SMSTC Geometry and Topology 2011–2012

Lecture 7

The classification of surfaces

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Manifolds

- An *n*-**dimensional manifold** M is a topological space such that each $x \in M$ has an open neighbourhood $U \subset M$ homeomorphic to n-dimensional Euclidean space $\mathbb{R}^n$

$$U \cong \mathbb{R}^n .$$

- Strictly speaking, need to include the condition that M be Hausdorff and paracompact = every open cover has a locally finite refinement.
- Called *n*-**manifold** for short.
- Manifolds are the topological spaces of greatest interest, e.g. $M = \mathbb{R}^n$.
- Study of manifolds initiated by Riemann (1854).
- A **surface** is a 2-dimensional manifold.
- Will be mainly concerned with manifolds which are compact = every open cover has a finite refinement.

Why are manifolds interesting?

- Topology.
- Differential equations.
- Differential geometry.
- Hyperbolic geometry.
- Algebraic geometry. Uniformization theorem.
- Complex analysis. Riemann surfaces.
- Dynamical systems,
- Mathematical physics.
- Combinatorics.
- Topological quantum field theory.
- Computational topology.
- Pattern recognition: body and brain scans.
- ...

Examples of n-manifolds

- The n-**dimensional Euclidean space** $\mathbb{R}^n$
- The n-**sphere** S^n.
- The n-**dimensional projective space**

$$\mathbb{RP}^n = S^n/\{z \sim -z\} .$$

- **Rank theorem in linear algebra**. If $J : \mathbb{R}^{n+k} \to \mathbb{R}^k$ is a linear map of rank k (i.e. onto) then $J^{-1}(0) = \ker(J) \subseteq \mathbb{R}^{n+k}$ is an n-dimensional vector subspace.
- **Implicit function theorem**. The solutions of differential equations are generically manifolds. If $f : \mathbb{R}^{n+k} \to \mathbb{R}^k$ is a differentiable function such that for every $x \in f^{-1}(0)$ the Jacobian $k \times (n+k)$ matrix $J = (\partial f_i/\partial x_j)$ has rank k, then

$$M = f^{-1}(0) \subseteq \mathbb{R}^{n+k}$$

  is an n-manifold.
- In fact, every n-manifold M admits an embedding $M \subseteq \mathbb{R}^{n+k}$ for some large k.

Manifolds with boundary

- An n-**dimensional manifold with boundary** $(M, \partial M \subset M)$ is a pair of topological spaces such that
  (1) $M \backslash \partial M$ is an n-manifold called the **interior**,
  (2) ∂M is an $(n-1)$-manifold called the **boundary**,
  (3) Each $x \in \partial M$ has an open neighbourhood $U \subset M$ such that
  $$(U, \partial M \cap U) \cong \mathbb{R}^{n-1} \times ([0, \infty), \{0\}) .$$
- A manifold M is **closed** if $\partial M = \emptyset$.
- The boundary ∂M of a manifold with boundary $(M, \partial M)$ is closed, $\partial \partial M = \emptyset$.
- **Example** (D^n, S^{n-1}) is an n-manifold with boundary.
- **Example** The product of an m-manifold with boundary $(M, \partial M)$ and an n-manifold with boundary $(N, \partial N)$ is an $(m+n)$-manifold with boundary
  $$(M \times N, M \times \partial N \cup_{\partial M \times \partial N} \partial M \times N) .$$

The classification of n-manifolds I.

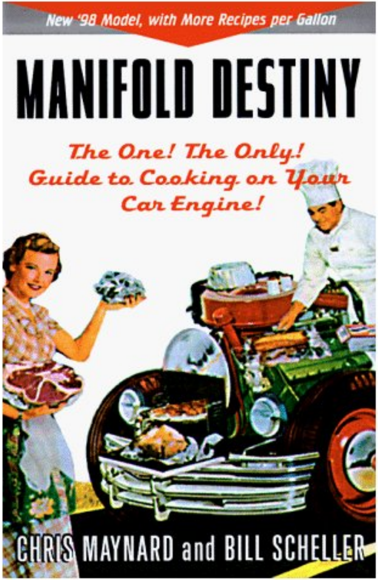
- Will only consider compact manifolds from now on.
- A function

$$i \ : \ \text{a class of manifolds} \to \text{a set} \ ; \ M \mapsto i(M)$$

  is a **topological invariant** if $i(M) = i(M')$ for homeomorphic M, M'. Want the set to be finite, or at least countable.
- **Example 1** The dimension $n \geqslant 0$ of an n-manifold M is a topological invariant (Brouwer, 1910).
- **Example 2** The number of components $\pi_0(M)$ of a manifold M is a topological invariant.
- **Example 3** The orientability $w(M) \in \{-1, +1\}$ of a connected manifold M is a topological invariant.
- **Example 4** The Euler characteristic $\chi(M) \in \mathbb{Z}$ of a manifold M is a topological invariant.
- A **classification** of n-manifolds is a topological invariant i such that $i(M) = i(M')$ if and only if M, M' are homeomorphic.

