THE THURSTON NORM, FIBERED MANIFOLDS AND TWISTED ALEXANDER POLYNOMIALS

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ABSTRACT. Every element in the first cohomology group of a 3-manifold is dual to embedded surfaces. The Thurston norm measures the minimal 'complexity' of such surfaces. For instance the Thurston norm of a knot complement determines the genus of the knot in the 3-sphere. We show that the degrees of twisted Alexander polynomials give lower bounds on the Thurston norm, generalizing work of Mc-Mullen and Turaev. Our bounds attain their most elegant form when interpreted as the degrees of the Reidemeister torsion of a certain twisted chain complex. Using these lower bounds we confirm the genus of all knots with 12 crossings or less, including the Conway knot and the Kinoshita–Terasaka knot which have trivial Alexander polynomial.

We also give obstructions to fibering 3-manifolds using twisted Alexander polynomials and detect all knots with 12 crossings or less that are not fibered. For some of these it was unknown whether or not they are fibered. Our work also extends the fibering obstructions of Cha to the case of closed manifolds.

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1. INTRODUCTION

1.1. Definitions and history. Let M be a 3-manifold. Throughout the paper we will assume that all 3-manifolds are compact, orientable and connected. Let

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 $\phi \in H^1(M)$ (integral coefficients are understood). The Thurston norm of ϕ is defined as

$$\begin{aligned} ||\phi||_T &:= \min\{\sum_{i=1}^k \max\{-\chi(S_i), 0\}| \quad S_1 \cup \dots \cup S_k \subset M \text{ properly embedded,} \\ \text{dual to } \phi, S_i \text{ connected for } i = 1, \dots, k\}. \end{aligned}$$

Thurston [Th86] showed that this defines a seminorm on $H^1(M)$ which can be extended to a seminorm on $H^1(M; \mathbb{R})$. As an example consider $X(K) := S^3 \setminus \nu K$, where $K \subset S^3$ is a knot and νK denotes an open tubular neighborhood of K in S^3 . Let $\phi \in H^1(X(K))$ be a generator, then it is easy to see that $||\phi||_T = 2 \operatorname{genus}(K) - 1$. It is a classical result of Alexander that

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$$2\text{genus}(K) \ge \deg(\Delta_K(t)),$$

where $\Delta_K(t)$ denotes the Alexander polynomial of a knot K. In recent years this was greatly generalized. Let M be a 3-manifold whose boundary is empty or consists of tori. Let $\phi \in H^1(M) \cong \operatorname{Hom}(H_1(M), \mathbb{Z})$ be primitive, i.e., the corresponding homomorphism $\phi: H_1(M) \to \mathbb{Z}$ is surjective. Then McMullen [Mc02] showed that if the Alexander polynomial $\Delta_1(t) \in \mathbb{Q}[t^{\pm 1}]$ of (M, ϕ) is non-zero, then

$$||\phi||_T \ge \deg(\Delta_1(t)) - (1 + b_3(M)).$$

This result has been reproved for closed manifolds by Vidussi [Vi99, Vi03] using results in Seiberg–Witten theory of Kronheimer–Mrowka [KM97] and Meng–Taubes [MT96]. We refer to [Kr98, Kr99] for more on the connection between the Thurston norm, Seiberg–Witten theory and 4–dimensional geometry.

Cochran [Co04] in the knot complement case and Harvey [Ha05] and Turaev [Tu02a, Tu02b] in the general case generalized McMullen's inequality. They studied maps $\mathbb{Z}[\pi_1(M)] \to \mathbb{K}[t^{\pm 1}]$ where \mathbb{K} is a skew field and $\mathbb{K}[t^{\pm 1}]$ is a skew Laurent polynomial ring. They showed that the degrees of corresponding *higher-order* Alexander polynomials give lower bounds on the Thurston norm.

We will show how the degrees of *twisted Alexander polynomials* give lower bounds on the Thurston norm. These bounds are easy to compute and remarkably strong.

1.2. Twisted Alexander polynomials and Reidemeister torsion. In the following let \mathbb{F} be a commutative field. Let $\phi \in H^1(M) \cong \operatorname{Hom}(\pi_1(M), \mathbb{Z})$ and $\alpha : \pi_1(M) \to \operatorname{GL}(\mathbb{F}, k)$ a representation. Then $\alpha \otimes \phi$ induces an action of $\pi_1(M)$ on $\mathbb{F}^k \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}] =: \mathbb{F}^k[t^{\pm 1}]$ and we can therefore consider the twisted homology $\mathbb{F}[t^{\pm 1}]$ module $H^{\alpha}_i(M; \mathbb{F}^k[t^{\pm 1}])$. We define $\Delta^{\alpha}_i(t) \in \mathbb{F}[t^{\pm 1}]$ to be its order; it is called *the i-th twisted Alexander polynomial of* (M, ϕ, α) and well–defined up to multiplication by a unit in $\mathbb{F}[t^{\pm 1}]$. We refer to Section 2 for more details.

The twisted Alexander polynomial of a knot was introduced by Lin [Lin01] in 1990 who used it to distinguish knots with the same Alexander polynomial. In this paper we use the above homological definition of Kirk and Livingston [KL99a].

If ∂M is empty or consists of tori and if $\Delta_1^{\alpha}(t) \neq 0$, then $H_i^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}] \otimes_{\mathbb{F}[t^{\pm 1}]} \mathbb{F}(t)) = 0$ for all *i* (see Corollary 4.3). Therefore the Reidemeister torsion $\tau(M, \phi, \alpha) \in \mathbb{F}(t)$

 $\mathbb{F}(t)$ is defined (cf. [Tu01] for a definition) and (cf. [Tu01, p. 20])

$$\tau(M,\phi,\alpha) = \prod_{i=0}^{2} \Delta_i^{\alpha}(t)^{(-1)^{i+1}} \in \mathbb{F}(t).$$

The equality holds up to multiplication by a unit in $\mathbb{F}[t^{\pm 1}]$. We will use this equality as a definition for $\tau(M, \phi, \alpha)$. For $f(t)/g(t) \in \mathbb{F}(t)$ we define $\deg(f(t)/g(t)) := \deg(f(t)) - \deg(g(t))$ for $f(t), g(t) \in \mathbb{F}[t^{\pm 1}]$. This allows us to consider $\deg(\tau(M, \phi, \alpha))$. In Section 3 we will point out that the Alexander polynomials and Reidemeister torsion can be computed efficiently using Fox calculus.

1.3. Lower bounds on the Thurston norm. The following is one of our main results.

Theorem 3.1 (Main Theorem 1). Let M be a 3-manifold whose boundary is empty or consists of tori. Let $\phi \in H^1(M)$ be non-trivial and $\alpha : \pi_1(M) \to GL(\mathbb{F}, k)$ a representation such that $\Delta_1^{\alpha}(t) \neq 0$. Then

$$||\phi||_T \ge \frac{1}{k} \deg(\tau(M, \phi, \alpha)).$$

Equivalently,

$$||\phi||_T \ge \frac{1}{k} \big(\deg \left(\Delta_1^{\alpha}(t) \right) - \deg \left(\Delta_0^{\alpha}(t) \right) - \deg \left(\Delta_2^{\alpha}(t) \right) \big).$$

The proof of Theorem 3.1 is partly based on ideas of McMullen [Mc02] and Turaev [Tu02b]. In Section 3 we will show that Theorem 3.1 generalizes McMullen's theorem [Mc02] and Turaev's abelian invariants in [Tu02a].

In Theorem 5.1 we show that the condition $\Delta_1^{\alpha}(t) \neq 0$ can sometimes be dropped. In [F05b] we will prove a version of Theorem 3.1 over skew fields, which combines our lower bounds from Theorem 3.1 with the lower bounds of Cochran, Harvey and Turaev [Co04, Ha05, Tu02b].

1.4. Fibered manifolds. Let $\phi \in H^1(M)$ be non-trivial. We say (M, ϕ) fibers over S^1 if the homotopy class of maps $M \to S^1$ induced by $\phi : \pi_1(M) \to H_1(M) \to \mathbb{Z}$ contains a representative that is a fiber bundle over S^1 . If K is a fibered knot, i.e., if X(K) fibers, then it is a classical result of Neuwirth that $2 \operatorname{genus}(K) = \operatorname{deg}(\Delta_K(t))$ and that $\Delta_K(t) \in \mathbb{Z}[t^{\pm 1}]$ is monic, i.e., its top coefficient is +1 or -1.

Theorem 6.1 (Main Theorem 2). Assume that (M, ϕ) fibers over S^1 and that $M \neq S^1 \times D^2, M \neq S^1 \times S^2$. Let $\alpha : \pi_1(M) \to GL(\mathbb{F}, k)$ be a representation. Then $\Delta_1^{\alpha}(t) \neq 0$ and

$$||\phi||_T = \frac{1}{k} \deg(\tau(M, \phi, \alpha)).$$

This result clearly generalizes the first classical condition on fibered knots. Mc-Mullen, Cochran, Harvey and Turaev prove corresponding theorems in their respective papers [Mc02, Co04, Ha05, Tu02b].

Let R be a Noetherian unique factorization domain (henceforth UFD), for example $R = \mathbb{Z}$ or a field. Given a representation $\pi_1(M) \to \operatorname{GL}(R,k)$ Cha [Ch03] defined a twisted Alexander polynomial $\Delta_1^{\alpha}(t) \in R[t^{\pm 1}]$, which is well-defined up to multiplication by a unit in $R[t^{\pm 1}]$. This is a generalization of the Alexander polynomial $\Delta_K(t) \in \mathbb{Z}[t^{\pm 1}]$ and coincides with the first twisted Alexander polynomial defined in Section 2 in the case that R is a field. We say a polynomial $\Delta_1^{\alpha}(t) \in R[t^{\pm 1}]$ is *monic*, if its top coefficient is a unit in R. Cha showed that for a fibered knot the polynomials $\Delta_1^{\alpha}(t)$ are monic [Ch03]. Using Theorem 6.1 we get the following theorem.

Theorem 6.4. Let M be a 3-manifold. Let $\phi \in H^1(M)$ be non-trivial such that (M, ϕ) fibers over S^1 and such that $M \neq S^1 \times D^2$, $M \neq S^1 \times S^2$. Let R be a Noetherian UFD and let $\alpha : \pi_1(M) \to GL(R,k)$ be a representation. Then $\Delta_1^{\alpha}(t) \in R[t^{\pm 1}]$ is monic and

$$||\phi||_T = \frac{1}{k} \deg(\tau(M, \phi, \alpha)).$$

In fact in Proposition 6.3 we show that if the fibering obstruction of Theorem 6.1 vanishes, then the conclusion of Theorem 6.4 holds. This shows the somewhat surprising fact that the obstructions of Theorem 6.1 contain Neuwirths's and Cha's [Ch03] obstructions for fibered knots. Note that Theorem 6.4 generalizes Cha's obstructions to closed 3-manifolds.

Goda, Kitano and Morifuji [GKM05] use the Reidemeister torsion corresponding to representations $\pi_1(X(K)) \to SL(\mathbb{F}, k)$, \mathbb{F} a field, to give fibering obstructions for a knot K. The precise relationship to our obstructions is not known. It would be interesting to generalize their obstructions to closed manifolds as well.

1.5. **Examples.** We give two main examples and more: we confirm the genus of knots with up to 12 crossings and detect all of non–fibered 12–crossing knots. To our knowledge, some of the examples of non–fibered 12–crossing knots are new. These examples and more are given in Section 7.

Consider the Conway knot $K = 11_{401}$ (knotscape notation, cf. [HT]). Gabai [Ga84] proved that the genus of K is 3 using geometric methods. We confirm this easily using Theorem 3.1. Note that for this knot $\Delta_K(t) = 1$, therefore the genus bounds of McMullen, Turaev, Cochran and Harvey vanish. The diagram is given in Figure 1. We found a representation $\alpha : \pi_1(X(K)) \to \operatorname{GL}(\mathbb{F}_{13}, 4)$ such that deg $(\tau(M, \phi, \alpha)) = 14$. These computations and all the following computations were done using the program *KnotTwister* [F05]. It follows from Theorem 3.1 that

$$2 \operatorname{genus}(K) - 1 = ||\phi||_T \ge \frac{14}{4}.$$



FIGURE 1. The Conway knot 11_{401} and a Seifert surface of genus 3 (from [Ga84]).

Hence $\text{genus}(K) \ge \frac{18}{8} = 2.25$. Since genus(K) is an integer we get $\text{genus}(K) \ge 3$. Since there exists a Seifert surface of genus 3 for K (cf. Figure 1) it follows that the genus of the Conway knot is 3.

