The algebraic surgery exact sequence

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- The algebraic surgery exact sequence is defined for any space X

$$\dots \to H_n(X;\mathbb{L}_\bullet) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(X)]) \to \mathbb{S}_n(X) \to H_{n-1}(X;\mathbb{L}_\bullet) \to \dots$$

  with A the L-theory assembly map. The functor $X \mapsto \mathbb{S}_*(X)$ is homotopy invariant.

- The 2-stage obstructions of the Browder-Novikov-Sullivan-Wall surgery theory for the existence and uniqueness of topological manifold structures in a homotopy type are replaced by single obstructions in the relative groups $\mathbb{S}_*(X)$ of the assembly map A.

Local and global modules

- The assembly map $A : H_*(X; \mathbb{L}_\bullet) \to L_*(\mathbb{Z}[\pi_1(X)])$ is induced by a forgetful functor

  $$A : \{(\mathbb{Z}, X)\text{-modules}\} \to \{\mathbb{Z}[\pi_1(X)]\text{-modules}\}$$

  where the domain depends on the local topology of X and the target depends only on the fundamental group $\pi_1(X)$ of X, which is global.

- In terms of sheaf theory $A = q_! p^!$ with $p : \widetilde{X} \to X$ the universal covering projection and $q : \widetilde{X} \to \{\text{pt.}\}$.

- The geometric model for the L-theory assembly A is the forgetful functor

  $$\{\text{geometric Poincaré complexes}\} \to \{\text{topological manifolds.}\}$$

  In fact, in dimensions $n \geq 5$ this functor has the same fibre as A.

Local and global quadratic Poincaré complexes

- (Global) The L-group $L_n(\mathbb{Z}[\pi_1(X)])$ is the cobordism group of n-dimensional quadratic Poincaré complexes (C, ψ) over $\mathbb{Z}[\pi_1(X)]$.

- (Local) The generalized homology group $H_n(X; \mathbb{L}_\bullet)$ is the cobordism group of n-dimensional quadratic Poincaré complexes (C, ψ) over $(\mathbb{Z}, X)$. As in sheaf theory C has stalks, which are $\mathbb{Z}$-module chain complexes $C(x)$ $(x \in X)$.

- (Local-Global) $\mathbb{S}_n(X)$ is the cobordism group of $(n-1)$-dimensional quadratic Poincaré complexes (C, ψ) over $(\mathbb{Z}, X)$ such that the $\mathbb{Z}[\pi_1(X)]$-module chain complex assembly $A(C)$ is acyclic.

The total surgery obstruction

- The total surgery obstruction of an n-dimensional geometric Poincaré complex X is the cobordism class

$$s(X) = (C,\psi) \in \mathbb{S}_n(X)$$

of a $\mathbb{Z}[\pi_1(X)]$-acyclic $(n-1)$-dimensional quadratic Poincaré complex (C,ψ) over $(\mathbb{Z},X)$. The stalks $C(x)$ $(x \in X)$ measure the failure of X to have local Poincaré duality

$$\dots \to H_r(C(x)) \to H^{n-r}(\{x\}) \to H_r(X, X\backslash\{x\}) \to H_{r-1}(C(x)) \to H^{n-r+1}(\{x\}) \to \dots$$

X is an n-dimensional homology manifold if and only if $H_*(C(x)) = 0$. In particular, this is the case if X is a topological manifold.

- Total Surgery Obstruction Theorem
$s(X) = 0 \in \mathbb{S}_n(X)$ if (and for $n \geq 5$ only if) X is homotopy equivalent to an n-dimensional topological manifold.

The proof of the Total Surgery Obstruction Theorem

The proof is a translation into algebra of the two-stage Browder-Novikov-Sullivan-Wall obstruction for the existence of a topological manifold in the homotopy type of a Poincaré complex X :

- The image $t(X) \in H_{n-1}(X;\mathbb{L}_\bullet)$ of $s(X)$ is such that $t(X) = 0$ if and only if the Spivak normal fibration $\nu_X : X \to BG$ has a topological reduction $\widetilde{\nu}_X : X \to BTOP$.