The classification of n-manifolds II. $n = 0, 1, 2, \ldots$

- **Classification of 0-manifolds** A 0-manifold M is a finite set of points. Classified by $\pi_0(M) =$ no. of points $\geqslant 1$.
- **Classification of 1-manifolds** A 1-manifold M is a finite set of circles S^1. Classified by $\pi_0(M) =$ no. of circles $\geqslant 1$.
- **Classification of 2-manifolds** Classified by $\pi_0(M)$, and for connected M by the fundamental group $\pi_1(M)$. Details to follow!
- For $n \leqslant 2$ homeomorphism $\Longleftrightarrow$ homotopy equivalence.
- It is theoretically possible to classify 3-manifolds, especially after the Perelman solution of the Poincaré conjecture.
- It is not possible to classify n-manifolds for $n \geqslant 4$. Every finitely presented group is realized as $\pi_1(M) = \langle S | R \rangle$ for a 4-manifold M. The word problem is undecidable, so cannot classify $\pi_1(M)$, let alone M.


New '98 Model, with More Recipes per Gallon
MANIFOLD DESTINY
The One! The Only! Guide to Cooking on Your Car Engine!
CHRIS MAYNARD and BILL SCHELLER

How does one classify surfaces?

- (1) Every surface M can be **triangulated**, i.e. is homeomorphic to a finite 2-dimensional cell complex

$$M \;\cong\; \bigcup_{c_0} D^0 \cup \bigcup_{c_1} D^1 \cup \bigcup_{c_2} D^2 \ .$$

- (2) Every connected triangulated M is homeomorphic to a normal form

$$M(g) \text{ orientable, genus } g \geqslant 0 \ ,$$

$$N(g) \text{ nonorientable, genus } g \geqslant 1$$

- (3) No two normal forms are homeomorphic.
- Similarly for surfaces with boundary, with normal forms $M(g,h)$, $N(g,h)$ with genus g, and h boundary circles.
- History: (2)+(3) already in 1860-1920 (Möbius, Clifford, van Dyck, Dehn and Heegaard, Brahana). (1) only in the 1920's (Rado, Kerékjártó). Today will only do (3), by computing π_1 of normal forms.

A page from Dehn and Heegaard's Analysis Situs (1907)

Jede geschlossene Fläche kann stets mit drei Elementarflächenstücken bedeckt werden. Jede nicht geschlossene Fläche und jede Kugelfläche kann mit zwei Elementarflächen bedeckt werden[95a]).

d) *Normalformen für geschlossene Flächen*[96]).

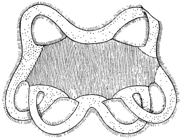


Fig. 8.

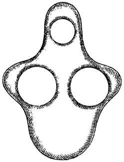


Fig. 9.

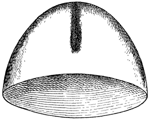


Fig. 10.

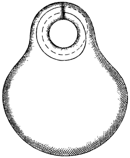


Fig. 11.

α) *Zweiseitige Flächen.* Eine Fläche, deren Restfläche p Doppelbänder hat, ist homöomorph mit einer Kugel mit p „*Henkeln*" (Fig. 9);

95) *Möbius*, Leipzig Ber. 15 (1863) = Werke 2, p. 450.

96) Diese Formen für geschlossene Flächen sind, soweit zweiseitige Flächen in Betracht kommen, als betrachtet worden von *Riemann* (cf. *Klein*, Über Riemanns Theorie . . . (1882), p. IV), *Möbius*, a. a. O. § 16, *Tonelli* (Rom Linc. Atti (2) 2 (1875), p. 594, vgl. Rom Linc. Rend. (5) 4^1 (1895), p 300; *W. K. Clifford* London Proc. Math. Soc. 8 (1877), p. 292). Normalformen für einseitige Flächen sind von *Dyck* a. a. O. aufgestellt.

The connected sum I.

- Given an n-manifold with boundary $(M, \partial M)$ with M connected use any embedding $D^n \subset M \backslash \partial M$ to define the **punctured n-manifold with boundary**

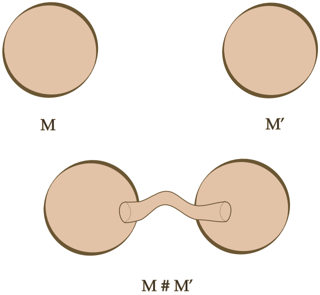
$$(M_0, \partial M_0) \; = \; (\text{cl.}(M \backslash D^n), \partial M \cup S^{n-1}) \; .$$

- The **connected sum** of connected n-manifolds with boundary $(M, \partial M)$, $(M', \partial M')$ is the connected n-manifold with boundary

$$(M \# M', \partial(M \# M')) \; = \; (M_0 \cup_{S^{n-1}} M'_0, \partial M \cup \partial M') \; .$$

  Independent of choices of $D^n \subset M \backslash \partial M$, $D^n \subset M' \backslash \partial M'$.
- If M and M' are closed then so is $M \# M'$.

The connected sum II.

- 

- The connected sum # has a neutral element, is commutative and associative:

$$\begin{array}{ll} \text{(i)} & M\#S^n \cong M' , \\ \text{(ii)} & M\#M' \cong M'\#M , \\ \text{(iii)} & (M\#M')\#M'' \cong M\#(M'\#M'') . \end{array}$$