We went over all knots with up to 12 crossings such that $2 \operatorname{genus}(K) \neq \operatorname{deg}(\Delta_K(t))$. In all cases we found representations $\alpha : \pi_1(X(K)) \to \operatorname{GL}(\mathbb{F}_{13}, k)$ which give the right genus bounds. Using *KnotTwister* this process just takes a few seconds. We also investigated the closed manifolds which are the result of 0-framed surgery along these knots. Again in all cases we found representations such that twisted Alexander polynomials give the right bound on the Thurston norm. In fact experience suggests that if $b_1(M) = 1$ then in most cases taking only a few non-trivial representations will give the correct bound on the Thurston norm, regardless of whether M is closed or not.

The situation for links is more complex. On the one hand in many interesting cases twisted Alexander polynomials give the correct bound. For example in Section 7.5 we reprove results of Harvey on the ropelength of a certain link [Ha05]. We also successfully apply our theory in Section 7.4. On the other hand boundary links have mostly vanishing twisted Alexander polynomials and therefore our lower bounds do not apply in general. But in Section 5 we show that in some cases we can still extract lower bounds from the degrees of twisted Alexander polynomials corresponding to the $\mathbb{F}[t^{\pm 1}]$ -torsion submodule of $H_1^{\alpha}(X(L); \mathbb{F}^k[t^{\pm 1}])$ where X(L) is the link complement in the 3-sphere (cf. Theorem 5.1).

It is known that a knot K with 11 or fewer crossings is fibered if and only if K satisfies

(1)
$$\Delta_K(t)$$
 is monic and $\deg(\Delta_K(t)) = 2 \operatorname{genus}(K)$.

Hirasawa and Stoimenow had started a program to find all non-fibered 12-crossing knots. Using methods of Gabai they showed that except for thirteen knots a 12-crossing knot is fibered if and only if it satisfies condition (1). Furthermore they

showed that among these 13 knots the knots 12_{1498} , 12_{1502} , 12_{1546} and 12_{1752} are not fibered even though they satisfy condition (1).

Using Theorem 6.1 we confirmed the non-fiberedness of these 4 knots and we showed that the remaining 9 knots are not fibered either. These 9 knots are:

 $12_{1345}, 12_{1567}, 12_{1670}, 12_{1682}, 12_{1771}, 12_{1823}, 12_{1938}, 12_{2089}, 12_{2103}.$

This result completes the classification of all fibered 12–crossing knots. Jacob Rasmussen confirmed our results using knot Floer homology which gives a fibering obstruction as well (cf. [OS02, Section 3]).

As we pointed out our fibering obstructions work for closed manifolds as well. If K is one of the 13 12-crossing knots in the previous paragraph, then we can easily show using Theorem 6.1 and *KnotTwister* that the zero surgery on K in S^3 is not fibered. (See Section 7.2.)

1.6. The twisted Alexander norm. McMullen [Mc02] also defined a (semi) norm $|| - ||_A$ on $H^1(M; \mathbb{R})$ called the Alexander norm, and showed that if $b_1(M) > 1$ then

$$||\phi||_T \ge ||\phi||_A$$

for all $\phi \in H^1(M; \mathbb{R})$. This norm is closely related to the degrees of the untwisted Alexander polynomials: McMullen shows that $||\phi||_A \ge \deg(\Delta_1(t)) - 1 - b_3(M)$ for all primitive $\phi \in H^1(M)$, and equality holds for almost all primitive $\phi \in H^1(M)$. In [FK05] the authors will introduce *twisted Alexander norms* which give lower bounds on the Thurston norm, extending the work of McMullen and work of Turaev [Tu02a]. The (twisted) Alexander norm can often be used to completely determine the Thurston norm ball of a link complement.

1.7. Conjectures and symplectic manifolds. It follows from Stallings' theorem [St62] together with the Poincaré conjecture that $\pi_1(M)$ contains enough information to decide whether M is fibered or not. We therefore conjecture that a converse to Theorem 6.4 holds. In fact we believe that representations corresponding to finite groups and their group rings suffice.

Conjecture 1.1. Let M be a closed 3-manifold and $\phi \in H^1(M)$ non-trivial. Then (M, ϕ) fibers over S^1 if and only if for all representations of the form $\alpha : \pi_1(M) \to G \to \mathbb{Z}[G]$, G a finite group, the twisted Alexander polynomial $\Delta_1^{\alpha}(t) \in \mathbb{Z}[t^{\pm 1}]$ is monic and

$$||\phi||_T = \frac{1}{|G|} \operatorname{deg}(\tau(M, \phi, \alpha)).$$

In a forthcoming paper of the first author and Stefano Vidussi [FV05] we will give further evidence for this conjecture. Furthermore based on work of Taubes [Ta94, Ta95] we will show in [FV05] that Conjecture 1.1 implies Taubes conjecture which states that if M is a closed 3-manifold and if $S^1 \times M$ is symplectic then (M, ϕ) fibers over S^1 for some $\phi \in H^1(M)$.

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1.8. Outline of the paper. In Section 2 we give a definition of twisted Alexander polynomials. In Section 3 we state Theorem 3.1 (Main Theorem 1) and discuss related theorems. We give a proof of Theorem 3.1 in Section 4. In Section 5 we show how in many important cases we can drop the assumption that $\Delta_1^{\alpha}(t) \neq 0$ in Theorem 3.1 and still get lower bounds on the Thurston norm. In Section 6 we consider fibered manifolds and give a proof of Theorems 6.1 (Main Theorem 2) and 6.4. We discuss a wealth of examples in Section 7.

Notations and conventions: We assume that all 3-manifolds are compact, oriented and connected. All homology groups and all cohomology groups are with respect to Z-coefficients, unless it specifically says otherwise. For a knot K in S^3 , we denote the result of zero framed surgery along K by M_K . For a link L in S^3 , X(L)denotes the exterior of L in S^3 . (That is, $X(L) = S^3 \setminus \nu L$ where νL is an open tubular neighborhood of L in S^3). An arbitrary (commutative) field is denoted by \mathbb{F} . We identify the group ring $\mathbb{F}[\mathbb{Z}]$ with $\mathbb{F}[t^{\pm 1}]$. We denote the permutation group of order k by S_k . For a 3-manifold M we use the canonical isomorphisms to identify $H^1(M) = \operatorname{Hom}(H_1(M), \mathbb{Z}) = \operatorname{Hom}(\pi_1(M), \mathbb{Z})$. Hence sometimes $\phi \in H^1(M)$ is regarded as a homomorphism $\phi : \pi_1(M) \to \mathbb{Z}$ (or $\phi : H_1(M) \to \mathbb{Z}$) depending on the context.

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2. The twisted Alexander Polynomials

Let M be a 3-manifold and $\phi \in H^1(M)$. Let $\alpha : \pi_1(M) \to \operatorname{GL}(\mathbb{F}, k)$ be a representation. We can now define a left $\mathbb{Z}[\pi_1(M)]$ -module structure on $\mathbb{F}^k \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}] =: \mathbb{F}^k[t^{\pm 1}]$ via $\alpha \otimes \phi$ as follows:

$$g \cdot (v \otimes p) := (\alpha(g) \cdot v) \otimes (\phi(g) \cdot p) = (\alpha(g) \cdot v) \otimes (t^{\phi(g)}p)$$

where $g \in \pi_1(M), v \otimes p \in \mathbb{F}^k \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}] = \mathbb{F}^k[t^{\pm 1}].$

Denote by \tilde{M} the universal cover of M. Then the chain groups $C_*(\tilde{M})$ are in a natural way right $\mathbb{Z}[\pi_1(M)]$ -modules. Therefore we can form the tensor product $C_*(\tilde{M}) \otimes_{\mathbb{Z}[\pi_1(M)]} \mathbb{F}^k[t^{\pm 1}]$. Now we define the *i*-th twisted Alexander module of (M, ϕ, α) to be

$$H^{\alpha\otimes\phi}_*(M;\mathbb{F}^k[t^{\pm 1}]):=H_*(C_*(\tilde{M})\otimes_{\mathbb{Z}[\pi_1(M)]}\mathbb{F}^k[t^{\pm 1}]).$$

Usually we drop the notation ϕ and write $H^{\alpha}_{*}(M; \mathbb{F}^{k}[t^{\pm 1}])$. Note that $H^{\alpha}_{i}(M; \mathbb{F}^{k}[t^{\pm 1}])$ is a finitely generated module over the PID $\mathbb{F}[t^{\pm 1}]$. Therefore there exists an isomorphism

$$H_i^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}]) \cong \mathbb{F}[t^{\pm 1}]^f \oplus \bigoplus_{i=1}^k \mathbb{F}[t^{\pm 1}]/(p_i(t))$$

for $p_1(t), \ldots, p_k(t) \in \mathbb{F}[t^{\pm 1}]$. We define

$$\Delta^{\alpha}_{M,\phi,i} := \begin{cases} \prod_{i=1}^{k} p_i(t), & \text{if } f = 0\\ 0, & \text{if } f > 0. \end{cases}$$

This is called the *i*-th twisted Alexander polynomial of (M, ϕ, α) . We furthermore define $\tilde{\Delta}^{\alpha}_{M,\phi,i} := \prod_{i=1}^{k} p_i(t)$ regardless of f. In most cases we drop the notations M and ϕ and write $\Delta^{\alpha}_i(t)$ and $\tilde{\Delta}^{\alpha}_i(t)$. It follows from the structure theorem of finitely generated modules over a PID that these polynomials are well-defined up to multiplication by a unit in $\mathbb{F}[t^{\pm 1}]$.

Remark. The twisted Alexander polynomial of a knot was introduced by Lin [Lin01] in 1990. Various versions of twisted Alexander polynomials have been successfully used in many situations to provide more information than can be extracted from the untwisted Alexander polynomial [JW93, Wa94, Kit96, KL99a, KL99b, Ch03, HLN04]. In particular we note that Kirk and Livingston [KL99a] first introduced the above homological definition of twisted Alexander polynomials for a finite complex. We refer to [KL99a, Section 4] for the relationship between our definition and the other definitions of twisted Alexander polynomials.

For an oriented knot K we always assume that ϕ denotes the generator of $H^1(X(K))$ given by the orientation. If $\alpha : \pi_1(X(K)) \to \operatorname{GL}(\mathbb{Q}, 1)$ is the trivial representation then the Alexander polynomial $\Delta_1^{\alpha}(t)$ equals the classical Alexander polynomial $\Delta_K(t) \in \mathbb{Q}[t^{\pm}]$ of the knot K.

If f = 0 then we write $\deg(f) = \infty$, otherwise, for $f = \sum_{i=m}^{n} a_i t^i \in \mathbb{F}[t^{\pm 1}]$ with $a_m \neq 0, a_n \neq 0$ we define $\deg(f) = n - m$. Note that $\deg(\Delta_i^{\alpha}(t))$ is well-defined. The following observation follows immediately from the classification theorem of finitely generated modules over a PID.

Lemma 2.1. $H_i^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}])$ is a finite-dimensional \mathbb{F} -vector space if and only if $\Delta_i^{\alpha}(t) \neq 0$. If $\Delta_i^{\alpha}(t) \neq 0$, then

$$\deg\left(\Delta_i^{\alpha}(t)\right) = \dim_{\mathbb{F}}\left(H_i^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}])\right).$$

Furthermore deg($\tilde{\Delta}_i^{\alpha}(t)$) = dim_{\mathbb{F}} $\left(\operatorname{Tor}_{\mathbb{F}[t^{\pm 1}]}(H_i^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}])) \right)$.

If ∂M is empty or consists of tori and if $\Delta_1^{\alpha}(t) \neq 0$, then $\Delta_i^{\alpha}(t) \neq 0$ for all i and hence $H_i^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}] \otimes_{\mathbb{F}[t^{\pm 1}]} \mathbb{F}(t)) = 0$ for all i (see Corollary 4.3). Furthermore, $\Delta_3^{\alpha}(t) = 1$ (see Lemma 4.1). Therefore the Reidemeister torsion $\tau(M, \phi, \alpha) \in \mathbb{F}(t)^* / \{rt^l | r \in \mathbb{F}^*, l \in \mathbb{Z}\}$ is defined. We refer to [Tu01] for an excellent introduction into the theory

of Reidemeister torsion. We will mostly use $\tau(M, \phi, \alpha)$ as a convenient way to store information.

Lemma 2.2. (cf. [Tu01, p. 20]) If $\Delta_1^{\alpha}(t) \neq 0$, then $\tau(M, \phi, \alpha)$ is defined and

$$\tau(M,\phi,\alpha) = \prod_{i=0}^{2} \Delta_{i}^{\alpha}(t)^{(-1)^{i+1}} \in \mathbb{F}(t).$$

3. MAIN THEOREM 1: LOWER BOUNDS ON THE THURSTON NORM

Our first main theorem gives a lower bound for the Thurston norm of a non-trivial element $\phi \in H^1(M)$.

