CORRIGENDUM TO
"GROUPOIDS, THE PHRAGMEN-BROUWER PROPERTY,
AND THE JORDAN CURVE THEOREM",
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(*communicated by Fred Cohen*)

Abstract

Omar Antolín Camarena pointed out a gap in the proofs in [**BT&G, Bro06**] of a condition for the Phragmen–Brouwer Property not to hold; this note gives the correction in terms of a result on a pushout of groupoids, and some additional background.

1.   Introduction

This note fills in two ways a gap in a proof in [**Bro06**], as explained in Section 2.

The paper [**Bro06**] shows how groupoid methods can be used to relate fundamental groups to a classic separation property of spaces, often called the Phragmen-Brouwer Poperty. However it turns out, as we explain in Section 2, that the applications to the Jordan Curve Theorem require the following stronger result on groupoids:

Theorem 1.1. *Suppose given a pushout of groupoids*

$$\begin{array}{ccc} C & \xrightarrow{\ j\ } & B \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle v} \\ A & \xrightarrow[\ u\ ]{} & G \end{array} \qquad\qquad (1)$$

such that i, j are bijective on objects, C is totally disconnected, and G is connected. Then G contains as a retract a free groupoid whose vertex groups are of rank

$$k = n_C - n_A - n_B + 1,$$

where n_P is the number of components of the groupoid P for $P = A, B, C$ (assuming these numbers are finite).

Further, if C contains distinct objects a, b such that $A(ia, ib), B(ja, jb)$ are nonempty, then F has rank at least 1.
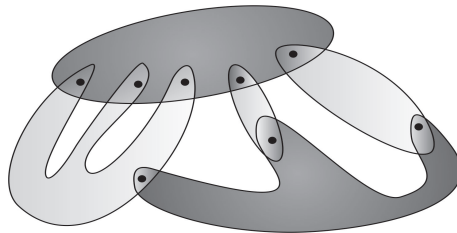
The proof is given in Section 3. It is easily seen how the Theorem applies to topological situations like the following:

for which covering space methods are more awkward.

2.   The Phragmen-Brouwer Property

One source of confusion is that there are two forms of this property, as follows.

Let X be a topological space. We say a subspace D of X *separates* X if $X \setminus D$ has more than one component. We say a subset D of X *separates the points a and b of X* if a and b lie in distinct components of $X \setminus D$.

A topological space X is said to have the *Phragmen-Brouwer Property I*, here abbreviated to PBI), if X is connected and the following holds:

PBI): If D and E are disjoint closed subsets of X and a, b are points of X not in $D \cup E$ such that neither D nor E separate a and b, then $D \cup E$ does not separate a and b.

We also consider:

PBI$'$): If D, E are disjoint closed subsets of X such that neither D nor E separate X, then $D \cup E$ does not separate X.

Such a space X is also called *unicoherent.* In [**Wil49**, pp. 47–49] it is proved that these two properties PBI), PBI$'$), are equivalent if X is connected and locally connected.

It is of interest and importance to relate these properties to the fundamental group of the space X; see [**Eil37**, Section3], and for further information on this area see [**Wil49, GMI89**].

Now we can state what is the gap in [**Bro06**]: Proposition 4.1 in that paper shows that if X is connected and does not satisfy PBI$'$), then the fundamental group of X contains the integers as a retract. However what is needed for the Jordan Curve Theorem is actually Corollary 3.1 in the next section, which is the same criterion but for X not satisfying PBI). This is required because the proof of Proposition 4.3 in [**Bro06**] uses PBI). The equivalence of PBI) and PBI$'$) fills the gap, but the proof in the next section fits better with the goal of the original proof of letting the algebra of groupoids do most of the work and minimizing the point-set topology input.

3.   Proof of Theorem 1.1

Let Fr : DirectedGraphs $\to$ Groupoids be the free groupoid functor. From [**BT&G**, Chapter 8][1] it follows that any groupoid G has a retraction $G \to \mathsf{Fr}\, W$ where W is a forest.

Let Z be the set of objects of C (and of A, B and G) regarded as a directed graph with no edges. Pick spanning forests X and Y of the underlying directed graphs of A and B. Then there are retractions

$$C \to \mathsf{Fr}\, Z, \; A \to \mathsf{Fr}\, X, \; B \to \mathsf{Fr}\, Y.$$

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[1]See also [**Hig05**].