The fundamental group of a connected sum

- If $(M, \partial M)$ is an n-manifold with boundary and M is connected then M_0 is also connected. Can apply the Seifert-van Kampen Theorem to
$$M \;=\; M_0 \cup_{S^{n-1}} D^n$$
to obtain
$$\pi_1(M) \;=\; \pi_1(M_0) *_{\pi_1(S^{n-1})} \{1\} \;=\; \begin{cases} \pi_1(M_0) & \text{for } n \geqslant 3 \\ \pi_1(M_0)/\langle \partial \rangle & \text{for } n = 2 \end{cases}$$
with $\langle \partial \rangle \triangleleft \pi_1(M_0)$ the normal subgroup generated by the boundary circle $\partial : S^1 \subset M_0$.
- Another application of the Seifert-van Kampen Theorem gives
$$\begin{aligned} \pi_1(M \# M') \;&=\; \pi_1(M_0) *_{\pi_1(S^{n-1})} \pi_1(M_0') \\ &=\; \begin{cases} \pi_1(M) * \pi_1(M') & \text{for } n \geqslant 3 \\ \pi_1(M_0) *_{\mathbb{Z}} \pi_1(M_0') & \text{for } n = 2\ . \end{cases} \end{aligned}$$

Orientability for surfaces

- Let M be a connected surface, and let $\alpha : S^1 \to M$ be an injective loop.
  - α is **orientable** if the complement is not connected, in which case it has 2 components.
  - α is **nonorientable** if the complement $M\backslash\alpha(S^1)$ is connected.
- **Definition** M is **orientable** if every $\alpha : S^1 \to M$ is orientable.
- **Jordan Curve Theorem** $\mathbb{R}^2$ is orientable.
- **Example** The 2-sphere S^2 and the torus $S^1 \times S^1$ are orientable.
- **Definition** M is **nonorientable** if there exists a nonorientable $\alpha : S^1 \to M$, or equivalently if Möbius band $\subset M$.
- **Example** The Möbius band, the projective plane $\mathbb{RP}^2$ and the Klein bottle K are nonorientable.
- **Remark** Can similarly define orientability for connected n-manifolds M, using $\alpha : S^{n-1} \to M$, $\pi_0(M\backslash\alpha(S^{n-1}))$.

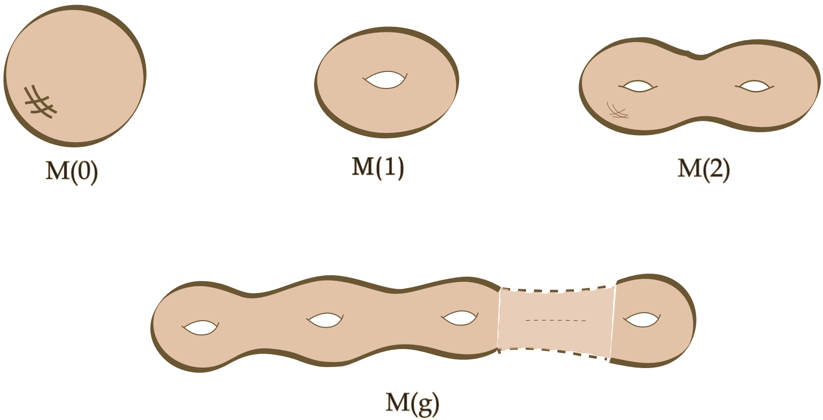
The orientable closed surfaces $M(g)$ I.

- **Definition** Let $g \geqslant 0$. The **orientable connected surface with genus** g is the connected sum of g copies of $S^1 \times S^1$

$$M(g) \ = \ \underset{g}{\#}(S^1 \times S^1)$$

- **Example** $M(0) = S^2$, the 2-sphere.
- **Example** $M(1) = S^1 \times S^1$, the torus.
- **Example** $M(2) =$ the 2-holed torus, by Henry Moore.

The orientable closed surfaces $M(g)$ II.



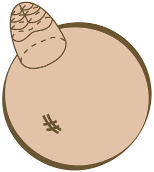
The nonorientable surfaces $N(g)$ I.

- Let $g \geqslant 1$. The **nonorientable connected surface with genus** g is the connected sum of g copies of $\mathbb{RP}^2$

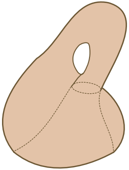
$$N(g) \;=\; \#_{g}\, \mathbb{RP}^2$$

- **Example** $N(1) = \mathbb{RP}^2$, the projective plane.
- Boy's immersion of $\mathbb{RP}^2$ in $\mathbb{R}^3$ (in Oberwolfach)

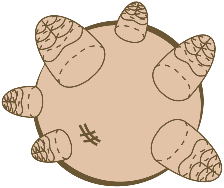
The nonorientable closed surfaces $N(g)$ II.



Projective plane = N(1)



Klein bottle = N(2)



N(g)

The Klein bottle

- **Example** $N(2) = K$ is the Klein bottle.
- The Klein bottle company

The classification theorem for closed surfaces

- **Theorem** Every connected closed surface M is homeomorphic to exactly one of

$$M(0)\ ,\ M(1)\ ,\ \dots\ ,\ M(g)=\#_g S^1\times S^1\ ,\ \dots\ \text{(orientable)}$$

$$N(1)\ ,\ N(2)\ ,\ \dots\ ,\ N(g)=\#_g \mathbb{RP}^2\ ,\ \dots\ \text{(nonorientable)}$$

- Connected surfaces are classified by the genus g and orientability.
- Connected surfaces are classified by the fundamental group :

$$\pi_1(M(g)) = \langle a_1,b_1,a_2,b_2,\dots,a_g,b_g\,|\,[a_1,b_1]\dots[a_g,b_g]\rangle$$

$$\pi_1(N(g)) = \langle c_1,c_2,\dots,c_g\,|\,(c_1)^2(c_2)^2\dots(c_g)^2\rangle$$

- Connected surfaces are classified by the Euler characteristic and orientability

$$\chi(M(g)) = 2-2g\ ,\ \chi(N(g)) = 2-g\ .$$

The punctured torus I.

