ON THE TOPOLOGICAL INVARIANTS OF MULTIDIMENSIONAL MANIFOLDS

HEINRICH TIETZE

Monatshefte für Mathematik und Physik, **19**, (1908), pp. 1–118

Translated by John Stillwell

In the domain of analysis situs Poincaré[1] has recently brought us an abundance of new results, but at the same time he has raised an abundance of new questions that still await settlement. Thus while we have known, for a long time, a set of necessary and sufficient conditions for the existence of a one-to-one continuous map between two two-dimensional manifolds, at present such a system of conditions for three- and higher dimensional manifolds is not known. Certainly one has a whole series of distinguishing features, thanks to the work of Poincaré in particular, of multidimensional manifolds (numbers, groups) which do not change under one-to-one continuous mappings, and which therefore can be described as topological invariants of the manifolds. These give necessary conditions for two manifolds to be related one-to-one and continuously, but we do not know whether agreement of the known invariants is also sufficient for the existence of a one-to-one and continuous relationship.

The following essay[2] supplements Poincaré's work and deals in particular with the mutual relationship between the known topological invariants, presenting in particular the fact (Section IV) that the "fundamental group" of the two-sided closed three-dimensional manifolds yields all the other known topological invariants (thus in addition to the Betti number, which has already been described by Poincaré, also the Poincaré torsion numbers and

[1]The following works come under consideration above: "Analysis situs", Journal d. l'École polytechnique, 2 ser., Cah.1; "Complément a l'Analysis situs", Rend.d.Circ.mat.d.Palermo, V.13; "Second Complément a l'Analysis situs" Proc. Lond. Math. Soc. 32; "Cinquieme Complément á l'Analysis situs", Rend. d. Circ. mat. d. Palermo, v.18. In what follows these works will be cited by "An.Sit.", "Compl. 1" etc. The 3rd and 4th compléments (Bull. d. l. Soc. Math. d. France v.30 and Liouv. J. 5 ser. v.8) as well as the work "Sur les périodes des intégrales doubles" (Liouv. J. 6 ser. v.2) have the object of applying analysis situs to algebraic surfaces.

[2]I have given a preview of some of the results in the Wiener Akad.d.Wiss (see Wr. Ber.115, IIa. p.841 and Anzeiger 1906, Math. nat. Kl. p.349.)

hence also the number Q introduced in Section III). This result follows in a simple way from the fact that each discrete finitely generated group has certain characteristic numbers, which may be called the Poincaré numbers of the group, and the torsion numbers of first order of a manifold are none other than the Poincaré numbers of its fundamental group.

Various other questions are discussed in the later sections V-VII. The first two sections, which serve as an introduction and foundation for what follows, contain the definitive treatment of the manifold concept on the basis of a certain presentation called a "cell system," followed by a sketch of the objectives and known facts. In this connection it should be remarked that the presentation of a manifold as a cell system in Section I is of particular theoretical interest in that it permits the construction of analysis situs free of the introduction of infinite point sets or function theoretic methods. This depends on the fact that a cell system is determined by a finite number of elements and a finite number of relations between them. This possibility of developing analysis situs purely combinatorially, so to speak, has achieved importance in the works of Dyck[3] and has been systematically presented by Dehn[4] in the most recent Enzyklopädie article. The presentation of the cell system and the pertinent definitions are guided by intuition up to three dimensions and are then continued by analogy[5].

§§15, 16 of this essay are indeed primarily concerned with continuous point manifolds, for which intuition is in large measure subordinate to deduction. Except for this part of the work, which is to be regarded only as preparation for the strict treatment of our questions, the emphasis is on the idea of a cell system independent of its relation to the concept of a point manifold. This applies in particular to the abovementioned results on the fundamental group of a cell system as an independent object of study, with the proof in §13 that the fundamental group is a topological invariant, and to the content of §19. Equally, it seems desirable to emphasize the relation between cell systems and point manifolds, indeed to introduce the cell system so that it has the character of a representation of a manifold, and to indicate its combinatorial character only incidentally. However, the cell system has been applied by Poincaré precisely as a method for the analysis situs of point manifolds, in a most productive way. Of course it turns out

[3]Math. Ann. 32 and 37.

[4]Dehn-Heegaard, Analysis situs, Enz. III AB3.

[5]See the above-cited Enzyklopädie article, Grundlagen, no.8.

that there are difficulties in carrying over many theorems, derived easily in the domain of combinatorial analysis situs, to the domain of point manifolds. Hence the distinction between "topological invariants of the schemata" and those of manifolds in §2; hence the replacement of the definition of Betti numbers first given in §6 by another based on the cell system. It is obvious that many of the difficulties could be avoided by restricting the class of manifolds considered, e.g., as Poincaré often does, by considering analytic manifolds. No doubt the method of approximation by analytic functions would cope with many of the difficulties considered by reduction to the case of analytic manifolds usually considered by Poincaré, while for other cases a deeper foundation may be necessary. However, we have confined ourselves to pointing out the difficulties and unsettled questions, especially since further intrusion of these questions into an essay devoted essentially to combinatorial analysis situs is to be avoided.

Now a few words concerning the use made of intuition in what follows. We have already said that it is used as at least a first step towards settling questions in certain developments in a few later sections.[6] On the other hand, intuition is also introduced for the sake of clarity in places where there is no difficulty in proceeding on a purely deductive basis.

I. The schemata of multidimensional manifolds

§1

Demarcation of the point manifolds to be considered

One can consider the most general purpose of analysis situs[1] to be the complete presentation and study of those properties of arbitrary point sets that are preserved when one passes to a homeomorphic image of the set. Following Poincaré,[2] two point sets are called homeomorphic when there is a one-to-one continuous correspondence between them.

[6]There are also a few places (particularly in Section II) in which certain assumptions are discussed in relation to their admissibility or probability. In a few of the examples claims are made by appeal to intuition, so that statements derived from them cannot be regarded as rigorously based, but only as plausible.

[1]This is the point of view expressed by Hurwitz at the Zürich Congress (Verh. Int. Math. Kongr. Zürich p.102).

[2]An. Sit.§2. In this instance it is only applied to continuous manifolds.

The enormous generality of the problem is reflected by the distance at which we find ourselves from our goal. Restricting consideration to connected point manifolds (or to their connected components) corresponds to the needs elsewhere in mathematics for a better view of the analysis situs of continuous manifolds.[3] Our first concern is therefore to circumscribe the concept of manifold to the extent appropriate for later applications. It seems most natural to do this by giving intrinsic features that one can prove to be preserved by one-one continuous transformations, so as to extract from the class of all point sets those for which the analysis situs of continuous point sets can be carried out[4]. Far-reaching results in this direction have so far been obtained only in the domain of plane point manifolds. As a result, it seems justified when investigating multidimensional manifolds to use a certain form of presentation as the definition of manifold (which leads in general to a certain restriction of the domain of manifolds considered), and to avoid in this way certain difficulties inherent in the first approach. By a manifold one then means a point set representable in the prescribed way, or one homeomorphic to it.

We begin by making a couple of general remarks on the different ways of presenting manifolds. The principal, and simplest, way to determine point sets and hence all point manifolds is to give a collection of points $(x_1, x_2, ..., x_n)$ in the space of n rectangular coordinates,[5] where each point is

[3]Thus it appears to be desirable to know the analysis situs of connected four-dimensional point manifolds in the theory of algebraic functions of two complex variables. In this connection we remark incidentally that the Cremona transformations of the space of two complex variables are certainly not always one-to-one, but may map two-dimensional point manifolds into points and conversely, and the same also holds for birational transformations of algebraic surfaces. As a result, it is possible that the topological invariants of the four-dimensional manifolds represented by algebraic surfaces change under birational transformations (See Picard, C.R.134, p.629 and the work of Poincaré cited above in Liouv. J. 6 ser., v.2). Thus for function-theoretic problems it may be necessary to use transformations that are not always invertible, and to study the invariants of such transformations.

[4]In order to grasp this idea it suffices to recall the well-known result of C. Jordan, supplemented by the series of works by A. Schoenflies which have appeared recently.

[5]Different forms of this presentation, such as equations $F_i(x_1, x_2, ..., x_n) = 0$ between the coordinates or by parametric representation, are given by Poincaré (An. Sit.§§1,3,15). He assumes, e.g., that the functions F_i are differentiable, even analytic. (It is clear that assuming them to be merely continuous leads to point sets that are far too general.) Admittedly, in the passage from a manifold defined in this way to a homeomorphic image the representability by equations with this property may be lost. The requirement that,

represented by a single n-tuple, and each n-tuple appearing corresponds to a point of the manifold. But if we look at, say, the points of a lemniscate and take the double point to be two points of a one-dimensional manifold, with the neighbourhoods of the latter consisting only of points that can be considered as neighbours on the same branch, then we have a representation of a simple closed curve of the same general type as the mode of representation first considered. Riemann surfaces are an example of this type of representation for two dimensional manifolds: each point (x, y) of the $(x + iy)$-plane represents not one, but a finite number of points of the manifold, and the manifold itself is determined by the number of points assigned to each pair (x, y) and the establishment of their neighbourhoods. Another way to determine a manifold is to view different n-tuples as representing the same point of the manifold, for example the corresponding points on opposite sides of the period parallelogram which defines the algebraic manifold of an elliptic curve. The general principle that arises here, namely the construction of two-dimensional manifolds by identification of the boundary segments of bounded surface pieces, has been especially stressed by Klein.[6] The generalization of the latter form of presentation, which Poincaré in particular has used several times, and which may be called a "cell system", will be used as the basis for what follows.

Generally one can supplement what has just been said about the different forms of representation of point manifolds by saying that the nature of a manifold is established on the one hand by giving its points, where, as we have said, each point is secured by a particular n-tuple $(x_1, x_2, ..., x_n)$, as well as saying what are considered to be the neighbourhoods of a particular point. The latter requires some measure of distance, which admittedly can be varied somewhat without changing the character of the manifold. As an example

for a point set to be regarded as a manifold, all its homeomorphic images must be also, is then expressed by saying that either a representation by functions with the prescribed property exists, or else a homeomorphic point set has such a representation. Such difficulties of definition are responsible for our basing the manifold concept on a certain form of presentation. Obviously it is also necessary, when a property is introduced topologically, to be able to prove it invariant under one-one continuous transformations of manifolds of the type considered, e.g., in the present case not merely for transformations realized by differentiable or analytic functions. It should be mentioned at the beginning that the satisfaction of this requirement for the presentation which follows, in terms of a "scheme", is closely connected with the proof of a later theorem (§2) that two schemata defining homeomorphic manifolds are themselves homeomorphic.

[6]See for example Math. Ann. 21 p.141

of the way such a prescription determines the nature of a manifold, consider how the plane can be regarded as a surface homeomorphic to the sphere: one adds a point ∞ to the usual points (x, y), and chooses the distance between the point ∞ and (x, y) to be, say, $\frac{1}{\sqrt{x^2+y^2}}$. The distance measure and neighbourhood concept then give the required notions of continuity and homeomorphism.[7]

Regarding the use of the word manifold, it may be said first of all to denote — in a given form of presentation — a certain particular set of points. However it is convenient to denote a totality of mutually homeomorphic manifolds as a manifold itself in an extended sense, so that a particular member of the totality can be considered as a particular presentation, or representative, of the manifold (when the word is taken in the extended sense).[8]

The description of the cell systems used in the following §§, or as we shall say, the schemata,[9] will be carried out only in the case of two (§2) and three dimensions (§3). In the case of higher-dimensional manifolds (§4) we can then confine ourselves to indications.[10]

[7]Apart from point manifolds, we can also consider manifolds of other elements (cf. Klein, Math.Ann. 9 p.480, and 21 p.154); however, provided the elements are determined by a finite number of coordinates, this yields nothing new.

[8]When speaking of a point of the manifold in this extended sense one needs to think of a specific one-to-one continuous relationship between any two representatives, chosen in such a way that if A, B, C are any three representatives of the manifold, the relationships between A and B and C respectively associate the same point of C with points of A and B that are associated with each other.

[9]Since the expression seems particularly appropriate for manifolds of three or more dimensions, we shall use the word Schema for the general case, though in a somewhat different sense from Poincaré (Compl.1, §2, p.290), whose systems we shall describe in what follows (see §5) as Poincaré relation systems. In order to avoid accumulation of new terms, I have absorbed this modification into the meaning.

[10]The representation in question is not taken as a foundation by Poincaré, but is obtained by decomposition of analytically defined manifolds. The development of the analysis situs of two-dimensional manifolds in terms of their construction from surface pieces first became known to me from the lectures of Professor Wirtinger (on Algebraic Functions, Vienna, Summer 1904), where the combinatorial side of this development was also indicated. It was these lectures, together with a later personal communication on the analogous presentation of three-dimensional manifolds, that stimulated the studies underlying the present essay. The combinatorial side of the following investigations, in particular the stepwise construction of schemata by increasing dimension, are also covered by Dehn in the Enzyklopädie article already cited.

§2
The schemata of two-dimensional manifolds

The description of each piece of information involved in the schema of a two-dimensional manifold begins with a special case, in which the schema is introduced to a certain extent in abstract form, then extended, just as one does in defining a point manifold.

Let a disk be given, with its perimeter divided into n parts, the dividing points of which will be called vertices, and the arcs, edges. In addition, let a rule by given that associates certain edges with certain other edges. The edges that do not occur in pairs of associated edges will be called free edges.

A particular orientation is chosen to be positive[1] for the perimeter of the disk, and correspondingly a positive and negative direction for each edge. The rule that associates pairs of edges must say in each case whether the edges are identified so that the positive direction of one matches the negative or positive direction of the other. The two types of edge identification will be called the first kind and the second kind respectively.

The correspondences between the edges and their directions determine correspondences between the vertices. Let $s_1, s_2, \ldots, s_n$ be the edges as they appear in order around the positively traversed circumference of the disk. The endpoints of s_i may be called A_{i1} and A_{i2}, so that the positive direction of s_i is from A_{i1} to A_{i2}.[2] Each vertex therefore carries two notations and we have

$$A_{n2} = A_{11}, \quad A_{i2} = A_{i+1,1}, \qquad (i = 1, 2, \ldots, n-1) \tag{1}$$

Now suppose two edges s_h, s_k are identified in the first way; then we derive the following identifications of vertices:

$$A_{h1} \quad \text{with} \quad A_{k2} \tag{2}$$

[1]The choice of this orientation is not an essential piece of information in the schema, only an aid in describing the correspondences that follow. However when we are dealing not with manifolds as such (even two-sided ones) but with their orientation or lack of it, then the choice of orientation of the schema becomes important. We discuss this in §4.

[2]In the case $n = 1$ one has a single subdivision point on the circle perimeter, which is at the same time the initial point and endpoint of the single edge S_1.

$$A_{h2} \quad \text{with} \quad A_{k1}$$

Conversely, an identification of the second kind matches A_{h1} with the vertex A_{k1} and A_{h2} with the vertex A_{k2}. By alternately applying the equations (1) and the identification relations (2) one obtains a grouping of all relations (1) and (2) into series of the following kind

$$\ldots\ A_{h-1,2} = A_{h1} \quad \text{associated with} \quad A_{k2} = A_{k+1,1}\ \ldots$$

and these series are continued to left and right until either a repetition occurs or the series breaks off. The latter obviously occurs at the endpoint of a free edge. In particular, a vertex that lies between two adjacent free edges s_{i-1} and s_i gives rise to a series consisting of the single equation

$$A_{i-1,2} = A_{i,1}$$

All the vertices appearing in a series may be associated with the vertex of a cycle, and the cycle may be called closed or open according as the relation series in question is periodic or breaks off.

A system of rules prescribing identifications of the kind described between the n edges of a subdivided circle perimeter represents the simplest case of the schema of a two-dimensional manifold.

The extent to which such a schema can be considered to define a two-dimensional manifold is most simply explained by reference to the corresponding properties of the fundamental domains of automorphic functions. We take for example the special case of a polygon with $4p$ edges that are circular arcs, with each pair of opposite edges related by a linear substitution. In the case of the algebraic manifold represented by the fundamental domain and the collection of automorphic functions defined on it, two points on opposite sides which correspond under the associated linear substitution represent a single point of the manifold, and similarly all $4p$ vertices represent a single point. It is quite analogous to consider a schema of the kind we have described to determine a two-dimensional manifold. Namely, one thinks of a one-to-one continuous relationship between the points of a pair of corresponding sides — correctly directed — and fuses two points related in this way by definition into a single point of the manifold. As a consequence, all vertices occurring within the same cycle represent the same point of the manifold. In the manifold so defined, points on a side that is paired with another side have the same character as points in the interior of the disk, as does a point represented by a closed cycle of vertices. On the other hand, the free edges and the vertices that fall into open cycles yield (closed) boundary lines of the manifold.

We append the following remark to the process just developed for deriving a two-dimensional manifold from a schema. The mere requirement of a one-to-one continuous relationship between the points of corresponding sides leaves a great deal of flexibility, apart from which the particular positions of the n subdivision points on the perimeter of the disk can be arbitrarily chosen, since only the number n and the order of the edges are expressed in the schema. It is obvious that no matter how one determines these arbitrary factors, all the point manifolds obtained are homeomorphic; the indeterminacy expressed is therefore without importance. It is likewise an inessential modification, which leads to a homeomorphic manifold, when one takes (provided $n > 2$) an ordinary rectilinear plane polygon with n sides in place of the disk with n points of subdivision. We shall therefore speak simply of a "polygon" in place of a "disk with perimeter divided into n parts", but without requiring the restriction $n > 2$.

A two-dimensional manifold defined by a schema in the way just described may be called two-sided, when all the identifications of edges are of the first kind, otherwise, one-sided.[3]

We shall consider a somewhat more general two-dimensional manifold in which we permit the defining schema to omit certain of the points represented by vertices of closed edges.

In such a case there are missing inner points of the manifolds which

[3]Cf. Poincaré, Compl. 5 p.52, 53. In his works Poincaré has gone back to the older and shorter terms "two-sided" and "one-sided" (bilatère, unilatère). However it should be noted that these terms presuppose an embedding of the two-dimensional (n-dimensional) manifold in a three-dimensional ($(n+1)$-dimensional) space, whereas they really express, not such a relative property, but an absolutely intrinsic property of the manifold. Klein (Math. Ann. 9 p.479) and Dyck (Math. Ann. 32, p.473) have replaced the notions of two-sidedness and one-sideness by "with non-reversing indicatrix" and "with reversing indicatrix". By an indicatrix of a two-dimensional manifold V we mean a small closed line drawn around an inner point A of V, upon which three points 1, 2, 3 are marked, e.g. a small triangle or a small circle with three distinguished points. Now if one carries this small closed line along a closed path in V, from A and back to A again, so as to bring it back into coincidence with its initial position with 1 on top of 1, then the points 2, 3 either return to their initial positions or come into coincidence with 3, 2 respectively. Correspondingly, closed paths in V are divided into those which "do not reverse the indicatrix" and those which do. The two-sided manifolds are distinguished by the fact that they have no paths in the latter category.

comprise the boundary together with the boundary lines.[4] A two-dimensional manifold is called closed when its schema contains neither boundary points of the kind just described, nor free edges.[5]

The general case of the schema of a two-dimensional manifold differs from the special case above only to the extent that there may be a finite number of polygons instead of one (again, disks with finitely subdivided perimeter). Again, certain edges are paired, with directions being observed. Just as in the case of a single polygon, cycles of vertices result. The schema may also admit deletion of certain cycles of vertices. What has been said concerning a schema involving only a single polygon, such as the definition of a closed manifold, carries over immediately to the general case. In the case of several polygons we call two different polygons directly connected when an edge of one is identified with an edge of the other, and indirectly connected when they comprise the first and last term of a finite series of directly connected polygons. If any two polygons in a schema are either directly or indirectly connected, then the schema itself and the manifold it defines will also be called connected. A connected manifold is called two-sided when one can give an orientation to each polygon so that all edge identifications are of the first kind; if that is not possible, then the manifold is called one-sided.

The polygons of a schema will be called surface pieces of the schema, and their number will be denoted α_2. An edge of the schema will be either a free edge or a pair of identified edges; a vertex of the schema is a cycle of identified polygon vertices. Let α_1 be the number of edges, and α_0 the number of vertices. A schema with a single surface piece, such as we first considered, will also be called a fundamental polygon of the manifold.

So far we have explained how a two-dimensional manifold is determined by a "schema", or as one may also say, a "system of surface pieces."[6] We now define a relation between schemata that will be called homeomorphism. This is done in terms of certain modifications of the schemata called "subdi-

[4]The $(n-1)$-dimensional boundary manifolds of an n-dimensional manifold may be distinguished from the others, following Poincaré (An. Sit. p.6) as proper boundary manifolds (véritables variétés frontières). The points represent so-called improper boundary manifolds of the two-dimensional manifold. The points on the proper boundary manifolds are reckoned to belong to the manifold itself.

[5]This represents a slight deviation from the definition of Poincaré (cf. An. Sit. p.7), wherein improper boundary manifolds are allowed to occur in closed manifolds. Cf. §15, note 9.

[6]Called a "polyhedron" by Poincaré (An. Sit., p.101).

visions", beginning with "elementary subdivisions", which are of two kinds for a two-dimensional schema.

1) One introduces a new subdivision point on an edge of the polygon from the original schema (say, dividing it in half), thus dividing it into two edges. If the edge being subdivided is paired with another edge, the latter is also subdivided at the corresponding point. The two subdivision points then constitute a two-termed closed cycle in the new schema.

2) One of the circles (polygons) is divided by a chord between any two vertices into two segments, each of which can again be deformed into the form of a circle. Thus one obtains two polygons in place of one. The two new edges that result from the chord are identified in such a way that vertices coinciding before the division are also identified. The identifications between the edges originally present remain unchanged.

These two types of elementary subdivision of a two-dimensional schema may be briefly described as division of an edge into two and division of a surface piece into two. A subdivision in general is understood to be the modification of a schema that results from a series of successive elementary subdivisions. The schema resulting from the subdivision will be called the subdivided or derived schema.[7]

Two schemata will be called homeomorphic when they have a common derived schema, thus when they have the property that subdivision of one schema yields a schema that can also be obtained by subdivision of the other. Two schemata homeomorphic to the same schema are homeomorphic to each other.[8] The schemata therefore fall into classes of mutually homeomorphic schemata and it is easy to see that the property of being closed, two- or one-sided, or connected either holds for all schemata in a class, or else fails for them all.

However, since the term "homeomorphic" already has a quite definite meaning in relation to manifolds, the above definition of homeomorphism requires some further explanation. Previously we have called two manifolds homeomorphic when there was a one-to-one continuous relationship between them. It is now clear that two manifolds defined by homeomorphic

[7]Called "derived polyhedron" by Poincaré (An. Sit. p.101).
[8]This theorem is the object of §19.

schemata are homeomorphic in the original sense, since this is obviously true of two manifolds related by an elementary subdivision. The converse fact, namely that if two manifolds defined by schemata are pointwise homeomorphic then their schemata are also homeomorphic, i.e., possess a common derived schema, can be made plausible by very simple considerations. Namely, one thinks of superimposing the images of the edges of one schema under the continuous mapping upon the other schema. Then under the assumption that each of these lines is divided into only a finite number of pieces by the latter schema, we have a subdivision of the original schema. The same holds for the schema of the other manifold when the converse construction is carried out, and the two schemata obtained are obviously the same, so the two given schemata are homeomorphic. However, since our assumption need by no means always be satisfied,[9] the proof of this theorem is still somewhat incomplete, since, although the two-dimensional case may be relatively simple, the higher-dimensional case is lacking a satisfactory treatment.

The following remark may be made in this connection. Suppose one had a process for deriving a certain property from the schema, e.g., a series of numbers, of such a kind that this series was the same for any two homeomorphic schemata. Then speaking of this series of numbers as a topological invariant is justified only if homeomorphic manifolds can never be defined by nonhomeomorphic schemata, in other words under the assumption that the questionable theorem above has been settled. We shall therefore retain the expression "topological invariant", but use it only under a clear understanding of the difference between the topological invariants of schemata and those of manifolds. Nevertheless, it is only theorems derived with the aid of the concept of homeomorphism of schemata, and therefore applicable only with some reservations to point manifolds, that are meaningful in the purely combinatorial development of analysis situs. The conception implied by such a development has already been mentioned in the introduction. One realizes

[9]When this assumption, which says that when the two-dimensional manifold is simultaneously defined by the two schemata the two edge systems meet in only a finite number of points, is not satisfied, then certain polygons are divided into infinitely many pieces. Still more complicated relations can occur between the schemata of homeomorphic three-dimensional manifolds, in which certain cells of a schema are divided by walls of the other schema into infinitely many pieces, some of which have infinite connectivity (i.e., which do not have a finite Betti number P_1). The above considerations also break down when dealing with manifolds of more than two dimensions with improper boundary manifolds (see §15, note 5).

immediately that the schemata carry a combinatorial imprint in terms of their information content (in the case of the schema of a two-dimensional manifold, the order of edges in the individual polygons and the pairings between them). The concept of a number continuum or functional relationship between real numbers does not intervene. The same is true, as one easily convinces oneself, for the schemata of three- and higher-dimensional manifolds.[10]

A two-dimensional schema will be called simply connected when it is homeomorphic to one of the following schemata, denoted by ε_2 and σ_2.

ε_2 denotes any schema consisting of a single polygon with two sides, with no pairing between the sides. The schema σ_2 consists of two polygons, each with two sides, where each side of one polygon is paired with a side of the other polygon in the first way. A schema homeomorphic to ε_2 resp. σ_2 is called a simply connected two-dimensional schema which is bounded or closed respectively.[11]

The manifold defined by σ_2, or a homeomorphic manifold, is called the two-dimensional spherical manifold. The manifold is obviously homeomorphic to the surface of a ball. The manifold defined by ε_2, or a homeomorphic manifold, may be called the two-dimensional element.

If one considers the schemata homeomorphic to the schema of a given two-dimensional manifold, then there is among them the so-called dual schema,[12] which is of particular interest. Its meaning becomes immediately clear when one uses the presentation to join together the individual polygons[13] into a closed surface. The edges of the schema then constitute a net of polygons

[10]For the sake of convenience in what follows we shall speak of a "two-dimensional schema" instead of the "schema of a two-dimensional manifold", and analogously on "n-dimensional schema". The justification for describing the schema itself as two-dimensional lies in the fact that the schema can be considered to have a meaning independent of the point manifold it defines.

[11]Schemata homeomorphic to ε_2 include all schemata consisting of a single polygon with an arbitrary number of edges, e.g., a single edge, with no pairings between them; and a simpler schema homeomorphic to σ_2 can be obtained from a single polygon with two sides, paired in the first way. The reason for choosing our specific ε_2, σ_2 was to bring them into line with the simply connected n-dimensional schemata of §4.

[12]Poincaré, Compl.1, §7 ("polyèdre reciproque").

[13]The actual execution of this process does not trouble us here. There is a question: how large is the smallest possible dimension $\psi(n)$ or $\psi^*(n)$ of a euclidean space containing a homeomorphic copy, without self-intersections, of each closed resp. bounded n-dimensional manifold? We know $\psi^*(2) = 3$, while it is open whether, as Poincaré assumes (An. Sit. §10 and §11, p.56), that $\psi^*(3) = 4$.

on the surface. One draws the corresponding polar of this figure by taking a point in the interior of each polygon, and crossing each edge k of the given figure by an edge $\overline{k}$ of the new figure, where $\overline{k}$ connects the chosen interior points of the two polygons which meet along k. Then each vertex, edge and surface piece of the given figure corresponds respectively to a surface piece, edge and vertex of the polar figure. If one cuts the surface along the edges of the new net, then one obtains the polygons of the desired dual schema.

This is the intuitive meaning of the dual schema. It corresponds to the following method for constructing the dual of a given schema, a method of a more abstract kind, like that used to define a schema at the beginning of this paragraph: one lets each vertex corresponding to a cycle γ_i of the given schema correspond to a polygon π_i of the new schema, namely a polygon with as many vertices as there are edges in the cycle. Now when (in terms of the notation explained earlier)

$$..., \; A_{h-1,2} = A_{h1} \quad \text{corr. to} \quad A_{k2} = A_{k+1,1} \; ...$$

denotes a relation series that generates the cycle γ_i, then the vertices of the polygon π_i may be provided with the notations $B_{h-1,2} = B_{h1}$, $B_{k2} = B_{k+1,1}, \ldots$. The edge $B_{h1}B_{k2}$ of the polygon π_i is now given one of the notations t_h, t_k e.g., t_h. The edge $B_{h2}B_{k1}$, which can belong to either the polygon π_i or to another polygon of the new schema is then called t_k. The pairing rule of the new schema then says that the edge t_h is paired with the edge t_k so that the vertices B_{h1}, B_{h2} correspond, and similarly B_{k1}, B_{k2}. In this way the new schema is completely determined by the old.

The schema dual to the dual itself is obviously the original schema. The proof of the fact that two schemata dual to each other are homeomorphic can be suppressed, since it is immediate when one sees the intuitive meaning of the dual schema.

§3
The schemata of three-dimensional manifolds

In order to explain the schemata of the three-dimensional manifolds and to carry over the various concepts defined for two-dimensional schemata, we again begin with a special case.

Consider a ball whose surface is divided into a finite number of simply connected polygons by a finite number of line segments.

A few explanatory remarks are in order immediately. By a line segment we mean a one-to-one continuous image of a straight line segment, and the simply connected polygons are point sets of the spherical surface that are in one-to-one and continuous correspondence with the set of points on a disk, so that boundary points correspond to boundary points. The polygons therefore constitute two-dimensional elements (see §2). The line segments on the spherical surface will be called edges of the polygon subdivision, the points at the ends of edges will be called vertices. When we regard each edge on the spherical surface as a pair of associated edges, the polygons on the spherical surface obviously form the schema Σ of a two-dimensional manifold. On the basis of the provisional theorem mentioned above, we can assume that the schema Σ is homeomorphic to the schema σ_2 (see §2). At any rate, we shall assume this to be true of the schema Σ.

Certain of the polygons that subdivide the spherical surface are identified. This is possible only if they have the same number of edges. Now let a certain orientation be designated as positive in each polygon, and let it be the same for all polygons from the point of view of an observer outside the sphere. As a result, each edge of a polygon has an orientation induced by the polygon itself.[1] The pairing of polygons will then also include the information whether the positive orientation of one matches the negative or positive orientation of the other (pairing of the first or second kind.) The edges of one polygon will also be paired with the edges of the other, and in the case of a polygon identification of the first (second) kind, the edges of one polygon, taken in the order of positive orientation, correspond to the edges of the other taken in the order of negative (positive) orientation,[2] and a positively directed edge of one polygon is identified with the negative (positive) direction of its partner in the other polygon. Such an edge identification will itself be said to be of the first or second kind in conformity with the polygon identification.

Due to the fact that each edge of the polygonal subdivision of the spherical surface appears doubly as a polygon side, cycles of identified edges can now be established, in complete analogy with the construction of cycles of identified vertices in a two-dimensional schema. The cycles of identified edges can be closed or open. Open cycles can only appear when there are free polygons, i.e., ones that are not paired with other polygons.

