Foundations of algebraic surgery

ANDREW RANICKI (Edinburgh)

http://www.maths.ed.ac.uk/~aar

- An n-dimensional manifold M determines an n-dimensional cellular f.g. free abelian group chain complex

$$C(M) : C_n(M) \xrightarrow{d} C_{n-1}(M) \xrightarrow{d} \dots \xrightarrow{d} C_0(M)$$

with a Poincaré duality chain equivalence

$$C(M)^{n-*} \simeq C(M)\ .$$

- The homology effect of a geometric surgery on a manifold M is given by an algebraic surgery on the chain complex $C(M)$.

The algebraic surgery machine

- The algebraic part of the machine studies chain complexes with Poincaré duality, which are quadratic forms on chain complexes. Algebraic surgery on such objects models the surgery of manifolds and normal maps.

- The geometric part reduces topological surgery problems to algebraic ones.

- For n-dimensional normal maps with $n \geq 5$ there is a one-one correspondence between algebraic and geometric surgery below and in the middle dimensions.

Symmetric and quadratic Poincaré duality

- General theory applies to chain complexes C over any ring with involution A. There are two types of Poincaré duality $C^{n-*} \simeq C$, corresponding to type I and type II symmetric forms.

- Use symmetric Poincaré duality for surgery on manifolds. In general, cannot realize symmetric Poincaré surgeries on manifolds.

- Use quadratic Poincaré duality for surgery on normal maps.

- <u>Theorem</u> The Wall surgery obstruction group $L_n(\mathbb{Z}[\pi])$ is the cobordism group of n-dimensional quadratic Poincaré complexes over $\mathbb{Z}[\pi]$.

Geometric surgery

- The n-dimensional manifold obtained from an n-dimensional manifold M by surgery on $S^i \times D^{n-i} \subset M$ is

  $$M' = (M \backslash S^i \times D^{n-i}) \cup D^{i+1} \times S^{n-i-1} .$$

  Call this the effect of the surgery.

- Can obtain M' from M by surgery on

  $$D^{i+1} \times S^{n-i-1} \subset M' .$$

- Example View $S^n = \partial(D^{i+1} \times D^{n-i})$ as

  $$S^n = S^i \times D^{n-i} \cup D^{i+1} \times S^{n-i-1} .$$

  Surgery on $S^i \times D^{n-i} \subset S^n$ gives

  $$D^{i+1} \times S^{n-i-1} \cup D^{i+1} \times S^{n-i-1} = S^{i+1} \times S^{n-i-1} .$$

Cobordism effect of surgery

- The trace of surgery on $S^i \times D^{n-i} \subset M$ with effect M' is the cobordism $(W; M, M')$ obtained from $M \times [0,1]$ by attaching an $(i+1)$-handle at $S^i \times D^{n-i} \times \{1\} \subset M \times \{1\}$

$$W = M \times [0,1] \cup D^{i+1} \times D^{n-i} .$$

- Theorem (Thom, Milnor) Every $(n+1)$-dimensional cobordism $(L; M, N)$ is a union of the traces of surgeries.
Proof A Morse function

$$f : (L; M, N) \to ([0,1]; \{0\}, \{1\})$$

determines a handle decomposition of L on M

$$L = M \times [0,1] \cup \bigcup_{k=0}^{n+1} \bigcup D^k \times D^{n+1-k}$$

with one k-handle for each critical point of index k.

Homotopy effect of surgery

- The mapping cone of a map $x : S^i \to M$ is the adjunction space $M \cup_x D^{i+1}$ obtained from M by <u>attaching an</u> $(i+1)$-cell.

- <u>Proposition</u> If $(W; M, M')$ is the trace of a surgery on $S^i \times D^{n-i} \subset M$ there are defined homotopy equivalences

  $$W \simeq M \cup_x D^{i+1} \simeq M' \cup_{x'} D^{n-i}$$

  with $x : S^i = S^i \times \{0\} \subset S^i \times D^{n-i} \subset M$.

- Surgery on an n-dimensional manifold attaches an $(i+1)$-cell and then detaches an $(n-i)$-cell.