- If $t(X) = 0$ then $s(X) \in \mathbb{S}_n(X)$ is the image of the surgery obstruction $\sigma_*(f,b) \in L_n(\mathbb{Z}[\pi_1(X)])$ of the normal map $(f : M \to X, b : \nu_M \to \widetilde{\nu}_X)$ determined by a choice of lift $\widetilde{\nu}_X : X \to BTOP$.

- $s(X) = 0$ if and only if there exists a reduction $\widetilde{\nu}_X : X \to BTOP$ for which $\sigma_*(f,b) = 0$.

The structure invariant

- The <u>structure invariant</u> of a homotopy equivalence $h : N \to M$ of n-dimensional topological manifolds is the cobordism class

$$s(h) = (C, \psi) \in \mathbb{S}_{n+1}(M)$$

  of a globally acyclic n-dimensional quadratic Poincaré complex (C, ψ). The stalks $C(x)$ $(x \in M)$ measure the failure of h to have acyclic point inverses, with

$$\begin{aligned} H_*(C(x)) &= H_*(h^{-1}(x) \to \{x\}) \\ &= \widetilde{H}_{*+1}(h^{-1}(x)) \ (x \in M) \ . \end{aligned}$$

- h has acyclic point inverses if and only if $H_*(C(x)) = 0$. In particular, this is the case if h is a homeomorphism.

- <u>Structure Invariant Theorem</u>
  $s(h) = 0 \in \mathbb{S}_{n+1}(M)$ if (and for $n \geq 5$ only if) h is homotopic to a homeomorphism.

The proof of the Structure Invariant Theorem (I)

The proof is a translation into algebra of the two-stage Browder-Novikov-Sullivan-Wall obstruction for the uniqueness of topological manifold structures in a homotopy type :

- the image $t(h) \in H_n(M;\mathbb{L}_\bullet)$ of $s(h)$ is such that $t(h) = 0$ if and only if the normal invariant can be trvialized

  $$(h^{-1})^*\nu_N - \nu_M \simeq \{*\} : M \to \mathbb{L}_0 \simeq G/TOP$$

  if and only if $1 \cup h : M \cup N \to M \cup M$ extends to a normal bordism

  $$(f,b) \; : \; (W;M,N) \to M \times ([0,1];\{0\},\{1\})$$

- if $t(h) = 0$ then $s(h) \in \mathbb{S}_{n+1}(M)$ is the image of the surgery obstruction

  $$\sigma_*(f,b) \in L_{n+1}(\mathbb{Z}[\pi_1(M)]) \; .$$

The proof of the Structure Invariant Theorem (II)

- $s(h) = 0$ if and only if there exists a normal bordism (f, b) which is a simple homotopy equivalence.

- Have to work with simple L-groups here, to take advantage of the s-cobordism theorem.

- Alternative proof. The mapping cylinder of $h : N \to M$

$$P = M \cup_h N \times [0,1]$$

defines an $(n+1)$-dimensional geometric Poincaré pair $(P, M \cup N)$ with manifold boundary, such that P is homotopy equivalent to M. The structure invariant is the rel ∂ total surgery obstruction

$$s(h) = s_\partial(P) \in \mathbb{S}_{n+1}(P) = \mathbb{S}_{n+1}(M) .$$

The simply-connected case

- For $\pi_1(X) = \{1\}$ the algebraic surgery exact sequence breaks up

$$0 \to \mathbb{S}_n(X) \to H_{n-1}(X;\mathbb{L}_\bullet) \to L_{n-1}(\mathbb{Z}) \to 0$$

- The total surgery obstruction $s(X) \in \mathbb{S}_n(X)$ maps injectively to the TOP reducibility obstruction $t(X) \in H_{n-1}(X;\mathbb{L}_\bullet)$ of the Spivak normal fibration ν_X. Thus for $n \geq 5$ a simply-connected n-dimensional geometric Poincaré complex X is homotopy equivalent to an n-dimensional topological manifold if and only if $\nu_X : X \to BG$ admits a TOP reduction $\widetilde{\nu}_X : X \to BTOP$.