[1] It may happen that a polygon meets itself along an edge. Of course such an edge represents two sides of the polygon, with two opposite orientations induced by the polygon itself.

[2] This condition is meaningful only for polygons of more than two sides.

Each individual edge of the polygon subdivision may itself be given an orientation.[3] (This is not to be confused with the induced orientation described earlier, and it naturally coincides with the positive direction of one of the two polygon sides forming the edge.) We shall now assume satisfaction of the condition[4] that in each closed cycle of identified edges the number of identifications of the second kind that occur is even, and then an association of orientations for the edges identified in the cycle can be derived from the association of directions of the polygon sides.

Just as in the case of a two-dimensional schema the identifications of the polygon vertices follow from those of the polygon sides. The identifications of the vertices here may be derived from the identifications of edges and their directions. Since a vertex in general appears in several ways as the endpoint of an edge, the construction of systems of associated vertices depends on the polygonal subdivision.

A further condition[5] that may be imposed on a three-dimensional schema concerns certain two-dimensional schemata derived from the present identification process. For this purpose we think of each vertex of the polygonal subdivision surrounded by a small circle. This circle is divided into a finite number of sides by the polygon edges radiating from the vertex in question. We take the positive direction of these sides to be that corresponding to a positive circuit around the vertex (i.e., one with the vertex on its left). Now let π_1, π_2 be two identified polygons, let A_1 a vertex of π_1, and let A_2 be the corresponding vertex of π_2. The arc of the small circle around A_1 that lies in π_1 may be called s_1, and the corresponding arc around A_2, lying in π_2, may be called s_2. The sides s_1 and s_2 are now identified, and in fact, when the identification of the polygons π_1 and π_2 is of the first (second) kind, the positive direction of s_1, is identified with the negative (positive) direction of s_2. The collection of all the small circles drawn around the vertices then constitutes a number of polygons, with certain identifications between their sides, and therefore gives us a two-dimensional schema. The latter divides

[3]The choice of this orientation, like that of the polygons on the spherical surface, is not an essential feature of the three-dimensional schema, but merely an aid in describing the identification process.

[4]This condition represents a special case of a condition mentioned in §4 for n-dimensional schemata, from which it follows that the identifications between the geometric figures considered (here: edges) are always of a nature such that a definite set of identifications of their boundary elements (here: the endpoints of edges) follows.

[5]cf. Poincaré, An. Sit. p.52-55.

into as many connected schemata as there are systems of identified vertices in the polygon subdivision.[6]

The condition for a schema to define a three-dimensional manifold is now that all these connected two-dimensional schemata be simply connected (with or without boundary). According as such a schema is closed or bounded, the corresponding system of identified vertices of the polygon subdivision is called closed or bounded.

To the extent that a three-dimensional manifold is defined at all by a schema, there is a complete analogy with the case of two dimensions. Thus the points of identified polygons are associated in a one-to-one and continuous fashion, with attention to orientation and side ordering, though only the closed cycles of identified edges of the polygon subdivision are easy to visualize.[7] Such sets of identified points are considered to be points of the manifold by definition. The interior points of identified polygons then have the same character as the points of the manifold in the interior of the ball, i.e., by means of a suitable one-one continuous map of the manifold onto itself they can be mapped to points in the interior of the ball, and the same holds for points that lie on the edge of a closed cycle, and any point that represents a closed system of identified vertices. In the latter case the condition of simple connectivity of the neighbourhood manifold, which excludes the appearance of certain singular points, is essential. The free polygons form themselves into "boundary surfaces" of the manifold, which contain the lines representing open cycles of identified edges, and the points representing open systems of identified vertices. The individual boundary surfaces have no common points, which is also guaranteed by the condition on neighbourhood manifolds.

One sees immediately that, just as the determination of two-dimensional manifolds by schemata involves a certain arbitrariness, the same is true for the above process, since it depends on a two-dimensional schema Σ dividing

[6]The manifold defined by these connected schemata may be called the neighbourhood manifold of the vertices in question in the three-dimensional schema (see text, two pages hence).

[7]The pointwise identification of polygons yields an identification of the points in any two successive edges in a cycle. Now if $k_1, k_2, \ldots, k_m$ are the edges in a closed cycle, a point P_1 of k_1 corresponds to a particular point P_2 of k_2, the latter corresponds to a point P_3 of k_3 and so on, finally with a point P_m of k_m corresponding to a point of k_1. The identifications now have to satisfy the condition that this last point is P_1 again. This condition is guaranteed by the above hypothesis on the number of side pairings of the second kind.

the spherical surface in a way that is by no means fully determined. The information expressed in a three-dimensional schema, namely the schema Σ and the identifications prescribed, is sufficient for the determination of a three-dimensional manifold.

More general three-dimensional schemata result by allowing certain points and lines represented by closed systems to be excluded. A particular case of interest is the one in which the line segments excluded from the manifold unite into closed lines.

The general case of a three-dimensional schema is obtained from the special case above by allowing the ball to be replaced by a finite number of balls whose surfaces are divided into polygons, then making identifications of these polygons in pairs. All that has been said for the special case carries over without further comment to the general case.

When the three-dimensional schema is so constituted that there are neither free polygons nor excluded points or lines, the manifold defined by the schema is called closed. The concept of connectedness is defined analogously as in the case of two dimensions. Each single ball of a three dimensional schema can be arbitrarily given a positive orientation by coherently orienting the polygons on its surface. If these orientations in a connected schema can be chosen so that all polygon identifications are of the first kind, then the manifold defined by the schema is called two-sided, otherwise one-sided.[8]

The individual balls will also be called cells of the three-dimensional schema. A wall[9] of the schema is either a free polygon or a pair of identified polygons. A cycle of identified edges of the polygon subdivision will be called an edge of the schema, a system of identified vertices will be called a vertex of the schema. The numbers of cells, walls, edges and vertices of the

[8]The indicatrix (cf. §2, note 3) of a three-dimensional manifold V can be taken to be a small tetrahedral surface surrounding an inner point A of V, or any other simply connected closed surface with a similar subdivision into four triangles, with vertices denoted 1, 2, 3, 4. If one now carries the indicatrix along a path from A to A and brings the tetrahedral edges and vertices back to their old positions so that 1 and 2 take exactly their original positions, then 3, 4 can either retain their old positions or exchange them. The paths in V divide into those that do not reverse the indicatrix and those that do.

The indicatrix for an n-dimensional manifold can be introduced in an analogous way.

[9]We introduce the word "wall" for a two-dimensional element in an n-dimensional schema, since the word "surface piece" will have another meaning in §5 (namely, the notation for a space piece in a manifold in the special case $m = 2$. The normal use of the word "surface piece" will only be retained in the case $n = 2$ (§2), for the two-dimensional elements of the schema.

schema will be denoted α_3, α_2, α_1, α_0.

The determination of a three-dimensional manifold by a schema or cell system has now been described. The concept of homeomorphism of schemata is now set up, just as in the two-dimensional case, in terms of elementary subdivisions, of which there are three types in the three-dimensional case:

1. One selects a cycle of identified edges from the schema and halves all the corresponding edges of the polygon subdivision, identifying the resulting half edges as they were before halving. We thus obtain, in place of the cycle of identified edges, two cycles and a system of identified vertices corresponding to the halving point, while the remainder of the schema is unchanged.[10]

2. One divides one of the polygons into two by a new edge of the polygon subdivision connecting two of its vertices. If the subdivided polygon is identified with another in the original schema, then the latter is also subdivided by a new edge connecting the corresponding vertices, and the two pieces of the first polygon are identified with the two pieces of the second in such a way that identifications between the original edges and vertices are preserved, while the two new edges of the polygon subdivision are identified with each other. The two new edges then constitute a cycle of identified edges.

3. One chooses a simple closed line consisting of edges on one of the balls of the schema and divides the ball into two by a simply connected surface spanning this line (e.g., by the surface comprised of the radius vectors to the closed line). The subdividing surface is therefore a two-dimensional elementary manifold, and it may be assumed that each of the two pieces can again be mapped one-to-one and continuously onto a ball K so that the points on the original spherical surface and the points on the subdividing surface map to the surface of K. The surfaces of the two pieces, which accordingly can also be regarded as balls, then consist of a polygon resulting from the subdividing surface together with polygons originally present. The two new polygons must then be identified in such a way that the identifications between the

[10]If the selected cycle is closed and there is a rule according to which the corresponding line is to be removed from the manifold, then this rule is carried over to the two new cycles.

original edges and vertices are preserved. The identifications between the polygons originally present remain unchanged.

The elementary subdivisions can be described as subdivision of an edge, a wall, or a cell of the three-dimensional schema into two edges, two walls, two cells respectively. The introduction of the general subdivision, the derived schema and homeomorphism now follows just as for two-dimensional schemata. Likewise the remarks concerning the relationship between the definition of homeomorphism for schemata and homeomorphism for manifolds carry over to three (and more) dimensions.

Bounded and closed simply connected three-dimensional schemata can now be introduced as those that are homeomorphic to schemata ε_3 and σ_3 respectively, where ε_3 consists of a single ball whose surface is divided by the equator into two hemispherical surfaces, which are not identified; and σ_3 consists of two balls each of which is divided by the equator into two 2-gons whose edges are a_1, a_2 and b_1, b_2 respectively, and the polygons of one ball are identified with those of the other in the first way so that a_1 is identified with b_1 and a_2 with b_2.[11]

The manifold defined by σ_3, and any manifold homeomorphic to it, is called the three-dimensional spherical manifold. We can represent it in euclidean space by closing the three-dimensional space by a point at infinity. The manifold defined by ε_3, or a manifold homeomorphic to it, is called the three-dimensional element.

The dual to the schema of a closed three-dimensional manifold is obtained by a process that is easily set up when one observes[12] that the relation between a schema and its dual in three dimensions is quite analogous to that in two dimensions, and in fact each vertex, edge, wall and cell of a schema corresponds respectively to a cell, wall, edge and vertex of the dual. It may also be remarked that each simply connected closed two-dimensional schema

[11]In place of σ_3 one could take a simpler schema with a single ball whose equator, divided into a certain number of arcs, divides the surface of the ball into two polygons that are identified in the first way and so that each arc on the equator corresponds to itself (a schema $(l, 0)$, see §20). In place of ε_3 one can choose any schema consisting of a single ball with an arbitrary subdivision and no identifications. The representation of ε_3, σ_3 in the text, like that for ε_2, σ_2 earlier, is chosen with an eye to the general case of ε_n, σ_n (§4). One notes that the polygonal subdivision of the ball surface used for ε_3 is exactly the schema σ_2.

[12]To see this done in detail in the three-dimensional case see Poincaré (Compl. 1, §7, p.314-316).

represented by the subdivided surface of a cell Z of the schema is none other than the neighbourhood schema (see §3, note 6) of the vertex dual to the cell Z in the dual schema. Thus the condition of simply-connectedness of the neighbourhood manifold of a vertex is again of importance here.[13]

§4

Schemata of arbitrary dimension. The meaning of cells and two-sided manifolds

From the development of §§2,3 one sees without difficulty how to obtain schemata of increasing dimension. Corresponding to the vertices, edges, walls and cells of the three-dimensional schema, the elements of the n-dimensional schema are cells of dimension m ($m = 0, 1, \ldots, n$). The number of these will be denoted by α_m.

One sees that the ascent to schemata of higher dimension takes place via the simply connected schemata ε_n, σ_n. One obtains σ_n by taking two copies of ε_n — whose elements carry corresponding notations — regarded as two n-dimensional cells, and identifies corresponding elements of $(n-1)^{\mathrm{th}}$ and lower dimensions in the two ε_n. Thus, from a purely combinatorial point of view, σ_n is not essentially different from ε_{n+1},[1] and the totality of schemata homeomorphic to σ_n yields the $(n+1)$-dimensional cells from which all $(n+1)$-dimensional schemata are constructed by identifications between their elements of ν^{th} dimension ($\nu < n+1$). These identifications have to satisfy conditions that generalize the simple-connectedness condition for neighbourhood manifolds described for three-dimensional schemata. Namely, in an $(n+1)$-dimensional schema each cell of dimension m has a neighbourhood manifold given by an $(n-m)$-dimensional schema. The condition that $(n+1)$-dimensional schema must satisfy is that all the schemata of neighbourhood manifolds be simply connected. A further condition, which expresses the fact that a geometric figure must always yield a definite arrangement of its boundary elements, is the generalization of the condition on the edges of

[13] Cf. the remark of Poincaré (Compl. 1, §10, p.336) on the assumptions underlying the "arithmetic" proofs.

[1] If one thinks of the cells of a schema written in rows, with the m-dimensional cells in the $(m+1)^{\mathrm{th}}$ row, then ε_{n+1} differs from σ_n only in the fact that ε_{n+1} contains an $(n+2)^{th}$ row of $(n+1)$-dimensional cells. However ε_{n+1} has the same identifications as σ_n.

a polygonal subdivision and their directions made for the three-dimensional schemata.[2]

The presentation of n-dimensional schemata according to increasing dimension given in this section begins with the case $n = 2$. It is clear that one could go back still further to ε_0 and σ_0, representing the zero-dimensional schemata of the single point and the point pair. The point pair is the boundary of the line segment. Identification of the endpoints of the line segment gives the one-dimensional schemata. One obtains only two non-homeomorphic classes of connected one-dimensional schemata, the open line and the closed line, the former represented by the line segment (schema ε_1 and the latter by the schema σ_1 consisting of two segments with corresponding endpoints identified. σ_1 and the schemata homeomorphic to it bound the surface piece, which is the building block for the two-dimensional schemata.

It should be mentioned what is meant by an m-dimensional cell and a two-sided manifold.[3] A point pair is given an orientation (direction) by designating one of the points as earlier, the other as later. This also gives an orientation to the one-dimensional cell, the line segment, in terms of the orientation of its pair of endpoints. Then if one chooses the orientation of each segment constituting a connected one-dimensional schema so that all identifications of the endpoints are of the first kind,[4] which is possible in two different ways, then the manifold defined by the schema receives an orienta-

[2]E.g., in the case of polygons (closed cycles of which occur in four-dimensional schemata) this condition reads as follows: if $\Pi_1, \Pi_2, \dots, \Pi_k$ are polygons (with the same number of sides, of course) each of which is identified with its successor, and the last with the first, so that any directed side of one corresponds to a particular side and direction in its successor, then the identification of Π_1 with itself induced via $\Pi_2, \dots, \Pi_k$ must in fact be the identity.

Since manifolds of more than three dimensions, and likewise one-sided manifolds, have been very little studied, the condition in question, which first becomes significant for three-dimensional manifolds, and then only for the one-sided manifolds among them, has previously not been noticed.

[3]Cf.P. Heegaard, Forstudier til en topologisk Teori for de algebraiske Fladers Sammenhaeng (Dissertation, Kopenhagen 1898), §10.

[4]The choice of an orientation for the constituent segments of the schema of a one-dimensional manifold determines whether identifications of their endpoints are of the first or second kind according as an earlier point is identified with a later, or two of the same orientation together, respectively. If one defines the concepts “two-sided” and “one-sided” for connected one-dimensional manifolds in the same way as for manifolds of higher dimension, then it is immediately clear that one-sided one-dimensional manifolds cannot occur.

tion as a result.[5] The orientation of a surface piece is then the orientation of the one-dimensional manifold that bounds it, and the two ways in which the surface pieces in the schema of a two-sided two-dimensional manifold can be oriented so that all side identifications are of the first kind give the two possible orientations of the manifold. One proceeds in this way to use the orientation of m-dimensional manifolds to give an orientation to $(m+1)$-dimensional cells, whose boundaries — which are closed simply connected m-dimensional manifolds — are indeed two-sided, and to thus obtain an orientation for any two-sided m-dimensional manifold.

II. The Betti numbers

§5

Homologies, the Poincaré relation system of a manifold

In order to present the Betti numbers, Poincaré introduced the concept of homology, which we shall develop in this paragraph. We begin with some notation for the m-dimensional space pieces lying in a manifold.

The points of a manifold will be denoted by small latin letters with superscript zero e.g., u^0, a^0_i.

The manifold of points of the segment $-1 \leq x \leq +1$ (i.e., the manifold defined by the schema ε_1 of §4) is denoted by E_1, and its endpoints $x=-1$, $x=+1$ by v^0, w^0. A line piece of a manifold is understood to be a set G of points of the manifold that can be mapped one-to-one and continuously on to E_1. The two points of G that correspond to v^0 and w^0 under this mapping may be denoted a^0, b^0 and will be called the end points of the line piece, the remaining points of G will be called inner points. The different one-to-one continuous maps of G onto E_1 fall into two classes, according as a^0 is mapped on to v^0 or w^0. If one particular class is chosen, this gives an orientation to the line segment. The point associated with v^0 will be called the earlier (negative) endpoint, the one associated with w^0 will be called the later (positive) endpoint. The one-to-oneness and continuity of the maps between G and E_1 may only apply to inner points. A point c^0 may correspond to both the points v^0, w^0 and its neighbourhoods will divide into two parts, one corresponding to a neighbourhood of v^0, the other to one of w^0. We

[5] In case the schema consists of a single line segment whose endpoints are not identified then the orientation of this manifold is the orientation of the segment.

can still speak of the orientation of the line piece, but the definition is now based on the two parts of the neighbourhood of c^0 and their association with v^0 and w^0. The point c^0 itself is both a positive and negative endpoint of the line piece. The oriented line pieces will be denoted by small latin letters with superscript 1. If a^0 is the positive endpoint of a line piece a^1 and b^0 the negative we shall write this symbolically[1,2]

$$a^1 \equiv +a^0 - b^0.$$

The manifold of points on the unit circle $x^2 + y^2 \leq 1$ (i.e., the manifold defined by the schema ε_2 of §2) will be denoted by E_2, while the circular line $x^2 + y^2 = 1$ will be denoted by K. By a surface piece lying in a manifold we mean a set G of points of the manifold that can be mapped one-to-one and continuously onto E_2. The distinction between the two types of one-to-one continuous maps of G onto E_2 takes place analogously with the case of the line piece, and is determined by the orientation in which the boundary line L of G is mapped onto the boundary of E_2. We shall assume that the circular line K is divided by points $k_1^0, k_2^0, \ldots, k_R^0$ $(R \geq 1)$ into line pieces $k_1^1, k_2^1, \ldots, k_R^1$. Correspondingly, the boundary L of the surface piece is divided by the image points $l_1^0, \ldots, l_R^0$ of the points k_i^0 into R line pieces $l_1^1, l_2^1, \ldots, l_R^1$. However the concept of a surface piece, like that of a line piece, may undergo an extension. Namely, it will be admitted that several of the line pieces into which L is divided may coincide. Then several line pieces k_i^1 correspond to a single line piece of G. The same applies to certain points l_i^0. The one-to-oneness and continuity of the mapping of E_2 on to G is preserved for the inner points, but it is modified as just described for certain points on the boundary line. The detailed investigation of relations corresponding to the line piece would be too tedious. The generalized surface piece can also be given an orientation in terms of the mapping from E_2 and a given orientation of K. The oriented surface pieces will be denoted by small latin letters with superscript 2. Let a^2 be a surface in the general sense and let $l_1^1, l_2^1, ...$ be the oriented individual line pieces of a^2, each of which corresponds to one or more line pieces k_i^1.

[1]See Poincaré, Compl. 1, p.291; cf. also the association of numbers with points, lines etc. of a manifold at the beginning of §18 of An. Sit. (p.114)

[2]If one wants to give the individual points u^0 of a manifold signs in full generality then one must write

$$a^1 \equiv \delta a^0 - \delta' b^0$$

when a^0 is the point at the positive end of a^1, with sign δ, and b^0 is the point at the negative end of a^1, with sign δ'. Use is made of this in §8 (note 8).

Moreover, let the orientations of the line pieces $k_1^1, k_2^1, \ldots, k_R^1$ be chosen so that they coincide with the positive orientation of K. Then if l_{λ_i} is the line piece of G corresponding to the line pieces k_i^1 and $\delta_i = +1$ or -1 according as the mapping of E_2 on to G establishes a correspondence between the positive orientation of k_i^1 and the positive or negative orientation of l_{λ_i}, then we shall symbolically write

$$a^2 \equiv \sum_{i=1}^{3} \delta_i l_{\lambda_i}^1$$

The order of the summands does not matter.

It is not difficult to infer from this what should be meant by an m-dimensional space piece[3] lying in a manifold (it will be denoted by a small latin letter with an upper index m, and by a symbolic congruence of the form

$$n^m \equiv \sum \delta_i u_i^{m-1} \tag{3}$$

(the δ_i are equal to $+1$ or -1).

The special congruences introduced here can be used in calculations subject to the following rules.

As we have already said, the terms on one side of a congruence may be permuted and grouped in the usual way of calculating with letters. Both sides of a congruence may be multiplied by a positive or negative integer. Two congruences yield a third by adding or subtracting the respective left and right sides[4] of the given congruences.[5]

---

[3] One could define the m-dimensional space piece to be a point set G upon which E_m is mapped continuously and one-to-one except for certain boundary elements, where E_m is the bounded simply connected m-dimensional manifold (defined by the schema ε_m of the previous paragraph) which can be taken to be the manifolds of points $x_1^2+x_2^2+\cdots+x_m^2 \leq 1$. The inner points of the m-dimensional space piece do not yield the most general set homeomorphic to the inner points of E_m, cf. the plane point set defined by $0 < x^2+y^2 < 1$, $(x - 3.2^{-\nu})^2 + y^2 - 2^{-2\nu} > 0$ $(\nu = 2, 3, \ldots)$.

Notice that, in the case of coincident endpoints, a line piece is characterized not just by the totality of its points, but also by the position of the point c^0. Analogously, the particular position in G of the coincident boundary elements is an essential part of the nature of the surface piece or space piece.

[4] It is not permitted to exchange the right and left side of a congruence in the process.

[5] When the collection of $m-1$-dimensional space pieces is so constituted that no two of them have inner points in common then the right hand sides of all congruences

$$\sum h_i u_i^m \equiv \sum k_i u_i^{m-1}$$

As a consequence of this first rule one can say: if u^1 is a line piece with coincident endpoints then $u^1 \equiv 0$, and in general we have the relation

$$u^m \equiv 0$$

when the points of u^m constitute a closed two-sided[6] m-dimensional manifold[7] in the sense of Section I.

More generally, the rules allow us to say the following: if $u_1^m, u_2^m, \dots, u_\alpha^m$ are m-dimensional space pieces in a manifold with no interior points in common,[8] and if the totality of their points comprise a closed two-sided m-dimensional manifold M (see Section I), and if in addition the orientation of the u_i^m is chosen to correspond to a particular orientation of M (see §4), then[9]

$$\sum_{i=1}^{\alpha} u_i^m \equiv 0.$$

A further theorem may be mentioned:

$$\text{when} \quad u^m \equiv \sum \delta_i u_i^{m-1}, \quad \text{then} \quad \sum \delta_i u_i^{m-1} \equiv 0.$$

The integral linear forms with the symbols for m-dimensional space pieces as variables to which we are led in this calculus, may be considered as m-dimensional figures, and in particular when the relation

$$\sum h_i u_i^m \equiv 0$$

holds, the m-dimensional figure on the left hand side may be considered closed and two-sided. The two-sided closed m-dimensional figures therefore include the (oriented) two-sided closed m-dimensional manifolds, when these appear as unions of space pieces without common interior points, as special cases.

The oriented m-dimensional cells of the schema of an n-dimensional manifold V which, following Poincaré, we shall denote by $a_i^m, a_2^m, \dots, a_{\alpha_m}^m$, represent a particularly important case of m-dimensional space pieces in V. The

are uniquely determined by the left hand sides, from which one infers that this holds for the congruences (3).

[6]As far as one-sided manifolds are concerned, see §9.

[7]For $m > 1$ the converse obviously does not hold.

[8]The assertion of the theorem assumes that the individual $(m-1)$-dimensional spare pieces comprising the boundary of the u_i^m have no common interior points.

[9]The converse obviously does not hold.

collection of all congruences that hold between them, expressing their spatial relations

$$a_j^m \equiv \sum_{j=1}^{k_{m-1}} \varepsilon_{ij}^m a_j^{m-1} \qquad (i = 1, 2, \ldots, \alpha_m; \quad m = 1, 2, \ldots, n)$$

constitute the Poincaré relation system[10] of V for the schema in question, which we shall often use in future.

On the basis of formal sums of space pieces and the symbolic congruences between them it is very simple to explain what a homology is in relation to a manifold V.[11] Namely, if u_1^{m+1}, $u_2^{m+1}, \ldots, u_\alpha^{m+1}$; $u_1^m, u_2^m, \ldots, u_\beta^m$ are space pieces in V and if there is a congruence

$$\sum_{i=1}^{\alpha} h_i u_i^{m+1} \equiv \sum_{i=1}^{\beta} k_i u_i^m \tag{4}$$

then it will be said that the figure $\sum_{i=1}^{\beta} k_i u_i^m$ is null homologous relative to V, in symbols

$$\sum_{i=1}^{\beta} k_i u_i^m \sim 0.$$

The existence of such a homology relation indicates the presence of space pieces u_i^{m+1} in V satisfying a congruence (4).

It will also be permitted to add the same symbol for an m-dimensional space piece to both sides of a homology between m-dimensional space pieces. It then follows from the rules for calculating with congruences[12] that it is in fact allowed to add homologies, or to multiply both sides of a homology by an integer.

It should be noted that the concept of homology is tied quite explicity to the underlying manifold V.

The following example may serve to clarify the above theory. We take a manifold V given by a schema that is a rectangle with sides identified in

[10] Poincaré described this relation system simply as the schema (compl. 1 p.291). Cf. in this connection note 9, §1.

[11] Cf. Poincaré, An. sit. §§5, 6.

[12] Poincaré distinguishes between "homologies with division" and "homologies without division" according as division of both sides of a homology by a common integral factor is allowed or not. In the text we shall use only "homologies without division".

pairs in the first way. The schema therefore has two edges and one vertex. Let A, B, C, D be the series of vertices of the rectangle. Let E be a point on AB, and F the corresponding point on DC. We draw the segments AF and EC and introduce the following notations for the points, line- and surface pieces which appear.

$$A = B = C = D = a^0, \quad E = F = b^0$$

$$AE = DF = a^1, \quad EB = FC = b^1, \quad AD = BC = c^1$$

$$AF = d^1, \quad EC = e^1$$

$$AFD = a^2, \quad EBC = b^2, \quad AECF = c^2$$

The following congruences hold:

$$a^1 \equiv b^0 - a^0, \quad b^1 \equiv a^0 - b^0, \quad c^1 \equiv 0,$$

$$d^1 \equiv b^0 - a^0, \quad e^1 \equiv a^0 - b^0;$$

$$a^2 \equiv d^1 - a^1 - c^1, \quad b^2 \equiv b^1 + c^1 - e^1, \quad c^2 \equiv a^1 + e^1 - b^1 - d^1.$$

This shows that, in conformity with the general remark on closed manifolds, $a^2 + b^2 + c^2 \equiv 0$ and that, in agreement with another general theorem, $d^1 - a^1 - c^1$, $b^1 + c^1 - e^1$ and $a^1 + e^1 - b^1 - d^1$ are congruent to null. The congruences also imply the homologies[13]

$$a^0 \sim b^0;$$

$$d^1 \sim a^1 + c^1, \quad e^1 \sim b^1 + c^1, \quad a^1 - b^1 \sim d^1 - e^1.$$

The last homology is obviously a consequence of the preceding two. If we set $a^1 + b^1 = f^1$, $d^1 + e^1 = g^1$, then f^1, g^1, c^1 represent three closed lines in V and we obtain the homology

$$g^1 \sim f^1 + 2c^1$$

between them.

Certain special homologies may also be noted. Let W^{m+1} be an $(m+1)$-dimensional two-sided[14] manifold lying in a manifold V (with respect to

[13] Any two points of a connected manifold are obviously homologous to each other.

[14] The condition of two-sidedness is obviously essential. This is particularly clear when one considers a Möbius band along an arbitrary closed line l^1 of an n-dimensional manifold V, with $n > 2$. Its boundary line is homologous to $2l^1$ whereas the homology $2l^2 \sim 0$ (relative to V) obviously does not hold in general.

which the homologies are calculated), whose boundary consists of the closed m-dimensional manifolds $U_1^{(m)}, U_2^{(m)}, \ldots, U_h^{(m)}, V_1^{(m)}, V_2^{(m)}, \ldots, V_k^{(m)}$. A particular orientation of the two-sided manifold $W^{(m+1)}$ then induces a particular orientation in each of the two-sided manifolds $U_m^{(i)}$, $V_m^{(i)}$. If this orientation coincides with the positive in each $U_i^{(m)}$, and with the negative in each $V_i^{(m)}$, then relative to V we have the homology

$$U_1^{(m)} + U_2^{(m)} + \ldots + U_h^{(m)} \sim V_1^{(m)} + V_2^{(m)} + \ldots + V_k^{(m)}.$$

However, not every homology expresses such a simple fact,[15] as the homology between g^1, f^1 and c^1 in the example above shows.[16]

§6
Definition of the Betti numbers

We proceed to the Poincaré definition of the Betti numbers of a manifold V.[1] Suppose there is a system of two-sided closed m-dimensional manifolds $W_1^{(m)}, W_2^{(m)}, \ldots, W_t^{(m)}$ in V between which there is no homology[2]

$$k_1 W_1^{(m)} + k_2 W_2^{(m)} + \cdots + k_t W_t^{(m)} \sim 0 \quad \text{(relative to } V\text{)}$$

while each closed m-dimensional manifold $W^{(m)}$ in V satisfies a homology

$$kW^{(m)} + k_1 W_1^{(m)} + k_2 W_2^{(m)} + \cdots + k_t W_t^{(m)} \sim 0 \quad (k \neq 0).$$

[15]Nevertheless the introduction of homologies preferably takes place with reference to these special homologies. Cf. the remarks of Heegaard in the dissertation already cited, pp. 64, 65.

[16]Consider also the example of a knotted closed line l^1 in the elementary three-dimensional manifold E_3, for which we have the homology

$$l^1 \sim 0 \quad \text{(relative to } V\text{)},$$

which is easily seen from the discussion in the text. It will be shown later that the discussion can also be extended to homologies involving manifolds with self-intersections.

[1]Connectivity numbers defined in various other ways are also known by the name of "Betti numbers" or "Riemann-Betti numbers" (in Picard-Simart, *Théorie des fonctions algébriques de deux variables indépendantes* vol.I, chap. 2) (looking back to the essay of Betti, Sugli spazi di un numero qualunque di dimensioni, Ann. di mat. ser. 2., vol. 4 and the Fragment 29 in Riemann's Ges. Werke, 2 Aufl. p.479). The only way of defining these numbers that is free of certain objections we shall come to discuss in §21 is due to Poincaré (illustrated in the definition on p.19 of An. Sit. and in more detail in Compl.1, §1), and the Betti numbers for us will be the connectivity numbers defined by his process.

[2]The case $k_1 = k_2 = \cdots = k_t = 0$ is naturally excluded.