- The computation of $\pi_1(M(g))$ for $g \geqslant 0$ will be by induction, using the connected sum

$$M(g+1) \ = \ M(g)\#M(1)$$

- So need to understand the fundamental group of the torus $M(1) = T = S^1 \times S^1$ and the puncture torus (T_0, S^1).
- Clear from $T = S^1 \times S^1$ that $\pi_1(T) = \mathbb{Z} \oplus \mathbb{Z}$.
- Can also get this by applying the Seifert-van Kampen theorem to $M(1) = M(1)\#M(0)$, i.e. $T = T_0 \cup_{S^1} D^2$.
- The punctured torus

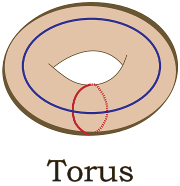
$$(T_0, \partial T_0) \ = \ (\mathrm{cl.}(S^1 \times S^1 \backslash D^2), S^1)$$

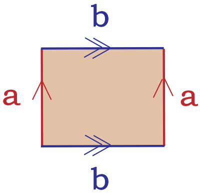
is such that $S^1 \vee S^1 \subset T_0$ is a homotopy equivalence.

The punctured torus II.

- The inclusion $\partial T_0 = S^1 \subset T_0$ induces

$$\pi_1(S^1) \ = \ \mathbb{Z} \to \pi_1(T_0) \ = \ \pi_1(S^1 \vee S^1) \ = \ \mathbb{Z} * \mathbb{Z} \ = \ \langle a, b \rangle \ ;$$
$$1 \mapsto [a, b] \ = \ aba^{-1}b^{-1} \ .$$

- 

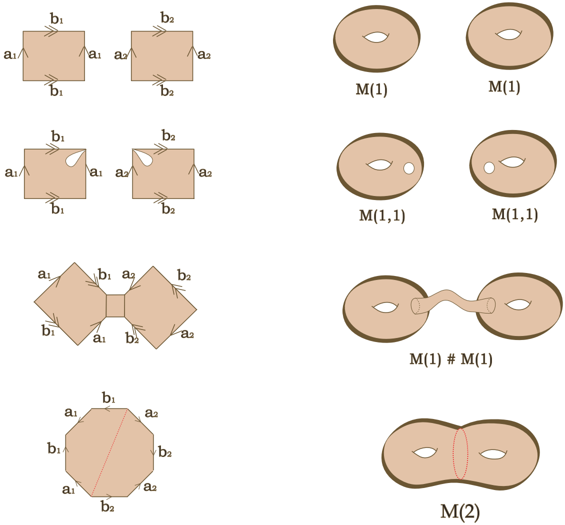


- The Seifert-van Kampen Theorem gives

$$\pi_1(T) \ = \ \pi_1(T_0) *_{\mathbb{Z}} \{1\} \ = \ \langle a, b \,|\, [a, b] \rangle \ = \ \mathbb{Z} \oplus \mathbb{Z} \ .$$

The calculation of $\pi_1(M(g))$ I.

- The initial case $g = 2$, using $M(2) = M(1)\#M(1)$



The calculation of $\pi_1(M(g))$ II. General case

- Assume inductively that
  - $\pi_1(M(g)) = \langle a_1, b_1, \dots, a_g, b_g \,|\, [a_1, b_1] \dots [a_g, b_g] \rangle$,
  - the punctured surface

    $$(M(g)_0, \partial M(g)_0) = (\text{cl.}(M(g) \backslash D^2), S^1)$$

    is such that $\bigvee_{2g} S^1 \subset M(g)_0$ is a homotopy equivalence,
  - the inclusion $\partial M(g)_0 = S^1 \subset M(g)_0$ induces

    $$\begin{aligned} \pi_1(S^1) = \mathbb{Z} \to \pi_1(M(g)_0) = \underset{2g}{*} \mathbb{Z} = \langle a_1, b_1, \dots, a_g, b_g \rangle \ ; \\ 1 \mapsto [a_1, b_1][a_2, b_2] \dots [a_g, b_g] \ . \end{aligned}$$

- Apply the Seifert-van Kampen Theorem to

  $$M(g+1) = M(g) \# M(1)$$

  to obtain

  $$\begin{aligned} \pi_1(M(g+1)) &= \pi_1(M(g)_0) *_{\mathbb{Z}} \pi_1(M(1)_0) \\ &= \langle a_1, b_1, \dots, a_{g+1}, b_{g+1} \,|\, [a_1, b_1] \dots [a_{g+1}, b_{g+1}] \rangle \end{aligned}$$

Cross-cap

- If M is a surface the connected sum

$$M' = M\#\mathbb{RP}^2$$

  is the surface obtained from M by forming a **crosscap** (*Kreuzhaube* in German).
- M' is homeomorphic to the identification space obtained from the punctured surface (M_0, S^1) by identifying $z \sim -z$ for $z \in S^1$

$$M' = M_0/\{z \sim -z\} .$$

- Equivalently, M' is obtained from M by punching out $D^2 \subset M$ and replacing it by a Möbius band.
- M' is nonorientable.
- **Example** If $M = S^2$ then $M' = \mathbb{RP}^2$.

The punctured projective plane I.

- The computation of $\pi_1(N(g))$ for $g \geqslant 1$ will be by induction, using the connected sum

$$N(g+1) \;=\; N(g)\#N(1)$$

with $N(1) = \mathbb{RP}^2$. Abbreviate $\mathbb{RP}^2 = P$.