- The structure invariant $s(h) \in \mathbb{S}_{n+1}(M)$ maps injectively to the normal invariant $t(h) \in H_n(M;\mathbb{L}_\bullet) = [M, G/TOP]$. Thus for $n \geq 5$ h is homotopic to a homeomorphism if and only if $t(h) \simeq \{*\} : M \to G/TOP$.

The geometric surgery exact sequence

- The structure set $\mathbb{S}^{TOP}(M)$ of a topological manifold M is the set of equivalence classes of homotopy equivalences $h : N \to M$ from topological manifolds N, with $h \sim h'$ if there exist a homeomorphism $g : N' \to N$ and a homotopy $hg \simeq h' : N' \to M$.

- Theorem (Quinn, R.) The geometric surgery exact sequence for $n = \dim(M) \geq 5$

$$\cdots \to L_{n+1}(\mathbb{Z}[\pi_1(M)]) \to \mathbb{S}^{TOP}(M) \to [M, G/TOP] \to L_n(\mathbb{Z}[\pi_1(M)])$$

is isomorphic to the relevant portion of the algebraic surgery exact sequence

$$\cdots \to L_{n+1}(\mathbb{Z}[\pi_1(M)]) \to \mathbb{S}_{n+1}(M) \to H_n(M; \mathbb{L}_\bullet) \xrightarrow{A} L_n(\mathbb{Z}[\pi_1(M)])$$

with $\mathbb{S}^{TOP}_\partial(M \times D^k, M \times S^{k-1}) = \mathbb{S}_{n+k+1}(M)$.

Example $\mathbb{S}^{TOP}(S^n) = \mathbb{S}_{n+1}(S^n) = 0$.

The image of the assembly map

- Theorem For any finitely presented group π the image of the assembly map

$$A \; : \; H_n(K(\pi,1);\mathbb{L}_\bullet) \to L_n(\mathbb{Z}[\pi])$$

is the subgroup of $L_n(\mathbb{Z}[\pi])$ consisting of the surgery obstructions $\sigma_*(f,b)$ of normal maps $(f,b) : N \to M$ of closed n-dimensional manifolds with $\pi_1(M) = \pi$.

- There are many calculations of the image of A for finite π, notably the Oozing Conjecture proved by Hambleton-Milgram-Taylor-Williams.

Statement of the Novikov conjecture

- The $\mathcal{L}$-genus of an n-dimensional manifold M is a collection of cohomology classes $\mathcal{L}(M) \in H^{4*}(M;\mathbb{Q})$ which are determined by the Pontrjagin classes of $\nu_M : M \to BTOP$. In general, $\mathcal{L}(M)$ is not a homotopy invariant.

- The Hirzebruch signature theorem for a $4k$-dimensional manifold M

  $$\text{signature}(H^{2k}(M), \cup) = \langle \mathcal{L}(M), [M] \rangle \in \mathbb{Z}$$

  shows that part of the $\mathcal{L}$-genus is homotopy invariant.

- The Novikov conjecture for a discrete group π is that the higher signatures for any manifold M with $\pi_1(M) = \pi$

  $$\sigma_x(M) = \langle x \cup \mathcal{L}(M), [M] \rangle \in \mathbb{Q} \ (x \in H^*(K(\pi, 1); \mathbb{Q}))$$

  are homotopy invariant.

Algebraic formulation of the Novikov conjecture

- Theorem The Novikov conjecture holds for a group π if and only if the rational assembly maps

$$A : H_n(K(\pi,1);\mathbb{L}_\bullet)\otimes\mathbb{Q} = H_{n-4*}(K(\pi,1);\mathbb{Q})$$
$$\to L_n(\mathbb{Z}[\pi_1(M)])\otimes\mathbb{Q}$$

  are injective.

- Trivially true for finite π.

- Verified for many infinite groups π using algebra, geometric group theory, C^*-algebras, etc. See Proceedings of 1993 Oberwolfach conference (LMS Lecture Notes 226,227) for state of the art in 1995, not substantially out of date.