Then if $V_1^{(m)}, V_2^{(m)}, \ldots, V_{t'}^{(m)}$ is another system of manifolds with the same property it is immediately clear that $t' = t$. The number t of such manifolds is therefore a characteristic number for V and it obviously has the same value for all manifolds homeomorphic to V. Thus t is a topological invariant, and the invariant $t+1$ is called the m^{th} Betti number, P_m, of the manifold V. One has to set $t = 0$, $P_m = 1$ when each two-sided closed m-dimensional manifold $W^{(m)}$ in V satisfies a homology $kW^{(m)} \sim 0$.[3]

The given definition of the Betti numbers is based — as Heegaard in particular[4] has pointed out — on the assumption that each manifold V in fact contains a (finite) system of m-dimensional manifolds $W_1^{(m)}, W_2^{(m)}, \ldots, W_t^{(m)}$ with the property in question. This assumption will be called Assumption I in what follows, and in order to retain the homology concept in the above form we need to surmount certain difficulties that may now be briefly described.

We consider an example, to which we shall also return later. One takes a cylindrical surface Z in three-dimensional space and draws a knotted line AB in its interior (see Fig. 1). One continues it with a congruent line BC in Z and continues to construct infinitely many congruent copies of AB on both sides, forming a single line K. By means of a projective transformation of the space, which sends the infinitely distant point of the cylinder to a finite point S, Z is transformed into a conical surface with apex S, and K is transformed into a line L in its interior (see Fig. 2). Let R be a point of contact of L with the conical surface, and connect R to S by a line s outside the cone, so as to form a closed line U^1 in conjunction with the part of L between R and S. We construct a sphere with U^1 in its interior and consider the simply connected manifold V of points in its interior. U^1 is a closed line lying in this manifold V.

Now it is usual to assign to the bounded simply connected manifold V the Betti number $P_1 = 1$, which means simply that for each closed line W^1 in V there is an integer k such that $kW^1 \sim 0$ relative to V. However it is incontestable that the closed line U^1 violates this theorem.

[3]If n is the dimension number of the manifold V, then one has to consider the values $1, 2, \ldots, n-1$ for m. However there is no difficulty in extending the definition of the Betti number P_m to the cases $m = 0$ and $m = n$ also. In the case $m = 0$, where the closed m-dimensional manifolds are point pairs, P_0 is obviously equal to the number of connected components of V. (For $n = 0$ this means $P_0 = \alpha_0 - 1$, where α_0 is the number of points in V.) P_n is however the number of two-sided manifolds among the connected components.

[4]Dissertation p.64

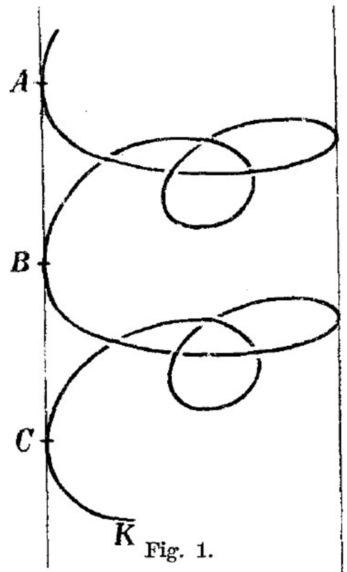

Fig. 1.

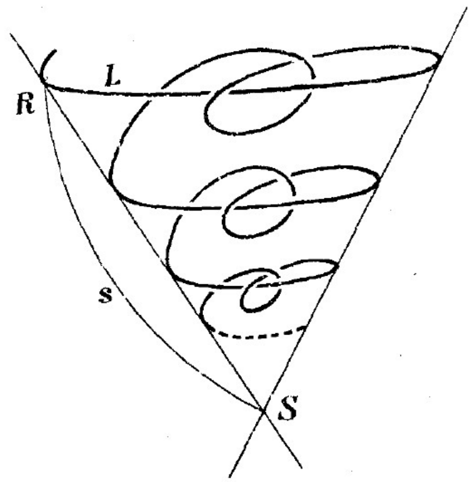

Fig. 2.

Moreover when one thinks of the multitude of infinitely knotted lines that can be constructed on the model of U^1, of new types not homologous to the original, then the admissibility of Assumption I (which is certainly unproved, on the face of it) seems highly questionable.

We shall not be concerned with extensions of the homology concept (say by means of infinite sums of homologies) that might serve to remove these difficulties. Rather, we shall replace the above definition of the Betti numbers by another, which is at least formally quite different, in that it depends entirely on the scheme of a manifold.

Let V be the n-dimensional manifold in question, which for the sake of simplicity we shall assume to possess only $(n-1)$-dimensional manifolds as boundary elements and let

$$a_i^m \equiv \sum_{j=1}^{a_m-1} \varepsilon_{ij}^m a_j^{m-1} \qquad (i = 1, 2, \ldots, \alpha_m; \quad m = 1, 2, \ldots, n)$$

be the Poincaré relation system of V. Then if γ_m denotes the rank of the matrix formed by the numbers ε_{ij}^m, then the m^{th} Betti number is defined by the equation[5] [6]

$$(5) \qquad P_m - 1 = \alpha_m - \gamma_m - \gamma_{m+1} \qquad (m = 1, 2, \ldots, n-1)$$

When the numbers P_m are defined in this way, so that they depend only on a particular scheme for the manifold V , then it is not difficult to prove (see §8) that they have the same value for homeomorphic schemata, so that they are in fact topological invariants.[7] The following may be said concerning the relationship between this definition and the earlier one.

The second and third of Poincaré's works on analysis situs[8] are based essentially on the latter definition of Betti numbers. Indeed the formula (5) appears as a means of calculating the Betti numbers (defined in the earlier way) from the schema of a manifold. However the problem immediately arises whether it is possible to prove that the numbers P_m are the same under the definition and the computation rule of formula (5), without certain extensions and modifications, and since no use is made of the original definition of Betti numbers in the works cited, the formula (5) can in fact be taken as a definition of the P_m.

The considerations[9] that lead from the earlier definition of Betti numbers to the formula (5), and constitute some motivation for considering the numbers defined by (5) to be Betti numbers, are first of all the assumption (from now on known as Assumption II) that each two-sided closed m-dimensional manifold in V is homologous to a two-sided closed figure $\sum k_i \alpha_i^m$ comprised of cells of the schema Σ of V.

Assumption II leads us to ask whether there is a system of two-sided closed m-dimensional figures $G_1^{(m)}, G_2^{(m)}, \ldots, G_s^{(m)}$ whose nature is such that there is no homology between the $G_i^{(m)}$ themselves, but there is between them and any other two-sided closed m-dimensional figure, which, like the $G_i^{(m)}$ is

[5] One proves easily that $\alpha_0 - \gamma_1$ is equal to the number of connected components of V (cf. note 1, §10) and $\alpha_n - \gamma_n$ is equal to the number of closed two-sided manifolds among them. Thus if γ_0 as well as γ_{n+1} is set equal to 1, the formula (5) can also serve to define P_0 and P_n. In the development §§6,7 the P_m are understood to mean only the numbers $P_1, P_2, \ldots, P_{n-1}$.

[6] Another form of this definition may be found on the last page of §9.

[7] Of course it is only shown that P_m is a topological invariant of schemata, not a topological invariant of manifolds. Cf. §2 in relation to this distinction.

[8] Compl. 1 and 2

[9] See Compl. 1

composed of cells of Σ.[10] It is easy to show that such a system can in fact be found, and that the number s of figures in such a system, which must obviously be independent of the choice of the system, is given by the right hand side of equation (5).

The Poincaré relation system can be transformed to a reduced form,[11] whose structure depends on well-known theorems from the theory of linear forms, involving linear homogeneous integral substitutions of determinant 1 that replace the a_i^m and the a_j^{m-1} by new series of variables

$$b_1^m, b_2^m, \ldots, b_{\alpha_m}^m \quad \text{and} \quad c_1^{m-1}, c_2^{m-1}, \ldots, c_{\alpha_{m-1}}^{m-1}$$

so that the relations

$$a_i^m \equiv \sum_{j=1}^{\alpha_{m-1}} \varepsilon_{ij}^m a_j^{m-1} \qquad (i = 1, 2, \ldots, \alpha_m)$$

transform to

$$\begin{aligned} &b_i^m \equiv \omega_i^m c_i^{m-1}, \qquad (i = 1, 2, \ldots, \gamma_m) \\ (6) \qquad &b_i^m \equiv 0 \qquad (i = \gamma_m + 1, \ldots, \alpha_m) \end{aligned}$$

Here $\omega_1^m, \omega_2^m, \ldots, \omega_{\gamma_m}^m$ denote the non-zero elementary divisors of the matrix of the numbers ε_{ij}^m.[12] The collection of relations (6) for $m = 1, 2, \ldots, n$ then constitute a reduced form of the Poincaré relation system.

It is now immediately clear that a figure

$$\sum_{i=1}^{\alpha_m} h_i a_i^m = \sum_{i=1}^{\alpha_m} k_i b_i^m$$

is two-sided and closed if and only if $k_1 = k_2 = \cdots = k_{\gamma_m} = 0$, so that it has the form

$$\sum_{i=\gamma_m+1}^{\alpha_m} k_i b_i^m,$$

[10] In case one considers the theorem B which appears in §7 under the discussion of Assumption II as obvious, it may be pointed out that such a system $G_1^{(m)}, G_2^{(m)}, \ldots, G_s^{(m)}$ has the property that each two-sided closed m-dimensional figure composed of cells in any schema for V satisfies a homology together with the $G_i^{(m)}$.

[11] Cf. Compl. 2, §2,3 (p.281ff.)

[12] Thus $\omega_{\gamma_m}^m = d_1$, $\omega_{\gamma_m - k}^m = d_{k+1}/d_k$ where d_k is the greatest common divisor of all k-rowed subdeterminants of $|\varepsilon_{ij}^m|$.

and conversely, that each figure of this form is two-sided and closed.

On the other hand there is no difficulty in presenting the homology which exists between the two-sided closed figures

$$b^m_{\gamma_m+1}, b^m_{\gamma_m+2}, \ldots, b^m_{\alpha_m}. \tag{7}$$

The Poincaré relation system in the original form yields the homologies

$$\sum_{j=1}^{\alpha_m} \varepsilon_{ij}^{m+1} a_j^m \sim 0, \qquad (i = 1, 2, \ldots, \alpha_{m+1}) \tag{8}$$

which must represent homologies between the figures (7), since the left hand sides of these homologies (cf. a theorem given in §5) are two-sided closed m-dimensional figures. But in general the α_{m+1} homologies written are not independent, i.e., a few of them serve to compute the rest. On the other hand, the system of homologies derived from the reduced form of the Poincaré system

$$\omega_i^{m+1} c_i^m \sim 0 \qquad (i = 1, 2, \ldots, \gamma_{m+1}) \tag{9}$$

is a system of independent homologies, since the c_i^m by hypothesis are linearly independent forms in the a_i^m and obviously all homologies (8) can be derived from the γ_{m+1} homologies (9). The homologies (9) now allow γ_{m+1} of the figures (7) to be expressed in terms of the remaining $s = \alpha_m - \gamma_m - \gamma_{m+1}$ figures, in other words: by suitably choosing s of the figures (7) one can construct γ_{m+1} new homologies from the homologies (9) so that the left sides of these new homologies are multiples of some non-selected figure, while the right hand side is each time a linear form in the s figures chosen. If we now make the assumption (called Assumption III in what follows) that the figures (7) satisfy no homologies other than those that follow from the homologies (8) and hence from the homologies (9), then the figures chosen constitute a system with the desired property, and the number of figures in such a system is $\alpha_1 - \gamma_m - \gamma_{m+1}$, as claimed.

Using Assumption II one can now say that each two-sided closed m-dimensional manifold in V is an integral linear form in the $s = \alpha_m - \gamma_m - \gamma_{m+1}$ selected figures, up to homology, so that the Betti number P_m (in the sense of the earlier definition) cannot be greater than $s+1$. But when we set the Betti number P_m exactly equal to $s+1$, on the basis of previous considerations, we are making a further assumption that a certain converse of Assumption II holds. This assumption (Assumption IV) says:

It is possible to construct s two-sided closed m-dimensional figures from the cells of Σ which satisfy no homology, but which are related by a homology to every other such figure, and which are also chosen so that each one of them is homologous to a two-sided closed m-dimensional manifold lying in V.

If one accepts this assumption, then it is clear that the s manifolds whose existence it asserts constitute a system with the property expressed by the $W_i^{(m)}$ in the definition of the Betti numbers, so that the Betti number P_m is in fact equal to $s + 1 = \alpha_m - \gamma_m - \gamma_{m+1} + 1$. This completes the path from the original definition of the Betti numbers to the formula (5).

§7

On the assumptions used in §6

We now give a by no means exhaustive discussion of the assumptions used in §6. It may be remarked to begin with that many of the assumptions used by Poincaré, which we consider to be not binding, become so with a suitable limitation of the manifold concept. Thus it appears that Poincaré has only analytic manifolds in mind. In this case many of the difficulties disappear, when one is confined to manifolds that are analytic overall or else composed of a finite number of analytic pieces.

As far as Assumption IV is concerned, it must be considered uncertain whether it will be proved correct.[1] In the evaluation of this question it is important to note that the manifold concept of Section I excludes manifolds with singularities and hence manifolds with self intersections.

The correctness of Assumption III can be regarded as highly probable. A proof of it was given by Poincaré[2] for the case of a three-dimensional manifold and $m = 1$ and 2. This would need some extensions in circumstances similar to those encountered with Theorem a) of §2 and in the discussion of the theorem on the homeomorphism of the schemata of homeomorphic manifolds, when complicated relationships can appear due to the higher dimension of V.

[1]In any case, not every two-sided closed m-dimensional figure in a manifold V is homologous to a two-sided closed m-dimensional manifold in V, e.g., the figure $2F^2$ is not, when V is the three-dimensional elementary manifold and F^2 is a spherical surface in V. The question in the note on the second page of my note in the Wien. Ber. (1906), cited above, is therefore to be disregarded.

[2]Compl. 1, §6.

The necessity for these extensions is mainly due to the fact, which we have already mentioned, that the manifolds considered in the text are more general than those of Poincaré, since he confines himself to analytic manifolds.

It remains to discuss Assumption II. This assumption derives from considerations something like the following.[3] At bottom there are the following two theorems:

A. For any m-dimensional manifold $W^{(m)}$ lying in a given manifold V one can always find a decomposition of V into a cell system (schema) such that $W^{(m)}$ consists of a collection of m-dimensional cells of this schema (i.e., there are a number of m-dimensional cells such that $W^{(m)}$ consists exactly of the points that belong to one or more of these cells.

B. When two decompositions Σ_1, Σ_2 of V are present in a cell system,[4] then the two-sided closed m-dimensional figures consisting of cells of Σ_1 are homologous to figures consisting of cells of Σ_2.

Assumption II is an immediate consequence of these theorems. Namely, if Σ_1 is the given schema of V and Σ_2 is a schema of V from whose m-dimensional cells $W^{(m)}$ can be built, then it follows that $W^{(m)}$ is homologous to a figure consisting of cells of Σ_1, and since V represents an arbitrarily selected m-dimensional manifold, Assumption II is confirmed.

In order to justify B, we introduce the following two theorems:

a) When Σ_1 and Σ_2 are decompositions of V into cell systems, then there is a decomposition of V into a schema Σ_3 such that Σ_3 can be derived from both Σ_1 and Σ_2 by subdivision.[5]

b) When a manifold is derived from the schema Σ by subdivision into a schema Σ', then all m-dimensional figures consisting of cells of Σ' are homologous to figures consisting of cells of Σ.

Theorem B follows immediately from these theorems. Any figure consist-

[3] A slightly different arrangement of these is given in Poincaré's Compl. 1 §§5,6.

[4] This is understood to mean the following: one thinks of the manifold V defined in any manner, say by a schema. Then a "decomposition of V by a schema Σ_1" is given, not only in terms of the information specifying Σ_1 itself, but also by giving the individual cells as point sets in V. The "decomposition of V by the schema Σ_2" is given in the same way. It is of course not excluded, e.g., that the decomposition of V by the schema Σ_1 coincides with the decomposition given by Σ (for which it is necessary, but not sufficient, that the schemata Σ and Σ_1, be formally identical, regardless of their "position" in V).

[5] This means (cf. note 4): 1. Σ_3 is a derived schema of Σ_1 (the cells of Σ_3 therefore divide into cells ζ' coinciding with cells ζ of Σ_1, or parts of them, and new cells ζ''); 2. If $\zeta'_1, \ldots, \zeta'_\mu$ make up the cell ζ_1, then ζ_1 consists exactly of the points of V belonging to one of the cells ζ'_i; 3. The same holds between the decompositions of V into Σ_3 and Σ_2.

ing of cells of Σ_2 can be considered to consist of cells of Σ_3, so by b) it is homologous to one consisting of cells of Σ_1.

It is easy to prove *b*) (see §8), but not *a*).[6] The following simple example suffices to show this. One lets V be the disc $x^2 + y^2 \leq 1$ and considers the two decompositions of V into two surface pieces that result from subdividing V along the respective lines $y = 0$ and $y = x \sin \frac{1}{x}$ (with $y = 0$ when $x = 0$). Since these two lines intersect each other infinitely often inside V, Theorem *a*) evidently fails here.[7]

A proof of Theorem B valid under all circumstances is therefore still lacking. On the other hand it seems that Theorem A, which we now turn to discuss, it quite untenable, however plausible it seems at first. In order to explain this we draw on the example used in §6 for the discussion of Assumption I. Since the closed line U^1 of this example is a line in the simply connected bounded manifold V, then according to Theorem A there must be decomposition of V into a cell system such that U^1 consists of edges of this cell system. Intuition says that such a cell system is impossible. Thus it is shown that Theorem A cannot be generally valid (at least without an extension of the concept of schemata to admit infinitely many cells).

The given example also shows that Assumption II itself is incorrect. Indeed it is intuitively clear that U^1 cannot be homologous to an edge path in a schema of V. This would only be possible if the homology concept were extended to admit sums of infinitely many homologies. We do not attempt such an extension here.

Thus the considerations that lead from the original definition of Betti numbers to the formula (5) depend on various assumptions that either (with retention of the adopted definition) turn out to be inadmissible or not yet rigorously proved. The obvious heuristic value that the original definition has in explaining the new one, based on the schema alone, is of course untouched by these objections. This is sufficient ground for departing from the original definition and using the definition of the numbers P_m based in formula (5) in what follows.

§8

On the topological invariants P_m

[6] If the theorem *a*) were available, the hypothetical theorem on the homeomorphism of schemata of homeomorphic manifolds, mentioned in §2, would be immediate.

[7] On the other hand, Theorem *b*) clearly retains its validity for this example.

This paragraph contains first of all an exposition of Poincaré's proof in Compl. 2 that the numbers $P_m = \alpha_m - \gamma_m - \gamma_{m+1}$ are topological invariants of the schemata. In addition, Poincaré's fundamental theorem on the Betti numbers of two-sided closed manifolds and the necessary and sufficient conditions for the homeomorphism of two-dimensional manifolds are discussed, and we finally make a few remarks on the invariant $N - \alpha_0 - \alpha_1 + \ldots \pm \alpha_n$ that is computable from the numbers P_m.

The first objective will be achieved when it is shown that the P_m do not change under elementary subdivision. Thus we consider an elementary subdivision of the schema Σ of an n-dimensional manifold V consisting of the replacement of an l-dimensional cell of Σ by two cells that meet along a new $(l-1)$-dimensional cell $(1 \leq l \leq n)$. The numbers α'_m, γ'_m for the derived schema Σ' have the same meaning as α_m, γ_m do for the original schema Σ. Then one has the first of all

$$\alpha'_l = \alpha_l + 1, \quad \alpha'_{l-1} = \alpha_{l-1} + 1$$

(10)

$$\alpha'_m = \alpha_m \quad \text{for} \quad m \neq l-1, l.$$

In addition, let a^l_r be the cell of Σ that is divided into two. The boundary of this cell is a spherical $(l-1)$-dimensional manifold that may be denoted by $S^{(l-1)}$. The subdivision in question then consists in dividing $S^{(l-1)}$ into two $(l-1)$-dimensional elementary manifolds $E^{(l-1)}_1$, $E^{(l-1)}_2$ [1] by means of a spherical $(l-2)$-dimensional manifold $S^{(l-2)}$ consisting of $(l-2)$-dimensional cells lying in $S^{(l-1)}$, and spanning $S^{(l-2)}$ by an $(l-1)$-dimensional elementary manifold a^{l-1}_s that divides a^l_r into two cells $a^l_{r_1}$ and $a^l_{r_2}$.[2] $S^{(l-2)}$ is therefore the boundary of a^{l-1}_s and the orientation of a^{l-1}_s is chosen, say, so that a^{l-1}_s together with $E^{(l-1)}_1$, and likewise $-a^{l-1}$ with $E^{(l-1)}_2$, each constitute a spherical $(l-1)$-dimensional manifold provided with an orientation.[3] If one now

[1] The manifolds $E^{(l-1)}_1$ and $E^{(l-1)}_2$ are provided with the orientation induced by that of $S^{(l-1)}$.

[2] In the case $l = 1$ this formulation undergoes an easy modification. $S^{(0)}$ is the pair of endpoints of the edge a^1_r, $E^{(0)}_1$ denotes one, and $E^{(0)}_2$ the other of these endpoints, and the subdivision is carried out simply by marking a point a^0_s on a^1_r.

[3] It may be noted that the terminology of the text needs to be made more precise. Namely, when we speak of the "spherical manifolds" $S^{(l-1)}$, $S^{(l-2)}$ and of the "elementary manifolds" $E^{(l-1)}_1$, $E^{(l-1)}_2$ then these expressions are, so to speak, not relative to V, but relative to the "l-dimensional elementary manifold represented by a^l_r". This means the

compares the Poincaré relation systems of Σ and Σ' then it is first of all obvious that the relations

$$a_i^m \equiv \sum_{j=1}^{\alpha_{m-1}} \varepsilon_{ij}^m a_j^{m-1} \qquad (i = 1, 2, \ldots, \alpha_m) \tag{11}$$

for $m > l+1$ and $m < l-1$ completely agree in both systems, whence $\gamma_m' = \gamma_m$ follows for these values of m. If in addition the positive orientation of $a_{r_1}^l$ and $a_{r_2}^l$ is chosen so as to coincide with that of a_r^l,[4] then it is clear that the numbers $\varepsilon_{ir_1}^{l+1}$, $\varepsilon_{ir_2}^{l+1}$ in the relation system of Σ are both equal to the number ε_{ir}^l in the relation system of Σ, while all the remaining numbers ε_{ij}^{l+1} of Σ also appear in Σ'. Consequently, $\gamma_{l+1}' = \gamma_{l+1}$ as well. The relations

$$a_i^l \equiv \sum \varepsilon_{rj}^l a_j^{l-1} \tag{12}$$

not only include such relations as completely coincide in both systems, but in the system of Σ there is also the relation

$$a_i^l \equiv \sum_{j=1}^{\alpha_{i-1}} \varepsilon_{rj}^l a_j^{l-1}$$

following. If one considers the collection of all incidence relations that make up the schema Σ of V, then it first of all defines elementary surfaces, each by giving a number of edges and identifying their endpoints so as to form a closed line, then taking this as the boundary line of an elementary surface. Later incidence relations introduce identifications between the edges of elementary surfaces, so that finally a whole series of elementary surfaces link up to form a single wall (= two-dimensional cell) a_i^2 of the schema. Any such elementary surface belonging to a_i^2 will be said to be represented by a_i^2. The further construction of the schema consists in assembling spherical two-dimensional manifolds by identifying edges of elementary surfaces. These spherical two-dimensional manifolds are then used as boundary surfaces of three-dimensional elementary manifolds. Any such elementary manifold that ends up in the cell a_i^3, as a a result of later identifications, is then a representative of the three-dimensional elementary manifold represented by a_i^3. Continuation of this reasoning shows what is to be understood in general by the m-dimensional elementary manifold represented by a_i^m. However it is clear that the collection of points of the cell a_i^m is not in general an elementary manifold lying in V, but only an m-dimensional space piece in V, since identifications between its boundary elements may occur as a result of later incidence relations in the schema of V. Thus in general $S^{(l-1)}$ and $S^{(l-2)}$ respectively are not spherical manifolds lying in V but only in the elementary manifold represented by a_r^l, and similarly for the "elementary manifolds" $E_1^{(l-1)}$, $E_2^{(l-1)}$. However, we omit carrying through the details of all these relations in order to avoid unnecessary complications in the presentation, which would be harmful to clarity.

[4]This assumption, which is made for the sake of simplicity, is of course inessential.

and in the system of Σ' there are the two relations

$$a^l_{r_1} \equiv \sum_{j=1}^{\alpha_{i-1}} \varepsilon^l_{r_1 j} a^{l-1}_j + a^{l-1}_s$$

(13)

$$a^l_{r_2} \equiv \sum_{j=1}^{\alpha_{i-1}} \varepsilon^l_{r_2 j} a^{l-1}_j + a^{l-1}_s$$

where $\varepsilon^l_{rj} = \varepsilon^l_{r_1 j} + \varepsilon^l_{r_2 j}$ and the linear forms

$$\sum_{j=1}^{\alpha_{i-1}} \varepsilon^l_{rj} a^{l-1}_j, \quad \sum_{j=1}^{\alpha_{i-1}} \varepsilon^l_{r_1 j} a^{l-1}_j, \quad \sum_{j=1}^{\alpha_{i-1}} \varepsilon^l_{r_2 j} a^{l-1}_j$$

in order are the expressions for $S^{(l-1)}$, $E^{(l-1)}_1$, $E^{(l-1)}_1$ in terms of the cells of Σ. Comparison of the numbers ε^l_{ij} in the two systems of relations yields $\gamma'_l = \gamma_l + 1$. Finally it is obvious that all relations

$$a^{l-1}_i \equiv \sum_{j=1}^{\alpha_{i-2}} \varepsilon^{l-1}_{ij} a^{l-2}_j \tag{14}$$

of the system Σ also stand in the system of σ', and the latter also has a relation

$$a^{l-1}_s \equiv \sum_{j=1}^{\alpha_{i-2}} \varepsilon^{l-1}_{sj} a^{l-2}_j \tag{15}$$

whose right hand side is the expression for $S^{(l-2)}$ in terms of the cells of Σ. Now since

$$a^{l-1}_s + \sum_{j=1}^{\alpha_{i-1}} \varepsilon^l_{r_1 j} a^{l-1}_j \equiv 0,$$

the right hand side of (15) must be a linear combination of the right hand sides of the relations (14), whence $\gamma'_{l-1} = \gamma_{l-1}$.

We now insert the following remark. A glance at the relations (12) and (13) shows that each two-sided closed l-dimensional figure consisting of cells of Σ' must provide the cells $a^l_{r_1}$, $a^l_{r_2}$ with the same coefficient, so that they can only occur in the combination $a^l_{r_1} + a^l_{r_2}$. Since $a^l_{r_1} + a^l_{r_2}$ can also be written for a^l_r, each such figure can be constructed from the cells of Σ. Since also

$$a_s^{l-1} \sim -\sum_{j=1}^{\alpha_{i-1}} \varepsilon^l_{r_1 j} a_j^{l-1},$$

each $(l-1)$-dimensional figure assembled from the cells of Σ' is homologous to one assembled from the cells of Σ. For all other dimensions the corresponding theorem is trivial and therefore Theorem *b*) of §7 is proved.

Putting together the equations derived above:

$$\gamma'_l = \gamma_l + 1, \qquad \gamma'_m = \gamma_m \quad \text{for} \quad m \neq l,$$

with the equations (10) proves the equality of the numbers P_m for the two schemata Σ and Σ' [5]

The following theorem has been proved by Poincaré:[6] if $P_1, P_2, \ldots, P_{n-1}$ are the Betti numbers of a two-sided closed n-dimensional manifold V, then[7]

$$P_m = P_{n-m} \qquad (m = 1, 2, \ldots, n-1).$$

The proof is carried out with the help of two dual schemata Σ and $\overline{\Sigma}$ of V. Suppose a_i^m, ε^m_{ij}, α_m, γ_m, P_m have the usual meanings for the schema Σ and the corresponding Poincaré relation system, and let a_i^m, $\overline{\varepsilon^m_{ij}}$, $\overline{\alpha_m}$, $\overline{\gamma_m}$, $\overline{P_m}$ have the analogous meanings for $\overline{\Sigma}$. In this connection, let $\overline{a_i^{n-m}}$ be the cell of $\overline{\Sigma}$ corresponding to the cell a_i^m of Σ. Then with suitable choice of the orientation of the cells of $\overline{\Sigma}$ [8] we have

$$\overline{\varepsilon^m_{ij}} = \varepsilon^{n-m+1}_{ji}$$

[5] One sees also, of course, that the numbers $\alpha_0 - \gamma_1$ and $\alpha_n - \gamma_n$, which can be defined in terms of P_0 and P_n, also have the same values for Σ and Σ'.

[6] Compl. 1, §8

[7] This equation is obviously also correct for $m = 0$.

[8] If we have chosen an orientation of the cells of V, the orientation of the cell $\overline{a_i^{n-m}}$ is derived from that of a_i^m by the following rule, where we shall use the term "m-dimensional generalized tetrahedron" for the m^{th} figure in a series that begins with the line segment, triangle and tetrahedron (cases $m = 1, 2, 3$): if M denotes the point of intersection of a_i^m and $\overline{a_i^{n-m}}$ (it is naturally assumed that only one such point exists) then the indicatrix J of V may be brought into such a position that the point 1 of J falls on M, the points $1, 2, \ldots, m+1$ on the boundary of J belong to a m-dimensional generalized tetrahedron T_m wholly inside the cell a_i^m, which otherwise has no points in common with J; the analogous generalized tetrahedron T_{n-m} lies wholly in the cell $\overline{a_i^{n-m}}$, which has no other points in common with J, and thus T_m and T_{n-m} respectively define positive indicatrices of a_i^m

and therefore $\overline{\gamma_m} = \gamma_{n-m+1}$ whence, since $\overline{\alpha_m} = \alpha_{n-m}$, we have

$$\overline{P_m} = \overline{\alpha_m} - \overline{\gamma_m} - \overline{\gamma_{m+1}} + 1 = \alpha_{n-m} - \gamma_{n-m+1} - \gamma_{n-m} + 1 = P_{n-m}.$$

But since $\overline{P_m} = P_m$ as a consequence of the homeomorphism of the schemata Σ and Σ', the assertion follows.

This is an appropriate place to review the well known necessary and sufficient conditions for the homeomorphism of two-dimensional manifolds. These are obtained with the help of the easily-proved theorem that the schema of a two-dimensional manifold V is homeomorphic to a fundamental polygon Π of one of the following types 1 and 2 according as V is two- or one-sided.