Theorem 3.1 (Main Theorem 1). Let M be a 3-manifold whose boundary is empty or consists of tori. Let $\phi \in H^1(M)$ be non-trivial and $\alpha : \pi_1(M) \to GL(\mathbb{F}, k)$ a representation such that $\Delta_1^{\alpha}(t) \neq 0$. Then

$$||\phi||_T \ge \frac{1}{k} \deg(\tau(M, \phi, \alpha)).$$

Equivalently,

$$||\phi||_T \ge \frac{1}{k} \big(\deg\left(\Delta_1^{\alpha}(t)\right) - \deg\left(\Delta_0^{\alpha}(t)\right) - \deg\left(\Delta_2^{\alpha}(t)\right) \big).$$

The proof of the above theorem is given in Section 4.

Remark. Given a presentation for $\pi_1(M)$ the polynomials $\Delta_1^{\alpha}(t)$ and $\Delta_0^{\alpha}(t)$ can be computed efficiently using Fox calculus (cf. e.g. [CF77, p. 98], [KL99a]). We point out that because we view $C_*(\tilde{M})$ as a *right* module over $\mathbb{Z}[\pi_1(M)]$ we need a slightly different definition of Fox derivatives. We refer to [Ha05, Section 6] for details. Proposition 4.13 allows us to compute $\Delta_2^{\alpha}(t)$ using the algorithm for computing the 0-th twisted Alexander polynomial. This shows that the lower bounds of Theorem 3.1 can be computed efficiently.

Remark. Not only does Reidemeister torsion give the most elegant formulation of our lower bounds on the Thurston norm. The functoriality of Reidemeister torsion also allows us to prove results in [FK05], [F05b] and Theorem 6.4 which would be much harder to prove if we only used Alexander polynomials.

Remark. Our restriction to closed manifolds or manifolds whose boundary consists of tori is not a significant restriction. Indeed, if ∂M has a spherical boundary component, then gluing in a 3-ball does not change the Thurston norm. Furthermore manifolds with a boundary component of genus greater than 1 have in most cases vanishing twisted Alexander polynomials.

Combining Theorem 3.1 with Proposition 4.13 we get the following important special case of Theorem 3.1: **Theorem 3.2.** Let M be a 3-manifold whose boundary is empty or consists of tori. Let $\phi \in H^1(M)$ be non-trivial. Let $\alpha : \pi_1(M) \to GL(\mathbb{F}, k)$ be a representation which is unitary with respect to any hermitian form on \mathbb{F}^k . If $\Delta_1^{\alpha}(t) \neq 0$, then

$$||\phi||_T \ge \frac{1}{k} \left(\deg\left(\Delta_1^{\alpha}(t)\right) - \left(1 + b_3(M)\right) \deg\left(\Delta_0^{\alpha}(t)\right) \right).$$

The following lemma shows that in most cases we can determine for a given $\phi \in H^1(M)$ whether $||\phi||_T$ is even or odd. This means that we can 'round up' the lower bounds from Theorem 3.1 to an even or odd number, depending on the parity of $||\phi||_T$. Recall that for a non-trivial $\phi \in H^1(M)$ the *divisibility* of ϕ equals the maximum natural number n such that $\frac{1}{n}\phi \in H^1(M)$.

Lemma 3.3. Let $\phi \in H^1(M)$ be primitive. If M is closed, then $||\phi||_T$ is even. Assume that ∂M consists of a non-empty collection of tori $N_1 \cup \cdots \cup N_s$. If $\phi|_{H_1(N_i)} = 0$ then let $n_i := 0$, otherwise define n_i to be the divisibility of $\phi|_{H_1(N_i)}$. Then

$$||\phi||_T \equiv \left(\sum_{i=1}^s n_i\right) \mod 2.$$

Proof. Let S be a Thurston norm minimizing surface dual to ϕ . If M is closed then S is closed, hence $\chi(S)$ is even. Now assume that ∂M is a collection of tori. Then

$$\chi_{-}(S) \equiv b_0(\partial S) \mod 2.$$

This follows from the observation that adding a 2-disk to each component of ∂S gives a closed surface, which has even Euler characteristic. Now consider N_i . Clearly $S \cap N_i$ is Poincaré dual to $\phi|_{H_1(N_i)}$. It follows from a standard argument that, modulo 2, $\partial S \cap N_i$ has n_i components.

In Section 7.2 we will see that Theorem 3.1 can be very successfully used to determine the genus of knots and the Thurston norm of closed manifolds. In particular we give many examples where the degrees of twisted Alexander polynomials give better bounds on the Thurston norm than the degree of the untwisted Alexander polynomial. But this is not always the case; there are situations when for a given manifold the degree of the twisted Alexander polynomial for some representation gives a worse bound than the degree of the untwisted Alexander polynomial. This should be compared to the situation of [Co04, Ha06, F05b]: Cochran's and Harvey's sequence of higher order Alexander polynomials gives a never decreasing sequence of lower bounds on the Thurston norm.

Remark. Let K_1 and K_2 be knots and assume there exists an epimorphism φ : $\pi_1(X(K_1)) \to \pi_1(X(K_2))$. Simon asked (cf. question 1.12 (b) on Kirby's problem list [Kir97]) whether this implies that genus(K_1) \geq genus(K_2). Let $\alpha : \pi_1(X(K_2)) \to$ $\operatorname{GL}(\mathbb{F}, k)$ be a representation. By [KSW04] $\Delta_{K_2,1}^{\alpha}(t)$ divides $\Delta_{K_1,1}^{\alpha\circ\varphi}(t)$. Together with Lemma 4.8 this shows that the genus bounds from Theorem 3.1 for K_1 are greater than or equal to the bounds for K_2 . Thus our results suggest an affirmative answer to Simon's question. This should also be compared to the results in [Ha06].

For one-dimensional representations it is easy to determine $\Delta_0^{\alpha}(t)$ and $\Delta_2^{\alpha}(t)$ (cf. Proposition 4.13). We immediately get the following theorem which contains Mc-Mullen's theorem [Mc02, Proposition 6.1] and results of Turaev [Tu02a].

Theorem 3.4. Let M be a 3-manifold whose boundary is empty or consists of tori, $\phi \in H^1(M)$ primitive, and $\alpha : \pi_1(M) \to H_1(M) \to GL(\mathbb{F}, 1)$ a one-dimensional representation such that $\Delta_1^{\alpha}(t) \neq 0$. If α is trivial on $Ker(\phi)$, then

$$||\phi||_T \ge \deg(\Delta_1^{\alpha}(t)) - (1 + b_3(M))$$

If α is non-trivial on $Ker(\phi)$, then

$$||\phi||_T \ge \deg\left(\Delta_1^{\alpha}(t)\right).$$

This simple abelian version of Theorem 3.1 can already be very useful. Using results of [FK05] one can show that for primitive $\phi \in H^1(M)$

$$||\phi||_A = \max\{ \deg\left(\Delta_1^{\alpha}(t)\right) | \alpha : \pi_1(M) \to H_1(M) / \operatorname{Tor}(H_1(M)) \to \operatorname{GL}(\mathbb{C}, 1)$$

non-trivial on Ker(ϕ) \},

where $||\phi||_A$ denotes McMullen's Alexander norm. Harvey [Ha05, Proposition 3.12] showed that the invariant $\bar{\delta}_0(\phi)$ in [Ha05] equals $||\phi||_A$. This shows that Alexander polynomials corresponding to one-dimensional representations contain all known lower bounds on the Thurston norm coming from abelian covers.

4. Proof of Main Theorem 1

4.1. Twisted Alexander polynomials of (M, ϕ) .

Lemma 4.1. Let $\phi \in H^1(M)$ be non-trivial and $\alpha : \pi_1(M) \to GL(\mathbb{F}, k)$ a representation. Then $H^{\alpha}_3(M; \mathbb{F}^k[t^{\pm 1}]) = 0$ and $H^{\alpha}_0(M; \mathbb{F}^k[t^{\pm 1}])$ is finite dimensional as a \mathbb{F} -vector space.

Proof. Both statements follow from an easy argument using a cell decomposition of M as in the proof of Proposition 6.3. Note also that Kirk and Livingston showed the second statement in [KL99a, Proposition 3.5].

Lemma 4.2. Assume that ∂M is empty or consists of tori and $\phi \in H^1(M)$ is non-trivial. Let $\alpha : \pi_1(M) \to GL(\mathbb{F}, k)$ be a representation. If $\Delta_1^{\alpha}(t) \neq 0$, then $H_2^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}])$ is $\mathbb{F}[t^{\pm 1}]$ -torsion. In particular $\Delta_2^{\alpha}(t) \neq 0$.

Proof. We know that $\Delta_i^{\alpha}(t) \neq 0$ for i = 0, 1, 3 by assumption and by Lemma 4.1. It follows from the long exact homology sequence for $(M, \partial M)$ and from duality that

 $\chi(M) = \frac{1}{2}\chi(\partial M)$. Hence $\chi(M) = 0$ in our case. It follows from Lemma 4.4 below (applied to the field $\mathbb{F}(t)$) that

$$\sum_{i=0}^{3} (-1)^{i} \dim_{\mathbb{F}(t)} \left(H_{i}^{\alpha}(M; \mathbb{F}^{k}[t^{\pm 1}] \otimes_{\mathbb{F}[t^{\pm 1}]} \mathbb{F}(t)) \right) = k \cdot \chi(M) = 0.$$

Note that $H_i^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}] \otimes_{\mathbb{F}[t^{\pm 1}]} \mathbb{F}(t)) = H_i^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}]) \otimes_{\mathbb{F}[t^{\pm 1}]} \mathbb{F}(t)$ since $\mathbb{F}(t)$ is flat over $\mathbb{F}[t^{\pm 1}]$. By assumption $H_i^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}]) \otimes_{\mathbb{F}[t^{\pm 1}]} \mathbb{F}(t) = 0$ for $i \neq 2$, hence $H_2^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}]) \otimes_{\mathbb{F}[t^{\pm 1}]} \mathbb{F}(t) = 0$ as well. \Box

We get the following corollary immediately from Lemmas 4.1 and 4.2.

Corollary 4.3. Let M be a 3-manifold whose boundary is empty or consists of tori. Let $\phi \in H^1(M)$ be non-trivial and $\alpha : \pi_1(M) \to GL(\mathbb{F}, k)$ a representation. If $\Delta_1^{\alpha}(t) \neq 0$ then $\Delta_i^{\alpha}(t) \neq 0$ for all i, and $\Delta_3^{\alpha}(t) = 1$.

A standard argument shows the following important lemma.

Lemma 4.4. Let X be an n-manifold, \mathbb{K} a field, and $\alpha : \pi_1(X) \to GL(\mathbb{K}, k)$ a representation. Then

$$\sum_{i=0}^{n} (-1)^n \dim_{\mathbb{K}}(H^{\alpha}_*(X; \mathbb{K}^k)) = k\chi(X).$$

4.2. Main argument. In this section we prove Theorem 3.1. Before beginning the proof we give relevant propositions and lemmas. We also need a delicate duality argument which we separately explain in detail in Section 4.3

Let M be a 3-manifold and $\alpha : \pi_1(M) \to \operatorname{GL}(\mathbb{F}, k)$ a representation. We will endow any subset $X \subset M$ with the representation given by $\pi_1(X) \to \pi_1(M) \xrightarrow{\alpha} \operatorname{GL}(\mathbb{F}, k)$. Note that because of base point issues this induced homomorphism is only defined up to conjugacy. But the homology groups $H^{\alpha}_*(X; \mathbb{F}^k)$ are isomorphic, and their dimensions over \mathbb{F} are well-defined. We will therefore suppress base points and the choice of paths connecting base points in our notation. Let $b^{\alpha}_n(X) := \dim_{\mathbb{F}}(H^{\alpha}_n(X; \mathbb{F}^k))$ for $n \geq 0$.

Proposition 4.5. Let $\phi \in H^1(M)$ and S a properly embedded surface dual to ϕ . Then

 $b_1^{\alpha}(S) \ge \dim_{\mathbb{F}} \left(\operatorname{Tor}_{\mathbb{F}[t^{\pm 1}]} \left(H_1^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}]) \right) \right).$

In particular if $\Delta_1^{\alpha}(t) \neq 0$, then $b_1^{\alpha}(S) \geq \deg(\Delta_1^{\alpha}(t))$.

