A PROOF OF JORDAN'S THEOREM ABOUT A SIMPLE CLOSED CURVE.*

BY J. W. ALEXANDER.

§ 1. We give below an easy proof of Jordan's theorem that a simple closed curve subdivides the Euclidean plane into two and only two regions.† The argument is based on elementary combinatorial properties of chains, or systems of polygons, and may be generalized at once to any number of dimensions, as will be shown in a subsequent paper.

The necessary properties of chains are somewhat weaker than the properties of polygons which are used as the starting point of most proofs. They are recalled very hastily in Part I, following the general lines exposed by Prof. O. Veblen in a paper "On the Decomposition of an n-Space by a Polyhedron" ‡ which deals with a more inclusive problem and to which we refer for fuller details.

I. Chains and Their Properties.

§ 2. A *chain* will be a sort of generalized polygon consisting of a finite number of non-intersecting *edges* (which may be either line segments or rays), and *vertices* (the end points of the edges), where at each vertex there end an even number of edges. A chain need not be connected.

Suppose we have a chain whose edges are all segments. Then if two vertices, Y and Z, may be joined by a broken line made up of elements of the chain, they may also be joined by a second broken line which has no edge in common with the first. For if we remove from the chain the edges of the first broken line, there will still remain an even number of edges abutting at every vertex except Y and Z where there will now remain an odd number. But, within each connected group of edges and vertices, the total number of times that edges abut on vertices is equal to twice the number of edges and is therefore even. Hence, the vertices Y and Z still belong to the same connected piece and may again be joined by a broken line.

A simple illustration of a chain would be a pair of broken lines con-

* Read before the American Mathematical Society, July, 1916.

† First correctly proved by O. Veblen, Trans. of Amer. Math. Soc., vol. VI, p. 83; 1906. Simplified proofs have been given by L. E. J. Brouwer and others. Cf. Brouwer, Mathematische Annalen, vol. 69, p. 169; 1910.

‡ Trans. of Amer. Math. Soc., vol. XIV, No. 1, pp. 65–72; Jan., 1913.

necting the same two points, Y and Z, and having only a finite number of points in common.

§ 3. A chain, k, like a simple polygon, has two "sides," though the sides are not in general connected regions. We may determine them as follows.*

We complete the lines to which the edges of the chain k belong and thus obtain a system of lines which subdivide one another into a finite number of line segments and rays, $b_1, b_2, \cdots, b_n$, while they subdivide the plane into a finite number of convex regions, $a_1, a_2, \cdots, a_m$. Now, the boundaries of the regions a_i are chains made up of sets of elements b_j and their end points. Out of the symbols for the elements in these sets, we shall form the expressions

$$a_i = b_{i1} + b_{i2} + b_{i3} + \cdots + b_{ik} \quad (i = 1, 2, \cdots, m), \tag{1}$$

which will be used to designate the boundaries of the various cells a_i. The expressions (1) will be combined by adding corresponding members, collecting terms, and reducing all coefficients modulo 2. In this way, we can obtain new combinations defining new chains whose edges can be read off from the right-hand members.

Now it can be shown without difficulty that any chain such as k composed of elements b_i and their end points can be derived from the elementary chains (1) in two and only two ways,

$$\Sigma_1 a_i = k \tag{2}$$

and

$$\Sigma_2 a_i = k, \tag{2'}$$

and that the boundary of each region a_i occurs in one and only one of the combinations. Therefore, the points of the plane fall into two classes according as they belong to the interior or boundary of a region occurring in the first combination or of a region occurring in the second. These two classes of points will be called the *sides* of the chain k.

§ 4. We point out one more theorem that will be used later. Suppose we have two chains, k_1 and k_2. Then the set of points which are on given sides both of the chain k_1 and of the chain k_2 may be subdivided into a finite number of convex regions. Therefore, the set is bounded by a chain composed of the sum, modulo 2, of the boundaries of the convex regions. By combining this chain with the chain k_1, we obtain a new chain k_3.

II. Jordan's Theorem.

§ 5. In the following discussion, a *region* will be a set of points each of which is interior to a triangle enclosing only points of the set while

* Cf. Veblen, loc. cit., for details.

any two may be joined by an arc made up of points of the set. The first condition is satisfied by the complementary set to any closed set of points in the plane. Moreover, when the second condition is also satisfied, two points, Y and Z, of the region may always be connected within the region by a broken line which may be so chosen as to have only a finite number of points in common with any preassigned finite system of lines, a property which we shall use later on.

To prove this property, we observe that about any point, P, of the arc joining the points Y and Z, we may place a triangle which encloses only points of the region. Therefore, within this triangle, we may find a sub-arc containing the point P, such that any two points of this sub-arc may be joined by a broken line of the required type and such that the point P is not an end point of the sub-arc unless it is an end point of the arc YZ itself. Since the whole arc is covered by these sub-arcs, it may be covered by a finite number of them, by the Heine-Borel theorem. We may therefore construct a broken line connecting the points Y and Z by piecing together a series of broken lines running from one sub-segment to an adjacent one, and so chosen that no two of them have more than a finite number of points in common. The broken line thus obtained may cross itself a finite number of times. When this is the case, we can clearly obtain a broken line without singularities by merely suppressing a certain number of supplementary loops, but for the purposes of this paper there will be no objection in allowing a broken line to have a finite number of singularities.