1. Π has $4p + 3r_1 + 2r_0$ sides, where the paired sides

   $$(s_{4k-3}, s_{4k-2}), (s_{4k-1}, s_{4k}) \quad \text{for } k = 1, 2, \ldots, p,$$

   and

   $$(s_{4p+3l-2}, s_{4p+3l}) \quad \text{for } l = 1, 2, \ldots, r_1,$$

   $$(s_{4p+3r_1+2m-1}, s_{4p+3r_1+2m}) \quad \text{for } m = 1, 2, \ldots, r_0$$

   are identified in the first way, and the points represented by the closed cycles of vertices

   $$A_{4p+3r_1+2m-1,2} = A_{4p+3r_1+2m,1}$$

   are deleted from the manifold.[9]

2. Π has $4q - 2 + 3r_1 + 2r_0$ sides, where the paired sides

   $$(s_{2k-1}, s_{4q-2k}) \quad \text{for } k = 1, 2, \ldots, q$$

   are identified in the second way, and

   $$(s_{2k}, s_{4q-2k-1}) \quad \text{for } k = 1, 2, \ldots, q-1,$$

   $$(s_{4q-2+3l-2}, s_{4q-2+3l}) \quad \text{for } l = 1, 2, \ldots, r_1,$$

and $\overline{a_i^{n-m}}$ when their vertices are taken in the order written. In addition, one thinks of the point $\overline{a_i^0}$ provided with a positive or negative sign (cf. §5, note 2) according as the orientation of a_i^n agrees with that of V or not, and the signs of the vertices a_i^0 are analogously determined by the orientations of the cells $\overline{a_i^n}$.

[9] Cf. §2 for the notation.

$$(s_{4q-2+3r_1+2m-1}, s_{4q-2+3r_1+2m}) \quad \text{for } m = 1, 2, \ldots, r_0$$

are identified in the first way, and the vertices

$$A_{4q-2+3r_1+2m-1,2} = A_{4q-2+3r_1+2m,1}$$

are the points omitted from the manifold.

The numbers r_1 and r_0 and q respectively are topological invariants of the schema, because r_1 is the number of boundary lines, r_0 is the number of isolated boundary points, and in the case of p and q this follows by first assuming $r_0 = 0$, when we have the formulae $P_1 = 2p + r_1$ resp. $2p + 1$ respectively when V is closed and $P_1 = q + r_1$.

Of course it is also true for $r_0 > 0$ that fundamental polygons with different p or q cannot be homeomorphic.[10] Consequently, equal values of r_1, r_0, P_1, i.e., of r_1, r_0, p resp. r_1, r_0, q is necessary and sufficient for the homeomorphism of two schemata. As is well-known, p is called the genus of the surface, and q is the number of independent paths that reverse the indicatrix.

To conclude this paragraph we consider the topological invariant

$$N = \alpha_0 - \alpha_1 + \alpha_2 - \cdots + (-1)^n \alpha_n$$

of schemata (it follows immediately from the formulae (10) that this is a topological invariant)[11] which is related to the Betti numbers via a generalization of the well-known Euler polyhedron formula[12]

$$\alpha_0 - \alpha_1 + \cdots + (-1)^n \alpha_n = P_0 - P_1 + \cdots + (-1)^n P_n + \frac{1}{2}(1 + (-1)^n)$$

[10]In the case $r_0 > 0$ we set $P_1 = 2p + r_1 + r_0$ resp. $q + r_1 + r_0$. Cf. the conditions on the last page of §14.

[11]It is easy to see that N is the only topological invariant obtained from the numbers α_i. This is because any such invariant $\varphi(\alpha_0, \alpha_1, \ldots, \alpha_n)$ can be written in the form $f(\eta_0, \eta_1, \ldots, \eta_n)$ where

$$\eta_i = \alpha_0 - \alpha_1 + \cdots + (-1)^i \alpha_i$$

and consideration of an elementary subdivision dividing an l-dimensional cell into two yields

$$f(\eta_0, \eta_1, \ldots, \eta_{l-2}, \eta_{l-1} \pm 1, \eta_l, \ldots, \eta_n) = (\eta_0, \eta_1, \ldots, \eta_n), \qquad (l = 1, 2, \ldots, n),$$

i.e., f depends only on $\eta_n = N$. One proves in an analogous way that, apart from $P_0, P_1, \ldots, P_n$, no other numbers formed from $\alpha_0, \alpha_1, \ldots, \alpha_n, \gamma_1, \ldots, \gamma_n$ are topological invariants, i.e., any other invariants formed from the numbers α_i, γ_i, which can obviously be written in the form $f(P_0, P_1, \ldots, P_n \gamma_1, \gamma_2, \ldots, \gamma_n)$, are independent of the last n arguments and are thus functions of the P_i.

[12]Cf. Poincaré, An. Sit. §18 and Compl. 1 §3 p.301

as follows immediately from the formulae (5) for $m = 0, 1, \ldots, n$, when it is remembered that $\gamma_0 = \gamma_{n+1} = 1$. This equation yields the theorem:[13]

For each closed manifold of odd dimension the invariant N is zero.

III. One-sided closed manifolds in a manifold

§9

The invariants Q_m analogous to the Betti numbers P_m

The original definition of the Betti numbers of a manifold V involved the two-sided closed manifolds in V and the homologies between them, as explained in §6.[1]

We shall now also consider the one-sided closed manifolds lying in a given manifold.

We begin with a simple example. Let T_3 be that two-sided closed three-dimensional manifold obtained from the ball $x^2 + y^2 + z^2 \leq 1$ by identifying pairs of diametrically opposite points on the spherical boundary surface. T_3 is then a finite image of projective three-dimensional space. One obtains a cell system Σ for T_3 by dividing the spherical surface by a meridian circle M into two hemispheres, and dividing the meridian circle into two arcs f, g by the north pole A and the south pole B, and regarding the ball as a single three-dimensional cell a^3 on which the two hemispherical boundary surfaces are identified in such a way as to bring diametrically opposite points

[13]Poincaré, An. Sit. p.114. Cf. also Dyck, Math. Ann. 37, p.295. Poincaré gave another proof of the simply-connectedness of the "neighbourhood manifolds" (see §3, note 6 and §4) based on this (An. Sit. §17). The theorem is obtained most simply as follows from properties of the dual schema (in the construction of which the condition of simply-connectedness of the neighbourhood manifold is of course implicit). Namely, if the dimension $n = 2\nu + 1$ then since $\overline{\alpha}_m = \alpha_{n-m}$ we have

$$N = \alpha_0 - \alpha_1 + \cdots - \alpha_{2\nu+1} = -\overline{\alpha}_0 + \overline{\alpha}_1 - \cdots + \overline{\alpha}_{2\nu+1}$$

but from topological invariance

$$N = \overline{\alpha}_0 - \overline{\alpha}_1 + \cdots - \overline{\alpha}_{2\nu+1}.$$

The remarks just made also allow us to see that the theorem is correct for one-sided closed manifolds, although the proof in the text based on $P_m = P_{n-m}$, because of the proof of this formula itself, assumes that the manifold is two-sided.

[1]This is also emphasized by Heegaard, Diss. p.64

into coincidence. (Since this identification is of the first kind, the manifold T_3 is two-sided.) In the process, the edge $f = AB$ is identified with the edge $g = BA$, and the vertex A with the vertex B. The two identified hemispherical surfaces constitute a wall a^2, the edges f, g an edge a^1 and the vertices A, B a vertex a^0 of the schema. The neighbourhood manifold of a^0 satisfies the condition of being simply connected.

The Poincaré relation system for the schema Σ of V is

$$a^3 \equiv 0,$$

$$a^2 \equiv 2a^1,$$

$$a^1 \equiv 0.$$

The two-dimensional figures ha^2 obtainable from the single wall of the schema clearly include none that are two-sided, and because of this we obtain the Betti number $P_2 = 1$ from $\alpha_2 = 1$, $\gamma_2 = 1$ and $\gamma_3 = 0$. On the other hand, the collection of points of a^2 obviously represents a closed one-sided surface lying in T_3, since it can be viewed as a projective plane lying in the projective space T_3.

If one considers yet another schema Σ', which results from Σ when one divides the cell a^3 into two cells a_1^3, a_2^3 by a new wall b^2 represented by a disc through M, then the Poincaré relation system becomes

$$a_1^3 \equiv a^2 - b^2, \quad a_2^3 \equiv -a^2 + b^2;$$

$$a^2 \equiv 2a^1, \quad b^2 \equiv 2a^2;$$

$$a^1 \equiv 0.$$

So b^2 as well as a^2 is a one-sided closed surface in T_3. The surface b^2 is related to a^2 by the homology

$$a^2 \sim b^2.$$

This leads us to the following question, which can be expressed quite generally and relates closely to the considerations upon which the original definition of Betti numbers was based.

Does each n-dimensional manifold V contain a system of closed (two- or one-sided) m-dimensional ($m < n$) manifolds connected by no homology, $\Phi_1^{(m)}, \Phi_2^{(m)} \ldots, \Phi_r^{(m)}$ such that any other closed m-dimensional manifold $\Phi^{(m)}$ in V, whether two-sided or one-sided, satisfies a homology

$$k\Phi^{(m)} + k_1\Phi_1^{(m)} + \ldots + k_r\Phi_r^{(m)} \sim 0 \quad (k \neq 0)?$$

If the answer is in the affirmative, then the number r is obviously independent of the choice of system, and its value plus one is a topological invariant Q_m of V completely analogous to the number P_m. But for the same reasons that led us away from the definition of Betti numbers originally given, we avoid this path in investigating the topological invariants associated with one-sided manifolds. Moreover, in complete analogy with the case of the numbers P_m we pass from one-sided closed manifolds in V to one-sided closed figures — it will be explained in a moment what they are — and indeed we pass directly from a schema of V to an unimpeachable definition of the desired topological invariants. For this purpose we introduce the following theorem, which is analogous to a theorem for two-sided closed manifolds given in §5.

If $u_1^m, u_2^m, \ldots, u_\alpha^m$ are m-dimensional space pieces without common interior points lying in a manifold and

$$\sum_{i=1}^{\alpha} u^m \equiv \sum h_i u_i^{m-1}, \tag{17}$$

and if the points in one or more of the space pieces u_i^m constitute a one-sided closed m-dimensional manifold, then all coefficients h_i of the u_i^{m-1} on the right hand side of (17) are divisible by 2,[2] which may be expressed in abbreviated form by

$$\sum_{i=1}^{\alpha} u^m \equiv 0. \quad [\text{mod } 2] \tag{18}$$

This assertion is clearly independent of the choice of sense of the individual u_i^m.

With the one-sided closed surface lying in the manifold T_3 we have already seen a simple example of this theorem. We note that (18) also holds when the points belonging to one or more u_i^m together constitute a two-sided closed manifold, and that just in this case the senses of the individual u_i^m may be chosen so that the coefficients h_i in (17) are exactly zero. We shall now — with the preceding theorem as motivation — call an m-dimensional figure

[2]As in the theorem of §5 we also assume here that the u_i^{m-1} comprising the boundary of u_i^m have no common interior points

$\sum h_i u_i^m$ closed when it satisfies the relation[3]

$$\sum h_i u_i^m \equiv 0 \quad [\text{mod } 2]$$

i.e., when the figure in question is congruent, in the sense of §5, to an $(m-1)$-dimensional figure, all coefficients of which are divisible by 2. If in fact we have the relation

$$\sum h_i u_i^m \equiv 0 \quad [\text{mod } 2]$$

then, as we have said, the figure is called two-sided and closed.

A remark is in order here. When the space pieces $u_1^m, u_2^m, \ldots, u_\alpha^m$, form a one-sided closed manifold then this manifold can be symbolized as one of the 2^α linear forms

$$\sum \delta_i u_i^m, \tag{19}$$

constructed from the u_i^m with coefficients ± 1, and all these linear forms satisfy the relation

$$\sum \delta_i u_1^m \equiv 0 \quad [\text{mod } 2].$$

On the other hand, if the space pieces $u_1^m, u_2^m, \ldots, u_\alpha^m$ form a two-sided closed manifold, so that we have the relation

$$\sum u_i^m \equiv 0,$$

then of all the remaining linear forms (19) only $\sum(-u_i^m)$ satisfies the relation

$$\sum \delta_i u_i^m \equiv 0.$$

Thus only the linear forms $\sum u_i^m$ and $\sum(-u_i^m)$ symbolize the same two-sided closed manifold, and they may be regarded as providing its two possible orientations. On these grounds, when we consider closed figures, which are possibly one-sided, then figures given by distinct linear forms $\sum h_i u_i^m$, $\sum h_i' u_i^m$ that satisfy the congruence

$$\sum h_i u_i^m \equiv \sum h_i' u_i^m \quad (\text{mod } 2)$$

[3]These symbolic congruences with the modulus enclosed in square brackets may be distinguished from congruences of the form

$$\sum h_i u_i^m \equiv \sum k_i u_i^m \quad (\text{mod } 2),$$

which, taken in the usual sense, express the divisibility of all numbers $(h_i - k_i)$ by 2.

will not be regarded as geometrically distinct, even though they are arithmetically distinct. Only in the case of two-sided closed figures will arithmetic distinctness be regarded as genuine distinctness.

Now let us recall the decisive definition of the numbers P_m of a manifold V in a form suitable for our future purposes.

One makes the assumption (called III above) that there are no homologies between the cells of the schema Σ of V other than those obtained from the Poincaré relation system of Σ, and then computes them from it. Under this assumption one obtains a system of figures, among which the collection Γ of two-sided closed m-dimensional figures comprised of cells of Σ is chosen with the property that no homology exists between these figures of Γ alone, although one exists between the figures of Γ and each other figure of the system. The number of figures in such a system (which is independent of the choice of system), plus 1, is called the number P_m for the manifold V.

The corresponding numbers Q_m for V are given by the same definition, except for omission of the word "two-sided". The inequality

$$Q_m \geq P_m$$

therefore follows immediately.

The definition of the Q_m just given needs clarification only to the extent of proving the existence of a system of closed figures with the required property. It is then clear that the number of figures in any such system is the same. The required proof will be carried out in the following paragraph in such a way as to simultaneously determine the difference $Q_m - P_m$ in terms of known topological invariants, namely the torsion numbers discovered by Poincaré.

§10

The Poincaré torsion numbers. Determination of Q_m

Continuing the notation of §§5,6, let ε_{ij}^m be the coefficients of the Poincaré relation system of a manifold schema and let $\omega_1^m, \omega_2^m, \ldots, \omega_{\gamma_m}^m$ be the nonzero elementary divisors of the matrix of the ε_{ij}^m. Then those of the numbers ω_h^m that are > 1 will be called the $(m-1)^{\text{th}}$ order torsion numbers[1] of the

[1]Called "coefficients de torsion" by Poincaré (Compl. 2 p.301). If one were to denote ε_{ij}^m, ω_h^m, γ_m, β_m instead by ε_{ij}^{m-1}, ω_h^{m-1}, γ_{m-1}, β_{m-1} this would agree better with the numbering of the orders of the torsion numbers introduced here (in my note in the Wiener Ber. cited above the torsion numbers of $(m-1)^{\text{th}}$ order were called torsion numbers

manifold. When, as in §8, one takes the Poincaré relation system for a schema Σ and derives a schema Σ' from Σ by elementary subdivision, then one sees that the torsion numbers derived from Σ agree with those derived from Σ', so that they are in fact topological invariants of the schemata.[2] We wish to show that a knowledge of P_m and the Poincaré torsion numbers of $(m-1)^{\text{th}}$ order suffices for the calculation of the Q_m.

In order to obtain a system of closed m-dimensional figures with the property expressed in the definition of the Q_m we go back to the Poincaré relation system of a schema Σ and its reduced form (6) described in §6. It follows from the latter that a figure

$$\sum_{i=1}^{\alpha_m} h_i a_i^m = \sum_{i=1}^{\alpha_m} k_i b_i^m$$

---

of m^{th} order) however we forego this to avoid deviating from the notation in Poincaré's fundamental works. On the other hand this numbering of the orders of the torsion numbers harmonises with the result of §14 that the fundamental group of a manifold V — which concerns the closed one-dimensional manifolds in V — determines P_1 as well as the torsion numbers of "first order", and its basis is ultimately the fact that the matrix of the numbers ε_{ij}^1 has no elementary divisors > 1 (see Poincaré, Compl. 2, p.307). This derives from the following property of the matrix: if not all numbers ε_{ij}^1 in the i^{th}th row are zero, then only two are non-zero, one of which $= +1$ and the other $= -1$. Suppose $\varepsilon_{ih}^1 = +1$ and $\varepsilon_{ik}^1 = -1$. One can assume $i = 1$, $h = 1$, $k = 2$, since this can always be arranged by reordering the rows and columns. Now if the elements of the first column are added to the corresponding elements of the second column, followed by multiplication of the first row by $-\varepsilon_{i1}^1$ and addition of it to the i^{th} row $(i = 2, 3, \ldots)$, then one obtains a matrix of the form

$$\begin{matrix} 1 & 0 & 0 & \cdots \\ 0 & \zeta_{11} & \zeta_{12} & \cdots \\ 0 & \zeta_{21} & \zeta_{22} & \cdots \\ \vdots & & & \end{matrix} \tag{20}$$

where the matrix of the numbers ζ_{ij}, which has one row and column fewer than the original matrix, has the same property as the matrix of the ε_{ij}^1, as one easily sees. But since the matrix (20) has the same elementary divisors as the matrix of the ε_{ij}^1 one sees by induction that elementary divisors > 1 cannot appear. (The same type of argument shows that $\gamma_1 = \alpha_0 - P_0$ which was used in note 5 of §6.)

[2]See Poincaré, Compl. 2. By introduction of the dual schema and the relation (16) one obtains the theorem (see Poincaré op. cit.): For a two-sided closed n-dimensional manifold the torsion numbers of m^{th} order coincide with those of $(n-m-1)^{\text{th}}$ order.

can be closed only when the numbers

$$k_1\omega_1^m, \quad k_2\omega_2^m, \quad \dots, \quad k_{\gamma_m}\omega_{\gamma_m}^m$$

are divisible by two. Now if β_m of the numbers ω_i^m are even, say $\omega_{\gamma_m-\beta_m+1}^m, \dots, \omega_{\gamma_m-1}^m, \omega_{\gamma_m}^m$, and the rest are odd, then $k_1, k_2, \dots, k_{\gamma_m-\beta_m}$ must be divisible by two if the figure in question is to be closed, so this figure must be congruent modulo 2 to a figure of the form

$$\sum_{i=\gamma_m-\beta_m+1}^{\alpha_m} k_i b_i^m$$

This necessary condition is clearly also sufficient.

Thus one sees first of all that all closed m-dimensional figures are linear combinations of $\alpha_m - \gamma_m + \beta_m$ closed figures

$$b_{\gamma_m-\beta_m+1}^m, \quad b_{\gamma_m-\beta_m+2}^m, \quad \dots, \quad b_{\alpha_m}^m \tag{21}$$

when one does not distinguish figures differing by linear forms divisible by two; these are not geometrically distinct in any case. But the figures (21) are not only all geometrically distinct from each other; because of the linear independence of these forms in the a_i^m, none of them can be congruent modulo 2 to a linear combination of the others. The last $\alpha_m - \gamma_m$ of the figures (21) are two-sided, the first β_m one-sided.

We now have the question of homologies between the figures (21) and indeed, since we are leaning on Assumption III, the question of deriving them from the Poincaré relation system. As already remarked in §6, these homologies may be represented as homologies between the two-sided closed figures $b_{\gamma_m+1}^m, \dots, b_{\alpha_m}^m$,[3] and in fact one obtains γ_m independent homologies, from which all the rest may be computed. It follows from this, analogously as in §6 for the corresponding case of two-sided closed figures, that one can exhibit a system of $t = \alpha_m - \gamma_m + \beta_m - \gamma_{m+1}$ closed m-dimensional figures,

[3] Thus when $g_1^m, g_2^m, \dots$ are closed m-dimensional figures, some of which are one-sided, and if there is a homology

$$\sum h_i g_i^m \sim 0$$

between them, then the linear form in the cells a_i^m on the left hand side may always be represented as a linear combination of suitably chosen two-sided closed figures. In the above example of the manifold T_3 this is immediately confirmed for the homology $a^2 - b^2 \sim 0$.

consisting of cells of the schema, connected by no homology but such that any other such figure does satisfy a homology together with these t figures. In the proof of the existence of such a system the required completion of the definition of the Q_m has to be carried out and at the same time we have found the formula

$$Q_m = t + 1 = \alpha_m - \gamma_m - \gamma_{m+1} + \beta_m + 1 = P_m + \beta_m.$$

β_m, the number of even numbers among $\omega_1^m, \omega_2^m, \ldots, \omega_{\gamma_m}^m$, is just the number of even torsion numbers of $(m-1)^{\text{th}}$ order of the manifold and we have the theorem:[4]

The difference $Q_m - P_m$ equals the number of even torsion numbers of $(m-1)^{\text{th}}$ order.

The determination of the number β_m does not necessarily require the determination of the elementary divisors $\omega_1^m, \omega_2^m, \ldots, \omega_{\gamma_m}^m$. Namely, let the Poincaré relation system be brought, by any means, into a form

$$b_i^m \equiv \tau_i^m c_i^{m-1}, \qquad (i - 1, 2, \ldots, \gamma_m) \quad (\tau_i^m > 0)$$

$$b_i^m \equiv 0 \qquad (i = \gamma_m, \ldots, \alpha_m)$$

where the $b_1^m, b_2^m, \ldots, b_{\alpha_m}^m$ and $c_1^{m-1}, c_2^{m-1}, \ldots, c_{\alpha_{m-1}}^{m-1}$ are linear homogeneous forms in the a_1^m and a_i^{m-1} respectively, with determinant 1. Then β_m is also the number of even numbers among the τ_i^m. This follows from the easily-proved theorem that the number of elementary divisors of the matrix

$$\begin{matrix} \tau_1^m & 0 & 0 & \ldots \\ 0 & \tau_2^m & 0 & \ldots \\ 0 & 0 & \tau_3^m & \ldots \\ \vdots & & & \end{matrix}$$

divisible by any prime p (2 in particular) is equal to the number of τ_i^m divisible by p.

IV. The fundamental group

§11

The Poincaré numbers of a discrete group

[4]See §1 of the note in the Wien. Ber (1906) cited above.

In this paragraph we introduce a few of the group-theoretic considerations relevant to the groups that will arise in connection with connected manifolds. We remark that the elements of these groups are not operations with a specific meaning, and we are concerned only with the laws of their combination, so we are dealing with the general group concept.

These laws of combination of the groups in question, and hence the presentations of the groups themselves, will be given in the following well-known way. First of all one gives a number[1] of elements (operations) $s_1, s_2, \ldots, s_n$ of the group, which will be called generating operations.[2] One then obtains all the elements of the group as products of the operations $s_1, s_2, \ldots, s_n$ and the inverse operations $s_1^{-1}, s_2^{-1}, \ldots, s_n^{-1}$. Thus each element of the group may be written in the form

$$s_1^{\alpha_1} s_2^{\alpha_2} \cdots s_n^{\alpha_n} s_1^{\beta_1} s_2^{\beta_2} \cdots \tag{22}$$

where the α_i, β_i are integers (positive, negative or zero) and each symbol (22) represents a particular element of the group. In general an element of the group will be representable in different ways in the form (22). In addition to giving the generating operations, the group must be defined by certain "defining relations" between them

$$F_i(s_1, s_2, \ldots, s_n) = 1 \qquad (i = 1, 2, \ldots, n) \tag{23}$$

whose left hand sides $F_i(s_1, s_2, \ldots, s_n)$ are expressions of the form (22) and which say that all the expressions $F_1(s_1, s_2, \ldots, s_n)$, $F_2(s_1, s_2, \ldots, s_n), \ldots,$ $F_m(s_1, s_2, \ldots, s_n)$ are symbols for the identity operation 1 of the group, and furthermore that any two expressions that can be proved equal with the help of the relations (23) are symbols for the same element of the group. In this way a group is completely determined by giving its generators and defining relations.

One observes immediately that two different systems of generators and defining relations can define groups that are isomorphic,[3] and hence the same in the sense of the general group concept. However we have neither solved the general problem of theoretically surveying all the possible ways in which

[1] We confine ourselves to groups that can be generated by a finite number of operations.

[2] The identity group can be thought of as the result of taking no generating operations, as well as none of the defining relations that are discussed later.

[3] By "isomorphic" we always mean "holoedrically isomorphic" [genuine isomorphism, rather than just homomorphism, Trans.]

the same group may be defined, nor found a means[4] for deciding in particular cases whether two groups given by different systems of generators and defining relations are the same, i.e., isomorphic. However, we shall give a necessary condition for two differently-defined groups to be isomorphic. It consists in the agreement of the Poincaré numbers derived from the systems of defining relations for the groups. But before giving the definition of these numbers we discuss the connection between two relation systems that define isomorphic groups.

First, a remark about the relations that follow from a given system of relations

$$A_1 = 1, \quad A_2 = 1, \quad \dots, \quad A_k = 1 \tag{24}$$

The $A_1, A_2, \dots, A_k$ are understood to be expressions of the kind (22) in the generating operations. Each of the relations (24) implies the relation obtained by replacing the left hand side by its inverse. Thus one obtains the relations[5]

$$A_1^{-1} = 1, \quad A_2^{-1} = 1, \quad \dots, \quad A_k^{-1} = 1. \tag{25}$$

Also, if L denotes any expression of the same kind as $A_1, A_2, \dots, A_k$, then any one of the relations (24), say $A_1 = 1$ has the consequence

$$L^{-1} A_1 L = 1$$

Another way of obtaining consequences of (24) is to take any two of them (which need not be distinct), say $A_1 = 1$ and $A_2 = 1$, and derive the relation

$$A_1 A_2 = 1.$$

If one assumes that the relation system (24) is constituted so that it already contains the relations (25), then one can say: successive applications of the last two types of construction allow one to derive all relations that follow from (24), if, each time one obtains a consequence relation, one adds this new relation to the system and proceeds with derivations from the extended system. This can also be summarized as follows:

[4]In the case of finite groups this decision is obviously always possible. However a criterion for deciding whether a group given by a particular presentation is in fact finite or not, is lacking.

[5]The inverse A_i^{-1} of A_i is defined so that $A_i A_1^{-1} = 1$ identically.

If the relation system (24) contains the inverse relations (25), then each consequence relation is of the form

$$L_1 A_{i_1} L_2 A_{i_2} L_3 \cdots A_{i_r} L_{r+1} = 1$$

where $i_1, i_2, \ldots, i_r$ are any numbers from the series $1, 2, \ldots, k$, and $L_1, L_2, \ldots, L_{r+1}$ are expressions in the generating operations of the form (22) that identically satisfy the equation

$$L_1 L_2 \cdots L_{r+1} = 1.$$

To deal with relation systems defining the same group we first introduce two special cases.

In the first case let $s_1, s_2, \ldots, s_n$ be the generating operations and let

$$F_1(s_1, s_2, \ldots, s_n) = 1, \quad \ldots, \quad F_m(s_1, s_2, \ldots, s_n) = 1 \tag{26}$$

be the defining relations of one system, while the second system has the same generating operations and, as well as (26), the additional defining relation

$$F_{m+1}(s_1, s_2, \ldots, s_n) = 1,$$

which is a consequence of the relations (26). It is evident that the two relation systems define the same group. The passage from the first system to the second will be called an extension of the first kind, while the passage from the second to the first, by omission of a consequence relation, will be called a reduction of the first kind.

In the second case let $s_1, s_2, \ldots, s_n$ again be the generating operations and let (26) be the defining relations of one system. The second system, on the other hand, has the elements $s_1, s_2, \ldots, s_n, t$ as generating operations, and the defining relations are (26) together with a relation of the form

$$A^{-1} t = 1,$$

where A is an expression in the elements $s_1, s_2, \ldots, s_n$ of the type (22). Since the second system differs from the first only in mentioning the operation $t = A$, which is already present in the first system, it is clear that the two relation systems here also define the same group. In this case the passage from the first system to the second by introduction of a superfluous generating operation is called an extension of the second kind. On the other hand, if the relation system contains a generator t that appears in only one of the relations, and if the latter is of the form (27), then omission of t and the relation in question will be called a reduction of the second kind.

We now consider the general case of two relation systems that define the same group.[6] Let the generating operations of one system be $s_1, s_2, \ldots, s_n$ and let

$$F_i(s_1, s_2, \ldots, s_n) = 1 \qquad (i = 1, 2, \ldots, m) \tag{28}$$

be the defining relations. For the second system let $t_1, t_2, \ldots, t_\nu$ be the generating operations and let

$$G_j(t_1, t_2, \ldots, t_\nu) = 1 \qquad (j = 1, 2, \ldots, \mu) \tag{29}$$

be the defining relations. Since we assume that the two systems define the same group, each operation t_h must be expressible as an operation of the first group in terms of the $s_1, s_2, \ldots, s_n$. There must therefore be relations

$$t_h = S_h(s_1, s_2, \ldots, s_n) \qquad (h = 1, 2, \ldots, \nu)$$

or

$$S_h^{-1} t_h = 1 \qquad (h = 1, 2 \ldots, \nu) \tag{30}$$

Also, the operations $S_h(s_1, s_2, \ldots, s_n)$ must satisfy the same relations in the first group as the operations t_h do in the second. The relations

$$G_j(S_1(s_1, s_2, \ldots, s_n), S_2(s_1, s_2, \ldots, s_n), \ldots, S_\nu(s_1, s_2, \ldots, s_n)) = 1 \tag{31}$$

$$(j = 1, 2, \ldots, \mu)$$

must therefore be consequence relations of (28). In the same way we must conversely have n relations

$$T_k^{-1} s_k = 1 \qquad (k = 1, 2, \ldots, n) \tag{32}$$

[6]Naturally it is not excluded that the groups defined by the two relation systems are isomorphic in more than one way. When we say that two relation systems define the same group we assume that a fixed isomorphism has been chosen between the two groups, and that the elements it pairs together will be identified for future purposes.

where the $T_k = T_k(t_1, t_2, \ldots, t_\nu)$ are products of the operations $t_1, t_2, \ldots, t_\nu$ and the relations

$$F_i(T_1(t_1, t_2, \ldots, t_\nu), T_2(t_1, t_2, \ldots, t_\nu), \ldots, T_n(t_1, t_2, \ldots, t_\nu)) = 1 \qquad (33)$$

$$(i = 1, 2, \ldots, m)$$

must be consequence relations of the relations (29).

These remarks permit a step-by-step passage from the first relation system to the second, the individual steps of which are extensions or reductions of the two kinds described. Namely, one first makes μ extensions of the first kind by adding the relations (31) followed by ν extensions of the second kind by introduction of the new generating operations t_h and the relations (30). The relations (29) are obviously consequence relations of the thus-extended relation system. By introducing these relations one obtains a relation system R_1, between the generating operations $s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_\nu$, consisting of all the relations (28), (31), (30), (29). The relations (32), (33) are clearly consequence relations of the system R_1. Addition of these extends R_1 to a relation system R. In the same way, however, we can obtain the relation system R from the system of defining relations (29) between the generating operations $t_1, t_2, \ldots, t_\nu$; first extending in the first way by adjoining the consequence relations (33), then in the second way by introducing new generators $s_1, s_2, \ldots, s_n$ with the help of the relations (32), then adding the consequences (28) of the relations now present, giving the relation system R_2 between the generators $s_1, s_2, \ldots, s_n, t_1, t_2, \ldots, t_\nu$, consisting of the relations (29), (33), (32), (28), which finally extends to the system R by adjoining the consequence relations (30) and (31). Consequently one can pass from the system of relations (28) between the generators $s_1, s_2, \ldots, s_n$ to the system R by a series of extensions, then to the system of relations (29) between the generators $t_1, t_2, \ldots, t_\nu$ by a series of reductions.