Proof. Denote the components of S by S_1, \ldots, S_l . Denote by N the result of cutting M along S. Denote by i_+ and i_- the two inclusions of S into ∂N induced by taking the positive and the negative inclusions of S into N. We use the same notations i_+ and i_- for the induced homomorphisms on homology groups. Note that ϕ vanishes on $\pi_1(N)$ and on every $\pi_1(S_i)$. Indeed, every curve in S_i can be pushed off into N, where ϕ vanishes. It follows that $H_i^{\alpha}(N; \mathbb{F}^k[t^{\pm 1}]) \cong H_i^{\alpha}(N; \mathbb{F}^k) \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}]$ and

 $H_i^{\alpha}(S; \mathbb{F}^k[t^{\pm 1}]) \cong H_i^{\alpha}(S; \mathbb{F}^k) \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}].$ Therefore we have a commutative diagram of exact sequences (2)

$$\begin{array}{cccc} \stackrel{'}{\to} & H_i^{\alpha}(S; \mathbb{F}^k[t^{\pm 1}]) & \xrightarrow{ti_- - i_+} & H_i^{\alpha}(N; \mathbb{F}^k[t^{\pm 1}]) & \to & H_i^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}]) & \to \\ & \downarrow \cong & & \downarrow = & & \downarrow = \\ \rightarrow & H_i^{\alpha}(S; \mathbb{F}^k) \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}] & \xrightarrow{ti_- - i_+} & H_i^{\alpha}(N; \mathbb{F}^k) \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}] & \to & H_i^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}]). & \to & \downarrow = & & \downarrow$$

Note that $\operatorname{Ker} \{ H_0^{\alpha}(S; \mathbb{F}^k[t^{\pm 1}]) \to H_0^{\alpha}(N; \mathbb{F}^k[t^{\pm 1}]) \} \subset H_0^{\alpha}(S; \mathbb{F}^k) \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}]$ is a (possibly trivial) free $\mathbb{F}[t^{\pm 1}]$ -module F. Therefore we get an exact sequence

$$H_1^{\alpha}(S; \mathbb{F}^k) \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}] \xrightarrow{ti_- - i_+} H_1^{\alpha}(N; \mathbb{F}^k) \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}] \to H_1^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}]) \xrightarrow{\partial} F \to 0.$$

Since $\mathbb{F}[t^{\pm 1}]$ is a PID the map ∂ splits, i.e., $H_1^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}]) \cong \text{Ker}(\partial) \oplus F$. Using appropriate bases the map $ti_- - i_+$, which represents the module $\text{Ker}(\partial)$, is presented by a matrix of size $\dim_{\mathbb{F}} (H_1^{\alpha}(N; \mathbb{F}^k)) \times \dim_{\mathbb{F}} (H_1^{\alpha}(S; \mathbb{F}^k))$ of the form At + B, A, Bmatrices over \mathbb{F} . It follows from an easy argument (cf. e.g. [Ha05, Proposition 9.1] in the harder non-commutative case) that

 $\dim_{\mathbb{F}} \left(\operatorname{Tor}_{\mathbb{F}[t^{\pm 1}]}(\operatorname{Ker}(\partial)) \right) \leq \min \{ \dim_{\mathbb{F}} \left(H_{1}^{\alpha}(N; \mathbb{F}^{k}) \right), \dim_{\mathbb{F}} \left(H_{1}^{\alpha}(S; \mathbb{F}^{k}) \right) \}.$

Combining with the above we get

$$\dim_{\mathbb{F}} \left(\operatorname{Tor}_{\mathbb{F}[t^{\pm 1}]}(H_{1}^{\alpha}(M; \mathbb{F}^{k}[t^{\pm 1}])) \right) = \dim_{\mathbb{F}} \left(\operatorname{Tor}_{\mathbb{F}[t^{\pm 1}]}(\operatorname{Ker}(\partial)) \right) \\ \leq \dim_{\mathbb{F}} \left(H_{1}^{\alpha}(S; \mathbb{F}^{k}) \right) \\ = b_{1}^{\alpha}(S).$$

The last part of this proposition is obvious by Lemma 2.1.

A weighted surface \tilde{S} in M is a collection of pairs $(S_i, w_i), i = 1, \ldots, k$ where $S_i \subset M$ are properly embedded, oriented, disjoint surfaces in M and w_i are positive integers. We denote the union $\bigcup_i S_i \subset M$ by $|\tilde{S}|$.

Every weighted surface \tilde{S} defines an element $\phi_{\tilde{S}} := \sum_{i=1}^{k} w_i \cdot PD([S_i]) \in H^1(M)$ where $PD(f) \in H^1(M)$ denotes the Poincaré dual of an element $f \in H_2(M, \partial M)$. By taking w_i parallel copies of S_i we get an (unweighted) properly embedded oriented surface $\tilde{S}^{\#}$ such that $\phi_{\tilde{S}} = PD(\tilde{S}^{\#})$. An example of a the surface $\tilde{S}^{\#}$ for a weighted surface \tilde{S} is given in Figure 2.

We need the following proposition proved by Turaev in [Tu02b].

Proposition 4.6. Let $\phi \in H^1(M)$. Then there exists a weighted surface \tilde{S} with

(1)
$$\phi_{\tilde{S}} = \phi,$$

(2) $\chi_{-}(\tilde{S}^{\#}) = ||\phi||_{T},$
(3) $M \setminus |\tilde{S}|$ connected,

Proposition 4.7. Let $\phi \in H^1(M)$ be primitive. Let \tilde{S} denote the weighted surface as in Proposition 4.6. Assume $\Delta_1^{\alpha}(t) \neq 0$. Then $S := \tilde{S}^{\#}$ is either connected or $b_0^{\alpha}(S_i) = 0$ for any component S_i of S.



FIGURE 2. Weighted surface in a handlebody.

Proof. Denote by N the result of cutting M along S. Consider the Mayer–Vietoris sequence (2) in Proposition 4.5

$$\to H_1^{\alpha}(S; \mathbb{F}^k) \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}] \xrightarrow{t_{i-i_+}} H_0^{\alpha}(N; \mathbb{F}^k) \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}] \to H_0^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}]) \to 0.$$

From $\Delta_1^{\alpha}(t) \neq 0$ it follows that $H_1^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}])$ is $\mathbb{F}[t^{\pm 1}]$ -torsion. By Lemma 4.1 $H_0^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}])$ is a finite-dimensional \mathbb{F} -vector space, hence $\mathbb{F}[t^{\pm 1}]$ -torsion. If we now consider the above exact sequence with $\mathbb{F}(t)$ -coefficients it follows that

(3)
$$H_0(S; \mathbb{F}^k) \cong H_0(N; \mathbb{F}^k).$$

Since we can arrange w_i parallel copies of S_i inside $\nu(S_i)$ in M, we see that $N \cong (M \setminus \nu|\tilde{S}|) \cup \bigcup_{i=1}^{l} \bigcup_{j=1}^{w_i-1} S_i \times [-1, 1]$. Therefore we have the following isomorphisms

(4)
$$H_0^{\alpha}(S; \mathbb{F}^k) \cong \bigoplus_{i=1}^l H_0^{\alpha}(S_i; \mathbb{F}^k) \oplus \bigoplus_{i=1}^l H_0^{\alpha}(S_i; \mathbb{F}^k)^{w_i - 1}$$
$$H_0^{\alpha}(N; \mathbb{F}^k) \cong H_0^{\alpha}(M \setminus \nu |\tilde{S}|; \mathbb{F}^k) \oplus \bigoplus_{i=1}^l H_0^{\alpha}(S_i; \mathbb{F}^k)^{w_i - 1}$$

where $H_0^{\alpha}(S_i; \mathbb{F}^k)^{w_i-1} := \bigoplus_{i=1}^{w_i-1} H_0^{\alpha}(S_i; \mathbb{F}^k)$. Note that the maps $i_+, i_- : \pi_1(S_i) \to \pi_1(M) \xrightarrow{\alpha} \operatorname{GL}(\mathbb{F}, k)$ factor through $\pi_1(M \setminus \nu |\tilde{S}|)$. Therefore

(5)
$$b_0^{\alpha}(S_i) \ge b_0^{\alpha}(M \setminus \nu|\tilde{S}|), i = 1, \dots, l$$

by Lemma 4.8 below.

First consider the case $b_0^{\alpha}(M \setminus \nu |\tilde{S}|) = 0$. In that case it follows from the isomorphisms in (3) and (4) that $\bigoplus_{i=1}^{l} H_0(S_i; \mathbb{F}^k) = 0$, hence $b_0^{\alpha}(S_i) = 0$ for all $i = 1, \ldots, l$.

Now assume that $b_0^{\alpha}(M \setminus \nu |\hat{S}|) > 0$. It follows immediately from the isomorphisms in (3) and (4) and from the inequality (5) that l = 1. But since ϕ is primitive it also follows that $w_1 = 1$, i.e., S is connected.

We will make use of the following lemma several times.

Lemma 4.8. Let V be an \mathbb{F} -vector space. Let A be a group and $\alpha : A \to GL(V)$ a representation. If $\varphi : B \to A$ is a homomorphism, then $H_0^{\alpha \circ \varphi}(B; V) \to H_0^{\alpha}(A; V)$ is surjective. Furthermore if φ is an epimorphism, then $H_0^{\alpha \circ \varphi}(B; V) \to H_0^{\alpha}(A; V)$ is an isomorphism.

Proof. The lemma follows immediately from the commutative diagram of exact sequences

and the observation that the vertical map on the left is injective (respectively an isomorphism). $\hfill \Box$

Proposition 4.9. Let $\phi \in H^1(M)$ be primitive and $\Delta_1^{\alpha}(t) \neq 0$. Let $S := \tilde{S}^{\#}$ denote the same surface as in Proposition 4.6. Then

$$b_0^{\alpha}(S) = \deg\left(\Delta_0^{\alpha}(t)\right).$$

Proof. Let N be M cut along S. Since $\Delta_1^{\alpha}(t) \neq 0$, we have $H_0^{\alpha}(S; \mathbb{F}^k) \cong H_0^{\alpha}(N; \mathbb{F}^k)$ as \mathbb{F} -vector spaces (see (2) in the proof of Proposition 4.7). First assume that $b_0^{\alpha}(S_i) = 0$ for every component S_i of S. Then $H_0^{\alpha}(S; \mathbb{F}^k) = H_0^{\alpha}(N; \mathbb{F}^k) = 0$. This implies that $H_0^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}]) = 0$ from the exact sequence (2) in the proof of Proposition 4.5, hence $\Delta_0^{\alpha}(t) = 1$.

Now assume that $b_0^{\alpha}(S_i) \neq 0$ for some *i*. By Proposition 4.7 *S* is connected. Hence *N* is connected. It follows from Lemma 4.8 that the maps $i_+, i_- : H_0^{\alpha}(S; \mathbb{F}^k) \to H_0^{\alpha}(N; \mathbb{F}^k)$ are surjective. Since $H_0^{\alpha}(S; \mathbb{F}^k) \cong H_0^{\alpha}(N; \mathbb{F}^k)$ it follows that i_+ and i_- induce isomorphisms on $H_0^{\alpha}(S; \mathbb{F}^k)$. Note that this argument uses that *S* is connected.

Let $b := b_0^{\alpha}(S) = b_0^{\alpha}(N)$. Picking appropriate bases for $H_0^{\alpha}(S; \mathbb{F}^k)$ and $H_0^{\alpha}(N; \mathbb{F}^k)$ the sequence (2) becomes

$$\mathbb{F}^b \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}] \xrightarrow{t \cdot I d - J} \mathbb{F}^b \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}] \to H_0^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}]) \to 0,$$

where $J : \mathbb{F}^b \to \mathbb{F}^b$ is an isomorphism. It follows that $H_0^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}]) \cong \mathbb{F}^b \cong H_0^{\alpha}(S; \mathbb{F}^k)$. The lemma now follows from Lemma 2.1.

We note that from Propositions 4.7 and 4.9 we immediately get the following useful corollary:

Corollary 4.10. If $\Delta_0^{\alpha}(t) \neq 1$ and $\Delta_1^{\alpha}(t) \neq 0$, then there exists a Thurston norm minimizing surface which is connected.

Proposition 4.11. Assume that ∂M is empty or consists of tori. Let $\phi \in H^1(M)$ be primitive and $\Delta_1^{\alpha}(t) \neq 0$. Let $S := \tilde{S}^{\#}$ denote the same surface as in Proposition 4.6. Then

$$b_2^{\alpha}(S) = \deg\left(\Delta_2^{\alpha}(t)\right).$$

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Proof. Let $\tilde{S} = (S_i, w_i)_{i=1,...,l}$ be the weighted surface with $w_i \neq 0$ for all *i* from Proposition 4.6. Let N be M cut along $S = \tilde{S}^{\#}$. Let $I' := \{i \in \{1, ..., l\} | S_i \text{ closed}\}$ and $I'' := \{i \in \{1, ..., l\} | S_i \text{ has non-empty boundary}\}$. Denote the union of w_i parallel copies of $S_i, i \in I'$, by $S' \subset S$. Clearly $b_2^{\alpha}(S) = b_2^{\alpha}(S')$.