- Need to understand the fundamental group of P and the punctured projective plane (P_0, S^1), i.e. the Möbius band.
- Clear from the universal double cover $p : S^2 \to P$ that

$$\pi_1(P) \;=\; \mathrm{Homeo}_p(P) \;=\; \mathbb{Z}_2 \;.$$

- Can also get this by applying the Seifert-van Kampen Theorem to $N(1) = N(1)\#M(0)$, i.e. $P = P_0 \cup_{S^1} D^2$.

The punctured projective plane II.

- The punctured projective plane

$$(P_0, \partial P_0) \; = \; (\text{cl.}(P \backslash D^2), S^1)$$

is a Möbius band, such that $S^1 \subset P_0 \backslash \partial P_0$ is a homotopy equivalence.

- The inclusion $\partial P_0 = S^1 \subset P_0$ induces

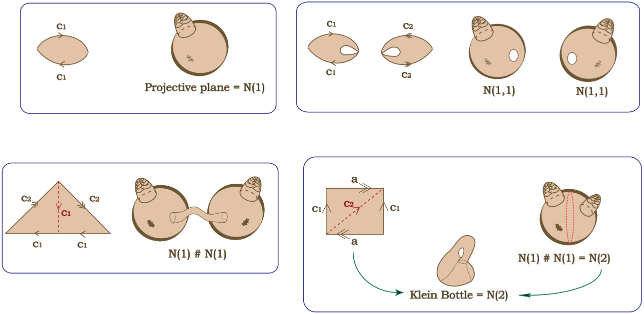
$$\pi_1(S^1) \; = \; \mathbb{Z} \to \pi_1(P_0) \; = \; \pi_1(S^1) \; = \; \mathbb{Z} \; ; \; 1 \mapsto 2 \; .$$

- The Seifert-van Kampen Theorem gives

$$\pi_1(P) \; = \; \pi_1(P_0) *_{\mathbb{Z}} \{1\} \; = \; \langle c \, | \, c^2 \rangle \; = \; \mathbb{Z}_2 \; .$$

The calculation of $\pi_1(N(g))$ I.

- The initial case $g = 2$, using $N(2) = N(1)\#N(1)$ and $(N(1)_0, S^1) = $ (Möbius band,boundary circle).



- By the Seifert-van Kampen Theorem, with $c_2 = (c_1')^{-1}$,

$$\begin{aligned}\pi_1(N(2)) &= \pi_1(N(1)\#N(1)) \\ &= \langle c_1, c_1' \,|\, (c_1)^2 = (c_1')^2 \rangle \;=\; \langle c_1, c_2 \,|\, (c_1)^2 (c_2)^2 \rangle \;.\end{aligned}$$

The calculation of $\pi_1(N(g))$ II.

- Assume inductively that
  - $\pi_1(N(g)) = \langle c_1, c_2, \dots, c_g \,|\, (c_1)^2(c_2)^2 \dots (c_g)^2 \rangle$,
  - the punctured surface

    $$(N(g)_0, \partial N(g)_0) = (\text{cl.}(N(g)\backslash D^2), S^1)$$

    is such that $\bigvee_g S^1 \subset N(g)_0$ is a homotopy equivalence,
  - the inclusion $\partial N(g)_0 = S^1 \subset N(g)_0$ induces

    $$\begin{aligned} \pi_1(S^1) = \mathbb{Z} \to \pi_1(N(g)_0) &= \mathop{*}_g \mathbb{Z} = \langle c_1, c_2, \dots, c_g \rangle ; \\ 1 &\mapsto (c_1)^2 \dots (c_g)^2 . \end{aligned}$$

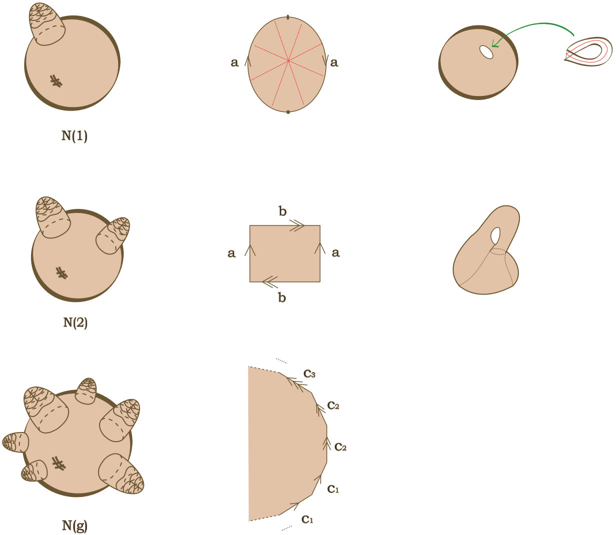
- Apply the Seifert-van Kampen Theorem to

  $$N(g+1) = N(g)\# N(1)$$

  to obtain

  $$\begin{aligned} \pi_1(N(g+1)) &= \pi_1(N(g)_0) *_{\mathbb{Z}} \pi_1(N(1)_0) \\ &= \langle c_1, \dots, c_{g+1} \,|\, (c_1)^2 \dots (c_{g+1})^2 \rangle . \end{aligned}$$

The calculation of $\pi_1(N(g))$ III.