The property of two relation systems defining the same group is therefore equivalent to the possibility of passing from one system to the other by a series of reductions and extensions of the first and second kind.

This suggests a way of defining certain numbers for each system of defining relations. Let $s_1, s_2, \ldots, s_n$ be the generators and (28) the defining relations of a group. Suppose that

$$F_i(s_1, s_2, \ldots, s_n) = s_1^{\kappa'_{i1}} s_2^{\kappa'_{i2}} \cdots s_n^{\kappa'_{in}} s_1^{\kappa''_{i1}} s_2^{\kappa''_{i2}} \cdots$$

and set

$$\lambda_{ij} = \kappa'_{ij} + \kappa''_{ij} + \cdots$$

The elementary divisors of the matrix of the numbers λ_{ij} that are > 1 may be denoted by $\pi_1, \pi_2, \ldots, \pi_\rho$, and we now prove that these numbers are the same for two defining systems of the same group.[7] These numbers $\pi_1, \pi_2, \ldots, \pi_\rho$, which are characteristic of the group, may be defined just as well for non-commutative groups as for abelian groups,[8] and are represented by the Poincaré torsion numbers[9] of a manifold in the case where the group in question is the fundamental group of a manifold.[10] The numbers $\pi_1, \pi_2, \ldots, \pi_\rho$ may therefore be called the Poincaré numbers of the group.

The fact that Poincaré numbers of a group indeed are independent of the relation system defining the group will follow from the remarks above when it is shown that these numbers do not change under an extension of the relation system of the first or second kind (nor of course under reduction of either kind). In order to see this for an extension of the first kind, note the following. One can think of the numbers λ_{ij} being obtained by regarding the group as commutative, then using the commutativity of the operations to express the left hand sides of the relations (28) in the form $s_1^{a_1} s_2^{a_2} \cdots s_n^{a_n}$ and determining the exponent of s_j in the i^{th} thus modified relation. An extension of the first kind now consists of addition of a consequence relation $F_{m+1}(s_1, \ldots, s_n) = 1$ to the original system, and two kinds of consequence relation are to be distinguished. The first consists in the construction of the inverse to the original relation, say $F_1 = 1$. It is clear that in this case

$$\lambda_{m+1,j} = -\lambda_{1j} \qquad (j = 1, 2, \ldots, n)$$

so the elementary divisors of the λ_{ij} matrix distinct from 1 remain the same.

[7]Naturally it can happen that no elementary divisors > 1 are present. In this case the proof shows that the property of the λ_{ij} matrix of having no elementary divisors > 1 likewise holds for all systems of defining relations for the same group.

[8]For finite abelian (commutative) groups these numbers correspond essentially to the invariants that are known to completely characterize the group. Namely, if one represents such a group by a basis so that the orders $\nu_1, \nu_2, \ldots, \nu_\sigma$ of the generating operations are such that each divides its predecessor, then the numbers ν_i (cf. Frobenius & Stickelberger, Crelle's J. 86, p.238), or the numbers one obtains by decomposition of the ν_i into relatively prime powers (cf. H. Weber, Algebra, 2. Aufl., Vol. 2 §12) are the invariants of the abelian group. One now finds $\rho = \sigma$ and $\pi_i = \nu_i$

[9]In fact the torsion numbers of first order.

[10]See §14.

The second kind of consequence relation has the form

$$L_1 F_{i_1} L_2 F_{i_2} \cdots F_{i_r} L_{r+1} = 1$$

where $L_1 L_2 \cdots L_{r+1} = 1$ identically. One can therefore completely ignore the expressions $L_1, L_2, \ldots, L_{r+1}$ since, as we have noted, all operations can be assumed to commute in constructing the numbers λ_{ij}. Now if h_i denotes the number of occurrences of F_i among the expressions $F_{i_1}, F_{i_2}, \ldots F_{i_r}$ then obviously

$$\lambda_{m+1,j} = h_1\lambda_{1j} + h_2\lambda_{2j} + \cdots + h_m\lambda_{mj} \qquad (j = 1, 2, \ldots, n),$$

whence the equality of the Poincaré numbers for the original and extended system is immediately clear.

Finally, if one is dealing with an extension of the second kind, then a new generator t appears in the original system together with a relation

$$St = 1,$$

where S contains only the operations $s_1, s_2, \ldots, s_n$. The matrix of the λ_{ij} thus receives a new $(m+1)^{\text{th}}$ row corresponding to this relation as well as a new $(n+1)^{\text{th}}$ column corresponding to the new operation t. This column contains only zeroes in the first to m^{th} places, and the number $\lambda_{m+1,n+1} = 1$ in the $(m+1)^{\text{th}}$ place. It follows that in addition to the elementary divisors of the original λ_{ij} matrix there is a new one, equal to 1. The Poincaré numbers therefore remain the same in this case also, and the assertion is proved. One can express it in the following form.

A necessary condition for the isomorphism of two groups is the equality of their Poincaré numbers.

It is possible to define another number that has the same value for isomorphic groups; in other words, it is independent of the special choice of defining relations. This result can be derived immediately from Poincaré's discussion of the fundamental groups of three-dimensional manifolds.[11] Namely, if n denotes the number of generators of a group and r the rank of the λ_{ij} matrix defined above (in other words, one obtains r when one assumes that the operations of the group commute, and under this assumption finds the number of independent defining relations) then the number $\zeta = n - r$ has the same value for all relation systems which define the same group, as one

[11]Note a remark in An. Sit. §13, p.65.

again sees easily by consideration of extensions of the first and second kind. If the group in question is the fundamental group of a manifold V, then the number $\zeta + 1$ is none other than the Betti number P_1 of V.

Thus for two groups to be isomorphic it is necessary that not only the numbers $\pi_1, \pi_2, \ldots, \pi_\rho$ but also the numbers ζ have the same value for both groups.[12] It is easy to see from examples that these conditions are not sufficient. One takes for example the group with generator s and defining relation $s^2 = 1$ — the cyclic group of order 2 — and the group with the generators t, u and the defining relations[13]

$$t^2 = 1, \quad u^2 t u^{-1} t^{-1} = 1, \quad u^3 = 1$$

which is obviously none other than the permutation group on three symbols, where

$$t = (1, 2), \quad u = (1, 2, 3).$$

Both groups have a Poincaré number $\pi_1 = 1$ and the same number $\zeta = 0$. Another example is provided by the relations

$$s^4 t s^{-1} t = 1, \quad t^{-2} s^{-1} t s^{-1} = 1$$

defining a group Γ on generators s, t which has $\zeta = 0$ and no Poincaré numbers.[14] The same obviously holds for the identity group, which is obviously not isomorphic to Γ.[15]

§12
Introduction of the fundamental group

The topological invariants discussed up to now have been whole numbers or, in the case of torsion numbers, systems of whole numbers. We shall now consider a topological invariant of a different kind, namely the fundamental group[1] introduced by Poincaré. When this group is referred to as a

[12]It is easy to convince oneself that there are always groups for which the numbers $\pi_1, \pi_2, \ldots, \pi_\rho$ and ζ have arbitrary prescribed values. E.g., the group with the generators $s_1, s_2, \ldots, s_\rho, t_1, t_2, \ldots, t_\zeta$ and the relations $s_1^{\pi_1} = 1$, $s_2^{\pi_2} = 1$, …, $s_\rho^{\pi_\rho} = 1$ obviously gives the desired values. For all finite groups $\zeta = 0$, hence $m \geq n$.

[13]The same is obviously true for the (infinite) group that has merely the defining relations $t^2 = 1$, $u^2 t u^{-1} t^{-1} = 1$.

[14]It is the fundamental group of the closed three-dimensional manifold given by Poincaré which has $P_1 = P_2 = 1$ and no torsion numbers, the same as the three-dimensional sphere without being homeomorphic to it (Compl. 5, p.109).

[15]Among other things, Poincaré on p.110 proves that adding the relation $s^{-1} t s^{-1} t = 1$ leads to the icosahedral group: $s^{-1} t s^{-1} t = 1$, $s^5 = 1$, $t^3 = 1$.

[1]An. sit. §12,13.

topological invariant of a connected manifold we mean that a group can be associated with each connected manifold, in such a way that its construction involves only properties common to all manifolds homeomorphic to the given manifold. However, just as with the previous topological invariants we shall confine ourselves to a definition of the fundamental group of a manifold based on its schema, and prove invariance by going to homeomorphic schemata. In so doing we shall touch on the relation between the fundamental group and many-valued but unbranched functions[2] defined on the manifold.

In this paragraph we repeat the procedure of Poincaré for determining the fundamental group from the schema of a manifold.[3] Let us take for example the case of a two-dimensional schema. Let $M^{(i)}$ be the centre of the disk representing the surface piece a_i^2 of the schema. We then mark a point N' on one of each pair of identified sides, and the corresponding point N'' on the other member of the pair; the two therefore represent a single point N on the edge of the manifold obtained when the sides are identified. Now in each polygon of the schema we draw radius vectors from $M^{(i)}$ to all the points N' or N'' on the sides of this polygon. We now think of any two radius vectors $M^{(i)}N'$ and $M^{(j)}N''$ to corresponding points N' and N'' (and of course j can equal i also) as a line $M^{(i)}N' + N''M^{(j)}$ from $M^{(i)}$ to $M^{(j)}$. The direction of each line can be chosen arbitrarily, but once chosen it is fixed. The directed lines constructed in this way will be denoted $S_1, S_2, S_3, \dots$ and called the "fundamental paths" connecting the points $M^{(i)}$.

One now considers all the polygon vertices which correspond to closed cycles, except those which correspond to points omitted from the manifold. Let A be such a vertex of the schema, say a μ-tuple edge endpoint.[4] A small closed line L_A is now drawn around A, cutting μ segments from the μ edges. We take a point on each of these segments, and give it the name $M_A^{(i)}$ when the segment lies in the surface piece a_i^2. (The possibility that the same name $M_A^{(i)}$ is given to several points is not excluded.) In addition, one considers one of the μ edges $k_1, k_2, \dots, k_\mu$ of the schema ending at A, say k_1, and considers

[2]Op. cit. p. 60,61.

[3]With trivial modifications. E.g. we drop the restriction to schemata which consist of a single n-dimensional cell (where n is the dimension number of the schema).

[4]This term can be briefly clarified as follows; when μ edges meet at A it should be remembered that there may be edges which lead from A to A. Such an edge contributes 2 to the number μ, so that μ is really the number of edge endpoints which come together at A, rather than the number of edges.

its point of intersection $N_{A,1}$ with L_A.[5] The point $N_{A,1}$ may be regarded as a pair of corresponding points on the pair of identified polygon sides that constitute the edge k_1, and these two points may be denoted $N'_{A,1}$ and $N''_{A,1}$. Now the two polygon sides carry points called N' and N'', and the notations of the points $N'_{A,1}$, $N''_{A,1}$, may be chosen so that the points $N'_{A,1}$ and N' lie on one of the sides, while $N''_{A,1}$ and N'' lie on the other. One then combines two half segments $M_A^{(i)}N'_{A,h}$ and $M_A^{(j)}N''_{A,h}$ that lead to the same point $N_{A,h}$ into a single line $M_A^{(i)}N'_{A,h}+N''_{A,h}M_A^{(j)}$ and gives it the same notation S_λ, when the corresponding direction is chosen, as the corresponding line $M^{(i)}N'+N''M^{(j)}$. Then a given sense of L_A determines a series of lines S_λ, say $S_{\lambda_1}, S_{\lambda_2}, \ldots, S_{\lambda_\mu}$ and exponents $\varepsilon_1 = 1$ or -1 according as S_{λ_i} is traversed in the positive or negative direction, and we write the relation:[6]

$$S_{\lambda_1}^{\varepsilon_1} S_{\lambda_2}^{\varepsilon_2} \cdots S_{\lambda_\mu}^{\varepsilon_\mu} = 1 \tag{34}$$

The correct cyclic order of the S_{λ_i} is obviously essential for this relation, though the choice of the initial S_{λ_1} is not. Each closed cycle of identified polygon edges yields one such relation between the fundamental paths.

As an example, we consider the two-dimensional manifold T_2 representing the projective plane (it is the surface characterized by $q = 1$, $r_1 = r_0 = 0$ according to §8), which is defined by the schema consisting of a single polygon with two sides identified in the second way. One obtains a single path $S_1 =$

[5]The line L_A, divided into a number of segments by the points $N_{A,i}$, is none other than the one- dimensional "neighbourhood manifold" of the vertex A (see §3, note 6 and §4).

[6]The fundamental group arises from many-valued but unbranched functions on the manifold. If $y_1^{(i)}, y_2^{(i)}, \ldots$ is the set of values of such a function y at the point $M^{(i)}$ and $y_1^{(j)}, y_2^{(j)}, \ldots$ is the corresponding collection at the point $M^{(j)}$ then the S_λ corresponding to a path from $M^{(i)}$ to $M^{(j)}$ can be viewed as the substitution

$$\begin{pmatrix} y_1^{(i)} & y_2^{(i)} & \ldots \\ y_{\nu_1}^{(j)} & y_{\nu_2}^{(j)} & \ldots \end{pmatrix}$$

where the given path carries the series $y_1^{(i)}, y_2^{(i)}, \ldots$ of function values into the series $y_{\nu_1}^{(j)}, y_{\nu_2}^{(j)}, \ldots$. We can think of the closed line L_A around A being expanded until it finally coincides with a series of paths $M^{(i)}N' + N''M^{(j)}$ in place of the segments $M_A^{(i)}N'_{A,h} + N''_{A,h}M^{(j)}$. The thus deformed L_A represents a closed path around A composed of fundamental paths, and the relation (34) then simply says that along this path the function returns to its initial values, which must indeed be the case for any unbranched function y.

$M^{(1)}N' + N''M^{(1)}$ and the line L_A around the single vertex of the schema consists of two segments, on each of which there is a point denoted $M_A^{(1)}$, and yields the relation $S_1^2 = 1$. On the basis of the details below, it follows that the fundamental group of T_2 is the cyclic group of order 2.

In the general case of an n-dimensional schema the determination of the fundamental paths and the relations between them follows quite analogously. A point $M^{(i)}$ is chosen in the interior of each n-dimensional cell, and a point N' in each $(n-1)$-dimensional boundary cell, provided it is identified with another $(n-1)$-dimensional cell, in which case the corresponding point in the latter cell is denoted by N''. The lines $M^{(i)}N + N''M^{(j)}$ then represent the fundamental paths S_λ. The $(n-2)$-dimensional boundary cells of the n-dimensional cells of the schema are determined by cycles. We then look at each closed cycle that determines an $(n-2)$-dimensional space piece, unless the latter is omitted from the manifold. Let A be an $(n-2)$-dimensional cell of the schema represented by one such cycle. A closed line L_A around A then meets each of the $(n-1)$-dimensional cells abutting at A, and these points of intersection divide L_A into a number of segments. In a completely analogous way as in the case of the two-dimensional schema we then arrive at the relation (34) between the S_λ by consideration of the lines L_A.

In order to obtain the fundamental group from the fundamental paths and the relations (34) between them, we select one of the points $M^{(1)}, M^{(2)}, \ldots,$ say $M^{(g)}$, and call it the basepoint. Then if $M^{(i)}$ is different from $M^{(g)}$ one can, since the schema is assumed connected, choose a sequence of fundamental paths (in many different ways of course)

$$S_{\nu_1}^{\delta_1} = M^{(g)}M^{(\mu_1)}, \quad S_{\nu_2}^{\delta_2} = M^{(\mu_1)}M^{(\mu_2)}, \quad \ldots, \quad S_{\nu_k}^{\delta_k} = M^{(\mu_{k-1})}M^{(\mu_i)}$$

$$(\delta_k = \pm 1)$$

which represents a path from $M^{(g)}$ to $M^{(i)}$. This particular type of path, which may be called an "approach path" will now be fixed, and described by the notation

$$S_{\nu_1}^{\delta_1} S_{\nu_2}^{\delta_2} \cdots S_{\nu_k}^{\delta_k} = U_{g,i}.$$

$U_{g,g}$ is understood to be the identity operation. Now if S_λ is any fundamental path from $M^{(h)}$ to $M^{(k)}$ we set

$$S_\lambda = U_{g,h}^{-1} s_\lambda U_{g,k} \tag{35}$$

so that s_λ is a closed path from $M^{(g)}$ to $M^{(g)}$. The paths s_λ will be called the closed fundamental paths.[7]

We now consider the collection $\mathfrak{R}$ of all the relations between the s_λ which can be obtained from the relations (34) and (35) by expressing the $U_{g,i}$ in the latter in terms of the S_λ and then eliminating the S_λ. Among the collection $\mathfrak{R}$ it is possible to choose a finite set $\mathfrak{F}$, as will turn out later, which has all the remaining relations of $\mathfrak{R}$ as consequences. The group defined by the generating operations s_λ and the relation system $\mathfrak{F}$ is called the fundamental group of the manifold.[8] This group is clearly independent of the choice of the system $\mathfrak{F}$ from the relation system $\mathfrak{R}$.

§13
Proof that the fundamental group is a topological invariant

In what follows we give a proof that the fundamental group is in fact a topological invariant,[1] though confining ourselves, as previously, to a proof that the fundamental group is a topological invariant of schemata. What we have to show is that, on the one hand, the fundamental group is independent of the choice of base point and "approach paths" $U_{g,i}$, and on the other hand, that it is the same for two homeomorphic schemata. When a schema is given, we therefore think first of the possible ways of choosing the basepoint and the approach paths $U_{g,i}$, and the effect this has on the above process for deriving the fundamental group. The different (i.e., nonisomorphic) groups obtained will be called the groups corresponding to the schema. We now prove that if Σ and Σ' are two schemata, one of which, Σ', results from an elementary subdivision of the other, Σ, then for each way of choosing the basepoint and approach paths, in one schema, the corresponding choice in the other yields isomorphic groups for Σ and Σ'. It follows that any two homeomorphic schemata have the same collections of groups.

[7] Called "contours fermés fondamentaux" by Poincaré (An. sit. §13, p.64).

[8] If one chooses the approach paths $U_{g,i}$ so that no closed paths result from the collection of S_{ν_h} comprising them [take a tree — Trans.] then one can take the defining relation system $\mathfrak{F}$ of the fundamental group to be simply the relations

$$s_{\lambda_1}^{\varepsilon_1} s_{\lambda_2}^{\varepsilon_2} \cdots s_{\lambda_\mu}^{\varepsilon_\mu} = 1$$

which result when one substitutes the expressions for the S_λ implied by (35) in the relations (34).

[1] In Poincaré's presentation this follows from the meaning of the fundamental group in terms of unbranched many-valued functions on the manifold.

Now it is easy to see that any connected n-dimensional schema homeomorphic to an n-dimensional fundamental polyhedron, i.e., a schema with only one n-dimensional cell, must by definition contain a cell a_i^n directly connected to any given n-dimensional cell a_j^n, i.e., a_j^n is such that at least one of its $(n-1)$-dimensional boundary cells is identified with an $(n-1)$-dimensional boundary cell of a_i^n. These two identified boundary cells of a_i^m and a_j^m represent an $(n-1)$-dimensional cell a_k^{n-1} of the schema. By the process inverse to elementary subdivision one can unite the cells a_i^n, a_j^n into a single cell along a_k^{n-1} and obtain a new schema from which the original is obtainable by elementary subdivision. If one continues this process, one finally obtains a fundamental polyhedron homeomorphic to the original schema. Now since essentially only one choice of basepoint is possible in a fundamental polyhedron, and approach paths do not appear, the latter has only a single group, namely that given by the relations (34) between the generators $s_\lambda = S_\lambda$. But then, for each schema homeomorphic to the fundamental polyhedron, the group collection must consist of this one group alone, and the independence of the fundamental group from its special construction in a schema, as well as from the choice of schema from its homeomorphism class is therefore proved. It also follows that, however the group is determined, the collection $\mathfrak{R}$ of relations described above is so constituted that it is the set of consequences of a finite relation system $\mathfrak{F}$.

We now go to the proof of the fact that when Σ' results from an elementary subdivision of Σ then for any group corresponding to Σ we can find an isomorphic group corresponding to Σ'.

The fundamental paths of Σ are denoted by S_λ, those of Σ' by S'_λ. The schema gives the relations (34) between the fundamental paths. The relations between the S_λ of Σ will be called the relations R, those for Σ' the relations R'. Approach paths will be called U for the first schema, U' for the second, and fundamental closed paths s_λ and s'_λ accordingly. The relations (35) between these will be called r for the first schema, r' for the second.

The elementary subdivision converting Σ into Σ' divides the l-dimensional cell a_r^l of Σ into two cells $a_{r_1}^l$, $a_{r_2}^l$ by means of a new $(l-1)$-dimensional cell a_s^{l-1}. The cases $l = 1, 2, \ldots, n-1$ may then be settled immediately. Namely, in all cases $l \leq n-2$ the subdivision does not change the collection of $(n-1)$-dimensional cells of the schema, and hence the corresponding paths S_k and S'_k in the two schemata correspond completely. Since the relations (34) between the fundamental paths depend on the $(n-2)$-dimensional cells of the schema, as long as these lie in the interior, and since these are not changed by the subdivision in the cases $l < n-2$, in these cases one obtains the relations R' from the relations R simply by replacing each S_λ by the corresponding S'_λ. In the case $l = n-2$ the relations R' result from the relations R, when they refer to $(n-2)$-dimensional cells other than a_r^{n-2}, by replacing each S_λ by the corresponding S'_λ. And in place of the the relation corresponding to the latter cell (such a relation is of course present only when a_r^{n-2} does not belong to the boundary of the manifold) we obtain two similar relations R' between the corresponding S'_λ. The choice of the basepoint and approach paths can be made the same for the two schemata when $l \leq n-2$, so that when such a choice is made the relations r' are obtained from r simply by replacement of S_λ, s_λ by S'_λ, s'_λ. Thus the claim is proved in the cases $l \leq n-2$, and the case $l = n-1$ now is scarcely more difficult.

Let S_ρ, S'_{ρ_1}, S'_{ρ_2} be the fundamental paths corresponding to the cells a_r^{n-1}, $a_{r_1}^{n-1}$, $a_{r_2}^{n-1}$. Each path S_λ $(\lambda \neq \rho)$ then corresponds to a path S'_λ and each S'_λ $(\lambda \neq \rho_1, \rho_2)$ to a path S_λ and the system of relations R' differs from the relations R in the appearance of a relation

$$S'_{\rho_1} S'^{-1}_{\rho_2} = 1,$$

which owes its existence to the new cell a_s^{n-2}, since every occurrence of the path S_ρ in a relation R becomes an occurrence of one of the paths S'_{ρ_1}, S'_{ρ_2} in a corresponding relation R', while all the remaining S_λ are replaced by the corresponding S'_λ. Now suppose a system of approach paths U' is chosen for the schema Σ', resulting in relations r'. Then the corresponding approach paths U will be chosen so that each S'_{ρ_1} or S'_{ρ_2} appearing in the expressions for the U' can be replaced by S_ρ, while the other S'_λ can be replaced by S_λ. One then sees that each relation r' corresponds to a relation r that results from the former by replacement of S'_λ, s'_λ $(\lambda \neq \rho_1, \rho_2)$ by S_λ, s_λ and of S'_{ρ_1} or S'_{ρ_2}, s'_{ρ_1} or s'_{ρ_2}, by S_ρ, s_ρ respectively. The two relations r' for s'_{ρ_1} and s'_{ρ_2} then correspond to the same relation r for s_ρ. Since the relation

$$s_{\rho_1} s_{\rho_2}^{-1} = 1$$

obviously holds, the isomorphism of the two groups is immediate. Conversely, if a system of approach paths U were given for Σ, then one could construct the corresponding approach paths U' for Σ' simply by replacing each S_ρ which appears by one or other of the paths S_{ρ_1} or S_{ρ_2}, arbitrarily.

It now remains to settle the case $l = n$. The schema Σ' then contains one point $M^{(i)}$ in addition to those present in the schema Σ, since a point

$M^{(r)}$ has been replaced by two points $M^{(r_1)}$, $M^{(r_2)}$. The schema Σ' contains a fundamental path S'_σ connecting the points $M^{(r_1)}$, $M^{(r_2)}$; it pierces the $(n-1)$-dimensional cell a_s^{n-1} and is positive, say, in the direction from $M^{(r_1)}$ to $M^{(r_2)}$. Apart from S'_σ, each fundamental path S'_λ has an obviously corresponding path S_λ. These corresponding paths S_λ and S'_λ may be divided into different categories. A first category comprises those S'_λ that begin and end at $M^{(r_1)}$. These paths will be indicated by a superscript α_1, thus by $S_\lambda^{(\alpha_1)'}$. The paths S_λ corresponding to them will be denoted $S_\lambda^{(\alpha_1)}$. These paths $S_\lambda^{(\alpha_1)}$ can be characterized as follows. The $(n-1)$-dimensional spherical manifold that forms the boundary of a_r^n is divided into two elementary manifolds $E_1^{(n-1)}$, $E_2^{(n-1)}$ by the $(n-2)$-dimensional boundary manifold of a_s^{n-1}. If one now considers each cell a_i^{n-1} of the schema that results from the pairing of two of the $(n-1)$-dimensional boundary cells of a_r^n in $E_1^{(n-1)}$, then the corresponding fundamental paths are none other than the $S_\lambda^{(\alpha_1)}$. Those paths that connect $M^{(r_1)}$ to a point $M^{(i)}$ different from $M^{(r_1)}$ or $M^{(r_2)}$ and for which the direction of $M^{(i)}$ to $M^{(r_1)}$ is chosen to be positive may be denoted $S_\lambda^{(\beta_1)'}$. Analogously, let $S_\lambda^{(\alpha_2)'}$ be the paths that connect $M^{(r_2)}$ to $M^{(r_2)}$, and let $S_\lambda^{(\beta_2)'}$ be those that connect $M^{(r_2)}$ to a point $M^{(I)}$, $(i \neq r_1, r_2)$, the latter being taken to be positive in the direction from $M^{(r_2)}$ to $M^{(i)}$. In addition, let $S_\lambda^{(\gamma)'}$ be the paths other than S'_σ connecting $M^{(r_1)}$ to $M^{(r_2)}$, taken positive in the direction $M^{(r_2)}$ to $M^{(r_1)}$, and let $S_\lambda^{(\delta)'}$ be the paths that connect two points other than $M^{(r_1)}$ and $M^{(r_2)}$. The paths of Σ corresponding to those just defined will be denoted by $S_\lambda^{(\beta_1)}$, $S_\lambda^{(\alpha_1)}$, $S_\lambda^{(\beta_2)}$, $S_\lambda^{(\gamma)}$, $S_\lambda^{(\delta)}$.

We now set

$$S_\lambda^{(\alpha_1)'} = T_\lambda^{(\alpha_1)}, \quad S'_\sigma S_\lambda^{(\alpha_2)'} S_\sigma'^{-1} = T_\lambda^{(\alpha_2)},$$
$$S_\lambda^{(\beta_1)'} = T_\lambda^{(\beta_1)}, \quad S'_\sigma S_\lambda^{(\beta_2)'} = T_\lambda^{(\beta_2)},$$
$$S_\lambda^{(\delta)'} = T_\lambda^{(\delta)}, \quad S'_\sigma S_\lambda^{(\gamma)'} = T_\lambda^{(\gamma)}.$$

We can then say that one obtains the relations R' from the relations R simply by replacing each path $S_\lambda^{(\varepsilon)}$ occurring in a relation R, where ε is one of the indices α_1, α_2, β_1, β_2, γ, δ by the expression $T^{(\varepsilon)}$. Conversely, if, the system of relations R' between the paths $S_\lambda^{(\alpha_1)'}$, $S_\lambda^{(\alpha_2)'}$, $S_\lambda^{(\beta_1)'}$, $S_\lambda^{(\beta_2)'}$, $S_\lambda^{(\gamma)'}$, $S_\lambda^{(\delta)'}$, S'_σ is given, then one easily sees that each relation R' must be expressible as a relation between the expressions $T_\lambda^{(\varepsilon)}$, i.e., when one expresses each $S_\lambda^{(\varepsilon)'}$ in terms of the corresponding $T_\lambda^{(\varepsilon)'}$ and S'_σ then S'_σ cancels out of the relation. The relations R then result from the relations R' when one substitutes the $S_\lambda^{(\varepsilon)}$ for these $T_\lambda^{(\varepsilon)}$.

The question we are dealing with is, given a choice of basepoint and approach paths for one of the schemata Σ, Σ', determine a suitable choice for the other. Suppose for example that the basepoint $M^{(g')}$ and the approach paths $U'_{g',i}$ are given for Σ'. Then the basepoint $M^{(g)}$ and the paths $U_{g,i}$ for Σ may be determined as follows. If g' is different from r_1 and r_2, so that the cell of Σ' containing $M^{(g')}$ is an unmodified cell of Σ, then the basepoint $M^{(g)}$ for Σ may be chosen in this cell, so $g = g'$. If on the other hand $M^{(g')}$ is one of the points $M^{(r_1)}$, M^{r_2} we can then suppose that the notation has been chosen so that $M^{(g')} = M^{(r_1)}$,[2] and set $M^{(g)} = M^{(r)}$. Now if $M^{(i)}$ is different from $M^{(r_1)}$ and $M^{(r_2)}$ then it is easy to see that the approach path $U_{g,i}$ is constructed from $U'_{g',i}$ by writing it in terms of the $T^{(\varepsilon)}$ alone and then replacing each $T^{(\varepsilon)}$ by the corresponding $S^{(\varepsilon)}$. The same process is used to derive $U_{g,r}$ from U'_{g',r_1}. However the approach path U'_{g',r_2}, which will also be denoted V', corresponds to no approach path of the schema Σ.

Conversely, if a basepoint $M^{(g)}$ and approach paths $U_{g,i}$ of Σ are given then, when $g \neq r$, the basepoint $M^{(g')}$ of Σ' may be set equal to $M^{(g)}$ and, when $M^{(g)} = M^{(r)}$, $M^{(g')} = M^{(r_1)}$. The approach paths $U'_{g',i}$ $(i \neq r_1, r_2)$ and U'_{g',r_1} may be constructed by replacing each $S_\lambda^{(\varepsilon)}$ in $U_{g,i}$ or $U_{g,r}$ by $T_\lambda^{(\varepsilon)}$. The approach path U'_{g',r_2} is taken to be an arbitrary path V' from $M^{(g')}$ to $M^{(r_2)}$.[3]

Our conventions are so arranged that obtaining the basepoint and approach paths for the derived schema from those for the original schema and conversely are quite similar processes, regardless of which schema provides them in the first place. The comparison of the relations r and r' can therefore be made without having to distinguish the two cases.