- <u>Example</u> The trace $(W; S^n, S^{i+1} \times S^{n-i-1})$ of the surgery on $S^i \times D^{n-i} \subset S^n$ has

  $$W \simeq S^n \vee S^{i+1} .$$

Homology effect of attaching a cell

- The algebraic mapping cone of a chain map $f : C \to D$ is the chain complex $C(f)$ with

$$C(f)_r \ = \ C_{r-1} \oplus D_r \ , \ d_{C(f)} \ = \ \begin{pmatrix} d_C & 0 \\ \pm f & d_D \end{pmatrix} \ .$$

- For $i \geq 0$ define the chain complex

$$S^i\mathbb{Z} \ : \ \cdots \to 0 \to \mathbb{Z} \to 0 \to \dots$$

concentrated in dimension i.

- The homology effect of attaching an $(i+1)$-cell to M at $x : S^i \to M$ is to attach an algebraic $(i+1)$-cell to $C(M)$: if $W = M \cup_x D^{i+1}$ then

$$C(W) = C(x : S^i\mathbb{Z} \to C(M))$$

- In particular, $H_i(W) = H_i(M)/\langle x \rangle$ is obtained from $H_i(M)$ by killing $x \in H_i(M)$.

Homology effect of surgery

- If $(W;M,M')$ is the trace of a surgery on $S^i \times D^{n-i} \subset M$ then there are defined chain equivalences

$$C(W) \simeq C(x : S^i\mathbb{Z} \to C(M))$$
$$C(W) \simeq C(x' : S^{n-i-1}\mathbb{Z} \to C(M')) \ .$$

- $C(M')$ obtained from $C(M)$ by an algebraic surgery which kills $x \in H_i(M)$, by first attaching an algebraic $(i+1)$-cell and then detaching an algebraic $(n-i)$-cell.

- Need Poincaré duality to describe the relationship between $x \in H_i(M)$ and $x' \in H_{n-i-1}(M')$.

Poincaré duality

- Cap product with fundamental class $[M] \in H_n(M)$ is a chain equivalence

$$[M] \cap - : C(M)^{n-*} \to C(M)$$

inducing the Poincaré duality isomorphisms

$$H^{n-*}(M) \cong H_*(M) .$$

- Poincaré-Lefschetz duality for any cobordism $(W; M, M')$

$$H^{n+1-*}(W, M) \cong H_*(W, M') .$$

- Can use Poincaré duality to decide which elements in $H_i(M)$ can be represented by $S^i \times D^{n-i} \subset M$ and so killed by surgery. Regarding duality as a quadratic form, can only kill isotropic elements.

Principle of Algebraic Surgery

- For <u>any</u> cobordism of n-dimensional manifolds $(W; M, M')$ the chain homotopy type of $C(M')$ and its Poincaré duality can be obtained from

  - the chain homotopy type of $C(M)$ and its Poincaré duality
  - the chain homotopy class of the chain map $j : C(M) \to C(W, M')$
  - $j[M] = 0 \in H_n(W, M')$ on the chain level

  using algebraic surgery on symmetric Poincaré complexes.

- An algebraic surgery corresponds to a sequence of geometric surgeries.

Symmetric Poincaré complexes

- An <u>n-dimensional symmetric Poincaré complex</u> (C, ϕ) is an n-dimensional f.g. free chain complex C with morphisms

  $\phi_s \ : \ C^r \ = \ \mathrm{Hom}_{\mathbb{Z}}(C_r, \mathbb{Z}) \to C_{n-r+s} \ (s \geq 0)$

  such that (up to signs)

  $d\phi_s + \phi_s d^* + \phi_{s-1} + \phi^*_{s-1} = 0 : C^r \to C_{n-r+s-1}$

  with $s \geq 0$, $\phi_{-1} = 0$ and

  $$\phi_0 \ : \ C^{n-*} \ = \ \mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{Z})_{n-*} \to C$$

  is a chain equivalence.

- Symmetric form on chain complex.