The Euler characteristic

- **Definition** The **Euler characteristic** of a finite cell complex

$$X = \bigcup_{c_0} D^0 \cup \bigcup_{c_1} D^1 \cup \bigcup_{c_2} D^2 \cup \cdots \cup \bigcup_{c_n} D^n$$

  with c_k k-cells is

$$\chi(X) = \sum_{k=0}^{n} (-1)^k c_k \in \mathbb{Z} .$$

- $\chi(D^n) = 1$, $\chi(S^n) = 1 + (-1)^n$
- If X is homotopy equivalent to Y then $\chi(X) = \chi(Y)$
- $\chi(X \cup Y) = \chi(X) + \chi(Y) - \chi(X \cap Y) \in \mathbb{Z}$.
- A punctured n-manifold has $\chi(M_0) = \chi(M) + (-1)^n$
- A connected sum of n-manifolds has

$$\chi(M \# M') = \chi(M) + \chi(M') - \chi(S^n)$$

- If $F \to \widetilde{X} \to X$ is a regular cover with finite fibre F then $\chi(\widetilde{X}) = \chi(F)\chi(X)$, with $\chi(F) = |F|$.

The Euler characteristic of $M(g)$

- The fundamental group of $M(g)$ determines the genus g.
- The first homology group of $M(g)$ is the free abelian group of rank $2g$

$$H_1(M(g)) \;=\; \pi_1(M(g))^{ab} \;=\; \bigoplus_{2g} \mathbb{Z}$$

- $M(g)$ is homotopy equivalent to the 2-dimensional cell complex

$$(\bigvee_{2g} S^1) \cup_{[a_1,b_1]\dots[a_g,b_g]} D^2 \;=\; D^0 \cup \bigcup_{2g} D^1 \cup_{[a_1,b_1]\dots[a_g,b_g]} D^2 \;.$$

- The Euler characteristic of $M(g)$ is

$$\chi(M(g)) \;=\; 2 - 2g \;.$$

- A closed surface M is homeomorphic to S^2 if and only if $\chi(M) = 2$.

The Euler characteristic of $N(g)$

- The fundamental group determines the genus g.
- The first homology group of $N(g)$ is direct sum of the free abelian group of rank $g-1$ and the cyclic group of order 2

$$H_1(N(g)) = \pi_1(N(g))^{ab} = (\bigoplus_g \mathbb{Z})/(2,2,\ldots,2) = (\bigoplus_{g-1} \mathbb{Z}) \oplus \mathbb{Z}_2$$

- $N(g)$ is homotopy equivalent to the 2-dimensional cell complex

$$(\bigvee_g S^1) \cup_{(c_1)^2(c_2)^2\ldots(c_g)^2} D^2 = D^0 \cup \bigcup_g D^1 \cup_{(c_1)^2\ldots(c_g)^2} D^2 .$$

- $N(g)$ has Euler characteristic

$$\chi(N(g)) = 2 - g .$$

The orientable surfaces with boundary $M(g,h)$

- Let $g \geqslant 0$, $h \geqslant 1$.
- **Definition** The **orientable surface of genus** g **and** h **boundary components** is

$$(M(g,h),\partial) \ = \ (\text{cl.}(M(g)\backslash\bigcup_h D^2),\bigcup_h S^1)\ .$$

- Cell structure $M(g,h) \ \simeq \ \bigvee_{2g+h-1} S^1 = D^0 \cup \bigcup_{2g+h-1} D^1$
- Fundamental group $\pi_1(M(g,h)) = \mathop{*}_{2g+h-1}\mathbb{Z}$
- Euler characteristic $\chi(M(g,h)) = 2-2g-h$
- **Classification Theorem** Every connected orientable surface with non-empty boundary is homeomorphic to exactly one of $(M(g,h),\partial M(g,h))$.
- Set $M(g,0) = M(g)$.

Examples of orientable surfaces with boundary

- $(M(0,1), \partial) = (D^2, S^1)$, 2-disk
- $(M(0,2), \partial) = (S^1 \times [0,1], S^1 \times \{0,1\})$, cylinder
- $(M(1,1), \partial) = ((S^1 \times S^1)_0, S^1)$, punctured torus.
- $(M(0,3), \partial) = (\text{pair of pants}, S^1 \cup S^1 \cup S^1)$.
- The pair of pants is an essential feature of topological quantum field theory, and so appeared in Ida's birthday cake for the 80th birthday of Michael Atiyah (29 April, 2009)

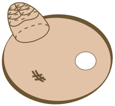
The nonorientable surfaces with boundary $N(g,h)$ I.

- Let $g \geqslant 1$, $h \geqslant 1$.
- **Definition** The **nonorientable surface with boundary with genus** g **with** h **boundary components** is

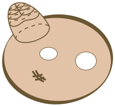
$$(N(g,h), \partial N(g,h)) \;=\; (\mathrm{cl.}(N(g)\backslash \bigcup_h D^2), \bigcup_h S^1) \;.$$

- Cell structure $N(g,h) \;\simeq\; \bigvee_{g+h-1} S^1 = D^0 \cup \bigcup_{g+h-1} D^1$.
- Fundamental group $\pi_1(N(g,h)) = \ast_{g+h-1} \mathbb{Z}$
- Euler characteristic $\chi(N(g,h)) = 2 - g - h$
- **Classification Theorem** Every connected nonorientable surface with non-empty boundary is homeomorphic to exactly one of $(N(g,h), \partial N(g,h))$.
- Set $N(g,0) = N(g)$.

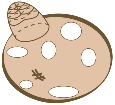
The nonorientable surfaces with boundary $N(g, h)$ II.