Since, with the exception of S'_σ, there is a one-one correspondence between the paths S_λ and s'_λ of Σ and Σ', the same is also true for the closed paths s_λ and s'_λ with the exception of the path

$$s'_\sigma = U'_{g',r_1} S'_\sigma V'^{-1}. \tag{36}$$

We consider each of the relations r and r' introduced by corresponding paths

[2]The way of introducing the expressions $T_\lambda^{(\varepsilon)}$ under which $M^{(r_1)}$ and $M^{(r_2)}$ indeed do not play the same role, has been arranged with this in mind.

[3]E.g., one can set $V' = U'_{g',r_1} S'_\sigma$. However in order to avoid separate consideration of the relations r and r' in the two cases where basepoint and approach paths are given, first for Σ, then for Σ', we shall not specialize V' further.

s_λ, s'_λ. Suppose for example $S_\lambda^{(\beta_2)'}$ is a path from $M^{(r_2)}$ to $M^{(i)}$ $(i \neq r_1, r_2)$, and $S_\lambda^{(\beta_2)}$ is the corresponding path from $M^{(r)}$ to $M^{(i)}$. Then the relations determined by $s_\lambda^{(\beta_2)}$ and $s_\lambda^{(\beta_2)'}$ are:

$$s_\lambda^{(\beta_2)} = U_{g,r} S_\lambda^{(\beta_2)} U_{g,r}^{-1}$$

$$s_\lambda^{(\beta_2)'} = U'_{g',r_2} S_\lambda^{(\beta_2)'} U'^{-1}_{g',i}.$$

Now if we set

$$s_\lambda^{(\alpha_1)'} = t_\lambda^{(\alpha_1)}, \quad s'_\sigma s_\lambda^{(\alpha_2)'} s'^{-1}_\sigma = t_\lambda^{(\alpha_2)},$$

$$s_\lambda^{(\beta_1)'} = t_\lambda^{(\beta_1)}, \quad s'_\sigma s_\lambda^{(\beta_2)'} = t_\lambda^{(\beta_2)},$$

$$s_\lambda^{(\delta)'} = t_\lambda^{(\delta)}, \quad s'_\sigma s_\lambda^{(\gamma)'} = t_\lambda^{(\gamma)},$$

then the relations in question can also be written in the form

$$s_\lambda^{(\beta_2)} = U_{g,r} S_\lambda^{(\beta_2)} U_{g,i}^{-1},$$

$$t_\lambda^{(\beta_2)} = U'_{g',r_1} T_\lambda^{(\beta_2)} U'^{-1}_{g',i},$$

and one sees that the second of these relations is obtained from the first when $s_\lambda^{(\beta_2)}$ is replaced by $t_\lambda^{(\beta_2)}$ and each $S_\lambda^{(\varepsilon)}$ by $T_\lambda^{(\varepsilon)}$.

Quite generally one obtains the relation r' by applying to relation r the operation Π that replaces $S_\lambda^{(\varepsilon)}$, $s_\lambda^{(\varepsilon)}$ by $T_\lambda^{(\varepsilon)}$, $t_\lambda^{(\varepsilon)}$ and adds the relations (36). Conversely, if the relations r' are given one can think of them being written in such a form that, apart from (36), every other relation r' has a $t_\lambda^{(\varepsilon)}$ on the left hand side, and on the right an expression in the $S_\lambda'^{(\varepsilon)}$ and S'_σ which can be expressed in terms of the $T_\lambda^{(\varepsilon)}$. One obtains the corresponding relation r from such an r' by the operation Π^{-1}.

In summary we can say: the collection ρ' of the relations R' and r' is obtained from the collection ρ of relations R and r by applying the operation Π to each relation ρ and adding the relations (36). The desired proof that the relation system ρ' for the schema Σ' defines a group Γ isomorphic to the group Γ defined by the relation system ρ for Σ is now easy to complete. The generators of Γ are the elements $s_\lambda^{(\varepsilon)}$ and Γ is characterized by the fact that the relations between these elements are just those which follow from the relations ρ (by elimination of the S_λ). The relations ρ' have the analogous meaning for the group Γ', whose generators are s'_σ and the $S_\lambda^{(\varepsilon)'}$. However we obviously can and will regard s'_σ and the $t_\lambda^{(\varepsilon)}$ as generators of Γ'. It is now easy to see that s'_σ, can be expressed in terms of the $t_\lambda^{(\varepsilon)}$. Namely, the paths S' appearing on the right hand side of the expression (36) for s'_σ represent a closed path beginning and ending at $M^{(g')}$, and this expression must therefore be representable in terms of the $T_\lambda^{(\varepsilon)}$, so that one may write

$$s'_\sigma = T_{\mu_1} T_{\mu_2} \cdots T_{\mu_\nu}.$$

The T_{μ_1} here are themselves paths between two points $M^{(i)}$ of Σ' (and in fact in such a way that none of these points is the point $M^{(r_2)}$) such that the final point of each $T_{\mu_{i-1}}$ coincides with the initial point of T_{μ_i}, and the initial point of T_{μ_1}, as well as the final point of T_{μ_ν}, is the point $M^{(g')}$. Consequently, the formulae

$$T_\lambda^{(\varepsilon)} = U'_{g',j} t_\lambda^{(\varepsilon)} U'_{g',k},$$

where $M^{(i)}$, $M^{(k)}$ are initial and final point of $T_\lambda^{(\varepsilon)}$, give the equation

$$s'_\sigma = t_{\mu_1} t_{\mu_2} \cdots t_{\mu_\nu}. \tag{37}$$

The collection $\mathfrak{R}'$ of relations between the $t_\lambda^{(\varepsilon)}$ and s'_σ, which includes (37), is therefore equivalent to the set $\overline{\mathfrak{R}'}$ of relations obtained when one replaces s'_σ by $t_{\mu_1} t_{\mu_2} \cdots t_{\mu_\nu}$ in each relation other than (37). And when this has been done, s'_σ can be omitted from the generators of Γ'. Γ' is therefore generated by the elements $t_\lambda^{(\varepsilon)}$, and is characterized by the relations between them, which are derivable from the relations of ρ'. And it is now obvious that for each relation between the generators $s_\lambda^{(\varepsilon)}$ of Γ we can construct the same relation between the generators $t_\lambda^{(\varepsilon)}$ of Γ', and conversely. Namely, let a relation between the $s_\lambda^{(\varepsilon)}$ be given, which may be written as a consequence relation of the relations ρ in the form

$$L_1 A_{i_1} L_2 A_{i_2} L_3 \cdots A_{i_r} L_{r+1} = 1,$$

where

$$A_1 = 1, \quad A_2 = 1, \quad \ldots$$

is the system of relations ρ and their inverses, and $L_1, L_2, \ldots$ are expressions in the $S_\lambda^{(\varepsilon)}$ and $s_\lambda^{(\varepsilon)}$ identically satisfying the condition

$$L_1 L_2 \cdots L_{r+1} = 1.$$

Then if A'_i, L'_i denote the expressions that result from A_i, L_i respectively by application of the operation Π, the relations

$$A'_1 = 1, \quad A'_2 = 1, \quad \ldots$$

together with (37) and the relations inverse to (37) are obviously none other than the relations ρ' and their inverses. The relation

$$L'_1 A'_{i_1} L'_2 A'_{i_2} L'_3 \cdots A'_{i_r} L'_{r+1} = 1$$

now obviously represents a consequence of the relations ρ' containing only the $t_\lambda^{(\varepsilon)}$, and it is obtainable from the given relation by the operation Π. Conversely, if a consequence of the relations ρ' is given that contains only the $t_\lambda^{(\varepsilon)}$, then one can obviously assume that it is derived without use of (37) and therefore expressible in the form

$$L'_1 A'_{i_1} L'_2 A'_{i_2} L'_3 \cdots A'_{i_r} L'_{r+1} = 1,$$

where the L'_i are expressions in the $t_\lambda^{(\varepsilon)}$, $T_\lambda^{(\varepsilon)}$. The operation Π^{-1} then leads to the corresponding consequence of the relations ρ.

The isomorphism of Γ and Γ' is therefore proved, and the proof that the fundamental group is a topological invariant is complete.

§14
Determination of P_1 and the first order torsion numbers from the fundamental group

If one considers the process given at the outset for determining the fundamental group in the special case of a closed manifold, then one sees that it is related to the construction of the dual schema. Each fundamental path S_λ that connects the points $M^{(i)}$, $M^{(j)}$ in the interior of the cells a_i^n, a_j^n and passes through a point N of a cell a_k^{n-1} in the process, can be regarded as the edge $\overline{a_k^1}$ connecting the vertices $M^{(i)} = \overline{a_i^0}$, $M^{(j)} = \overline{a_j^0}$ of the dual schema. Then if

$$S_{\lambda_1}^{\varepsilon_1} S_{\lambda_2}^{\varepsilon_2} \cdots S_{\lambda_\mu}^{\varepsilon_\mu}$$

is the relation that follows from consideration of the $(n-2)$-dimensional cell $A = a_l^{n-2}$, one can also express this as follows: when $\overline{a_l^2}$ is the wall in the dual schema corresponding to the cell a_l^{n-2} then the perimeter of $\overline{a_l^2}$ is described by the sequence of edges of the dual schema represented by the S_{λ_i}, each in the direction given by ε_i.

Thus one obtains the fundamental group of a closed manifold by constructing the schema $\overline{\Sigma}$ dual to Σ, and taking the closed perimeters of walls to obtain the relations (34) between the edges denoted by S_λ, then using a vertex of $\overline{\Sigma}$ as basepoint and taking approach paths along the edges to the remaining vertices to get closed paths s_λ to serve as generators of the fundamental group, and to obtain the defining relations between them. But since the fundamental groups of homeomorphic schemata are isomorphic, one can obviously apply this process to the original schema Σ instead of $\overline{\Sigma}$. This process is particularly useful for the actual determination of the fundamental group of a manifold with a given schema.

If one applies this process to the example of the schema σ_3 of the spherical three-dimensional manifold, then one has in this case two edges S_1, S_2 from the vertex a_1^0 to the vertex a_2^0, and circuits around both walls yield the same relation $S_1S_2^{-1} = 1$. If we now take the path U_{12} as approach path from a_1^0 to a_2^0, then one obtains the closed fundamental paths $s_1 = S_1S_2^{-1}$ and $s_2 = 1$ and hence the defining relations $s_1 = s_2 = 1$, which show that the fundamental group is the identity. For the first schema (denoted by Σ) of the projective three-dimensional manifold T_3 in §9 one obtains the relation $s_1^2 = 1$ for one of the edges $S_1 = a^1$, and since the schema has only one vertex, $S_1 = s_1$ is the single generator of the fundamental group, which is therefore the cyclic group of order 2.

One is easily convinced that the process just given also applies to bounded manifolds with proper boundaries. If one considers, e.g., the schema of an annulus as a rectangle $ABCD$ in which the sides AD, BC are identified: $A = B = a_1^0$, $C = D = a_2^0$, $AB = S_1$, $CD = S_2$, $AD = BC = S_3$, $U_{12} = S_3$, $s_1 = S_1$, $s_2 = S_3S_2S_3^{-1}$, $s_3 = 1$. So one obtains the relation $S_1S_2S_2S_3^{-1} = 1$ and hence for the fundamental group the relations $s_1s_3s_2s_3^{-1} = 1$, $s_3 = 1$, which define the infinite cyclic group.

The above process now affords a simple means of deriving the first Betti number P_1, and also the torsion numbers of first order from the fundamental group.

For this purpose we think of the manifold being based on a schema for which the number α_0 of vertices is 1. Such a schema may always be found, since one need only go from a fundamental polyhedron to its dual schema. Now in such a schema the closed fundamental paths coincide with the fundamental paths S_λ represented by the edges of the schema. The relations derived from the walls then already represent the system of defining relations between the generators S_λ of the fundamental group. Now let S_j be the fundamental path represented by the edge a^1_j and let

$$S_1^{k'_{i1}} S_2^{k'_{i2}} \cdots S_n^{k'_{in}} S_1^{k''_{i1}} S_2^{k''_{i2}} \cdots = 1$$

be the relation derived from the perimeter of the wall a^2_i. Then $k'_{ij}+k''_{ij}+\cdots$ is obviously none other than the coefficient ε^2_{ij} of the Poincaré relation system for the schema in question. Since α_1 is the number of generators of the fundamental group, γ_2 is the rank of the matrix of the numbers $\varepsilon^2_{ij} = k'_{ij} + k''_{ij} + \cdots$, hence the characteristic number of the fundamental group denoted by ζ in §11 is equal to $\alpha_1 - \gamma_2 = P_1 - 1$. One has only to recall that $\gamma_1 = \alpha_0 - P_0$ for the schema in question.

The Betti number P_1 is therefore equal to the number ζ from the fundamental group, plus one.[1]

But the meaning of the matrix of the ε^2_{ij} for the relation system of the fundamental group also yields the theorem.

The torsion numbers of first order of a connected manifold are the Poincaré numbers of its fundamental group.

The proof of this theorem for the case of bounded manifolds with proper boundary may be omitted. For manifolds with improper boundary manifolds (i.e., those of dimension less than $n-1$), for which a definition of Betti and torsion numbers has not been given, the theorem in question may be taken as a definition of these numbers.

Now for closed n-dimensional manifolds the Betti numbers and torsion numbers are the only known topological invariants apart from the fundamental group. The number N and the numbers Q_m of §3 of course depend on them. The above theorem therefore shows that, since $P_1 = P_2$ for two-sided closed three-dimensional manifolds, all the known topological invariants derive from the fundamental group in this case.

On the other hand, since Poincaré has shown[2] that there are manifolds with the same Betti and torsion numbers but different fundamental groups, the fundamental group serves as a better characterization of two-sided closed three-dimensional manifolds than the topological invariants previously known. However, one qualification must be made to this statement. While the equality of two series of numbers can always be effectively decided,

[1] Cf. Poincaré, An.Sit. p.65.
[2] Compl. 5

the question whether two groups are isomorphic (cf. §11) is not solvable in general. Thus in contrast to the other topological invariants, the fundamental group is one whose agreement or disagreement for two manifolds cannot be decided in all cases.

V. Theorems and problems on developability and transformations

§15
Developable manifolds

While the previous discussion has dealt mainly with multidimensional manifolds in terms of their schemata, and is thus to be counted as combinatorial analysis situs, in this section the concept of a point manifold will prevail. However, this does not mean going to the most general point sets, for which these theorems are not valid, and the examples discussed often call on intuition. Thus in this Section V we are merely hinting at ways some important questions of analysis situs may be settled.

A manifold W is said to be developable on a manifold V when W has the same dimension as V and is homeomorphic to a part of the manifold V or to the manifold V itself. Manifolds are simply called developable if they are developable on a spherical manifold.[1] A manifold W developable on a manifold V will also be called a submanifold of V, and a proper submanifold when there is a proper part of V homeomorphic to W. If one chooses, from the collection of all submanifolds of V, a complete system of nonhomeomorphic manifolds, then this system may be denoted by $\Delta(V)$. The notation for a corresponding system of proper submanifolds of V is $\Delta^*(V)$. Obviously we can have $\Delta(V) = \Delta^*(V)$ (e.g., for a disc or an annulus) only for bounded manifolds. If S_n denotes the n-dimensional spherical manifold, then Δ_n denotes the set $\Delta(S_n)$ of all developable n-dimensional manifolds and Δ_n^* denotes the set $\Delta^*(S_n)$ of all bounded developable n-dimensional manifolds. We have the equation $\Delta(E_n) = \Delta^*(E_n) = \Delta^*(S_n)$ where E_n is the n-dimensional element, whence it follows that, when V is any n-dimensional manifold, Δ^* is contained in $\Delta(V)$ and $\Delta^*(V)$.

Two manifolds U, V will be called coextensive when $\Delta^*(U) = \Delta^*(V)$. U will be called subordinate to V, and V superordinate to U, when $\Delta^*(U)$ is a

[1]This terminology is due to Poincaré (Compl. 5, §5, p.90)

proper subset of $\Delta^*(V)$. When U, V are manifolds of the same dimension and the common elements of $\Delta^*(U)$, $\Delta^*(V)$ form a proper subset of both $\Delta^*(U)$ and $\Delta^*(V)$, then U, V may be called incomparable. The ordering implied by these relations satisfies the well-known properties of partial orders. E.g., two manifolds which are coextensive with a third are coextensive with each other, so that one can divide the manifolds into coextensivity classes. One such class consists of all manifolds in Δ_n. Each manifold developable on a manifold V is either coextensive with, or subordinate to, V. The projective plane T_2 and the two-sided closed surface of genus $p = 1$ constitute an example of an incomparable pair of manifolds. It is not improbable that in three dimensions there are already incomparable pairs among the two-sided manifolds.

When the manifold U is superordinate to, subordinate to, or incomparable with the manifold V, the same may be said of the systems of manifolds coextensive with U and V. A system S of coextensive manifolds is called μ-tuply superordinate to another system T when it is possible to find $\mu - 1$ and no more systems $S_1, S_2, \ldots, S_{\mu-1}$ of coextensive manifolds such that S_i is superordinate to S_{i+1}, S to S_1 and $S_{\mu-1}$ to T.

When the system S is μ-tuply superordinate to the system Δ_n we call μ the order of the system S and each of manifolds belonging to it. The genus p of two-sided surfaces may be taken as an example. All surfaces of the same genus constitute a system of coextensive manifolds, and the system of surfaces of genus $p + q$ is q-tuply superordinate to the system of surfaces of genus p.

In the order relation just defined we have a topological invariant (the order) of which we admittedly know nothing in the case of more than two dimensions. We do not even have an overview of the collection Δ_3 of developable three-dimensional manifolds beyond a few simple cases. Consider for example the developable manifolds bounded by a single surface of genus 1. The simplest example of such a manifold is the part of $\mathfrak{R}_3$ bounded by a torus surface. The fundamental group of this manifold is the infinite cyclic group. One obtains a manifold homeomorphic to this one by boring a cylindrical canal out of a ball. But if one were to take instead a knotted canal as shown in Fig. 3, the fundamental group of the resulting manifold has two generators with the defining relation $sts = tst$, so that this manifold cannot be homeomorphic to the former.[2]

[2] Cf. my note in the Wr. Ber., point 3.

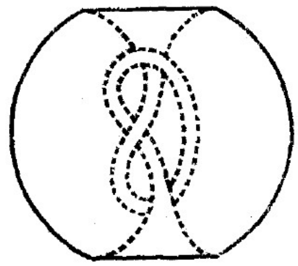

Fig. 3.

If one were to take the canal knotted in a more complicated way, then the manifolds obtained would be different again. The manifolds obtained in this way can also be viewed as the result of taking a closed line L, knotted arbitrarily, in the spherical three-dimensional manifold, letting a ball move along this line, and removing the space swept out. Whether or not two such manifolds can be homeomorphic only when the line L for one manifold is knotted "in the same way" as the line L for the other manifold or its mirror image (so that one line can be deformed into the other or its mirror image) is not investigated. Indeed, I do not know how to prove that any developable three-dimensional manifold bounded by a single surface of genus 1 is homeomorphic to one of the manifolds derived in the above manner, so that the manifolds representable in this way exhaust all manifolds bounded by a surface of genus 1.

It may be remarked at this point that two-dimensional developable manifolds have a characteristic property that in higher dimensions can also occur for other than developable manifolds. It is obvious that each developable n-dimensional manifold is separated by each closed $(n-1)$-dimensional manifold in its interior. Conversely, each two-dimensional manifold which is separated by each closed line is also developable. This converse is already incorrect for $n = 3$, as witnessed by Poincaré's example (Compl. 5, §6) of a closed three-dimensional manifold V with Betti number $P_2 = 1$, which shows that every closed surface separates V. But V is obviously not developable, since V is closed and different from the spherical manifold.

We now make a few remarks about n-dimensional manifolds that possess only improper boundary manifolds.[3] Such a manifold W is defined by an n-dimensional schema in which the points of certain m-dimensional cells

[3]See §2, note 4. The expression boundary manifold in the present section refers to the "complex" consisting of the boundary points (which is thus not a manifold in the sense of Section I).

($m = 0, 1, \ldots, n-2$) are declared to be omitted. Apart from these omissions, one has the schema of a closed manifold V on which W is developable. It is now possible to give a few theorems for the case $n = 3$ showing that, among the manifolds with improper boundary developable on the same closed manifold V, there are some that are homeomorphic despite quite different appearance and being defined by nonhomeomorphic schemata.

Suppose for example that the schema of the manifold W is obtained from that of the closed three-dimensional manifold V by omitting a number of edges from the schema of V, without removing any isolated vertices (i.e., vertices other than those on the edges removed from V). The collection of these edges constitutes a so-called one-dimensional complex K, and we shall assume it to be connected, so that one can travel from any edge a of the complex to any other along edges of K itself. A vertex of the schema V is called a free endpoint of the complex K when it lies in only one edge of the complex, and in this it is an endpoint. We let $V(K)$ denote the bounded manifold that results from V by removing the one-dimensional complex K. One can now prove the theorem.

If K is a connected one-dimensional complex, then the three-dimensional manifold $V(K)$ is homeomorphic to a manifold $V(L)$ for which L either has no free endpoints or consists of a single edge.

Namely, if k is an edge of K with a free endpoint A, while the other endpoint B lies on other edges of K, then one takes A as the midpoint of a small ball and lets this ball move in V so that its midpoint describes the edge $k = AB$, while its radius tends to zero continuously as it approaches B. The piece of V swept out by the variable ball then is a simply connected space R enclosing the edge k, with A in its interior and B on its boundary. The moving ball can be made so small in each position that R has no points in common with K outside k. Let R_1 denote the manifold obtained from R (including its boundary points) by omitting the points of k (including A, B). Let R_2 denote the collection of all points of R with the exception of B, and let S be the collection of all boundary points of R with the exception of B.

It is now possible to map R_1 and R_2 one-one and continuously onto each other so that each point of S is fixed. In order to see this one thinks of R being deformed into the point manifold M defined by the inequalities

$$x^2 + y^2 + z^2 \leq 4, \quad z \geq 0$$

in such a way that the points A, B and the edge k go to the points $A' = (0, 0, 1)$, $B' = (0, 0, 0)$ and the segment $A'B'$. Let M_1, M_2, N denote the manifolds to which this map sends R_1, R_2, S. Now the formulae

$$\rho = \rho', \quad \varphi = \varphi', \quad z = \frac{z'}{\sqrt{\rho'^2 + z'^2}} + \frac{z'}{2},$$

where $\rho = \sqrt{x^2 + y^2}$, $\varphi = \arctan \frac{y}{x}$ and ρ, φ, z' and ρ', φ', z' are the coordinates on M_1 and M_2 respectively, define a one-one continuous map t of M_1 onto M_2 that leaves the points of N fixed, and therefore utu^{-1} is the desired mapping of R_1, onto R_2, where u denotes the transformation of R into M. But the possibility of this mapping shows that $V(K)$ is homeomorphic to $V(K_1)$ when K_1 denotes any complex that results from K by omitting an edge such as k. Repeated application of such mappings removes all edges of the complex with one free endpoint.[4]

Another mapping, also showing that different complexes K, L lead to homeomorphic manifolds $V(K)$, $V(L)$, is obtained as follows. One removes an edge $k = AB$ from a complex K in which at least two other edges of K end at A as well as B. One now introduces a small moving ball so that its midpoint describes k and its radius tends to 0 on approaching A as well as B. We think of the space R swept out by the ball being mapped onto the cylindrical space M defined by

$$\rho^2 = x^2 + y^2 \leq 1, \quad -1 \leq z \leq +1,$$

and in such a way that the edge k corresponds to the piece of the z-axis in M. Let R_1 denote the points of R minus k, and let M_1 consist of the points of M for which $\rho > 0$. We set $\varphi = \arctan \frac{y}{x}$. Then the mapping of the points of M_1, given by the formulas

$$\rho' = \rho, \quad \varphi' = \varphi, \quad z' = \rho z,$$

onto the set of points

$$0 < \rho'^2 \leq 1, \quad z'^2 \leq \rho'^2$$

shows the possibility of mapping $V(K)$ onto $V(L)$ one-to-one and continuously, where L is the complex which results from K by continuously contracting k until A and B coincide. By mappings of an inverse character one can

[4] If the process leads finally to $V(L)$, where L consists of a single edge with two free endpoints, then a further application of the mapping described leads to a manifold $V(B)$ that results from V by omitting a single point B.

show that when K contains a vertex e in which r (≥ 4) edges $k_1, k_2, \ldots, k_r$ end, then $V(K)$ is homeomorphic to a manifold $V(K')$, where K' results from K by replacing e by a new edge k connecting vertices e_1, e_2 such that $k_1, k_2, \ldots, k_s, k$ end in e_1 and $k_{s+1}, \ldots, k_r, k$ end in e_2. Thus one obtains the theorem:

Each three-dimensional manifold $V(K)$ is homeomorphic to a manifold $V(L)$ such that at most three edges of L end at each of its vertices.[5]

Thus, e.g., the two manifolds $V(K)$ and $V(L)$ are homeomorphic when one chooses V to be the three-dimensional spherical manifold S_3, which one can think of as being represented by euclidean space with a point at infinity, and takes K to be two touching circles in a plane α of S_3 and takes L to be the set of points of two disjoint circles in α together with a connecting segment.

One can collect all the one-dimensional complexes K in a manifold V into classes consisting of those that lead to homeomorphic $V(K)$. If one chooses V to be the spherical manifold S_3, then one can ask in particular, for each closed line in S_3, what other lines belong to the same class. If, as above, one says lines in S_3 are knotted "the same way" when they may be deformed into each other in S_3 (for the concept of deformation see §16),[6] then it is clear that $S_3(L_1)$ and $S_3(L_2)$ are homeomorphic when L_1 is knotted the same way as L_2 or its mirror image, and the same is true when L_1, L_2 are two systems of closed lines linked in the same, or mirror image, fashion. The question arises, whether conversely, if two closed lines L_1, L_2 (or systems of closed lines) in S_3 belong to the same class then L_1 always knotted (linked) the same way as L_2 or its mirror image.[7])

It may also be remarked that each three-dimensional[8] manifold W with

[5]The schemata of the two manifolds in the above theorems are in general not homeomorphic. Thus, due to the fact that improper boundary manifolds appear, the theorem on the homeomorphism of schemata of homeomorphic manifolds is no longer valid (see §2, note 9). The situation is otherwise for proper boundary manifolds, since it is stipulated that their points are to be counted among those of the manifold itself (§2, note 4).

[6]The concept of the "same way" for linking of lines of for two one-dimensional complexes in general is defined analogously.

[7]In my note already cited I have answered this question affirmatively, but under the mistaken assumption (as the details in the text show) of the theorem that the homeorphism of $S_3(K)$ and $S_3(L)$ for any one-dimensional complexes K, L implies that K is knotted in the same way as L or its mirror image.

[8]The same holds for two-dimensional manifolds with isolated boundary points, as one may easily convince oneself.

improper boundary manifolds only — which may be obtained from the closed manifold V by removal of certain edges and vertices — is homeomorphic to a manifold U developable on V with proper boundary manifolds, although these boundary points are not to be counted as belonging to U.[9]

Namely, let A be an excluded isolated vertex. One then surrounds A with a small ball of radius 2ρ. If r, φ, ϑ are polar coordinates with A as origin, then the transformation

$$\varphi' = \varphi, \quad \vartheta' = \vartheta, \quad r' = \frac{1}{2}r + \rho$$

maps the collection of points $0 < r \leq 2\rho$ one-to-one and continuously on to the collection of points $\rho < r' \leq 2\rho$. The boundary point A is therefore replaced by the boundary surface $r = \rho$. If we are dealing with a one-dimensional complex K as the improper boundary manifold, then to replace it by a proper boundary manifold, one must first consider the points at the ends of edges. A small ball of radius 2ρ is centred on each such point B, and we assume that the edge ending at B runs linearly through this ball, as can indeed be arranged by a deformation. If we again apply the above transformation to the points $0 < r \leq 2\rho$, then the endpoint of each edge is replaced by a ball of radius ρ whose points are excluded from the manifold. Each single edge k of K now connects two points P, Q, each lying in one of these balls σ. Let the manifold obtained, homeomorphic to W, be W_1. Now if one lets a small ball move so that its centre describes the edge k, then the part of W_1 swept out is a small cylindrical piece, from which the points of the line k are excluded, and it may be mapped one-to-one and continuously onto the manifold M defined by

$$0 < R^2 = x^2 + y^2 \leq 4, \quad -1 < z < +1.$$

If we set $\varphi = \arctan \frac{y}{x}$ and consider the mapping

$$R' = \frac{1}{2}R + 1, \quad \varphi' = \varphi, \quad z' = z$$

of M onto the manifold $1 < R' \leq 2$, $-1 < z' < +1$, then it is clear that the boundary edge $k = PQ$ of W_1 has been replaced by a cylindrical surface,

[9]Cf. notes 4 and 5 of §2. Despite the fact that all improper boundary manifolds can be replaced by proper ones, it is nevertheless useful, to consider the improperly bounded manifolds, since, e.g., they play a role in the form of representation discussed in §18.

whose points are not counted in the manifold. The balls σ together with these cylindrical surface pieces yield as many closed boundary surfaces as there were connected one-dimensional complexes excluded from W.

Finally it may be mentioned that the developability concept can be extended by saying that a manifold W that is homeomorphic to a part of V is also developable on V when W has lower dimension than V. One can go even further and consider the developability of arbitrary point sets on an n-dimensional manifold V or on another point set. Thus, e.g., when any point set P is given one can ask what is the smallest dimension number m for which P is developable on an elementary manifold E_m.[10]

§16
Self-transformations and deformations of manifolds[1]

A one-to-one and continuous map of a manifold V onto itself will be called a self-transformation of the manifold. Let t', t'' be two self-transformations of V, P a point of V, and P', P'' its images under t', t'' respectively. We shall say that the two transformations differ by less than ε when the distance between the image points P', P'' is less than ε whatever the choice of the point P.[2] Two transformations t_1, t_2 of V will be said to be of the same type when there is a sequence of transformations $t(a)$, so that for each value of the parameter a, $0 \leq a \leq 1$, there is a transformation $t(a)$, $t(0) = t_1$ and $t(1) = t_2$, and $t(a)$ depends continuously on a. By this we mean that for any a_0 $(0 \leq a_0 \leq 1)$ and each $\varepsilon > 0$ there is a δ such that for $|a - a_0| < \delta$ the transformations $t(a_0)$ and $t(a)$ differ by less than ε. The transformations of the same type as the identity will be called deformations of V.

We illustrate these definitions, which may also be carried over from manifolds to arbitrary point sets,[3] briefly with the example of the two-sided closed two-dimensional manifold of genus 1, which we can think of as being realized by a torus surface. In this case intuition gives us a clear survey of the state

[10] The question raised in §2, note 13 may be placed in this general area.

[1] The relations about to be discussed between the self-transformations of a manifold and isomorphisms of the fundamental group have already been mentioned in my note cited above.

[2] It has already been mentioned in §1 that in applying the concept of manifold and continuous mappings to the analysis situs of point manifolds, the notion of distance is decisive.