§ 6. Lemma. *Let ACB be a simple arc passing through a point C and ending at the points A and B, and let Y and Z be any two points of the plane not on the arc ACB. Then, if the points Y and Z are not separated by either of the sub-arcs AC or CB, either are they separated by the arc ACB itself.*

For the points Y and Z may be connected by a pair of broken lines, a and b, such that the first does not meet the arc AC nor the second the arc CB. Moreover, by § 5, the broken line b may be so chosen as to meet the broken line a in at most a finite number of points, in which case, it may be combined with a to form a chain k.

Now, consider such points of the arc CB as are either on the chain k, (that is, on the broken line a), or on the opposite side of the chain k from the point C. Each of these points may be enclosed within a triangle which neither meets nor encloses a point of the arc AC or of the broken line b, and since the set of all such points is closed, they may all be enclosed within a finite number of these triangles, by the Heine-Borel theorem.

Following § 4, let us add modulo 2 to the chain k the boundaries of the finite set of convex regions made up of points which are both interior to one of the triangles and on the opposite side of the chain k from the point C. We thus obtain a new chain, k', which still contains the broken line b, as well as a supplementary piece, a', made up of segments which neither meet nor end on the arc ABC. Therefore, by § 2, the points Y and Z may be joined by a broken line within the piece a' and consequently by one which does not meet the arc ACB.

§ 7. THEOREM. *The points of the plane not on a simple arc AB do not form more than one connected region.* For we shall prove that any two such points, Y and Z, may be joined by a broken line which does not meet the arc AB.

About any point, C, of the arc AB, we may place a triangle with respect to which Y and Z are exterior points. Therefore, by remaining within this triangle, we may find a sub-arc of the arc AB which does not separate the points Y and Z, which contains the point C, and which ends at the point C only when C is one of the points A or B. The arc AB may thus be covered by a set of overlapping sub-arcs, and consequently by a finite set of overlapping sub-arcs, such that no one of them separates the points Y and Z. But the end points of this last set subdivide the arc AB into a still smaller finite set of non-overlapping sub-arcs. Therefore, since the arc AB may be built up by piecing together these sub-arcs, it cannot separate the points Y and Z, by § 6.

§ 8. THEOREM. *The points of the plane not on a simple closed curve do not form more than two connected regions.* For we shall prove that, given any three such points, X, Y, and Z, two of them, at least, may always be connected by a broken line which does not meet the curve.

Let A, B, and C be any three distinct points of the curve. Then, by § 7, the points X and Y, Y and Z, and Z and X may be joined by three broken lines, a, b, and c, respectively which do not meet the arcs CAB, ABC, and BCA respectively. Moreover, the broken lines a, b, and c may be so chosen that no two of them have more than a finite number of points in common, so that they may be combined to form a chain, k.

Now, of the three points A, B, and C, two, at least, must be on the same side of the chain k, and we can assume without loss of generality that the points B and C are. It then follows by a direct paraphrase of the reasoning in § 6 that the points Y and Z can be joined by a broken line which does not meet the curve. For we have only to substitute the arc CAB for the arc AC and the broken line bc for the broken line b and repeat the rest of the argument word for word.

§ 9. THEOREM. *The points of the plane not on a simple closed curve form at least two connected regions.*

Choose any two points, A and B, on the curve and denote by AB and

BA respectively the two arcs of the curve bounded by these points. Then any line, l, which separates the points A and B meets the arcs AB and BA in two closed sets of points.

Now, every point of the first set is interior to some interval of the line l which contains no point of the arc BA, hence, by the Heine-Borel theorem, the entire set may be covered by a finite number of such intervals. Moreover, by combining intervals when necessary, we may so arrange that no two of them either touch or overlap. We shall prove that the end points of this last set of intervals, i, are not all within the same region by showing that, however we may connect them in pairs by a system of broken lines, one or more of the broken lines will always meet the curve.

Consider such a system of broken lines, assuming, as we may, that no one of them meets the line l or another broken line of the system in more than a finite number of points. Then the system of lines may be combined with the intervals i to form a chain, k. Moreover, if we add to the chain k the boundary of one side of the line l (that is, the line l itself), we shall obtain a second chain k' made up of the broken lines combined with the segments, s, of the line l complementary to the intervals i. By the definition of the sides of a chain (Relations (2) and (2′), § 4), it is clear that one or other of the chains k and k' separates the points A and B.

Now, if the chain k separates the points A and B, it surely meets the arc BA. But the arc BA cannot meet the intervals i and must therefore meet one of the broken lines of the system. Similarly, if the net k' separates the points A and B, the other arc AB must meet one of the broken lines, since it cannot meet the segments s. Therefore, in either case, the curve meets one of the broken lines, proving that the ends of the intervals i do not all belong to the same region.

§ 10. Corollary. *A point Z not on a simple closed curve may be connected to any arc of the curve AB by a broken line z which, except for one end point, lies wholly within the same region as the point Z.*

For let Y be any point on the opposite side of the curve from Z. Then, by § 6, the points Y and Z may be connected by a broken line which does not meet the other arc BA of the curve and which therefore meets the arc AB. This broken line evidently includes the required broken line z connecting the point Z to a point of the arc AB.

Since the point Z and broken line z may be chosen on either side of the curve, and since the arc AB may be made arbitrarily small, we also have at once the following proposition:

Corollary. In the neighborhood of any point, A, of a simple closed curve, there are points from each side of the curve.

This would also have followed at once from § 9 if we had used an arbitrarily small triangle about the point A instead of the line l.