Note that we can write $\partial N = S'_+ \cup S'_- \cup W$ for some surface W where S'_- and S'_+ are the images of the two canonical inclusion maps of $S' \to N$. It follows from Lemmas 4.1 and 4.2 that the long exact sequence (2) becomes

$$0 \to H_2^{\alpha}(S'; \mathbb{F}^k) \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}] \xrightarrow{ti_- - i_+} H_2^{\alpha}(N; \mathbb{F}^k) \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}] \to H_2^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}]) \to 0.$$

Clearly we are done once we show that $i_{-}, i_{+} : H_{2}^{\alpha}(S'; \mathbb{F}^{k}) \to H_{2}^{\alpha}(N; \mathbb{F}^{k})$ are isomorphisms. Considering the sequence with $\mathbb{F}(t)$ -coefficients it follows that $H_{2}^{\alpha}(S'; \mathbb{F}^{k})$ and $H_{2}^{\alpha}(N; \mathbb{F}^{k})$ have the same dimension as \mathbb{F} -vector spaces. It is therefore enough to show that i_{-} and i_{+} are injections, or equivalently that the maps $H_{2}^{\alpha}(S'_{\pm}; \mathbb{F}^{k}) \to H_{2}^{\alpha}(N; \mathbb{F}^{k})$ are injections.

Consider the short exact sequence

$$H_3^{\alpha}(N, S_+; \mathbb{F}^k) \to H_2^{\alpha}(S'_+; \mathbb{F}^k) \to H_2^{\alpha}(N; \mathbb{F}^k).$$

By Poincaré duality and by Lemma 4.12 in Section 4.3 we have

$$H_3^{\alpha}(N, S'_+; \mathbb{F}^k) \cong H_{\alpha}^0(N, S'_- \cup W; \mathbb{F}^k) \cong \operatorname{Hom}_{\mathbb{F}}(H_0^{\overline{\alpha}}(N, S'_- \cup W; \mathbb{F}^k), \mathbb{F}).$$

Here $\overline{\alpha}$ is the adjoint representation of α which is defined in the first paragraph of Section 4.3.

Claim.

$$H_0^{\overline{\alpha}}(N, S'_- \cup W; \mathbb{F}^k) = 0.$$

Recall that

$$N \cong M \setminus \nu |\tilde{S}| \cup \bigcup_{i \in I'} \bigcup_{j=1}^{w_i - 1} S_i \times [0, 1] \cup \bigcup_{i \in I''} \bigcup_{j=1}^{w_i - 1} S_i \times [0, 1]$$

which equals the decomposition of N into connected components. Clearly there exists a surjective map

$$\varphi : \{ \text{components of } S'_{-} \cup W \} \to \{ \text{components of } N \},$$

such that $S_0 \subset \partial(\varphi(S_0))$ for every component S_0 of $S'_- \cup W$. Therefore it follows from Lemma 4.8 that $H_0^{\overline{\alpha}}(S'_- \cup W; \mathbb{F}^k[t^{\pm 1}]) \to H_0^{\overline{\alpha}}(N; \mathbb{F}^k[t^{\pm 1}])$ is surjective. The claim now follows from the long exact homology sequence.

Now we can conclude the proof of Theorem 3.1.

Proof of Theorem 3.1. Without loss of generality we can assume that ϕ is primitive since the Thurston norm and the degrees of twisted Alexander polynomials are

homogeneous. Let \tilde{S} be the weighted surface from Proposition 4.6. Let $S := \tilde{S}^{\#}$. By Lemma 4.4 we have

$$||\phi||_{T} = \max\{0, b_{1}(S) - (b_{0}(S) + b_{2}(S))\}$$

$$\geq b_{1}(S) - (b_{0}(S) + b_{2}(S))$$

$$= \frac{1}{k} (b_{1}^{\alpha}(S) - (b_{0}^{\alpha}(S) + b_{2}^{\alpha}(S))).$$

The theorem now follows immediately from Propositions 4.5, 4.1, 4.9, 4.11 and Lemma 2.2.

4.3. **Duality arguments.** In this section we clarify a delicate duality argument. Since this is perhaps of independent interest, and since we need it in [F05b] we will explain this in the non-commutative setting.

In this section let R be a (possibly non-commutative) ring with involution $r \mapsto \overline{r}$ such that $\overline{ab} = \overline{b} \cdot \overline{a}$. Let V be a right R-module together with a map $\beta : \pi_1(M) \to GL(V, R)$. This representation β can be used to define a left $\mathbb{Z}[\pi_1(M)]$ -module structure on V which commutes with the R-module structure. Pick a non-singular Rsesquilinear inner product $\langle , \rangle : V \times V \to R$. This means that for all $v, w \in V$ and $r \in R$ we have

$$\langle vr, w \rangle = \langle v, w \rangle r, \quad \langle v, wr \rangle = \overline{r} \langle v, w \rangle$$

and \langle , \rangle induces via $v \mapsto (w \mapsto \langle v, w \rangle)$ an *R*-module isomorphism $V \cong \operatorname{Hom}_R(V, R)$. Here we view $\operatorname{Hom}_R(V, R)$ as right *R*-module homomorphisms where *R* gets the right *R*-module structure given by involuted left multiplication. Furthermore consider $\operatorname{Hom}_R(V, R)$ as a right *R*-module via right multiplication in the target *R*.

There exists a unique representation $\beta : \pi_1(M) \to \operatorname{GL}(V, R)$ such that

$$\langle \beta(g^{-1})v, w \rangle = \langle v, \beta(g)w \rangle$$

for all $v, w \in V, g \in \pi_1(M)$. Note that $\overline{\beta}$ induces a left $\mathbb{Z}[\pi_1(M)]$ -module structure on V (which is possibly different from that induced from β) which commutes with the R-module structure. To clarify which $\mathbb{Z}[\pi_1(M)]$ -module structure we use, we sometimes denote V with the $\mathbb{Z}[\pi_1(M)]$ -module structure induced from β (respectively $\overline{\beta}$) by $V(\beta)$ (respectively $V(\overline{\beta})$). Note that they are the same viewed as R-modules.

Lemma 4.12. [KL99a, p. 638] Let X be an n-manifold, V an R-module and β : $\pi_1(X) \to GL(V)$ a representation. Let $\langle , \rangle : V \times V \to R$ be a non-singular sesquilinear inner product as above. If R is a PID then

$$H_{n-i}^{\beta}(X;V(\beta)) \cong Hom_{R}(H_{i}^{\overline{\beta}}(X,\partial X;V(\overline{\beta})),R) \oplus Ext_{R}(H_{i-1}^{\overline{\beta}}(X,\partial X;V(\overline{\beta})),R)$$

as *R*-modules.

Here we equip $H_*(-, V)$, $H^*(-, V)$ with the right *R*-module structures given on *V*. Also for a right *R*-module *H* we view $\operatorname{Hom}_R(H, R)$ as a right *R*-module homomorphisms where *R* gets the right *R*-module structure given by involuted left multiplication. We consider $\operatorname{Hom}_R(H, R)$ as a right *R*-module via right multiplication in the target *R*.

Proof. Let $\pi := \pi_1(X)$. Let $V(\beta)' = V(\beta)$ as *R*-modules equipped with the right $\mathbb{Z}[\pi_1(M)]$ -module structure given by $v \cdot g := \beta(g^{-1})v$ for $v \in V(\beta)$ and $g \in \pi$. By Poincaré duality

$$H_{n-i}^{\beta}(X;V(\beta)) \cong H^{i}(X,\partial X;V(\beta)') := H_{i}\big(\operatorname{Hom}_{\mathbb{Z}[\pi]}(C_{*}(\tilde{X},\partial \tilde{X}),V(\beta)')\big),$$

where \tilde{X} denotes the universal cover of X. Using the inner product we get a map

$$\operatorname{Hom}_{\mathbb{Z}[\pi]}(C_*(\tilde{X},\partial\tilde{X}),V(\beta)') \to \operatorname{Hom}_R(C_*(\tilde{X},\partial\tilde{X})\otimes_{\mathbb{Z}[\pi]}V(\bar{\beta}),R)$$
$$f \mapsto ((c\otimes w)\mapsto \langle f(c),w\rangle).$$

Note that this map is well-defined since $\langle \beta(g^{-1})v, w \rangle = \langle v, \overline{\beta}(g)w \rangle$. It is now easy to see that this defines in fact an isomorphism of right *R*-module chain complexes.

Now we can apply the universal coefficient theorem for chain complexes over the PID R to $C_*(\tilde{X}, \partial \tilde{X}) \otimes_{\mathbb{Z}[\pi]} V(\bar{\beta})$. The lemma is now immediate. \Box

Now assume that the field \mathbb{F} has a (possibly trivial) involution. We equip \mathbb{F}^k with a hermitian inner product, denoted by \langle , \rangle .

Proposition 4.13. Let M be a 3-manifold whose boundary is empty or consists of tori and let $\phi \in H^1(M)$ be non-trivial. Let $\alpha : \pi_1(M) \to GL(\mathbb{F}, k)$ be a representation such that $\Delta_1^{\alpha}(t) \neq 0$.

(1) If M is closed, then

$$\Delta_2^{\alpha}(t) = \Delta_0^{\overline{\alpha}}(t^{-1}).$$

(2) If M has non-empty boundary, then $\Delta_2^{\alpha}(t) = 1$.

In particular deg $(\Delta_2^{\alpha}(t)) = b_3(M) \deg (\Delta_0^{\overline{\alpha}}(t))$. Furthermore, if α is unitary, i.e. $\alpha = \overline{\alpha}$, then deg $(\Delta_2^{\alpha}(t)) = b_3(M) \deg (\Delta_0^{\alpha}(t))$.

Proof. We extend the involution on \mathbb{F} to $\mathbb{F}[t^{\pm 1}]$ by taking $t \mapsto t^{-1}$. Now equip $\mathbb{F}^k[t^{\pm 1}]$ with the hermitian inner product defined by $\langle vt^i, wt^j \rangle := \langle v, w \rangle t^i t^{-j}$ for all $v, w \in \mathbb{F}^k$. To simplify the notation we denote $\mathbb{F}^k[t^{\pm 1}](\alpha \otimes \phi)$ and $\mathbb{F}^k[t^{\pm 1}](\overline{\alpha \otimes \phi})$ just by $\mathbb{F}^k[t^{\pm 1}]$. The $\mathbb{Z}[\pi_1(M)]$ -module structure on $\mathbb{F}^k[t^{\pm 1}]$ will always be clear from the context.

Note that $\mathbb{F}[t^{\pm 1}]$ is a PID. We apply Lemma 4.12 with $R = \mathbb{F}[t^{\pm 1}], V = \mathbb{F}^k[t^{\pm 1}]$ and $\beta = \alpha \otimes \phi$, and get

$$\begin{aligned} H_2^{\alpha \otimes \phi}(M; \mathbb{F}^k[t^{\pm 1}]) &\cong & \operatorname{Hom}_{\mathbb{F}[t^{\pm 1}]} \left(H_1^{\overline{\alpha \otimes \phi}}(M, \partial M; \mathbb{F}^k[t^{\pm 1}]), \mathbb{F}[t^{\pm 1}] \right) \\ &\oplus & \operatorname{Ext}_{\mathbb{F}[t^{\pm 1}]} \left(H_0^{\overline{\alpha \otimes \phi}}(M, \partial M; \mathbb{F}^k[t^{\pm 1}]), \mathbb{F}[t^{\pm 1}] \right) \end{aligned}$$

as $\mathbb{F}[t^{\pm 1}]$ -modules. Since $\Delta_1^{\alpha}(t) \neq 0$, $H_2^{\alpha \otimes \phi}(M; \mathbb{F}^k[t^{\pm 1}])$ is $\mathbb{F}[t^{\pm 1}]$ -torsion by Lemma 4.2. Hence the first summand on the right hand side is zero.

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By Lemma 4.1 $H_0^{\overline{\alpha \otimes \phi}}(M; \mathbb{F}^k[t^{\pm 1}])$ is $\mathbb{F}[t^{\pm 1}]$ -torsion. From the long exact homology sequence of the pair $(M, \partial M)$ it follows that $H_0^{\overline{\alpha \otimes \phi}}(M, \partial M; \mathbb{F}^k[t^{\pm 1}])$ is also $\mathbb{F}[t^{\pm 1}]$ torsion. Since $H_0^{\overline{\alpha \otimes \phi}}(M, \partial M; \mathbb{F}^k[t^{\pm 1}])$ is a finitely generated $\mathbb{F}[t^{\pm 1}]$ -torsion module and $\mathbb{F}[t^{\pm 1}]$ is a PID, $\operatorname{Ext}_{\mathbb{F}[t^{\pm 1}]}(H_0^{\overline{\alpha \otimes \phi}}(M, \partial M; \mathbb{F}^k[t^{\pm 1}]), \mathbb{F}[t^{\pm 1}]) \cong H_0^{\overline{\alpha \otimes \phi}}(M, \partial M; \mathbb{F}^k[t^{\pm 1}]).$

If M is closed then we get $H_2^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}]) \cong H_0^{\overline{\alpha \otimes \phi}}(M; \mathbb{F}^k[t^{\pm 1}])$. Note that $\overline{\alpha \otimes \phi} = \overline{\alpha} \otimes (-\phi)$. Therefore we deduce that $\Delta_2^{\alpha}(t) = \underline{\Delta_0^{\overline{\alpha}}(t^{-1})}$.