[3] The definition given for continuous manifolds in my note obviously agrees with the definition of deformation given in the text.

of affairs. If x denotes a meridian circle and y a latitude circle on the torus, each provided with an orientation, and if ω_x, ω_y denote the periods of an elliptic integral of the first kind over the surface, taken along the paths x, y respectively, then each deformation of the surface carries x into a simple path x' and y into a simple path y' such that the periods of the integral along x' and y' are again ω_x and ω_y. However, the periods $\omega_{x'}$, $\omega_{y'}$ along paths x', y' resulting from x, y by an arbitrary self-transformation of the surface are given by equations of the form

$$\omega_{x'} = \alpha \cdot \omega_x + \beta\omega_y$$

(38)

$$\omega_{y'} = \gamma \cdot \omega_x + \delta\omega_y$$

where α, β, γ, δ are integers for which $\alpha\delta - \beta\gamma$ equals $+1$ or -1. Conversely, for each quadruple of integers α, β, γ, δ of determinant $+1$ or -1 there is a transformation of the surface for which the periods $\omega_{x'}$, $\omega_{y'}$ are given by (38), and all self-transformations with the same values of the numbers α, β, γ, δ are of the same type. If one provides the surface with an orientation, say by an indicatrix, then a transformation of the surface either preserves this orientation or reverses it according as $\alpha\delta - \beta\gamma$ equals $+1$ or -1. The group T^* of all self-transformations of the surface that preserve orientation is a distinguished subgroup of index 2 of the group T of all self-transformations of the surface. The group D of all deformations is a distinguished subgroup of T as well as of T^*, and the quotient group T/D is the group A of all integral binary linear homogeneous substitutions of determinant $+1$ or -1. The group T^*/D is the group B of integral binary linear homogeneous substitutions of determinant $+1$.

If we now consider the fundamental group F of the two-dimensional manifold in question, which is given by two generators s, t connected by the relation $sts^{-1}t^{-1} = 1$, and therefore represents the commutative group on two generators, then A obviously represents the group of all isomorphisms of F onto itself (when we again, using the point of view of abstract group theory, do not regard isomorphic groups as distinct).

Just as in this example, entirely analogous relations hold between the groups mentioned in the general case. The group D of all deformations of a manifold V is a distinguished subgroup of the group of all self-transformations of D and similarly, when the manifold is two-sided, it is a distinguished subgroup of the group T^* of all orientation preserving self-transformations of the manifold. If the manifold is connected, then the quotient groups $G = T/D$, $G^* = T^*/D$ are identical with the groups T^0/D^0, T^{*0}/D^0 respectively, where D^0, T^0, T^{*0} respectively denote the groups of deformations, self-transformations and orientation-preserving self-transformations of the manifold that leave an arbitrarily chosen inner point M_0 of the manifold fixed. Now we can choose a system of closed paths $a_1, a_2, \ldots$ emanating from M_0 as generators of the fundamental group F of V, and consider the nature of F to be determined by the laws of combination of these paths.[4]

Each transformation in T^0 now corresponds to a permutation of the paths a_i of such a kind that the relations between the original paths also hold between the permuted paths. The transformations therefore correspond to automorphisms of F. All transformations of the same type correspond to the same permutation, the deformations to the identity, and therefore each operation of $G = T^0/D^0 = T/D$ corresponds to an operation of the group J of all automorphisms of F. Now if τ, τ', τ'' are operations of T^0 which satisfy the relation $\tau\tau' = \tau''$ then obviously the corresponding isomorphisms j, j', j'' of F satisfy the relation $jj' = j''$. The same is true when τ, τ', τ'' are understood to be operations of G. Thus G, and similarly G^*, is homomorphic to a subgroup H, and H^* respectively, of J, in general in a many-to-one manner. There can be, say μ operations of G that correspond to the same operation of J. If we consider the further example of the two-dimensional manifold represented by the annulus, for which G is the Klein four group, G^* the cyclic group of order 2, F the infinite cyclic group and J therefore the cyclic group of order two, then one obtains $H = H^* = J$ and hence $\mu = 2$, while G^* is isomorphic to H^*. Note that H coincides with J in both examples.

The groups G, G^*, H, H^* obviously represent topological invariants, about which we admittedly know little at present. As already remarked in connection with the fundamental group, these invariants do not have the same significance as the previous topological invariants as long as no method is known for deciding the equality of two groups, but at least in the case of the fundamental group we can always obtain a presentation when the schema of the manifold is known. Not even this can be asserted for the groups G, G^*, H, H^*. Moreover the same is already true for the group J, whose deter-

[4]The product of two paths in succession obviously represents a closed path. A relation $a_i a_k = a_l$ between closed paths then means simply that composition of the values of an arbitrary unbranched function in V (see §12, note 6) satisfies this relation.

mination from F is a purely group theoretic question.

When it is a question of determining the groups just mentioned for a given manifold, we can pose the problem as one of carrying out this determination from the schema of the manifold. One is therefore involved in defining groups[5] for schemata that reflect the deformations and self-transformations of the manifold. Then groups based on these combinatorial concepts can be defined, analogous to the groups mentioned above, and proofs for the old groups carried over to the new. We would also want to prove theorems such as the following: a simply connected manifold admits no orientation preserving self-transformations other than deformations, and apart from these no self-transformations other than deformations composed with reflections. A reflection of the closed or open n-dimensional simply connected manifold or the bounded $(n+1)$-dimensional manifold

$$x_1^2 + x_2^2 + \cdots + x_n^2 + x_{n+1}^2 = 1 \text{ or } \leq 1 \text{ respectively}$$

is understood to be the transformation:

$$x_1' = x_1, \quad x_2' = x_2, \quad \ldots, \quad x_n' = x_n, \quad x_{n+1}' = -x_{n+1}.$$

The above-mentioned theorems on the relations between the transformation groups in question and the isomorphisms of the fundamental group may require combinatorial analysis situs based on the notion of schema not only for their introduction and development, but also in another respect. Not only the relations derived in the above examples by appeal to intuition, but also the general theorems, need a sharper formulation and basis, requiring a determination of the most general hypotheses one must impose on point sets in order to retain the desired theorems for their self-transformations and deformations. The definitions of self-transformation and deformation certainly apply, like homeomorphism, just as well to arbitrary point sets as to continuous ones.

The above discussion of self-transformations of a manifold V may be extended in a certain sense. Namely, suppose that V is bounded and that

[5] In order to see how this might happen one may refer to the investigations of C. Jordan (Recherches sur les polyèdres, Crelles J., vol. 66) on the concept of the "aspect" of a polyhedron, and the question whether different aspects of the same polyhedron can be similar to each other, also to the Dehn-Heegaard Enzyklopädie article (IIIAB3, Grundlagen T) already mentioned for a concept of deformation based on the schema, applicable to figures lying in a manifold V.

$W_1, W_2, \ldots$ are the boundary manifolds of V. A self-transformation of V then permutes the manifolds W_i. For a deformation of V this permutation is the identity, and one can therefore speak of the permutation of the W_i induced by a particular operation of G. The permutations of the W_i associated with the operations of G in this way constitute a group P homomorphic to G which is intransitive in general. The intransitivity system of this permutation group consists of the systems of boundary manifolds that can be carried into each other by self-transformations of V. We shall assume further that all the W_i are proper boundary manifolds, so that the points of W_i are to be counted among those of V. Those manifolds W_i belonging to the same system must therefore necessarily be homeomorphic to each other.[6]

If one chooses a particular manifold W_i, then those operations of T or G that map W_i into itself constitute a subgroup T_i of T, or a subgroup G_i of G respectively. Each transformation of V contained in T_i corresponds to a transformation of W_i, and similarly each operation of G_i corresponds to an operation of a group Γ_i that plays the same role for self-transformations W_i as G does for self-transformations of V. In general those operations of Γ_i that correspond to operations of G_i do not exhaust the group Γ_i but constitute a subgroup Γ_i'. Γ_i' is homomorphic to G_i. When W_j, W_k belong to the same system of mutually transformable boundary manifolds then G_j and G_k are conjugate subgroups of G and not only are the groups Γ_j, Γ_k isomorphic (which is simply a consequence of the homeomorphism of W_j and W_k), but so are the groups Γ_j', Γ_k'.

If V is two-sided and we confine ourselves to orientation-preserving self-transformations, then one comes to consider the group P^* of permutations of the W_i induced by operations of T^*, and groups T_i^*, G_i^* in T^*, $G*$ respectively that map W_i into itself and the corresponding subgroup ${\Gamma^*}_i'$ of the group Γ_i^* of all operations of Γ_i that preserve the orientation of W_i.

A simple example may clarify some of these general concepts and the-

[6]However this condition is not sufficient. To see this one can consider the manifold shown by Fig. 3 in §15 and cut from a space piece bounded by a torus surface. The resulting manifold V is bounded by two surfaces W_1, W_2 of genus $p = 1$. Its fundamental group is generated by three operations s, t, u satisfying the relation $sts = tst$. There are two closed paths on W_1 that generate all others, and which correspond to the operations s and $tst^{-1}st$ of the fundamental group, and two such paths on W_2, which correspond to the operations 1 and u. One sees that there is no closed non-separating curve on W_1 corresponding to the identity operation, so that W_1 and W_2 cannot be transformed into each other by any self- transformation of V.

orems. We choose V to be a three-dimensional space enclosed by a torus surface. The single boundary manifold W_1 is therefore the torus surface considered above and Γ_1 is accordingly the group of integral linear transformations

$$x' = \alpha x + \beta y,$$
$$y' = \gamma x + \delta y,$$

where $\alpha\delta - \beta\gamma = \pm 1$. But in order to determine Γ_1' one notes that, of all the closed paths $w = \alpha x + \beta y$ (α, β relatively prime), only the paths $w = +x$ and $w = -x$ satisfy the homology

$$w \sim 0 \quad \text{(relative to } V\text{)},$$

where x again denotes a meridian circle and y a circle of latitude, each with a definite orientation. But then it follows that an operation of γ_1' can transform the meridian circle x only into $+x$ or $-x$, and it therefore must have the form

$$\begin{aligned} x' &= \varepsilon x \\ y' &= \gamma x + \eta y, \end{aligned} \tag{39}$$

where ε, η are each either $+1$ or -1. On the other hand it is easy to see that each operation of this form is in fact an operation of Γ_1', and Γ_1' is therefore isomorphic to the group of these substitutions. Namely, if ρ denotes the distance of a point from the axis of the ring, φ the latitude, and ϑ the meridian angle (geographic latitude and longitude) so that φ is constant on meridians, and ϑ on latitudes, then the transformations of V represented by the equations

$$\rho' = \rho, \quad \varphi' = \eta\varphi, \quad \theta' = \varepsilon\vartheta - \frac{\gamma}{2\pi}\varphi$$

induce the self-transformations of V represented by (39).[7]

At this point we may insert a remark concerning the combinatorial construction of analysis situs. We recall that in §4 the $(n+1)$-dimensional cells from which $(n+1)$-dimensional manifolds were constructed were obtained from simply connected closed n-dimensional manifolds. The geometric idea underlying this stepwise construction of manifolds of ever greater dimension is that the simply connected n-dimensional manifold.

$$x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1$$

[7]This may already be found in Heegaard, Diss. p.56.

bounds the $(n+1)$-dimensional manifold

$$x_1^2 + x_2^2 + \cdots + x_{n+1}^2 \leq 1$$

which we take as the $(n+1)$-dimensional cell. The construction of $(n+1)$-dimensional manifolds therefore depends on the previous introduction of the concepts of "simply connected", as well as "elementary subdivision" and "homeomorphism" for n-dimensional manifolds. One can ask whether it may not be possible to use a two-sided closed n-dimensional manifold $V^{(n)}$ for the construction of $(n+1)$-dimensional manifolds without restricting it to be simply connected, merely taking $V^{(n)}$ to lie in n-dimensional euclidean space $\mathfrak{R}_{n+1}$ and using the part $\Xi^{(n+1)}$ that it bounds in this space as the $(n+1)$-dimensional cell.

But apart from the fact that it seems questionable whether each closed two-sided n-dimensional manifold can be embedded continuously as a point manifold $X^{(n)}$ of n-dimensional euclidean space[8] (without singularities and self-intersections, which would violate the embedding requirement), and apart from the fact that there may be essentially different embeddings $X^{(n)}$ of the same manifold $V^{(n)}$, so that the parts of $\mathfrak{R}_{n+1}$ they bound are not homeomorphic,[9] thus requiring a particular embedding to be selected, the example just discussed shows that a usable $(n+1)$-dimensional element is still not completely determined, because of the fact that $X^{(n)}$ can admit self-transformations that do not correspond to self-transformations of $\Xi^{(n+1)}$. This is seen immediately in the example.

Now suppose one proceeds to use closed two-sided surfaces to determine three-dimensional cells, in the case $p = 1$ letting it be represented by a torus surface $X^{(2)}$, whose interior is understood to be the cell. Now if ξ, η are independent closed paths on the surface which generate all the others then we map the surface one time on to $X^{(2)}$ so that ξ goes to the meridian circle x, and η to the latitude circle y, and another time so that ξ goes to a line $x' = \alpha x + \beta y$, and η goes to a line $y' = \gamma x + \delta y$, where $\alpha\delta - \beta\gamma = \pm 1$. We then have two different interpretations of the surface with $p = 1$ as a surface $X^{(2)}$, and the interior of $X^{(2)}$, which is regarded as the three-dimensional cell, accordingly has two different interpretations $\Xi_1^{(3)}$, $\Xi_2^{(3)}$. Now the $p = 1$ surface is thought of as divided into polygons that are identified with each other or

[8]For $n = 2$ this embedding is always possible, as is well known. For $n > 2$ cf. the question raised in §2, note 13.

[9]Cf. the example given by fig. 3 in §15.

with polygons on the boundary surfaces of other cells. The three-dimensional manifold defined by such a pairing of boundary surfaces of cells will in general be different when one takes the space piece $\Xi_1^{(3)}$ rather than the space piece $\Xi_2^{(3)}$, and we can only be assured of equality when the substitution

$$x' = \alpha x + \beta y$$
$$y' = \gamma x + \delta y$$

has the form (39). Consequently, further information is necessary if the surface with $p = 1$ is to be used to determine three-dimensional cells, say the specification of a particular closed, non-separating curve to wrap to the meridian circle x of $X^{(2)}$. It follows that the restriction to simply connected $(n+1)$-dimensional manifolds is the most natural in using n-dimensional cells to build up n-dimensional manifolds. Difficulties of the kind just discussed do not then appear, since all parts $\Xi^{(n+1)}$ of $\mathfrak{R}_{n+1}$ bounded by simply connected n-dimensional manifolds $X^{(n)}$ are homeomorphic, and each transformation t of $X^{(n)}$ can be induced by a self-transformation of $\Xi^{(n+1)}$.[10]

The groups P, P^*, G_i, G_i^*, Γ'_i, $\Gamma^{*\prime}_i$ represent topological invariants of bounded manifolds which seem important, inasmuch as there seem to be manifolds they distinguish when all the topological invariants previously investigated (connectivity numbers, torsion numbers, fundamental group, number and character of boundary manifolds) do not.

L_1 L_2

Fig. 4.

Consider for example[11] the three-dimensional manifold V_1 one obtains by removing two unlinked closed lines L_1, L_2 (see Fig. 4), which are mirror image simple knots, from the three-dimensional spherical manifold.

[10]These theorems appear intuitively clear, and one can pose the problem of proving them rigorously in the domain of point manifolds, or perhaps (cf. notes) in the domain of combinatorial analysis situs, where the manifolds are presented by schemata, as was done in Section I.

[11]In the examples that follow only the groups P, P^* are considered, since only these groups are defined for improperly bounded manifolds (see §2, note 4), and the examples concern such manifolds.

Thus the line L_1 cannot be deformed into L_2 within the three-dimensional spherical manifold without intersecting itself.[12] One obtains the fundamental group of V_1 [13] from four generators s_1, s_2, s_3, s_4 with the relations $s_1s_2s_1 = s_2s_1s_2$ and $s_3s_4s_3 = s_4s_3s_4$, whence $P_1 = 3$ and there are no torsion numbers. One obtains the same fundamental group for the manifold V_2 obtained by removing two unlinked closed lines Λ_1, Λ_2 which are simply knotted in the same sense, e.g., like L_1. The boundary manifolds of both V_1 and V_2 are the same in number and homeomorphic, but V_1 and V_2 themselves are not homeomorphic.[14]

For both manifolds the permutation group P consists of the permutation group on two letters, the two boundary elements being L_1, L_2 or Λ_1, Λ_2 respectively. However for V_2 the group $P^* = P$ since there is an orientation-preserving self-transformation of V_2 that exchanges Λ_1 and Λ_2, whereas each orientation- preserving self-transformation of V_1 must fix the lines L_1, L_2, so that P^* is the identity group for V_1. We may also consider the example of two manifolds V_1, V_2 resulting from the three-dimensional spherical manifold by removal of three unlinked closed lines Λ_1, Λ_2, Λ_3 and K_1, K_2, K_3 respectively. Each of these lines is a simple knot, but Λ_1, Λ_2, Λ_3, K_1, K_2, are the same as the line L_1 of Fig. 4, whereas K_3 is the same as L_2. The group P of the manifold V_1 obviously consists of the group of all permutations on three letters Λ_1, Λ_2, Λ_3. But, apart from the identity, the group P of V_2 contains only the permutation

$$\begin{pmatrix} K_1 & K_2 & K_3 \\ K_2 & K_1 & K_3 \end{pmatrix}.$$

---

[12] This is also intuitively based, a fact of topological experience if I may use the term, for which a stricter proof is not known to me.

[13] Cf. the determination of the fundamental group of Ψ_1, in §18.

[14] One could regard this as intuitively evident. Or one could proceed from the standpoint that if V_1 and V_2 were homeomorphic there would be a one-one continuous correspondence between V_1 and V_2, from which a one-one continuous correspondence between the boundary points could be derived, extendible to a self- mapping of the whole spherical manifold. But since, on the basis of the above theorem, the spherical manifold has no self-transformations other than deformations or deformations in combination with reflections, the system of lines L_1, L_2 would be knotted in the same way as Λ_1, Λ_2 or its mirror image, i.e., each member of the system would be deformable into its mirror image, which is obviously out of the question.

VI. Special ways of representing closed multidimensional manifolds

§17

Closed manifolds are obtained: 1. by identification of boundary manifolds, 2. by double covering of a "ground form"

In this section we shall generalize the known ways of representing closed two-dimensional manifolds to more dimensions (especially three). As a result it will be shown that certain of these forms of representation, which suffice to obtain each closed two-dimensional manifold, are not adequate for the representation of all closed manifolds in higher dimensions (so that the possibility of such a representation is a special topological property of a manifold), or else the uniqueness of the presentation is lost.

The first form of representation to be discussed, proceeds, in the case of two-dimensional manifolds,[1] from a "ground form," i.e., from a developable surface with r boundary lines, which one may represent as circles. If one imposes an orientation on the developable surface, this induces a positive direction for each of the r boundary lines. Identification of the circles in pairs[2] now serves to define a closed surface.

Three kinds of identification come into consideration:

1. The points of one circle K_1, may be mapped on to the points of another, K_2, so that the positive orientation of K_1, corresponds to the negative orientation of K_2.

2. The points of two circles K_1, K_2 may be related so that the positive orientation of K_1 corresponds to the positive orientation of K_2.

3. The points of a circle K are identified with their diametric opposites. If one now considers two identified arcs of K, then the positive orientation of one corresponds to the positive orientation of the other.

One divides the r circles into s pairs that are identified in the first or second way, and t individual circles identified with themselves in the third way. The

[1] Cf. Dyck, Math. Ann. 32

[2] The way to interpret this is immediate from what was said in Section I on the identification of boundary elements.

resulting closed manifold[3] is obviously two-sided if only identifications of the first kind are present. Each two-dimensional manifold admits a representation of the kind described. It is unique only for the two-sided surfaces and the one-sided surface with invariant $q = 1$ (see §8) (i.e., apart from topologically inessential modifications of the ground form).[4]

The generalization of the form of representation just discussed is immediate. One proceeds from a developable n-dimensional manifold, bounded by a number of $(n-1)$-dimensional manifolds and induces orientations on the boundary manifolds by choosing an orientation of the developable manifold. Then the boundary manifolds are identified in the following three ways:

1. Two boundary manifolds R_1, R_2 are identified so that the positive orientation of R_1 corresponds to the negative orientation of R_2, or

2. So that the positive orientation of R_1 corresponds to the positive orientation of R_2.

3. The points of a boundary manifold R are identified so that the positive orientation of a piece of R corresponds to the positive or negative orientation of its partner according as n is even or odd. This includes, for example, the identification of diametrically opposite points of the manifold
$$x_1^2 + x_2^2 + \cdots + x_n^2 = 1.$$

Identified point pairs are regarded as single points of the manifold being represented. The resulting n-dimensional manifold is two-sided in the case of even n just when all identifications are of the first kind, and in the case of odd n only when no identifications of the second kind appear.

As an example, identification of diametrically opposite points on the surface of the ball yields the two-sided three-dimensional manifold T_3 (see §9), and the result of identification of points on the same radius vector for the space between two spheres is likewise a two-sided manifold U (whose schema is given later). A further example is the three-dimensional manifold defined by the following schema with two cells. The two cells may be given as cylinders in the spaces with coordinates x, y, z and x', y', z' respectively by

$$x^2 + y^2 + z^2 \leq 1, \quad +1 \geq z \geq -1,$$

[3]Naturally one also obtains bounded surfaces by the same process when not all circles are subject to identification, in fact each bounded surface is representable in this way.

[4]Dyck presents these relations op.cit. p.480.

$$x'^2 + y'^2 + z'^2 \leq 1, \quad +1 \geq z' \geq -1$$

respectively, with their boundary surfaces divided into three polygons each by the perimeters of the end surfaces and the lines $x = 1$, $y = 0$ and $x' = 1$, $y' = 0$ respectively. Let the two vertical surfaces be identified according to the formulae

$$x' = x, \quad y' = y, \quad z' = -z$$

and the end surfaces according to the formulae

$$x' = -x, \quad y' = -y, \quad z' = -z.$$

It is easy to see that the manifold defined in this way may be expressed in the above form, once as the space between two concentric spherical surfaces, where diametrically opposite points on each sphere are identified, and also as the space bounded by a torus surface whose points are identified according to the formula[5]

$$\varphi' = 2\pi - \varphi, \quad \vartheta' = \vartheta + \pi$$

where φ, ϑ are understood to be geographic longitude and latitude on the torus surface as in §16. Thus the form of representation is no longer unique for two-sided manifolds of three dimensions.

When $n = 3$ it already happens that not all closed manifolds admit representations of the kind described. Namely, since the set of points on the manifold resulting from an identification of boundary surfaces R_1, R_2 in the first or second way, or from the identification of a boundary surface R with itself yields a (two- or one-sided) closed non-separating surface in the manifold, the number P_2 or Q_2 must be greater than 1. But there are closed two-sided three-dimensional manifolds, not homeomorphic to the spherical manifold, for which $P_2 = Q_2 = 1$. Examples are the manifolds $[2m+1, \lambda]$ considered in §20, which have the cyclic group of order $2m+1$ as fundamental group.

The second way we shall describe for representing closed multi-dimensional manifolds concerns two-sided manifolds. In the two-dimensional case one recalls the well known normal form of the surface as a sphere with p handles.

[5]The collection of points of the manifold, each of which is represented by two points on the torus, constitute a closed one-sided surface with invariant $q = 2$ (see §8). In general, the result of identification of the third kind on a surface of genus p is a one-sided surface in the three-dimensional manifold with invariant $q = p + 1$, as one sees by consideration of the number N (see §8) which, for the surface of genus p, must be double that for the one-sided surface.

Now if one projects say the torus surface, the normal form of the surface of genus 1, onto a suitably positioned plane, then one obtains an annulus, whose boundary circles may be called K_1 and K_2, and each point between them corresponds to two points of the torus surface. Thus if we think of the annulus as consisting of two sheets connected along K_1 and K_2, we have a way of representing the closed surface of genus 1. The following should then be noted as far as the orientation of the manifold is concerned. If one chooses an indicatrix for the annular surface between K_1 and K_2, then this orientation is the same as that of the torus surface on one sheet, and the opposite on the other sheet. Thus if one wants to coherently orient the closed surface, the orientation has to be reversed in passing from one sheet to the other across K_1 or K_2.

Quite analogously as in the case $p = 1$, the general closed surface of genus p is represented by a developable surface with $p+1$ boundary lines, covered by two sheets joined along these boundaries. One obtains the generalization of this form of representation for two-sided closed manifolds of more dimensions when one thinks of a developable n-dimensional manifold, bounded by a number of $(n-1)$-dimensional manifolds, being doubly covered and then joining the two sheets along the boundary manifolds.[6]

But while this form of representation is possible in only one way in the two-dimensional case, it is not longer unique for three dimensions. This is shown, for example, by the following manifold U. The schema consists of a single cell, which one represents as a cylinder.

$$x^2 + y^2 \leq 1, \quad +1 \geq z \geq -1$$

whose vertical surface is divided into two rectangular pieces by its lines of intersection with the plane $y = 0$; these are identified with each other by the formulae

$$x' = x, \quad y' = -y, \quad z' = z$$

and the two end surfaces are identified by the formulae

$$x' = x, \quad y' = y, \quad z' = -z.$$

Now if one considers firstly the space D' enclosed by a torus surface, and secondly the space D'' between two concentric spheres, and thinks of each

[6]One can of course generalize this form of representation by taking $2m$ sheets and joining a pair of them at each boundary manifold.

of the developable manifolds D', D'' doubly covered, with the sheets joined along the boundary surfaces, then it is easy to see that one obtains the same manifold U in each case. Furthermore, if one considers the representations, characterized by numbers p, q, that one obtains by taking the normal form of the closed surface of genus p, i.e., the sphere with p handles, in three-dimensional space, removing q balls from its interior, covering the remaining developable space with two sheets and joining them along the $q+1$ boundary surfaces, then one obtains two different representations of the same two-sided closed three-dimensional manifold when $p+q$ has the same value for each.[7]

It appears that this form of representation is also special for $n \geq 3$, and by no means all two-sided closed manifolds admit it. Thus it is not improbable that not only all developable manifolds, but also closed manifolds obtainable by double covering of developable manifolds, have no torsion numbers. However, no proof is known to me in either case.

§18
Riemann spaces[1]

A third form of representation of multi-dimensional manifolds is a generalization of Riemann surfaces. We confine ourselves to discussing this generalization in the case of three dimensions.

One can obtain Riemann surfaces by beginning with a spherical surface from which n points $a_1, a_2, \dots, a_n$ are removed,[2] so that one obtains a surface Φ bounded by n points. If one draws non-intersecting lines from a point O of the surface to the points a_i, then one can associate an operation s_i with the crossing of each line $Oa_1, Oa_2, \dots, Oa_n$ in a positive circuit around O. If one crosses all lines in the positive orientation this corresponds to a closed circuit around O, and since the point O by no means occupies a special position on the surface Φ, the operation $s_1 s_2 \cdots s_n$ associated with this circuit, like a circuit which crosses none of the lines Oa_i, must be the identity. Thus we have the relation

$$s_1 s_2 \cdots s_n = 1.$$

[7]The manifolds considered are identical with those represented as Riemann spaces in ex. 2, 3 of Heegaard's dissertation §14.

[1]Cf. Appell, Math. Ann. 30, Sommerfeld, Proc. Lond. Math. Soc. 28 and §§13,14 of the dissertation of Heegaard.

[2]Thus representing improper boundary manifolds, cf. §2, note 4.

This requirement obviously corresponds to the idea of an arbitrary unbranched function on the surface Φ, when the s_i are understood to be substitutions of the function values along closed paths. In fact, the group generated by $s_1, s_2, \ldots, s_n$ with the relation (40) is the fundamental group of Φ.

It is now a question of covering the surface Φ with a finite number of sheets, say m, so that the resulting m-sheeted surface has branch points only at the positions $a_1, a_2, \ldots, a_n$, which means that one need only take $s_1, s_2, \ldots, s_n$ to be permutations of the n letters $1, 2, \ldots, n$ that denote the corresponding permutations of the sheets, so that these permutations satisfy the relation (40). Then if the group generated by the permutations $s_1, s_2, \ldots, s_n$ is transitive one obtains a connected m-sheeted Riemann surface over the sphere.[3]

Analogously, one can consider a three-dimensional manifold Ψ resulting from the spherical three-dimensional manifold, viewed as $\mathfrak{R}_3$ closed by a point at infinity, by removing a number of closed lines $a, b, c, \ldots$ [4]

One considers the surface F consisting of all radius vectors from a point O of Ψ to the points of these lines. Then one considers the projections of the lines $a, b, c, \ldots$ from O onto a plane E not containing O. The resulting curves in E cross each other at certain points $N_1, N_2, \ldots$. We assume that only two pieces of curve cross at each such point N_i. Then two of the radius vectors of the surface F lie in the same direction ON_i, leading to two points A_i, B_i of the system of lines $a, b, c, \ldots$ where the length $OA_i < OB_i$ say. Thus the surface F passes through itself along OA_i. The points A_i divide the lines $a, b, c, \ldots$ (or a single one of them) into pieces and the pieces of a may be denoted by $a_1, a_2, \ldots$, the pieces of b by $b_1, b_2, \ldots$, etc. The orientation of a piece is determined by choosing a positive orientation of the line. Each of the pieces $a_i, b_i, c_i, \ldots$ corresponds to a piece of the surface F consisting of the radius vectors leading to this piece. If A_h is the negative end, and A_k the positive end, of a line piece l_i, then the circuit OA_hA_kO around the corresponding surface piece may be taken as positive. This orientation determines a positive and negative side of the surface piece.

[3] As is well-known, one can apply the same method to obtain many-sheeted covering surfaces over surfaces other than the sphere.

[4] The process that follows, which uses the cone F as the branching surface, together with the conditions for joining the sheets where the lines from O have apparent double points with the lines $a, b, c, \ldots$, has been obtained from Herr Wirtinger, who developed it and used it in investigations to be mentioned later. The same process, with a restriction to first order branching, already occurs in Heegaard op. cit.

We associate an operation with the piercing of a surface piece of F from the positive or negative side, which is denoted by the same letter as the corresponding line piece. Now if the two line pieces $u_1 = A_h A_i$ and $u_2 = A_i A_j$ meet at the point A_i, while the point B_i lies on the line piece v, then U_1, u_2, v satisfy the relation

$$(41) \qquad vu_1v^{-1}u_2^{-1} = 1 \quad \text{or} \quad v^{-1}u_1vu_2^{-1} = 1$$

according as u_1 ends on the positive or negative side of the surface piece corresponding to v. One also obtains this by considering a small loop around the radius vector OA_i, which must correspond to the identity operation. The group defined by these generators and the relations (41) is obviously the fundamental group of Ψ.[5]

A many-sheeted covering of the spherical manifold, with the lines $a, b, c, \ldots$ as branch lines, is obtained in analogy with the two-dimensional case, by choosing the operations $a_i, b_i, c_i, \ldots$ to be permutations of m sheets so that the relations (41) are satisfied.[6]

While each closed two-sided surface is homeomorphic to one represented by a Riemann surface, it is not known whether each two-sided closed three-dimensional manifold is homeomorphic to a "Riemann space" of the kind described. In any case (just as with two dimensions) the same manifold may admit different representations as a Riemann space, as the following examples show.