Statement of the Borel conjecture

- An <u>n-dimensional Poincaré duality group</u> π is a discrete group such that the classifying space $K(\pi, 1)$ is an n-dimensional Poincaré complex.

- π must be infinite and torsion-free.

- The <u>Borel conjecture</u> is that for every n-dimensional Poincaré duality group π there exists an aspherical n-dimensional manifold $M \simeq K(\pi, 1)$ with

  $$\mathbb{S}^{TOP}(M) = 0 .$$

  This is topological rigidity: every homotopy equivalence $h : N \to M$ is (conjectured) to be homotopic to a homeomorphism. The conjecture also predicts higher rigidity

  $$\mathbb{S}_{\partial}^{TOP}(M \times D^k, M \times S^{k-1}) = 0 \ (k \geq 0) .$$

Algebraic formulation of the Borel conjecture

- Theorem For $n \geq 5$ the Borel conjecture holds for an n-dimensional Poincaré group π if and only if $s(K(\pi, 1)) = 0 \in \mathbb{S}_n(K(\pi, 1))$ and the assembly map

  $$A : H_{n+k}(K(\pi, 1); \mathbb{L}_\bullet) \to L_{n+k}(\mathbb{Z}[\pi_1(M)])$$

  is injective for $k = 0$ and an isomorphism for $k \geq 1$.

- Verified for many Poincaré duality groups π, with $\mathbb{S}_n(K(\pi, 1)) = L_0(\mathbb{Z}) = \mathbb{Z}$.

- True in the classical case $\pi = \mathbb{Z}^n$, $K(\pi, 1) = T^n$, which was crucial in the extension due to Kirby-Siebenmann (ca. 1970) of the 1960's Browder-Novikov-Sullivan-Wall surgery theory from the differentiable and PL categories to the topological category.

The 4-periodic algebraic surgery exact sequence

- The 4-periodic algebraic surgery exact sequence is defined for any space X

$$\cdots \to H_n(X;\overline{\mathbb{L}}_\bullet) \xrightarrow{\overline{A}} L_n(\mathbb{Z}[\pi_1(X)]) \to \overline{\mathbb{S}}_n(X) \to H_{n-1}(X;\mathbb{L}_\bullet) \to \dots$$

with $\overline{\mathbb{L}}_0 = L_0(\mathbb{Z}) \times G/TOP$ and $\overline{A}$ the L-theory assembly map.

- Exact sequence

$$\cdots \to H_n(X;L_0(\mathbb{Z})) \to \mathbb{S}_n(X) \to \overline{\mathbb{S}}_n(X) \to \dots$$

- The 4-periodic total surgery obstruction $\overline{s}(X) \in \overline{\mathbb{S}}_n(X)$ of an n-dimensional geometric Poincaré complex X is the image of $s(X) \in \mathbb{S}_n(X)$.

Homology manifolds (I)

- Every n-dimensional compact ANR homology manifold M is homotopy equivalent to a finite n-dimensional geometric Poincaré complex (West)

- The total surgery obstruction $s(M) \in \mathbb{S}_n(M)$ of an n-dimensional compact ANR homology manifold M is the image of the Quinn resolution obstruction $i(M) \in H_n(M; L_0(\mathbb{Z}))$. The 4-periodic total surgery obstruction is $\overline{s}(M) = 0 \in \overline{\mathbb{S}}_n(M)$.

- The homology manifold structure set $\mathbb{S}^H(M)$ of a compact ANR homology manifold M is the set of equivalence classes of simple homotopy equivalences $h : N \to M$ from topological manifolds N, with $h \sim h'$ if there exist an s-cobordism $(W; N, N')$ and an extension of $h \cup h'$ to a simple homotopy equivalence $(W; N, N') \to N \times ([0,1]; \{0\}, \{1\})$.