If $\partial M \neq 0$, then by Lemma 4.8 the map $H_0^{\overline{\alpha \otimes \phi}}(\partial M; \mathbb{F}^k[t^{\pm 1}]) \to H_0^{\overline{\alpha \otimes \phi}}(M; \mathbb{F}^k[t^{\pm 1}])$ is surjective, hence $H_0^{\overline{\alpha \otimes \phi}}(M, \partial M; \mathbb{F}^k[t^{\pm 1}]) = 0$. This shows that $H_2^{\alpha}(M; \mathbb{F}^k[t^{\pm 1}]) = 0$ and hence $\Delta_2^{\alpha}(t) = 1$.

5. The case of vanishing Alexander Polynomials

Let L be a boundary link (for example a split link). It is well–known that the multivariable Alexander polynomial of L has to vanish (cf. [Hi02]). Most of the twisted multivariable and twisted one–variable Alexander polynomials vanish as well. (See [FK05] for the definition of twisted multivariable Alexander polynomials.) Therefore Theorem 3.1 can in most cases not be applied to get lower bounds on the Thurston norm.

It follows clearly from Propositions 4.5 and 4.11 that the condition $\Delta_1^{\alpha}(t) \neq 0$ is only needed to ensure that there exists a surface S dual to ϕ with $b_0^{\alpha}(S) = \deg(\Delta_0^{\alpha}(t))$ and $b_2^{\alpha}(S) = \deg(\Delta_2^{\alpha}(t))$. The following theorem can often be applied in the case of link complements.

Theorem 5.1. Let M be a 3-manifold such that $H^1(M) \xrightarrow{i^*} H^1(\partial M)$ is an injection where i^* is the inclusion-induced homomorphism. Let N be a torus component of ∂M and $\phi \in H^1(N) \cap Im(i^*)$ primitive, and $\alpha : \pi_1(M) \to GL(\mathbb{F}, k)$ a representation. Then

$$||(i^*)^{-1}(\phi)||_{T,M} \ge \frac{1}{k} \deg(\tilde{\Delta}_1^{\alpha}(t)) - 1.$$

It is not hard to show that we can find a Thurston norm minimizing surface dual to $(i^*)^{-1}(\phi)$ which is connected and has boundary (cf. e.g. [Ha05, Corollary 10.4] or Turaev [Tu02b, p. 14]). The theorem now follows from the proof of Theorem 3.1.

We will apply this theorem later to the complement of a link $L = L_1 \cup \cdots \cup L_m \subset S^3$. In this case we can take ϕ to be dual to the meridian of the i^{th} component L_i . Then it follows from the proof of Theorem 5.1 and a standard argument that $||(i^*)^{-1}(\phi)||_T = 2 \operatorname{genus}(L_i) - 1$, where $\operatorname{genus}(L_i)$ denotes the minimal genus of a surface in X(L) bounding a parallel copy of L_i . Similar results were obtained by Turaev [Tu02b, p. 14] and Harvey [Ha05, Corollary 10.4].

The following observation will show that in more complicated cases there is no immediate way to determine $b_0(S)$: if $L = L_1 \cup L_2$ is a split oriented link, and $\phi : H_1(X(L)) \to \mathbb{Z}$ given by sending the meridians to 1, then a Thurston norm

minimizing surface S dual to ϕ is easily seen to be the disjoint union of the Seifert surfaces of L_1 and L_2 . In particular $b_0(S) = 2$. On the other hand if L_1 and L_2 are parallel copies of a knot with opposite orientations and $\phi : H_1(X(L)) \to \mathbb{Z}$ is again given by sending the meridians to 1, then the annulus S between L_1 and L_2 is dual to ϕ with Euler characteristic zero. In particular it is connected, hence $b_0(S) = 1$. Summarizing, we have two situations in which the first twisted Alexander polynomials vanish, ϕ is of the same type, but $b_0(S)$ differs.

6. Main theorem 2: Obstructions to fiberedness

Theorem 6.1 (Main Theorem 2). Let M be a 3-manifold and $\phi \in H^1(M)$ such that (M, ϕ) fibers over S^1 and such that $M \neq S^1 \times D^2$, $M \neq S^1 \times S^2$. If $\alpha : \pi_1(M) \to GL(\mathbb{F}, k)$ is a representation, then $\Delta_1^{\alpha}(t) \neq 0$ and

$$\begin{aligned} ||\phi||_T &= \frac{1}{k} \operatorname{deg}(\tau(M,\phi,\alpha)) \\ &= \frac{1}{k} \big(\operatorname{deg}\left(\Delta_1^{\alpha}(t)\right) - \operatorname{deg}\left(\Delta_0^{\alpha}(t)\right) - \operatorname{deg}\left(\Delta_2^{\alpha}(t)\right) \big). \end{aligned}$$

If α is unitary, then also

$$||\phi||_T = \frac{1}{k} \left(\deg \left(\Delta_1^{\alpha}(t) \right) - \left(1 + b_3(M) \right) \deg \left(\Delta_0^{\alpha}(t) \right) \right).$$

Proof. Let S be a fiber of the fiber bundle $M \to S^1$. Clearly S is dual to $\phi \in H^1(M)$ and it is well-known that S is Thurston norm minimizing. Denote by \hat{M} the infinite cyclic cover of M corresponding to ϕ . Then an easy argument shows that $H^{\alpha}_i(M; \mathbb{F}^k[t^{\pm 1}]) \cong H^{\alpha}_i(\hat{M}; \mathbb{F}^k)$ (cf. also [KL99a, Theorem 2.1]). In particular $H^{\alpha}_i(M; \mathbb{F}^k[t^{\pm 1}]) \cong H^{\alpha}_i(S; \mathbb{F}^k)$.

By assumption $S \neq D^2$ and $S \neq S^2$. Therefore by Lemmas 4.4 and 2.1 we get

$$\begin{aligned} ||\phi||_{T} &= \chi_{-}(S) \\ &= b_{1}(S) - b_{0}(S) - b_{2}(S) \\ &= \frac{1}{k} \left(b_{1}^{\alpha}(S) - b_{0}^{\alpha}(S) - b_{2}^{\alpha}(S) \right) \\ &= \frac{1}{k} \left(\dim_{\mathbb{F}} \left(H_{1}^{\alpha}(M; \mathbb{F}^{k}[t^{\pm 1}]) \right) - \dim_{\mathbb{F}} \left(H_{0}^{\alpha}(M; \mathbb{F}^{k}[t^{\pm 1}]) \right) - \dim_{\mathbb{F}} \left(H_{2}^{\alpha}(M; \mathbb{F}^{k}[t^{\pm 1}]) \right) \\ &= \frac{1}{k} \left(\deg \left(\Delta_{1}^{\alpha}(t) \right) - \deg \left(\Delta_{0}^{\alpha}(t) \right) - \deg \left(\Delta_{2}^{\alpha}(t) \right) \right) \\ &= \deg(\tau(M, \phi, \alpha)). \end{aligned}$$

The unitary case follows now immediately from Proposition 4.13.

Since $||\phi||_T$ might be unknown for a given example the following corollary gives a more practical fibering obstruction.

Corollary 6.2. Let M be a 3-manifold and $\phi \in H^1(M)$ such that (M, ϕ) fibers over S^1 and such that $M \neq S^1 \times D^2, M \neq S^1 \times S^2$. Let \mathbb{F} and \mathbb{F}' be fields. Consider the untwisted Alexander polynomial $\Delta_1(t) \in \mathbb{F}[t^{\pm 1}]$. For any representation $\alpha : \pi_1(M) \to \mathbb{F}[t^{\pm 1}]$.

 $GL(\mathbb{F}',k)$ we have

$$\deg(\Delta_1(t)) - \left(1 + b_3(M)\right) = \frac{1}{k} \left(\deg\left(\Delta_1^{\alpha}(t)\right) - \deg\left(\Delta_0^{\alpha}(t)\right) - \deg\left(\Delta_2^{\alpha}(t)\right)\right).$$

Proof. The corollary follows immediately from applying Theorem 6.1 to the trivial representation $\pi_1(M) \to \operatorname{GL}(\mathbb{F}, 1)$ and to the representation α .

Let $\alpha : \pi_1(M) \to \operatorname{GL}(R,k)$ be a representation where R is a Noetherian UFD, for example $R = \mathbb{Z}$ or a field \mathbb{F} . Recall that in this situation Cha [Ch03] defined the twisted Alexander polynomial $\Delta_1^{\alpha}(t) \in R[t^{\pm 1}]$ which is well–defined up to multiplication by a unit in $R[t^{\pm 1}]$. Given a prime ideal $\mathfrak{p} \subset R$ we denote the quotient field of R/\mathfrak{p} by $\mathbb{F}_{\mathfrak{p}}$. Furthermore we denote by $\alpha_{\mathfrak{p}}$ the representation $\pi_1(M) \xrightarrow{\alpha}$ $\operatorname{GL}(R,k) \to \operatorname{GL}(\mathbb{F}_{\mathfrak{p}},k)$ where $\operatorname{GL}(R,k) \to \operatorname{GL}(\mathbb{F}_{\mathfrak{p}},k)$ is induced from the canonical map $\pi_{\mathfrak{p}} : R \to R/\mathfrak{p} \to \mathbb{F}_{\mathfrak{p}}$.

Proposition 6.3. Let M be a 3-manifold whose boundary is empty or consists of tori and let R be a Noetherian UFD. Let $\phi \in H^1(M)$ be non-trivial and $\alpha : \pi_1(M) \to$ GL(R,k) a representation. Then $\Delta_1^{\alpha_p}(t)$ is non-trivial and

$$||\phi||_T = \frac{1}{k} \operatorname{deg}(\tau(M, \phi, \alpha_{\mathfrak{p}}))$$

for all prime ideals \mathfrak{p} if and only if $\Delta_1^{\alpha}(t) \in R[t^{\pm 1}]$ is monic and

$$||\phi||_T = \frac{1}{k} \deg(\tau(M, \phi, \alpha))$$

We will prove Proposition 6.3 at the end of this section. By combining Theorem 6.1 and Proposition 6.3 we immediately get the following theorem.

Theorem 6.4. Let M be a 3-manifold. Let $\phi \in H^1(M)$ be non-trivial such that (M, ϕ) fibers over S^1 and such that $M \neq S^1 \times D^2, M \neq S^1 \times S^2$. Let R be a Noetherian UFD and $\alpha : \pi_1(M) \to GL(R, k)$ a representation. Then $\Delta_1^{\alpha}(t) \in R[t^{\pm 1}]$ is monic and

$$||\phi||_T = \frac{1}{k} \deg(\tau(M, \phi, \alpha)).$$

Remark. (1) Theorem 6.4 shows that the fibering obstructions from Theorem 6.1 contain Neuwirth's theorem that $\Delta_K(t) \in \mathbb{Z}[t^{\pm 1}]$ is monic for a fibered knot.

- (2) Cha's methods in [Ch03] can be used to show that if (M, ϕ) fibers over S^1 , $\partial M \neq \emptyset$ and if $\alpha : \pi_1(M) \to \operatorname{GL}(R, k)$, R a Noetherian UFD, is a representation factoring through a finite group G, then the corresponding Alexander polynomial $\Delta_1^{\alpha}(t) \in R[t^{\pm 1}]$ is monic. Thus Theorems 6.1 and 6.4 generalize Cha's results.
- (3) The main significance of our results lies in the fact that they also give fibering obstructions for *closed* manifolds.

(4) Our approach in Theorem 6.1 also have advantages from a computational point of view since it is easier to compute Alexander polynomials over several PID's $\mathbb{F}_p[t^{\pm 1}]$ than over the ring $\mathbb{Z}[t^{\pm 1}]$.

Proof of Proposition 6.3. We only prove this proposition in the case that M is closed. The case that ∂M is a non-empty collection of tori is very similar. Note that in either case $\chi(M) = 0$.