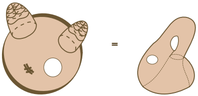
N(1,1)



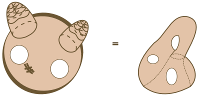
N(1,2)



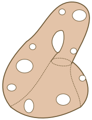
N(1, h)



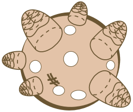
N(2,1)



N(2,1)



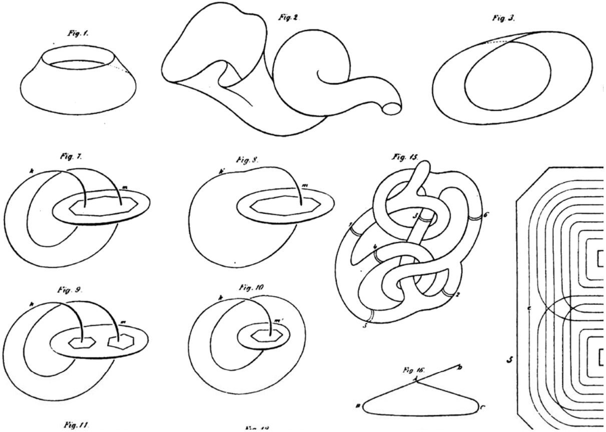
N(2, h)



N(g, h)

The Möbius band

- The Möbius band $(N(1,1), \partial N(1,1)) = ((\mathbb{RP}^2)_0, S^1)$.
- The first drawing of a Möbius band, from Listing' s 1862 Census der Räumlichen Complexe



The orientation double cover

- A **double cover** of a space N is a regular cover $\widetilde{N} \to N$ with fibre $F = \{0, 1\}$. Connected double covers of connected N are classified by index 2 subgroups $\pi_1(\widetilde{N}) \triangleleft \pi_1(N)$.
- A surface N has an **orientation double cover** $p : \widetilde{N} \to N$, with $\widetilde{N}$ an orientable surface. For connected N classified by the kernel of the orientation character group morphism

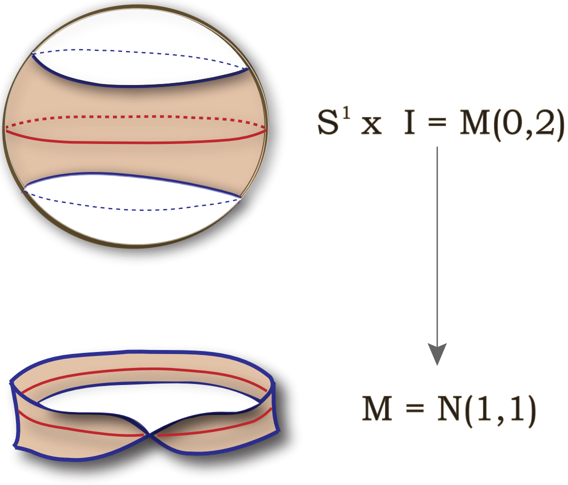
$$w \ : \ \pi_1(N) \to \mathbb{Z}_2 = \{+1, -1\}$$

  sending orientable (resp. nonorientable) α to $+1$ (resp. -1).
- If N is orientable $\widetilde{N} = N \cup N$ is the trivial double cover of N.
- If N is nonorientable w is onto, $\pi_1(\widetilde{N}) = \ker w$. Pullback along nonorientable $\alpha : S^1 \to N$ is the nontrivial double cover

$$q \ = \ \alpha^* p \ : \ S^1 \to S^1 \ ; \ z \mapsto z^2$$

$$\begin{array}{ccc} S^1 & \xrightarrow{\widetilde{\alpha}} & \widetilde{N} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ S^1 & \xrightarrow{\alpha} & N \end{array}$$

The orientation double cover of a Möbius band is a cylinder



$M(g-1,2h)$ is the orientation double cover of $N(g,h)$

- **Proposition** The orientation double cover of $N(g,h)$ is

$$\widetilde{N(g,h)} \;=\; M(g-1,2h)\ (g \geqslant 1, h \geqslant 0)$$

- **Proof** Let N be a connected nonorientable surface with orientation double cover $\widetilde{N}$. The boundary circle of $N_0 = \mathrm{cl.}(N \backslash D^2)$ is orientable. The orientation double cover of N_0 is the twice-punctured $\widetilde{N}$, $\widetilde{N}_{00} = \mathrm{cl.}(\widetilde{N} \backslash D^2 \cup D^2)$. The orientation double cover of $N' = N \# \mathbb{RP}^2$ is

$$\widetilde{N}' \;=\; \widetilde{N}_{00} \cup_{S^1 \cup S^1} S^1 \times I\ .$$

with $\chi(\widetilde{N}') = \chi(\widetilde{N}_{00}) = \chi(\widetilde{N}) - 2$. This gives the inductive step in checking that $\widetilde{N(g,h)} = M(g-1,2h)$.

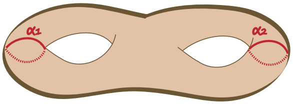
- **Example** For $h = 0$, $g \geqslant 1$ have $\widetilde{N(g)} = M(g-1)$. Simply-connected for $g = 1$. For $g \geqslant 2$ universal cover $\mathbb{R}^2$.