To present such an example, one begins with the manifold Ψ_1, which results from the spherical manifold by removal of a knotted line L_1 (cf. Fig. 4). L_1 divides into three pieces s, t, u satisfying the relations

$$sts^{-1} = u, \quad tut^{-1} = s, \quad usu^{-1} = t,$$

the last of which is a consequence of the first two. If one omits the superfluous generator u, then one obtains the fundamental group of Ψ_1 by the relation

$$sts = tst$$

between the generators s, t.

[5]The form of the relations (41) shows the manifolds Ψ in question have no torsion numbers. This follows from the same considerations as were presented in §10, note 1.

[6]Naturally one can again replace the covering of the spherical manifold by one of an arbitrary closed manifold.

One obtains a three-sheeted Riemann space branched along L_1 when one chooses the permutations s, t, u to be:

$$s = (2,3), \quad t = (1,3), \quad u = (1,2).$$

In this way one obtains a manifold[7] obtained by Wirtinger[8] in an investigation of the algebraic function of two complex variables represented by the Cardano formula. It is noteworthy that this Riemann space is three-sheeted but has branching only of first order along L_1.

A two-sheeted Riemann space branched along L_1 is obtained when one sets[9]

$$s = t = u = (1,2)$$

We now go to the manifold Ψ_2 that results from the spherical manifold by removal of two unknotted lines a, b which are simply linked, e.g., the circles $x^2 + y^2 = 1$, $z = 0$ and $x^2 - 2x + z^2 = 0$, $y = 0$. The operations a, b satisfy the relation

$$aba^{-1}b^{-1} = 1,$$

so that the fundamental group of Ψ_2, is the abelian group on two generators. If one takes a three-sheeted covering of Ψ_2 and

$$a = b = (1,2,3)$$

then the resulting Riemann space may be denoted by R_2. One can then show that R_1 and R_2 are representations of the same three-dimensional manifold, and in fact the one denoted by $[3,1]$ in §20.

VII. Some supplementary material

§19

On a theorem from the foundations of combinatorial analysis situs

[7] The same Riemann space had previously been considered by Heegaard (op. cit. p.84, ex. 4) and recognized to be simply connected.

[8] Erste Sitzung d. Math. Ges. in Vienna on 22 Jan. 1904 and Jber. d. Deutsch. Math. Ver. Meran, Sept. 1905 (see Jber. 14, p.517).

[9] Heegaard, op. cit. p.84, ex. 5.

Combinatorial[1] analysis situs rests on the division of schemata into homeomorphism classes. Two schemata are called homeomorphic if they have a derived schema in common (see §2). The division into classes then results from the fact that if two schemata are both homeomorphic to a third then they are homeomorphic to each other. This theorem seems obvious, but it is necessary to provide a proof and this will be carried out for the case of two dimensions. As far as the position of this theorem in the systematic construction of analysis situs is concerned, it may be remarked that it is possible to dispense with it. One can describe two schemata that have a common derived schema as homeomorphic in the strict sense, and call two schemata homeomorphic in the extended sense if a sequence of schemata may be interpolated between them so that each schema in the sequence is strictly homeomorphic to its predecessor. In this way one obtains a division of the schemata into homeomorphism classes without depending on a subdivision lemma, and the properties of schemata that are topologically invariant, i.e., the same for homeomorphic schemata, are just those that are invariant under elementary subdivisions.

The theorem we are concerned with will obviously be proved when we have established the following special case: two schemata Q, R derived from the same schema P by subdivision are homeomorphic. This in turn will be settled when it is established in the particular case when one of the schemata Q, R is derived from P by an elementary subdivision. Suppose for example that Q is derived from P by an elementary subdivision and that $P, P_1, P_2, \ldots, P_n$ ($P_n = R$) is a sequence of schemata, each of which is derived from its predecessor by an elementary subdivision. It is a question of finding, for each P_i, a schema Q_i derived from it and also derivable from Q. Q_n is then the common derived schema of Q and R.

In the first instance, where R as well as Q results from P by an elementary subdivision, it is possible to give a common derived schema S. In the case of two dimensions there are then two kinds of elementary subdivisions to be distinguished: division of an edge into two edges and division of a surface piece into two surface pieces. We call these subdivisions of first and second order respectively and the subdivided edge or surface piece may be called the element of the schema subject to subdivision.

Now when the elements of the schema P subject to subdivision, whether it be of first or second order, are different, then it is immediately clear that

[1] See the introduction.

the two schemata Q, R derived from P possess a common derived schema S, obtainable from Q as well as from R by an elementary subdivision. If on the other hand the same element of P is subject to the two subdivisions that yield Q, R, and if both subdivisions are of first order, then Q and R are identical and we can set $S = Q = R$. Finally, if each subdivision is of second order and the same surface piece a^2 is subject to subdivision, then the perimeter of the surface piece, which one can regard as a circle, can be marked with the endpoints A_Q, B_Q and A_R, B_R respectively of the two new edges c_Q, c_R realizing the subdivisions Q and R respectively. Now when the point A_Q is followed by one of the points A_R, B_R and then the point B_Q in the cyclic order on the circle, one derives the schema S from Q by a series of three elementary subdivisions: first dividing c_Q into two edges by a new vertex C and then subdividing the two surface pieces into which a^2 is divided by c_Q by further edges connecting C with A_R and B_R. This schema S may be derived from R in an analogous way and it is therefore a common derived schema of Q and R. When the point pair A_Q, B_Q coincides with the point pair A_R, B_R, so that Q is equal to R, we take $S = Q = R$.

In all the remaining cases of second order subdivisions that apply to the same surface piece one can give a schema S derivable from Q as well as R by an elementary subdivision. Thus in all cases we have a process which, given two schemata Q, R derived from a schema P by elementary subdivision, finds a common derived schema S for both Q and R using a finite number of elementary subdivisions (zero, one or three).

If $T, T', T'', \ldots, T^{(m)}$ is a series of schemata in which $T^{(i)}$ results from $T^{(i-1)}$ by subdivision of an element e_{i-1} of $T^{(i-1)}$ into two pieces e_i' and e_i'' by a new element $\overline{e_i}$ (of dimension one lower than that of e_{i-1}) $(i = 1, 2, \ldots, m)$, and if none of the later subdivisions apply to one of the elements $\overline{e_i}$, e_i', e_i'',[2] then the sequence of subdivisions that yield T' from T, T'' from T', …, $T^{(M)}$ from $T^{(M-1)}$ may be called a sequence of independent elementary subdivisions of the schema T. It is obvious that when Q, R result from the same schema by elementary subdivisions then the sequence of elementary subdivisions used in the above process to obtain the common derived schema S from R is a sequence of independent elementary subdivisions of R.

If we now suppose that a sequence of schemata

$$P_i, P_{i1}, P_{i2}, \ldots, P_{i,m_i-1}, P_{i,m_i} \quad (P_{i,m_i} = Q_i) \tag{42}$$

[2]By virtue of this hypothesis the elements $e, e_1, e_2, \ldots, e_{m-1}$ can all be regarded as elements of the schema T.

has been found such that each results from its predecessor by an elementary subdivision and the sequence of these elementary subdivisions is independent, and the last schema Q_i is a derived schema of Q, then it can be easily shown that there is a sequence.

$$P_{i+1}, \quad P_{i+1,1}, \quad P_{i+1,2}, \quad \dots, \quad P_{i+1,m_{i+1}-1}, \quad P_{i+1,m_{i+1}}$$

(43)

$$(P_{i+1,m_{i+1}} = Q_{i+1})$$

with exactly the same properties as the sequence (42). When $i = 1$ we can find a sequence with the stated properties by the process described above, which finds a common derived schema $S = Q_1$ for the two schemata Q and P_1 obtained from P by elementary subdivision, so the desired demonstration can be achieved by complete induction.

As far as finding the sequence (43) is concerned, it can be assumed that $P_{i+1} \neq P_{i1}$, otherwise one need only take $m_{i+1} = m_i - 1$, $P_{i+1,k} = P_{i,k+1}$. We consider each element e of the schema P_i that is subject to elementary subdivisions leading from P_i to P_{i+1}. If none of the elementary subdivisions leading from P_i to Q_i apply to the element e, then $P_{i+1,1}$ is taken to be any schema obtainable from P_{i1} as well as from P_{i+1} by an elementary subdivision (and on the basis of our assumption there is such a schema). In general we take $P_{i+1,k}$ to be any schema obtainable from P_{ik} as well as $P_{i+1,k-1}$ by an elementary subdivision. The sequence of schemata $P_{i+1}, P_{i+1,1}, P_{i+1,2}, \dots, p_{i+1,m_i-1}, P_{i+1,m_i}$ is then the desired sequence (43).

If on the other hand one of the subdivisions leading from P_i to Q_i, say yielding $P_{i,h+1}$ from P_{ih}, applies to the element e, then for $k \leq h$ we take $P_{i+1,k}$ to be any schema derivable from $P_{i+1,k-1}$ as well as from $P_{i,k}$ by an elementary subdivision. Then the schemata $P_{i,h+1}$ and $P_{i+1,h}$ each result from $P_{i,h}$ by an elementary subdivision, and these two subdivisions both apply to the element e of P_{ih}. Suppose that e is a surface piece and that the two subdivisions in question are situated so that the endpoint pairs of the new edges separate each other. The subdivision yielding $P_{i,h+1}$ from P_{ih} consists of the subdivision of e by a new edge $\overline{e}$ into the surface pieces e', e''; the subdivision yielding P_{i+1} from P_i, and hence also $P_{i+1,h}$ from P_{ih}, consists of the subdivision of e by the edge $\overline{e_1}$ into the surface pieces e_1', e_1''.

On the basis of the process set out above there is then a common derived schema S for $P_{i,h+1}$ and $P_{i+1,h}$ which, since we are assuming that $\overline{e}$ and $\overline{e_1}$

cross, is obtained from each of these schemata by three elementary subdivisions. The three applied to $P_{i+1,h}$ involve in turn the elements $\overline{e_1}$, e_1', e_1'', and the three schemata that result may be denoted $P_{i+1,h+1}$, $P_{i+1,h+2}$, $P_{i+1,h+3}$, where $P_{i+1,h+3} = S$. The three applied to $P_{i,h+1}$ involve $\overline{e}$, e', e'' and lead in order to the schemata

$$P_{i,h+1,1}, \quad P_{i,h+1,2}, \quad P_{i+1,h+3}.$$

Suppose that the elementary subdivisions yielding $P_{i1}, P_{i2}, \ldots, P_{ih}$ from P_i are of the elements $f_1, f_2, \ldots, f_h$, and that those yielding $P_{i,h+2}, P_{i,h+3}, \ldots, P_{i,m_i}$ from $P_{i,h+1}$ are of the elements $g_{h+2}, g_{h+3}, \ldots,\ g_{m_i}$. By hypothesis the elements f_k, g_k are all different from each other and from e. One now determines any schema that can be obtained from both $P_{i,h+2}$ and $P_{i,h+1,1}$ by an elementary subdivision and calls it $P_{i,h+2,1}$, then any common derived schema $P_{i,h+2,2}$ of $P_{i,h+2,1}$ and $P_{i,h+1,2}$, and any obtainable from $P_{i,h+2,2}$ and $P_{i+1,h+3}$, where in each case we take any common derived schema obtainable from the two given schemata by an elementary subdivision. The last common derived schema, of $P_{i,h+2,2}$ and $P_{i+1,h+3}$, is called $P_{i+1,h+4}$.

The elementary subdivisions yielding $P_{i,h+2,1}$ from $P_{i,h+1,1}$, $P_{i,h+2,2}$ from $P_{i,h+1,2}$, and $P_{i+1,h+4}$ from $P_{i+1,h+3}$, all apply to the element g_{h+2}. The subdivisions yielding $P_{i,h+2,1}$, $P_{i,h+2,2}$, $P_{i+1,h+4}$ from $P_{i,h+2}$ apply in turn to $\overline{e}$, e', e''. Now if in general we have found a sequence of schemata $P_{i,h+k,1}$, $P_{i,h+k,2}$, $P_{i+1,h+k+2}$ resulting in turn from $P_{i,h+k}$ by subdivisions of $\overline{e}$, e', e'', and where $P_{i+1,h+k+2}$ is a derived schema of P_{i+1}, then by applying the elementary subdivisions of g_{h+k+1} that produce $P_{i,h+k+1}$ from $P_{i,h+k}$, to the schemata in this sequence, one may obtain the schemata $P_{i,h+k+1,1}$, $P_{i,h+k+1,2}$, $P_{i+1,h+k+3}$ that also result from $P_{i,h+k+1}$ by subdivisions of $\overline{e}$, e', e''. The resulting sequence of schemata $P_{i+1,h+3}, P_{i+1,h+4}, P_{i+1,h+5}, \ldots$ ends with a schema P_{i+1,m_i+2} that is a derived schema of $P_{I,m}$, and the sequence of schemata

$$P_{i+1}, \quad P_{i+1,1}, \quad \ldots, \quad P_{i+1,m_i+2}$$

that results by a sequence of successive elementary subdivisions therefore ends with a schema derivable from Q. The successive elementary subdivisions are independent, since they apply to distinct elements $f_1, f_2, \ldots, f_h, \overline{e_1}, e_1', e_1''$, $g_{h+1}, g_{h+2}, \ldots, g_{m_i}$ of P_{i+1}. The sequence is therefore the desired sequence (43). We have just discussed the case where the element e is a surface piece and the edges $\overline{e}$, $\overline{e_1}$ cross. In all the remaining cases the determination of the sequence (43) is simpler.

The proof of the theorem is therefore complete for the two-dimensional case.

§20
An example

We shall consider an example (in fact not a single manifold, but a whole series of manifolds) that is in a certain sense the simplest possible type of two-sided closed three-dimensional manifold.

The schema of this manifold consists of a single three-dimensional cell a^3, thus a single fundamental polyhedron, and since the manifold is to be closed, this cell is bounded by an even number of boundary polygons. One obtains the simplest case by dividing the spherical surface of the given cell a^3 by an equatorial circle which is itself divided into l equal parts. The two-sidedness condition still leaves open l different ways of associating the two hemispherical polygons. These identifications of the two hemispheres are expressible by the formulae

$$\varphi' = \varphi + \frac{2\pi\lambda}{l}, \quad \vartheta' = -\vartheta, \qquad (\lambda = 0, 1, 2, \dots, l-1)$$

where φ, ϑ are the geographical longitude and latitude of a point on the upper hemisphere ($\vartheta > 0$), and ϑ', ϑ' are the coordinates of the corresponding point on the lower hemisphere. The identification is completely determined by the numbers l and λ, so the schema may be denoted by (l, λ). The schemata $(l, 0)$ all represent the spherical manifold defined by $(1, 0)$. In general, if l, λ have a common divisor, so that $l = kl_1$, $\lambda = k\lambda_1$, then the manifold defined by (l, λ) is no different from the one defined by (l_1, λ_1). Thus for each value of l we need only consider the $\varphi(l)$ members of the series $1, 2, \dots, l-1$ relatively prime to l.

Suppose l, λ are relatively prime. The l edges of the polygon in the schema (l, λ) then constitute a single closed cycle, the l vertices on the equator a single system of corresponding vertices of the polygon subdivision. The neighbourhood manifold of the vertex of the schema corresponding to this system is, as one easily realizes, the two-dimensional sphere, so the condition presented in §3 is satisfied. For this schema, $\alpha_3 = \alpha_2 = \alpha_1 = \alpha_0 = 1$, and when a^3, a^2, a^1, a^0 denote the cell, surface piece, edge and vertex of the schema the Poincaré relation system reads

$$a^3 \equiv 0; \quad a^2 \equiv la^1; \quad a^1 \equiv 0.$$

The manifold defined by the schema in question, which may be called $[l, \lambda]$,[1] therefore has the Betti numbers

$$P_1 = P_2 = 1$$

and, when $l > 1$, a torsion number equal to l. One obtains the cyclic group of order l as fundamental group.[2]

I have Herr Wirtinger to thank for an oral communication giving a simple representation of the manifold $[l, \lambda]$ as a branched covering over the spherical manifold, and thus a representation as a "Riemann Space", (§18). In order to obtain this one takes the two poles N and S of the ball representing the cell a^3 and l subdivision points $M_0, M_1, \ldots, M_{l-1}$ on the equator, and draws each meridian from N to S through a point M_i. One cuts the cell a^3 into l pieces by discs through these meridians and the NS axis, and calls them $a_0^3, a_1^3, \ldots, a_{l-1}^3$. Each of these pieces a_i^3 can be deformed into a tetrahedron. The vertices of this tetrahedron, which result from M_i, M_{i+1}, N, S by the deformation, may be denoted by A_i, B_i, C_i, D_i respectively. To obtain a schema of the manifold $[l, \lambda]$ one has to define the following correspondence between the boundary surfaces of the cells a_i^3: the triangle $B_iC_iD_i$ is associated with the triangle $A_{i+1}C_{i+1}D_{i+1}$, and the triangle $A_iB_iC_i$ with the triangle $A_{i+\lambda}B_{i+\lambda}D_{i+\lambda}$.

Now one observes that one obtains a schema of the manifold Ψ_2 considered in §18 when one takes a tetrahedron $ABCD$, identifies the triangle BCD with the triangle ACD, the triangle ABC with the triangle ABD, and leaves out the lines $a = AB$ and $b = CD$. (Each of the tetrahedron edges AB, CD obviously represents a "closed cycle of identified edges of

[1] When using this symbol we always assume that λ is relatively prime to l and that $0 < \lambda < l$. In this connection (as well as in connection with the considerations of §22) the "diagrams" of Heegaard's dissertation may be recalled. The diagram (p.57) of a torus with the curve $[m\beta + n\lambda]$ yields a manifold homeomorphic to $[m, n']$ when $n' \equiv n \bmod m$.

[2] Functions lying on the manifold without branching can therefore be at most l-valued. When one forms the corresponding l-sheeted cover of the manifold, on which l-valued functions become single-valued, the result is a manifold homeomorphic to the sphere. In general, one finds that a μ-sheeted cover of $[l, \lambda]$ is a manifold homeomorphic to $[l/t, r/t]$, where r is the residue of $\mu\lambda$ mod l, and t is the greatest common divisor of l and r. We remark that we have finite groups other than the identity here, whereas all two-sided two-dimensional manifolds with the exception of the simply connected manifold, whose group is trivial, have infinite fundamental groups (see Poincaré An. sit. §14). The question whether there are closed three- dimensional manifolds, other than the sphere, with trivial fundamental group (Poincaré, Compl. 5, p.110) is undecided.

the polygonal decomposition".) If the manifold Ψ_2 is now covered with l sheets $0, 1, 2, \ldots, l-1$ so that a circuit around b corresponds to the cyclic permutation $(0, 1, 2, \ldots, l-1)$ and a circuit around a corresponds to the permutation $(0, \lambda, 2\lambda, \ldots, \lambda(l-1))$, then one obviously obtains the schema of the resulting manifold when one represents each sheet by a tetrahedron $A_iB_iC_iD_i$ and identifies the triangle $B_iC_iD_i$ with the triangle $A_{i+1}C_{i+1}D_{i+1}$, and the triangle $A_iB_iC_i$ with the triangle $A_{i+\lambda}B_{i+\lambda}C_{i+\lambda}$. But this is exactly the above schema of $[l, \lambda]$, so that the latter manifold may be represented as an l-sheeted cover branched over two linked curves.[3]

§21
A definition of Betti numbers different from Poincaré's

The preceding example of the manifolds $[l, \lambda]$ provides a suitable background for a short discussion of the different ways of defining Betti numbers, from which it becomes clear that, with the exception of Poincaré's, which is found to be the simplest and most natural, all these definitions require a supplementary statement in order to be correct.

We first recall the Poincaré definition and suppose that the difficulties pointed out in §§6,7 have been removed, say by a suitable definition of homology. This definition of the Betti number P_m of a manifold V depends on the existence of a system of two-sided closed m-dimensional manifolds $W_1^{(m)}, W_2^{(m)}, \ldots, W_t^{(m)}$ lying in V, between which there is no homology

$$k_1 W_1^{(m)} + k_2 W_2^{(m)} + \ldots + k_t W_t^{(m)} \sim 0,$$

whereas every other two-sided closed m-dimensional manifold $W^{(m)}$ which lies in V is connected with $W_1^{(m)}, W_2^{(m)}, \ldots, W_t^{(m)}$ by a homology. Assuming the existence of such a system, the number t must be the same for all such systems. For if $V_1^{(m)}, V_2^{(m)}, \ldots, V_{t'}^{(m)}$ were a second system with the same

[3]The l-sheeted covering of $[l, \lambda]$ mentioned in the previous footnote corresponds to an l^2-sheeted covering of the spherical manifold S_3 branched along the closed lines a and b, which one obtains when one joins the l^2 sheets $\{i, k\}$, $(i, k = 0, 1, 2, \ldots, l-1)$, in such a way that circuits around b and a bring one from sheet $\{i, k\}$ to sheets $\{i, k+1\}$ and $\{i+1, k+\lambda\}$ respectively. This l^3-sheeted covering of S_3 is homeomorphic to S_3 itself. One can represent each sheet $\{i, k\}$ by a cell in the form of a tetrahedron and thus obtain a schema of S_3 consisting of l^2 tetrahedra with identifications between their faces. In this way one obtains a decomposition of the three dimensional spherical manifold into l^2 tetrahedral regions.

property, and if say $t' < t$ then each manifold $W_i^{(m)}$, multiplied by a suitable integer h_i, would be homologous to a linear combination of the $V_i^{(m)}$, and these t linear combinations could not be linearly independent. But a linear relation between them would give a homology between the $h_i W_i^{(m)}$ and hence between the $W_i^{(m)}$, contrary to hypothesis.

Definitions of the Betti numbers P_m of V other than that of Poincaré are based on systems of (two-sided, closed) m-dimensional manifolds lying in V.[1] However for these the independence of the particular choice of manifolds in the system is by no means guaranteed, and this is the point to which we shall apply the above example.

We consider first of all the definition of Betti numbers in the section with this object in the book of Picard and Simárt: *Théorie des fonctions algébriques de deux variables indépendantes.*[2] The considerations presented to prove the independence of the number from the special choice of manifold system are based essentially on the above definition of Poincaré. However, the definition given does not fully coincide with that of Poincaré but is based on a system of two-sided closed manifolds $W_1^{(m)}, W_2^{(m)}, \ldots, W_t^{(m)}$ lying in V such that

A) no homology of the form (44) holds between the manifolds $W_i^{(m)}$ and

B) every other two-sided closed m-dimensional manifold $W^{(m)}$ that lies in V satisfies a homology

$$W^{(m)} \sim \sum_{i=1}^{t} k_i W_i^{(m)},$$

where the coefficients k_i are integers.[3]

But it can be seen that such a system is by no means always present in a manifold V. E.g., there is no such system when one takes V to be any one of the manifolds $[l, \lambda]$, $(l > 1)$, since not every line $W^{(1)}$ in the latter satisfies the homology $W^{(1)} \sim 0$, although it always satisfies the homology $lW^{(1)} \sim 0$. We can find systems of manifolds $W_1^{(m)}, W_2^{(m)}, \ldots, W_t^{(m)}$ lying in V which, in addition to B, also satisfy the condition A' (in place of A) saying that none

[1] In each case P_m is defined to be one less than the number of manifolds in such a system. As a result we always have to consider the case where the system does not contain a single manifold, in which case P_m is set equal to 1.

[2] Vol. 1, p.28ff.

[3] The case $t = 0$, in which the system contains no manifold at all, here corresponds to the case where each manifold $W^{(m)}$ is null-homologous.

of the manifolds $W_i^{(m)}$ is expressible in terms of the others in the form

$$W_i^{(m)} \sim \sum k_J W_j^{(m)} \quad (j = 1, 2, \ldots, i-1, i+1, \ldots, t)$$

where the k_j are integers.

In order to show that one can sometimes find two such systems, in the same manifold, for which the t values are different we consider the manifold [6,1] and the closed path represented by the edge a^1 of the schema (6,1). We have $6a^1 \sim 0$ and for each closed line l in [6,1] there is a homology $l \sim ka^1$. Each of the systems (a^1), (l_2^1, l_3^1), (l_3^1, l_4^1), (l_5^1) is then a system of closed lines satisfying the conditions A', B, where l_k^1 denotes a closed line satisfying the homology $l_k^1 \sim ka^1$ and consisting of k parts lying close to a^1. Thus [6,1] contains systems of one, and also two, closed lines with the property required. Thus if the definition of Betti number is to be based on such systems, P_m must be defined as one less than the number of members in the smallest system with this property;[4] and this supplementary condition is therefore essential.

The same supplement proves to be necessary for the other definition corresponding to the original conception of Betti numbers.[5] The latter is based on a system of two-sided closed m-dimensional manifolds $W_1^{(m)}, W_2^{(m)}, \ldots, W_t^{(m)}$ in V such that these manifolds (or no single one of them) do not bound any two-sided $(m+1)$-dimensional manifold lying in V, while the manifolds $W_i^{(m)}$ (or a single one of them) in combination with every other two-sided closed m-dimensional manifold $W^{(m)}$, do bound a two-sided $(m+1)$-dimensional manifold in V. These conditions on the system $W_1^{(m)}, W_2^{(m)}, \ldots, W_t^{(m)}$ may be expressed precisely with the help of the homology concept as follows: there is no homology of the form

$$W_{i_1}^{(m)} + W_{i_2}^{(m)} + \ldots + W_{i_r}^{(m)} \sim W_{j_1}^{(m)} + W_{j_2}^{(m)} + \ldots + W_{j_s}^{(m)}$$

whereas for each $W^{(m)}$ there is a homology

$$W^{(m)} \sim W_{k_1}^{(m)} + W_{k_2}^{(m)} + \ldots + W_{k_\rho}^{(m)} - W_{l_1}^{(m)} - W_{l_2}^{(m)} - \ldots - W_{l_\sigma}^{(m)}$$

[4] Of course one could also, instead of a single number P_m, take all those numbers that, when reduced by one, are possible sizes for such a system $W_1^{(m)}, W_2^{(m)}, \ldots$. But then the question arises whether these numbers can be determined from Poincaré's Betti numbers and torsion numbers, or at least from the fundamental group.

[5] See the work of Betti and the fragment of Riemann cited in note 1 of §6.

where $i_1, i_2, \dots, i_r, j_1, j_2, \dots, j_s$ and similarly $k_1, k_2, \dots, k_\rho, l_1, l_2, \dots, l_\sigma$ are understood to be all distinct numbers in the series $1, 2, \dots, t$. The example of the manifold $[8, 1]$ shows that one and the same manifold can contain two or more such systems $W_i^{(m)}$ with different t values; namely, the systems (a^1, l_2^1, l_4^1), (a^1, l_3^1), (l_2^1, l_3^1, l_4^1) all have the required property, where a^1 is the closed line represented by the edge of the schema [8,1] and l_k^1 denotes a closed line satisfying the homology $l_k^1 \sim ka^1$ as above. In this case the Betti numbers are also determined by the supplementary requirement that the system with the given property be the smallest possible.

§22
Manifolds that agree with respect to the topological invariants previously considered

By way of conclusion we would like to point out that the manifolds $[l, \lambda]$ of §20 appear to hold a certain interest for the basic problem of analysis situs, that of obtaining necessary and sufficient conditions for the homeomorphism of two manifolds. The ultimate solution of this problem would be a means of determining a system of topological invariants for a given manifold such that agreement of all invariants in the system for two manifolds enables us to conclude that they are homeomorphic. Now two manifolds $[l, \lambda]$ with the same value of l agree in their fundamental groups, and hence in all topological invariants that we presently have at our disposal. The question therefore arises, whether two manifolds $[l, \lambda]$ with the same l value are always homeomorphic.

One observation is immediate, namely that $[l, \lambda]$ and $[l, l-\lambda]$ are homeomorphic manifolds, since one obviously obtains the schema (l, λ) from the schema $(l, l-\lambda)$ by reflection of the cell a^3. For $l = 2$, in which case one has in [2,1] the projective space T_3 of §9, this remark is trivial. For $l > 2$ the remark implies that at most $\varphi(l)/2$ of the manifolds in question can be different: $l = 5$ is therefore the smallest value of l for which one obtains two manifolds, [5,1] and [5,2] whose homeomorphism is questionable.

One can view the question of homeomorphism between [5,1] and [5,2] as follows.[1] In [5,1] one considers the points of the geometric figure consisting

[1] One knows no process for deciding such questions, which are also difficult to settle intuitively. E.g., the homeomorphism between the schema: tetrahedron $ABCD$ with identifications $BCD = ACD = a_1^2$, $ABC = DAB = a_2^2$ and the schema σ_3 is not immediately obvious because of the complicated position of the two-dimensional complex a_2^2.

of the surface a^2. One obtains this from a pentagon $M_0M_1M_2M_3M_4$ when one identifies the sides M_0M_1, M_1M_2, ..., M_4M_1 in the order written. In this way one obtains a "two-dimensional complex" characterized by the property that the boundary of the surface piece describes a closed line five times. It is a question of the position of this two-dimensional complex in the manifold [5,1].

One obtains a better view when one draws the meridian semicircles through the points M_0, M_1, ..., M_4 that divide a^2 into five triangles $NM_0M_1 = SM_1M_2$, $NM_1M_2 = SM_2M_3$, etc., which may be called, in order, d_0, d_1, d_2, d_3, d_4. Two sides of such a triangle lead from the point $N = S$ to a^0, the third side consists of the edge a^1 leading from a^0 to a^0. Let A be a point on a^1. In the neighbourhood of A one draws a small loop L in the manifold [5,1] around the edge a^1. This line pierces the triangles d_i, and indeed in the same order d_0, d_1, d_2, d_3, d_4 as these triangles are placed around the point $N = S$.

Now if [5,1] and [5,2] are homeomorphic it must be possible to place a two-dimensional complex in [5,2] in such a way that

1. it is homeomorphic to the complex a^2 in [5,1] and therefore consists of a surface piece whose boundary wraps five times around a closed line,

2. it lies in [5,2] the same way the previous complex lay in [5,1] and

3. it dissects [5,2] into a simply connected space.

Condition 1 is obviously satisfied by the complex represented by a^2 in [5,2], but not condition 2. Namely, if one carries out the decomposition of a^2 into five triangles in [5,2] the same way as in [5,1], and again denotes them d_0, d_1, d_2, d_3, d_4 in the order in which they lie around $N = S$, then a small loop around the edge a^1 pierces them in the order d_0, d_2, d_4, d_1, d_5. Thus a two-dimensional complex in [5,2] satisfying the conditions 1, 2, 3 must be placed differently from a^2.

We do not go further into this question, seeing only that it essentially involves certain order relations. The fundamental group is already distinguished from the other topological invariants (Betti numbers, torsion numbers) by the fact that it reflects certain order relations to a higher degree. However, one obtains these other topological invariants by the process of letting the operations of the fundamental group commute. The reflections on the manifolds [5,1] and [5,2], which both have the cyclic group of order 5 as fundamental group, show that certain order relations in the schemata are also not expressed in the fundamental group.