By "span" in a category we mean a pair of arrows $U \leftarrow W \rightarrow V$; this is the shape of a diagram whose pushout you can take. Then the following diagram in Groupoids commutes and its rows are spans:

$$
\begin{array}{ccc}
\mathsf{Fr}\,X & \longleftarrow \mathsf{Fr}\,Z \longrightarrow & \mathsf{Fr}\,Y \\
\downarrow & \downarrow & \downarrow \\
A & \longleftarrow\quad C \quad\longrightarrow & B \\
\downarrow & \downarrow & \downarrow \\
\mathsf{Fr}\,X & \longleftarrow \mathsf{Fr}\,Z \longrightarrow & \mathsf{Fr}\,Y
\end{array}
$$

So the span $\mathsf{Fr}\,X \leftarrow \mathsf{Fr}\,Z \rightarrow \mathsf{Fr}\,Y$ is a retract of the span $A \leftarrow C \rightarrow B$. This implies that the pushout, say F, of $\mathsf{Fr}\,X \leftarrow \mathsf{Fr}\,Z \rightarrow \mathsf{Fr}\,Y$ is a retract of G (which is the pushout of $A \leftarrow C \rightarrow B$). Since the span of free groupoids is actually the image under Fr of the obvious span $X \leftarrow Z \rightarrow Y$ of graphs, and since Fr is a left adjoint, this pushout F is actually just $\mathsf{Fr}\,W$ where W is the pushout in the category of directed graphs of $X \leftarrow Z \rightarrow Y$.

This graph W is connected because G is connected, so, denoting by $e(\mathrm{Q})$ and $v(\mathrm{Q})$ the number of vertices of a graph, the vertex groups in $\mathsf{Fr}\,W$ are free of rank $k = e(W) - v(W) + 1$. We have $v(W) = v(X) = v(Y) = v(Z) = n_C$; and, since Z has no edges, $e(W) = e(X) + e(Y)$. Also, since X is a spanning forrest we have $e(X) = v(X) - n_A = n_C - n_A$, and similarly, $e(Y) = n_C - n_A$. Putting this all together, the vertex groups in F have rank $(n_C - n_A) + (n_C - n_B) - n_C + 1 = n_C - n_A - n_B + 1$, as claimed.

For the last part of the theorem, we choose X, Y so that the elements α, β of $A(ia, ib), B(jb, ja)$ respectively are parts of $\mathsf{Fr}\,X, \mathsf{Fr}\,Y$ respectively. These map to elements α', β' in F and the element $\alpha'\beta'$ will be nontrivial in F; so F has rank at least 1. This completes the proof.

The next Corollary is an essential part of the proof of the Jordan Curve Theorem. It appears, without the retraction condition, as part of [**Mun75**, Theorem 63.1], and also as [**Bro06**, Proposition 4.1],[**BT&G**, 9.2.1].

Corollary 3.1. *If the space X is path connected and does not have the PBI), then its fundamental group at any point contains the infinite cyclic group as a retract.*

Proof. The proof now follows the methods of [**Bro06**], replacing that paper's Corollary 3.5 with Theorem 1.1. □

References

[**Bro67**]   Brown, R. 'Groupoids and van Kampen's theorem'. *Proc. London Math. Soc.* **17** (3) (1967) 385–401.

[**Bro06**]   Brown, R. 'Groupoids, the Phragmen-Brouwer property and the Jordan curve theorem'. *J. Homotopy and Related Structures* **1** (2006) 175–183.

[**BT&G**]   Brown, R. *Topology and Groupoids*, Booksurge LLC, S. Carolina, (2006).

[**BRa84**]   Brown, R. and Razak, A., 'A van Kampen theorem for unions of non-connected spaces', *Archiv. Math.* 42 (1984) 85-88.

[**Eil37**]   Eilenberg, S. 'Sur les espaces multicoherents II', *Fund. Math.* **29** (1937) 101–122.

[**GMI89**]   García-Máynez, A. and Illanes, A. 'A survey of multicoherence', *An. Inst. Autonóma Mexico* **29** (1989) 17-67.

[**Hig05**]   Higgins, P. J. 'Categories and groupoids'. *Van Nostrand Mathematical Studies, 1971, Reprints in Theory and Applications of Categories* **7** (2005) 1–195.

[**Hun74**]   Hunt, J.V. 'The Phragmen-Brouwer Theorem for separated sets', *Bol. Soc. Sei. Mexicana* 19 (1974) 26–35.

[**Kam33**]   Kampen, E. H. van 'On the connection between the fundamental groups of some related spaces'. *Amer. J. Math.* **55** (1933) 261–267.

[**Mun75**]   Munkres, J. R. *Topology: a first course.* Prentice-Hall, Englewood Cliffs (1975).

[**Wil49**]   Wilder, R. L. *Topology of manifolds*, *AMS Colloquium Publications*, Volume 32. American Mathematical Society, New York (1949).

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