- <u>Theorem</u> (Mishchenko) An n-dimensional manifold M determines an n-dimensional symmetric Poincaré complex $(C(M), \phi)$, with

  $$\phi_0 = [M] \cap - : C(M)^{n-*} \xrightarrow{\simeq} C(M)$$

Symmetric algebraic surgery

- An <u>algebraic surgery</u> on (C,ϕ) has input a chain map $j : C \to D$ with chain homotopy

$$\delta\phi_0 : j\phi_0 j^* \simeq 0 : D^{n-*} \to D \ .$$

The <u>effect</u> is the n-dimensional symmetric Poincaré complex (C',ϕ') with

$$\begin{aligned} C'_r &= C_r \oplus D_{r+1} \oplus D^{n-r+1} \ , \\ d_{C'} &= \begin{pmatrix} d_C & 0 & \pm\phi_0 j^* \\ \pm j & d_D & \delta\phi_0 \\ 0 & 0 & (d_D)^* \end{pmatrix} \end{aligned}$$

- Generalization of the operation which replaces a symmetric form $(K, \lambda = \lambda^* : K \to K^*)$ for any $x \in K$ with $\lambda(x)(x) = 0$ (isotropic) by the subquotient form

$$(K', \lambda') = (\{y \in K \mid \lambda(x)(y) = 0\}/\langle x\rangle, [\lambda])$$

Algebraic and geometric surgery

- Surgery on $S^i \times D^{n-i} \subset M$ determines algebraic surgery on $(C(M), \phi)$ with input $j : C(M) \to S^{n-i}\mathbb{Z}$ a cocycle representing the Poincaré dual $j \in H^{n-i}(M)$ of $x = [S^i] \in H_i(M)$, and $\delta\phi$ determined by framing of $S^i \subset M$.

- <u>Theorem</u> The symmetric Poincaré complex $(C(M'), \phi')$ of the geometric effect M' is the effect of the algebraic surgery on $(C(M), \phi)$.

- <u>Exercise</u> Work out the algebraic surgery corresponding to the geometric surgery on $S^i \times D^{n-i} \subset S^n$ with effect $S^{i+1} \times S^{n-i-1}$.

The algebraic effect of a surgery

- The Theorem is an example of the Algebraic Surgery Principle in action.

- If $(W; M, M')$ is the trace of surgery on $S^i \times D^{n-i} \subset M$ then

$$j : C(M) \to C(W, M') \simeq S^{n-i}\mathbb{Z} \ .$$

- Write $C(M) = C$, and let $x \in C_i$ be cycle being killed, $y = j \in C^{n-i}$ the dual cocycle.

- The chain complex $C(M')$ is chain equivalent to

$$C' : \cdots \to C_{n-i} \xrightarrow{d \oplus y} C_{n-i-1} \oplus \mathbb{Z} \xrightarrow{d \oplus 0} C_{n-i-2}$$
$$\to \cdots \to C_{i+2} \xrightarrow{d \oplus 0} C_{i+1} \oplus \mathbb{Z} \xrightarrow{d \oplus x} C_i \to \ldots$$

Composition of surgeries

- Suppose given a surgery on $S^i \times D^{n-i} \subset M$ with effect M' and trace $(W; M, M')$, and then a surgery on $S^k \times D^{n-k} \subset M'$ with effect M'' and trace $(W'; M', M'')$. Can apply the Principle to the union cobordism

  $$(W''; M, M'') = (W; M, M') \cup (W'; M', M'')$$

  to recover $C(M'')$ from $C(M)$ and $C(W'', M'')$.

- If $k = i + 1$ then $C(W'', M'')$ is chain equivalent to

  $$\cdots \to 0 \to \mathbb{Z} \xrightarrow{\lambda} \mathbb{Z} \to 0 \to \ldots$$

  concentrated in dimensions $n-i, n-i+1$, with $\lambda \in \mathbb{Z}$ the algebraic intersection number of the cocore $\{0\} \times S^{n-i-1} \subset M'$ and the core $S^{i+1} \times \{0\} \subset M'$ of the surgeries.

Handle cancellation

- Surgery on

  $S^i \times D^{n-i} \subset S^i \times D^{n-i} \cup D^{i+1} \times S^{n-i-1} = S^n$

  has trace $(W; S^n, S^{i+1} \times S^{n-i-1})$ a punctured $S^{i+1} \times D^{n-i}$. Reverse the roles of $i+1, n-i-1$. Surgery on

  $$S^{i+1} \times D^{n-i-1} \subset S^{i+1} \times S^{n-i-1}$$

  has trace $(W'; S^{i+1} \times S^{n-i-1}, S^n)$ a punctured $D^{i+2} \times S^{n-i-1}$.