Homology manifolds (II)

- Theorem (Bryant-Ferry-Mio-Weinberger)
(i) The 4-periodic total surgery obstruction of an n-dimensional geometric Poincaré complex X is $\bar{s}(X) = 0 \in \overline{\mathbb{S}}_n(X)$ if (and for $n \geq 6$ only if) X is homotopy equivalent to a compact ANR homology manifold.
(ii) For an n-dimensional compact ANR homology manifold M with $n \geq 6$ the 4-periodic rel ∂ total surgery obstruction defines a bijection

$$\mathbb{S}^H(M) \to \overline{\mathbb{S}}_{n+1}(M) \; ; \; (h : N \to M) \mapsto \bar{s}(h)$$

- $\overline{\mathbb{S}}_{n+1}(S^n) = \mathbb{S}^H(S^n) = L_0(\mathbb{Z})$, i.e. there exists a non-resolvable compact ANR homology manifold Σ^n homotopy equivalent to S^n, with arbitrary resolution obstruction $i(\Sigma^n) \in L_0(\mathbb{Z})$.

The simply-connected surgery spectrum $\mathbb{L}_\bullet$

- What is $\mathbb{L}_\bullet$? Required properties

$$\pi_n(\mathbb{L}_\bullet) \;=\; L_n(\mathbb{Z}) \;,\; \mathbb{L}_0 \;\simeq\; G/TOP \;.$$

- What are the generalized homology groups $H_*(X;\mathbb{L}_\bullet)$? Will construct them as cobordism groups of combinatorial sheaves over X of quadratic Poincaré complexes over $\mathbb{Z}$.

- Confession: so far, have only worked out everything for a (locally finite) simplicial complex X, using simplicial homology. In principle, could use singular homology for any space X, but this would be even harder. In any case, could use nerves of covers to get Čech theory.

The $(\mathbb{Z}, X)$-module category

- X = simplicial complex.

- A $(\mathbb{Z}, X)$-module is a based f.g. free $\mathbb{Z}$-module B with direct sum decomposition

$$B = \sum_{\sigma \in X} B(\sigma) .$$

- A $(\mathbb{Z}, X)$-module morphism $f : B \to C$ is a $\mathbb{Z}$-module morphism such that

$$f(B(\sigma)) \subseteq \sum_{\tau \geq \sigma} C(\tau) .$$

- Proposition (Ranicki-Weiss) A $(\mathbb{Z}, X)$-module chain map $f : D \to E$ is a chain equivalence if and only if the $\mathbb{Z}$-module chain maps

$$f(\sigma, \sigma) : D(\sigma) \to E(\sigma) \quad (\sigma \in X)$$

are chain equivalences. (This illustrates the X-local nature of the $(\mathbb{Z}, X)$-category).

Assembly for $(\mathbb{Z}, X)$-modules

- Use the universal covering $p : \widetilde{X} \to X$ to define the assembly functor

$$A \ : \ \{(\mathbb{Z}, X)\text{-modules}\} \to \{\mathbb{Z}[\pi_1(X)]\text{-modules}\} \ ;$$
$$B \mapsto B(\widetilde{X}) \ = \ \sum_{\widetilde{\sigma} \in \widetilde{X}} B(p(\widetilde{\sigma})) \ .$$

- In order to extend A to L-theory need involution on the $(\mathbb{Z}, X)$-category. Unfortunately, it does not have one! The naive dual of a $(\mathbb{Z}, X)$-module morphism $f : B \to C$ is not a $(\mathbb{Z}, X)$-module morphism $f^* : C^* \to B^*$.

- Instead, have to work with a chain duality, in which the dual of a $(\mathbb{Z}, X)$-module B is a $(\mathbb{Z}, X)$-module chain complex $T(B)$. Analogue of Verdier duality in sheaf theory.

Dual cells

- The barycentric subdivision X' of X is the simplicial complex with one n-simplex $\widehat{\sigma}_0\widehat{\sigma}_1\dots\widehat{\sigma}_n$ for each sequence of simplexes in X

$$\sigma_0 < \sigma_1 < \cdots < \sigma_n \ .$$

- The dual cell of a simplex $\sigma \in X$ is the contractible subcomplex

$$D(\sigma, X) \ = \ \{\widehat{\sigma}_0\widehat{\sigma}_1\dots\widehat{\sigma}_n \,|\, \sigma \leq \sigma_0\} \subseteq X' \ ,$$

with boundary

$$\partial D(\sigma, X) \ = \ \{\widehat{\sigma}_0\widehat{\sigma}_1\dots\widehat{\sigma}_n \,|\, \sigma < \sigma_0\} \subseteq D(\sigma, X) \ .$$

- Introduced by Poincaré to prove duality.

