.$$
- A $\mathbb{Z}_4$-quadratic enhancement $q : V \to \mathbb{Z}_4$ is such that $q(V) \subseteq 2\mathbb{Z}_2 \subset \mathbb{Z}_4$ if and only if (V,λ) is isotropic, and
$$h ~=~ q/2 ~:~ V \to \mathbb{Z}_2 ~;~ x \mapsto h(x) ~=~ q(x)/2$$
is a $\mathbb{Z}_2$-quadratic enhancement as above.
- **Example** For the symmetric form $A_2 \in \mathrm{Sym}_2(\mathbb{Z}_4)$
$$(V,\lambda,q) = (\mathbb{Z}_2 \oplus \mathbb{Z}_2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, q(x,y) = 2(x^2+xy+y^2))$$
$$\mathrm{BK}(V,\lambda,q) ~=~ 4 \in \mathbb{Z}_8 ~,~ \mathrm{Arf}(V,\lambda,h) ~=~ 1 \in \mathbb{Z}_2 ~.$$

Carmen Rovi's Edinburgh thesis I.

- **Theorem** (CR 2015)
  (i) Let (V, λ) be a nonsingular symmetric form over $\mathbb{Z}_2$ with a $\mathbb{Z}_4$-quadratic enhancement $q : V \to \mathbb{Z}_4$, and characteristic element $v \in V$.
  The Brown-Kervaire invariant $\text{BK}(V, \lambda, q) \in \mathbb{Z}_8$ has mod 4 reduction $[\text{BK}(V, \lambda, q)] = 0 \in \mathbb{Z}_4$ if and only if $q(v) = 0 \in \mathbb{Z}_4$. In this case $\lambda(v, v) = 0 \in \mathbb{Z}_2$ and the maximal isotropic nonsingular subquotient of (V, λ, q)

  $$(V', \lambda', q') \ = \ (\{x \in V \,|\, \lambda(x, v) = 0 \in \mathbb{Z}_2\}/\{v\}, [\lambda], [q])$$

  has $\mathbb{Z}_2$-quadratic enhancement $h' = q'/2 : V' \to \mathbb{Z}_2$ such that

  $$\begin{aligned}\text{BK}(V, \lambda, q) \ &= \ \text{BK}(V', \lambda', q') \ = \ 4\,\text{Arf}(V', \lambda', h') \\ &\in \ \text{im}(4 : \mathbb{Z}_2 \to \mathbb{Z}_8) \ = \ \ker(\mathbb{Z}_8 \to \mathbb{Z}_4)\ .\end{aligned}$$

Carmen Rovi's Edinburgh thesis II.

- (ii) For any fibre bundle $F^{2m} \to E^{4k} \to B^{2n}$

  $$(\sigma(E) - \sigma(B)\sigma(F))/4 \ = \ \mathrm{Arf}(V', \lambda', h') \in \mathbb{Z}_2$$

  with

  $$\begin{aligned} &(V, \lambda, q) \\ &= \ (H^{2k}(E;\mathbb{Z}_2), \lambda_E, q_E) \oplus (H^{2k}(B \times F;\mathbb{Z}_2), -\lambda_{B\times F}, -q_{B\times F}) \ . \end{aligned}$$

- (iii) If the action of $\pi_1(B)$ on $(H_m(F;\mathbb{Z})/\mathrm{torsion}) \otimes \mathbb{Z}_4$ is trivial then the Arf invariant in (ii) is 0 and

  $$\sigma(E) \ \equiv \ \sigma(B)\sigma(F) \ \bmod 8 \ .$$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