- The union $W \cup W'$ is a double punctured

  $$S^{n+1} = S^{i+1} \times D^{n-i} \cup D^{i+2} \times S^{n-i-1}$$

- In this case $\lambda = 1$, and there is an isomorphism

  $$(W \cup W'; S^n, S^n) \ \cong \ S^n \times ([0,1]; \{0\}, \{1\})$$

  The two surgeries have cancelled each other out. Draw the picture for $i = 0$, $n = 1$!

Surgery and the orthogonal group

- Can vary extension of $x : S^i \subset M$ to $S^i \times D^{n-i} \subset M$ by maps $\omega : S^i \to O(n-i)$. Corresponds to different $\delta\phi_0$'s. Given $\omega : S^i \to O(n-i)$ define embedding

$$e_\omega : S^i \times D^{n-i} \to S^n = S^i \times D^{n-i} \cup D^{i+1} \times S^{n-i-1};$$
$$(x,y) \mapsto (x, \omega(x)(y)) \ .$$

- Surgery on S^n killing e_ω gives sphere bundle $S(\omega)$ of the $(n-i)$-plane vector bundle over S^{i+1} with clutching map ω

$$\omega \in \pi_i(O(n-i)) = \pi_{i+1}(BO(n-i)) \ ,$$
$$S^{n-i-1} \to S(\omega) \to S^{i+1} \ ,$$
$$S(\omega) \ = \ D^{i+1} \times S^{n-i-1} \cup_\omega D^{i+1} \times S^{n-i-1} \ .$$

The trace of the surgery is $\mathrm{cl}(E(\omega)\backslash D^{n+1})$, with $D^{n-i} \to E(\omega) \to S^{i+1}$ the disk bundle.

- Exercise Work out the algebraic effect!

Quadratic Poincaré complexes

- An <u>n-dimensional quadratic Poincaré complex</u> (C,ψ) is an n-dimensional f.g. free chain complex C with morphisms

$$\psi_s \ : \ C^r \to C_{n-r-s} \ (s \geq 0)$$

  such that (up to signs)

$$d\psi_s + \psi_s d^* + \psi_{s+1} + \psi^*_{s+1} = 0 : C^r \to C_{n-r-s-1}$$

  and

$$\psi_0 + (\psi_0)^* \ : \ C^{n-*} \to C$$

  a chain equivalence.

- Quadratic form on chain complex.

- Can define algebraic surgery as in the symmetric case.

Cobordism of quadratic Poincaré complexes

- The n-dimensional quadratic Poincaré complexes (C,ψ), (C',ψ') are <u>cobordant</u> if there exists an algebraic surgery on $(C \oplus C', \psi \oplus -\psi')$ with effect (C'',ψ'') such that $H_*(C'') = 0$. (This is equivalent to the Poincaré-Lefschetz duality definition).

- <u>Theorem</u> The Wall surgery obstruction group $L_n(A)$ of a ring with involution A is isomorphic to the group of cobordism classes of n-dimensional quadratic Poincaré complexes over A.

- $L_{2i}(A)$ is the Witt group of nonsingular $(-)^i$-quadratic forms over A. Such forms are precisely the $2i$-dimensional quadratic Poincaré complexes (C,ψ) over A with $C_r = 0$ for $r \neq i$. Similarly for $L_{2i+1}(A)$ and formations, with $C_r = 0$ for $r \neq i, i+1$.

The algebraic surgery obstruction

- Theorem An n-dimensional normal map $(f,b):M\to X$ determines an n-dimensional quadratic Poincaré complex (C,ψ) over $\mathbb{Z}[\pi_1(X)]$ with homology the kernel modules

  $$H_*(C) = K_*(M) = \ker(\widetilde{f}_*: H_*(\widetilde{M})\to H_*(\widetilde{X})) .$$

  The surgery obstruction of (f,b) is the cobordism class of (C,ψ)

  $$\sigma_*(f,b) = (C,\psi)\in L_n(\mathbb{Z}[\pi_1(X)]) .$$

  Thus $(C,\psi)=0$ if (and for $n\geq 5$ only if) (f,b) is normal bordant to a homotopy equivalence.

- Half the proof is by the normal map version of the Principle of Algebraic Surgery: a surgery on (f,b) determines a quadratic Poincaré surgery on the kernel complex (C,ψ). The other half is by the geometric realization for $n\geq 5$ of quadratic surgeries below and in the middle dimension.