We first make use of an argument in the proof of [Mc02, Theorem 5.1]. Choose a triangulation τ of M. Let T be a maximal tree in the 1-skeleton of τ and let T'be a maximal tree in the dual 1-skeleton. We collapse T to form a single 0-cell and join the 3-simplices of T' to form a single 3-cell. Denote the number of 1-cells by n. It follows from M closed that $\chi(M) = 0$, hence there are n 2-cells. From the CW structure we obtain a chain complex $C_*(\tilde{M})$ of the following form

$$0 \to C_3(\tilde{M}) \xrightarrow{\partial_3} C_2(\tilde{M}) \xrightarrow{\partial_2} C_1(\tilde{M}) \xrightarrow{\partial_1} C_0(\tilde{M}) \to 0$$

where $C_i(\tilde{M}) \cong \mathbb{Z}[\pi_1(M)]$ for i = 0, 3 and $C_i(\tilde{M}) \cong \mathbb{Z}[\pi_1(M)]^n$ for i = 1, 2. Let $A_i, i = 0, \ldots, 3$ over $\mathbb{Z}[\pi_1(M)]$ be the matrices corresponding to the boundary maps $\partial_i : C_i \to C_{i-1}$ with respect to the bases given by the lifts of the cells of M to \tilde{M} . We can arrange the lifts such that

$$A_3 = (1 - g_1, 1 - g_2, \dots, 1 - g_n)^t, A_1 = (1 - h_1, 1 - h_2, \dots, 1 - h_n),$$

where $\{g_1, \ldots, g_n\}$ and $\{h_1, \ldots, h_n\}$ are generating sets for $\pi_1(M)$. Since ϕ is nontrivial there exist r, s such that $\phi(g_r) \neq 0$ and $\phi(h_s) \neq 0$. Let B_3 be the *r*-th row of A_3 . Let B_2 be the result of deleting the *r*-th column and the *s*-th row from A_2 . Let B_1 be the *s*-th column of A_1 . Note that

$$\det((\alpha \otimes \phi)(B_3)) = \det(\mathrm{id} - (\alpha \otimes \phi)(g_r)) = \det(\mathrm{id} - \phi(g_r)\alpha(g_r)) \neq 0$$

since $\phi(g_r) \neq 0$. Similarly det $((\alpha \otimes \phi)(B_1)) \neq 0$ and det $((\alpha_{\mathfrak{p}} \otimes \phi)(B_i)) \neq 0$, i = 1, 3 for any prime ideal \mathfrak{p} . We need the following theorem which can be found in [Tu01].

Theorem 6.5. [Tu01, Theorem 2.2, Lemma 2.5, Theorem 4.7] Let S be a Noetherian UFD. Let $\beta : \pi_1(M) \to GL(S, k)$ be a representation and $\varphi \in H^1(M)$.

- (1) If det($(\beta \otimes \varphi)(B_i)$) $\neq 0$ for i = 1, 2, 3, then $H_i^{\beta}(M; S^k[t^{\pm 1}])$ is $S[t^{\pm 1}]$ -torsion for all i.
- (2) If $H_i^{\beta}(M; S^k[t^{\pm 1}])$ is $S[t^{\pm 1}]$ -torsion for all i, and if $\det((\beta \otimes \varphi)(B_i)) \neq 0$ for i = 1, 3, then $\det((\beta \otimes \varphi)(B_2)) \neq 0$ and

$$\prod_{i=1}^{3} \det((\beta \otimes \varphi)(B_i))^{(-1)^{i+1}} = \prod_{i=0}^{3} \left(\Delta_i^{\beta}(t)\right)^{(-1)^{i+1}} = \tau(M,\varphi,\beta).$$

First assume that $\Delta_1^{\alpha_p}(t) \neq 0$ and

$$||\phi||_{T} = \frac{1}{k} \left(\deg \left(\Delta_{1}^{\alpha_{\mathfrak{p}}}(t) \right) - \deg \left(\Delta_{0}^{\alpha_{\mathfrak{p}}}(t) \right) - \deg \left(\Delta_{2}^{\alpha_{\mathfrak{p}}}(t) \right) \right)$$

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for all prime ideals \mathfrak{p} . By Corollary 4.3 we get $\Delta_i^{\alpha_\mathfrak{p}}(t) \neq 0$ for all i, in particular $H_i^{\alpha_\mathfrak{p}}(M; \mathbb{F}_\mathfrak{p}^k[t^{\pm 1}])$ is $\mathbb{F}_\mathfrak{p}[t^{\pm 1}]$ -torsion for all i and all prime ideals \mathfrak{p} . It follows from Theorem 6.5 that $\det((\alpha_\mathfrak{p} \otimes \phi)(B_2)) \neq 0$. Clearly this also implies that $\det((\alpha \otimes \phi)(B_2)) \neq 0$ for i = 1, 3 it follows from Theorem 6.5 that $H_i^{\alpha}(M; \mathbb{R}^k[t^{\pm 1}])$ is $\mathbb{R}[t^{\pm 1}]$ -torsion for all i.

It follows from [Tu01, Lemma 4.11] that $\Delta_0^{\alpha}(t)$ divides $\det((\alpha \otimes \phi)(B_1)) = \det(\operatorname{id} - \phi(h_s)\alpha(h_s))$ which is a monic polynomial in $R[t^{\pm 1}]$ since $\phi(h_s) \neq 0$ and since $\det(\alpha(h_s))$ is a unit. But then $\Delta_0^{\alpha}(t)$ is monic as well. The same argument (again using [Tu01, Lemma 4.11]) shows that $\Delta_2^{\alpha}(t)$ is monic. It follows from Lemma 4.1 that $H_3^{\alpha}(M; R^k[t^{\pm 1}]) = 0$, hence $\Delta_3^{\alpha}(t) = 1$.

Denote the map $R \to R/\mathfrak{p} \to \mathbb{F}_{\mathfrak{p}}$ by $\pi_{\mathfrak{p}}$. We also denote the induced map $R[t^{\pm 1}] \to \mathbb{F}_{\mathfrak{p}}[t^{\pm 1}]$ by $\pi_{\mathfrak{p}}$. It follows from Theorem 6.5 that

$$\prod_{i=0}^{3} \pi_{\mathfrak{p}} \left(\Delta_{i}^{\alpha}(t)^{(-1)^{i+1}} \right) = \prod_{i=1}^{3} \pi_{\mathfrak{p}} \left(\det((\alpha \otimes \phi)(B_{i}))^{(-1)^{i+1}} \right)$$

$$= \prod_{i=1}^{3} \det((\alpha_{\mathfrak{p}} \otimes \phi)(B_{i}))^{(-1)^{i+1}}$$

$$= \prod_{i=0}^{3} \Delta_{i}^{\alpha_{\mathfrak{p}}}(t)^{(-1)^{i+1}}$$

for all prime ideals \mathfrak{p} . By assumption we get

$$\frac{1}{k} \sum_{i=0}^{3} (-1)^{i+1} \deg \left(\pi_{\mathfrak{p}} \left(\Delta_{i}^{\alpha}(t) \right) \right) = \frac{1}{k} \sum_{i=0}^{3} (-1)^{i+1} \deg \left(\Delta_{i}^{\alpha_{\mathfrak{p}}}(t) \right) = ||\phi||_{T}$$

for all \mathfrak{p} . Since $\Delta_i^{\alpha}(t)$ is monic for i = 0, 2, 3 it follows that

$$\deg\left(\pi_{\mathfrak{p}}\left(\Delta_{1}^{\alpha}(t)\right)\right) = \deg\left(\pi_{\mathfrak{q}}\left(\Delta_{1}^{\alpha}(t)\right)\right)$$

for all prime ideals \mathfrak{p} and \mathfrak{q} . Since R is a UFD it follows that $\Delta_1^{\alpha}(t)$ is monic. Hence $\deg(\pi_{\mathfrak{p}}(\Delta_i^{\alpha}(t))) = \deg(\Delta_i^{\alpha}(t))$ for all i and all prime ideals \mathfrak{p} and clearly

$$||\phi||_T = \frac{1}{k} \big(\deg \left(\Delta_1^{\alpha}(t) \right) - \deg \left(\Delta_0^{\alpha}(t) \right) - \deg \left(\Delta_2^{\alpha}(t) \right) \big).$$

Now assume that $\Delta_1^{\alpha}(t) \in R[t^{\pm 1}]$ is monic and

$$||\phi||_T = \frac{1}{k} \left(\deg\left(\Delta_1^{\alpha}(t)\right) - \deg\left(\Delta_0^{\alpha}(t)\right) - \deg\left(\Delta_2^{\alpha}(t)\right) \right).$$

The same argument as above shows that $\Delta_i^{\alpha}(t)$, i = 0, 2, 3, are monic as well. Recall that $\det(\alpha \otimes \phi)(B_i)$, i = 1, 3, are monic polynomials. It follows from Theorem 6.5 that

$$\det(\alpha \otimes \phi)(B_2) = \det(\alpha \otimes \phi)(B_1) \, \det(\alpha \otimes \phi)(B_3) \, \prod_{i=0}^3 \left(\Delta_i^{\alpha}(t)\right)^{(-1)^{i+1}}$$

is a quotient of monic non-zero polynomials. In particular det $(\alpha_{\mathfrak{p}} \otimes \phi)(B_2) = \pi_{\mathfrak{p}}(\det(\alpha \otimes \phi)(B_2)) \neq 0$. It now follows immediately from Theorem 6.5 that $H_i^{\alpha_{\mathfrak{p}}}(M; \mathbb{F}_{\mathfrak{p}}^k[t^{\pm 1}]))$ is

 $\mathbb{F}_{\mathfrak{p}}[t^{\pm 1}]$ -torsion for all *i*. In particular $\Delta_1^{\alpha_{\mathfrak{p}}}(t) \neq 0$. Using arguments as above we now see that

$$deg(\tau(M,\phi,\alpha_p)) = \frac{1}{k} \left(deg\left(\Delta_1^{\alpha_p}(t)\right) - deg\left(\Delta_0^{\alpha_p}(t)\right) - deg\left(\Delta_2^{\alpha_p}(t)\right) \right)$$
$$= \frac{1}{k} \sum_{i=0}^{3} (-1)^{i+1} deg\left(\Delta_i^{\alpha_p}(t)\right)$$
$$= \frac{1}{k} \sum_{i=0}^{3} (-1)^{i+1} deg\left(\pi_p\left(\Delta_i^{\alpha}(t)\right)\right)$$
$$= \frac{1}{k} \sum_{i=0}^{3} (-1)^{i+1} deg\left(\Delta_i^{\alpha}(t)\right)$$
$$= ||\phi||_T.$$

Remark. Let $\alpha : \pi_1(M) \to \operatorname{GL}(\mathbb{Z}, k)$ be a representation. Then it is in general not true that $\Delta_1^{\alpha_p}(t) = \pi_p(\Delta_1^{\alpha}(t)) \in \mathbb{F}_p[t^{\pm 1}]$ (we use the notation of Proposition 6.3), not even if (M, ϕ) fibers over S^1 . Indeed, let K be the trefoil knot and $\varphi : \pi_1(X(K)) \to S_3$ the unique epimorphism. Consider the representation $\alpha(\varphi) : \pi_1(X(K)) \to \operatorname{GL}(\mathbb{Z}, 2)$ as in Section 7.1. Then deg $(\pi_3(\Delta_1^{\alpha}(t))) = 2$, but deg $(\Delta_1^{\alpha_3}(t)) = 3$.

7. Examples

We applied our results to many explicit situations. In all reasonable situations we found the correct Thurston norm bounds and we found whether a manifold fibers or not.

7.1. Representations of 3-manifold groups. In our applications we first find homomorphisms $\pi_1(M) \to S_k$, and then find a representation of S_k . The first choice of a representation for S_k that comes to mind is $S_k \to \operatorname{GL}(\mathbb{F}, k)$ where S_k acts by permuting the coordinates. But S_k leaves the subspace $\{(v, v, \ldots, v) | v \in \mathbb{F}\} \subset \mathbb{F}^k$ invariant, hence this representation is 'not completely non-trivial'. To avoid this we prefer to work with a slightly different representation of S_k . If $\varphi : \pi_1(M) \to S_k$ is a homomorphism then we consider

$$\alpha(\varphi): \pi_1(M) \xrightarrow{\varphi} S_k \to \mathrm{GL}(V_{k-1}(\mathbb{F})),$$

where

$$V_l(\mathbb{F}) := \{ (v_1, \dots, v_{l+1}) \in \mathbb{F}^{l+1} | \sum_{i=1}^{l+1} v_i = 0 \}.$$

Clearly $\dim_{\mathbb{F}}(V_l(\mathbb{F})) = l$ and S_{l+1} acts on it by permutation. Since $\alpha(\varphi)$ is a subrepresentation of a unitary representation, $\alpha(\varphi)$ is unitary itself. These representations are easy to find and remarkably useful for our purposes.