- A simplicial map $f : M \to X'$ has acyclic point inverses if and only if

$$(f|)_* : H_*(f^{-1}D(\sigma, X)) \cong H_*(D(\sigma, X)) \ (\sigma \in X) \ .$$

Where do $(\mathbb{Z}, X)$-module chain complexes come from?

- For any simplicial map $f : M \to X'$ the simplicial chain complex $\Delta(M)$ is a $(\mathbb{Z}, X)$-module chain complex:

$$\Delta(M)(\sigma) = \Delta(f^{-1}D(\sigma, X), f^{-1}\partial D(\sigma, X))$$

- The simplicial cochain complex $\Delta(X)^{-*}$ is a $(\mathbb{Z}, X)$-module chain complex with:

$$\Delta(X)^{-*}(\sigma)_r = \begin{cases} \mathbb{Z} & \text{if } r = -|\sigma| \\ 0 & \text{otherwise.} \end{cases}$$

The $(\mathbb{Z}, X)$-module chain duality

- The additive category $\mathbb{A}(\mathbb{Z}, X)$ of $(\mathbb{Z}, X)$-modules has a chain duality with dualizing complex $\Delta(X)^{-*}$

  $$T(B) \;=\; \mathrm{Hom}_{\mathbb{Z}}(\mathrm{Hom}_{(\mathbb{Z},X)}(\Delta(X)^{-*}, B), \mathbb{Z})$$

- $T(B)_r(\sigma) \;=\; \begin{cases} \sum\limits_{\tau \geq \sigma} \mathrm{Hom}_{\mathbb{Z}}(B(\tau), \mathbb{Z}) & \text{if } r = -|\sigma| \\ 0 & \text{if } r \neq -|\sigma| \end{cases}$

- the dual of a $(\mathbb{Z}, X)$-module chain complex C is a $(\mathbb{Z}, X)$-module chain complex $T(C)$ with

  $$T(C) \simeq_{\mathbb{Z}} \mathrm{Hom}_{(\mathbb{Z},X)}(C, \Delta(X'))^{-*} \simeq_{\mathbb{Z}} \mathrm{Hom}_{\mathbb{Z}}(C, \mathbb{Z})^{-*}$$

- $T(\Delta(X')) \simeq_{(\mathbb{Z},X)} \Delta(X)^{-*}$.

The construction of the algebraic surgery exact sequence

- The generalized $\mathbb{L}_\bullet$- homology groups are the cobordism groups of adjusted n-dimensional quadratic Poincaré complexes over $(\mathbb{Z}, X)$
$$H_n(X;\mathbb{L}_\bullet) \;=\; L_n(\mathbb{Z}, X)\ .$$
Require adjustments to get $\mathbb{L}_0 \simeq G/TOP$. Unadjusted L-theory is the 4-periodic $H_n(X;\overline{\mathbb{L}}_\bullet)$ with $\overline{\mathbb{L}}_0 \simeq L_0(\mathbb{Z}) \times G/TOP$. Adjust to kill $L_0(\mathbb{Z})$.

- The assembly map A from $(\mathbb{Z}, X)$-modules to $\mathbb{Z}[\pi_1(X)]$-modules induces
$$A \;:\; L_n(\mathbb{Z}, X) \to L_n(\mathbb{Z}[\pi_1(X)])$$

- The relative groups $\mathbb{S}_n(X) = \pi_n(A)$ are the cobordism groups of $(n-1)$-dimensional quadratic Poincaré complexes (C, ψ) over $(\mathbb{Z}, X)$ with assembly $C(\widetilde{X})$ an acyclic $\mathbb{Z}[\pi_1(X)]$-module chain complex.

Reference

- Algebraic L-theory and topological manifolds
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