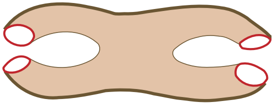
The genus measures connectivity
I. The orientable case

- The genus g of an orientable surface M is the maximum number of disjoint loops $\alpha_1, \alpha_2, \ldots, \alpha_g : S^1 \to M$ such that the complement $M \backslash \bigcup_{i=1}^{g} \alpha_i(S^1)$ is connected. The complement is homeomorphic to $M(0,2g) \backslash \partial M(0,2g)$.
- **Example** For $M = M(2)$ let $\alpha_1, \alpha_2 : S^1 \to M$ be disjoint loops which go round as in the diagram.
  The complement

$$M \backslash (\alpha_1(S^1) \cup \alpha_2(S^1)) \;=\; M(0,4) \backslash \partial M(0,4)$$

  is the sphere $M(0) = S^2$ with 4 holes punched out.





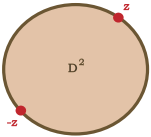
The genus measures connectivity
II. The nonorientable case

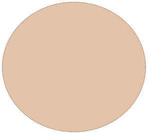
- The genus g of a nonorientable surface N is the maximum number of disjoint nonorientable loops $\beta_1, \beta_2, \ldots, \beta_g : S^1 \to N$ such that the complement $N \backslash \bigcup\limits_{i=1}^{g} \beta_i(S^1)$ is connected.
  The complement is homeomorphic to $M(0,g)\backslash \partial M(0,g)$.
- **Example** Let $N = \mathbb{RP}^2 = D^2/\{z \sim -z \,|\, z \in S^1\}$ and

$$\beta \ : \ S^1 \ = \ \mathbb{RP}^1 \to \mathbb{RP}^2 \ ; \ z \mapsto [\sqrt{z}] \ .$$

  The complement is

$$\mathbb{RP}^2 \backslash \beta(S^1) \ = \ M(0,1)\backslash \partial M(0,1) \ = \ D^2 \backslash S^1 \ = \ \mathbb{R}^2 \ .$$





Morse theory

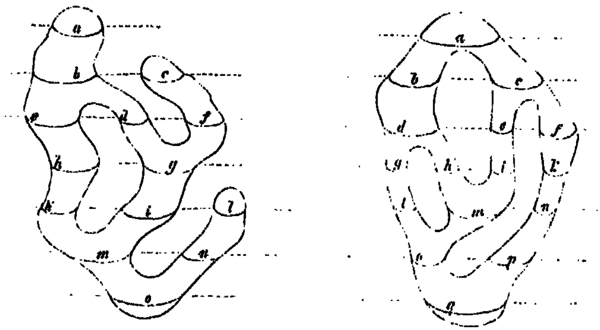
- For an orientable surface $M \subset \mathbb{R}^3$ in general position the height function

$$f \ : \ M \to \mathbb{R} \ ; \ (x, y, z) \mapsto z$$

  has the property that the inverse image $f^{-1}(c) \subset M$ is a 1-dimensional submanifold for all except a finite number $c \in \mathbb{R}$ called the critical values of f.
- Can recover the genus g of M by looking at the jumps in the number of circles in $f^{-1}(a)$ and $f^{-1}(b)$ for $a < b < c$.
- Morse theory developed (since 1926) is the key tool for studying n-manifolds for all $n \geqslant 0$.

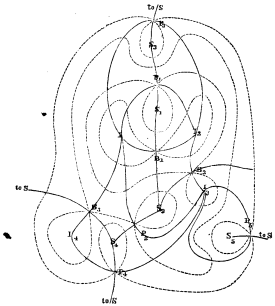
An early exponent of Morse theory on a surface

- August Ferdinand Möbius
  Theorie der elementaren Verwandschaften (1863)
- Fill a surface shaped bathtub with water, and recover the genus of the surface from a film of the cross-sections.



Another early exponent of Morse theory on a surface

- James Clerk Maxwell (1870) On hills and dales
- Reconstruct surface of the earth ($= S^2$) from contour lines.



- Mountaineer's equation for surface of Earth

$$\text{no. of peaks} - \text{no. of pits} + \text{no. of passes} \;=\; \chi(S^2) \;=\; 2\ .$$

Modern account in Chapter 8 of Surfaces (CUP, 1976) by H.B.Griffiths

Complex algebraic curves

- The **complex projective space** $\mathbb{CP}^2$ is the space of 1-dimensional complex linear subspaces $L \subset \mathbb{C}^2$. A closed 4-manifold. Homogeneous coordinates $[x, y, z] \in \mathbb{CP}^2$.
- For a degree d homogeneous complex polynomial $P(x, y, z)$ let

$$M(P) = \{[x, y, z] \in \mathbb{CP}^2 \mid P(x, y, z) = 0\}$$

- **Theorem** (Special case of the Riemann-Hurwitz formula) If $(\partial P/\partial x, \partial P/\partial y, \partial P/\partial z) \neq (0, 0, 0)$ for all $(x, y, z) \in M(P)$ then $M(P)$ is a closed orientable surface with genus

$$g = (d-1)(d-2)/2$$

  0 or one of the triangular numbers

$$1, 3, 6, 10, 15, 21, 28, 36, 45, 55, \ldots$$

- **Complex algebraic curves** by Frances Kirwan (CUP, 1992)

Further reading

- Google for "Classification of Surfaces" (147,000 hits)
- An Introduction to Topology. The classification theorem for surfaces by E.C. Zeeman (1966)
- A Guide to the Classification Theorem for Compact Surfaces by Jean Gallier and Dianna Xu (2011)
- Home Page for the Classification of Surfaces and the Jordan Curve Theorem Online resources, including many of the original papers.