We point out that the fundamental groups of 3-manifolds for which the geometrization conjecture holds are residually finite (cf. [Th82] and [He87]). In particular most

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(or perhaps all) 3–manifolds have many homomorphisms to finite groups, and in particular they have many interesting finite representations.

7.2. Knots with up to 12 crossings: genus bounds and fiberedness. In this section we confirm bounds on the genus of all knots with 12 crossings or less. Also we detect all non–fibered knots with 12 crossings or less, some of which are new discoveries to our knowledge.

I. Knot genus:

There are 36 knots with 12 crossings or less for which $\text{genus}(K) > \frac{1}{2} \text{deg } \Delta_K(t)$. The most famous and interesting examples are $K = 11_{401}$ (the Conway knot) and 11_{409} (the Kinoshita–Terasaka knot). Here we use the *knotscape* notation. We will show that in all cases the correct genus is detected by twisted Alexander polynomials.

Using geometric methods Gabai [Ga84] showed that the genus of the Conway knot is 3 and that the genus of the Kinoshita–Terasaka knot is two. The computation of the genus for all 11–crossing knots was done by Jacob Rasmussen, using a computer assisted computation of the Oszváth–Szabó knot Floer homology (cf. also [OS04a] and [OS04b]).

We first consider the Conway knot $K = 11_{401}$ whose diagram is given in Figure 3. This knot has Alexander polynomial one, i.e., the degree of $\Delta_K(t)$ equals zero. Furthermore this implies that $\pi_1(X(K))^{(1)}$ is perfect, i.e., $\pi_1(X(K))^{(n)} = \pi_1(X(K))^{(1)}$ for any n > 1. (For a group G, $G^{(n)}$ is defined inductively as follows; $G^{(0)} := G$ and $G^{(n+1)} := [G^{(n)}, G^{(n)}]$.) Therefore the genus bounds of Cochran [Co04] and Harvey [Ha05] vanish as well. The fundamental group $\pi_1(X(K))$ is generated by the meridians



FIGURE 3. The Conway knot 11_{401} and the Kinoshita–Terasaka knot 11_{409} .

 a, b, \ldots, k of the segments in the knot diagram of Figure 3. The relations are

Using the program *KnotTwister* [F05] we found the homomorphism $\varphi : \pi_1(X(K)) \to S_5$ given by

$$\begin{array}{rclrcl} A &=& (142), & B &=& (451), & C &=& (451), & D &=& (453), \\ E &=& (453), & F &=& (351), & G &=& (351), & H &=& (431), \\ I &=& (351), & J &=& (352), & K &=& (321), \end{array}$$

where we use cycle notation. The generators of $\pi_1(X(K))$ are sent to the element in S_5 given by the cycle with the corresponding capital letter. We then consider $\alpha := \alpha(\varphi) : \pi_1(X(K)) \xrightarrow{\varphi} S_5 \to \operatorname{GL}(V_4(\mathbb{F}_{13}))$. Using *KnotTwister* we compute $\operatorname{deg}(\Delta_0^{\alpha}(t)) = 0$ and we compute the twisted Alexander polynomial to be

$$\Delta_1^{\alpha}(t) = 1 + 6t + 9t^2 + 12t^3 + t^5 + 3t^6 + t^7 + 3t^8 + t^9 + 12t^{11} + 9t^{12} + 6t^{13} + t^{14} \in \mathbb{F}_{13}[t^{\pm 1}].$$

Note that α is unitary and we can therefore apply Theorem 3.2 which says that if $\Delta_1^{\alpha}(t) \neq 0$, then

$$\operatorname{genus}(K) \ge \frac{1}{2k} \left(\operatorname{deg} \left(\Delta_1^{\alpha}(t) \right) - \operatorname{deg} \left(\Delta_0^{\alpha}(t) \right) \right) + \frac{1}{2}$$

Therefore in our case we get

genus(K)
$$\ge \frac{1}{8} \cdot 14 + \frac{1}{2} = \frac{18}{8} = 2.25.$$

Since genus(K) is an integer we get genus(K) ≥ 3 . Since there exists a Seifert surface of genus 3 for K (cf. [Ga84] and Figure 1) it follows that the genus of the Conway knot is 3.

For the Kinoshita–Terasaka knot K we found a map $\varphi : \pi_1(X(K)) \to S_5$ such that $\Delta_1^{\alpha(\varphi)}(t) \in \mathbb{F}_{13}[t^{\pm 1}]$ has degree 12 and deg $(\Delta_0^{\alpha(\varphi)}(t)) = 0$. It follows from Theorem 3.2 that genus $(K) \geq \frac{1}{8} \cdot 12 + \frac{1}{2} = 2$. A Seifert surface of genus two is given in [Ga84]. Note that in this case our inequality becomes equality, hence 'rounding up' is not necessary. Our table below shows that this is surprisingly often the case.

Table 1 gives all knots with 12 crossings or less for which $\deg(\Delta_K(t)) < 2 \operatorname{genus}(K)$. We obtained the list of these knots from Alexander Stoimenow's knot page [Sto]. One can also find the genus of all these knots on his knot page. We compute twisted Alexander polynomials using *KnotTwister* and 4–dimensional representations of the form $\alpha(\varphi) : \pi_1(X(K)) \xrightarrow{\varphi} S_5 \to \operatorname{GL}(V_4(\mathbb{F}_{13}))$. Our genus bounds from Theorem 3.2 give (by rounding up if necessary) the correct genus in each case.

Using *KnotTwister* it takes only a few seconds to find such representations and to compute the twisted Alexander polynomial.

II. Fiberedness:

Consider the knot $K = 12_{1345}$. Its Alexander polynomial equals $\Delta_K(t) = 1 - 2t + 3t^2 - 2t^3 + t^4$ and its genus equals two, therefore K satisfies condition (1). It follows from Corollary 6.2 that if K were fibered, then for any field \mathbb{F} and any representation

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Knotscape name	11_{401}	11_{409}	11_{412}	11_{434}	11_{440}	11_{464}
genus bound from $\Delta_K(t)$	0	0	2	1	2	1
genus bound from $\Delta_1^{\alpha}(t)$	2.25	2.00	3.00	2.00	3.00	2.00
Knotscape name	11_{519}	12_{1311}	12_{1316}	12_{1319}	12_{1339}	12_{1344}
genus bound from $\Delta_K(t)$	2	1	2	1	1	2
genus bound from $\Delta_1^{\alpha(\varphi)}(t)$	3.00	2.00	2.50	3.00	2.00	3.00
Knotscape name	12_{1351}	12_{1375}	12_{1412}	12_{1417}	12_{1420}	12_{1509}
genus bound from $\Delta_K(t)$	2	2	1	1	2	2
genus bound from $\Delta_1^{\alpha(\varphi)}(t)$	3.00	3.00	2.00	3.00	3.00	3.00
Knotscape name	12_{1519}	12_{1544}	12_{1545}	12_{1552}	12_{1555}	12_{1556}
genus bound from $\Delta_K(t)$	2	2	2	2	2	1
genus bound from $\Delta_1^{\alpha(\varphi)}(t)$	3.00	3.00	3.00	3.00	3.00	2.00
Knotscape name	12_{1581}	12_{1601}	12_{1609}	12_{1699}	12_{1718}	12_{1745}
genus bound from $\Delta_K(t)$	1	0	1	1	0	1
genus bound from $\Delta_1^{\alpha(\varphi)}(t)$	2.00	1.25	2.00	2.00	2.00	2.00
Knotscape name	12_{1807}	12_{1953}	12_{2038}	12_{2096}	12_{2100}	12_{2118}
genus bound from $\Delta_K(t)$	1	2	2	2	2	2
conversion hound from $\Lambda^{\alpha(\varphi)}(t)$	9.00	2.00	2.00	2.00	2.00	2.00

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TABLE 1. Computation of degrees of twisted Alexander polynomials.

 $\alpha : \pi_1(M) \to \operatorname{GL}(\mathbb{F}, k)$ the following would hold:

$$\deg(\Delta_K(t)) = \frac{1}{k} \left(\deg\left(\Delta_1^{\alpha}(t)\right) - \deg\left(\Delta_0^{\alpha}(t)\right) \right) + 1.$$

We found a representation $\alpha : \pi_1(X(K)) \to S_4$ such that for the canonical representation $\alpha : \pi_1(X(K)) \to S_4 \to \operatorname{GL}(\mathbb{F}_3, 4)$ given by permuting the coordinates, we get $\operatorname{deg}(\Delta_1^{\alpha}(t)) = 7$ and $\operatorname{deg}(\Delta_0^{\alpha}(t)) = 1$. We compute

$$\frac{1}{4} \deg \left(\Delta_1^{\alpha}(t) \right) - \frac{1}{4} \deg \left(\Delta_0^{\alpha}(t) \right) + 1 = \frac{10}{4} \neq 4 = \deg(\Delta_K(t)).$$

Hence K is not fibered.

Now consider $\alpha : \pi_1(X(K)) \to S_4 \to \operatorname{GL}(\mathbb{Z}, 4)$, the second map being the canonical representation induced from permutation on the basis elements. Then according to Proposition 6.3 our computation can also be interpreted as saying that $\Delta_1^{\alpha}(t) \in \mathbb{Z}[t^{\pm 1}]$ is not monic.

Similarly we found altogether 13 12–crossings knots which satisfy condition (1) but which are not fibered; we list them in Table 2. As we mentioned in the introduction, Stoimenow and Hirasawa showed that the remaining 12–crossing knots are fibered if and only if they satisfy condition (1). Corollary 6.2 completes the classification of all fibered 12–crossing knots.

Knotscape name	12_{1345}	12_{1498}	12_{1502}	12_{1546}	12_{1567}	12_{1670}	12_{1682}
Order of permutation group k	4	5	5	3	5	5	4
Order p of finite field	3	2	11	2	3	2	3
Knotscape name	12_{1752}	12_{1771}	12_{1823}	12_{1938}	12_{2089}	12_{2103}	
Order of permutation group k	3	3	5	5	5	4	
Order p of finite field	2	7	7	11	2	3	

TABLE 2. Alexander polynomials of non–fibered knots

7.3. Closed manifolds. In this short section we intend to show that twisted Alexander polynomials are also very useful for studying closed manifolds.

Let $K \subset S^3$ be a non-trivial knot and ϕ a generator of $H^1(X(K))$. Since $H^1(X(K)) \cong H^1(M_K)$ we will denote the corresponding generator of $H^1(M_K)$ by ϕ as well. Let S be a minimal Seifert surface for K. Adding a disk to S along the boundary clearly gives a closed surface \hat{S} dual to $\phi \in H^1(M_K)$, hence $||\phi||_{T,M_K} \leq ||\phi||_{T,X(K)} - 1$. Gabai [Ga87, Theorem 8.8] showed that \hat{S} is in fact norm minimizing. In particular for a non-trivial knot K

$$||\phi||_{T,M_K} = ||\phi||_{T,X(K)} - 1 = 2 \operatorname{genus}(K) - 2.$$

If K fibers, then clearly (M_K, ϕ) fibers over S^1 as well. Gabai [Ga87] showed the converse; a knot K is fibered if and only if M_K is fibered.

We confirm Gabai's results for some cases. We applied our theory together with KnotTwister to M_K where K is one of the 36 knots with 12 crossings or less with $genus(K) > \frac{1}{2} deg(\Delta_K(t))$. In each case we found the correct Thurston norm bound for M_K . Furthermore, if K is one of the 13 non-fibered knots with 12 crossings with monic Alexander polynomial and $deg(\Delta_K(t)) = 2 genus(K)$, then using Corollary 6.2 and KnotTwister we could show that twisted Alexander polynomials detect that these manifolds are not fibered.

7.4. Satellite knots. We will show how to find lower bounds for the genus of satellite knots. We will see that even though we are interested in the genus of a knot we sometimes have to study the Thurston norm of a link complement.

Let K and C be knots in S^3 . Let $A \subset S^3 \setminus K$ be a simple closed curve, unknotted in S^3 . Then X(A) is a solid torus. Let $\psi : \partial X(A) \to \partial X(C)$ be a diffeomorphism which sends a meridian of A to a longitude of C, and a longitude of A to a meridian of C. The space

$$X(A) \cup_{\psi} X(C)$$

is a 3-sphere and the image of K is denoted by S := S(K, C, A). We say S is the satellite knot with companion C, orbit K and axis A. Note that we replaced a tubular neighborhood of C by a knot in a solid torus, namely $K \subset X(A)$.

In [FT05] the first author and Peter Teichner study examples where K is the knot 6_1 , C is an arbitrary knot, and A is a knot as in Figure 4. In [FT05] they show that



FIGURE 4. Knot 6_1 with choice of A and considered as a knot in the solid torus X(A).

all these knots are topologically slice, but it is not known whether they are smoothly slice or not. Chuck Livingston asked what the genus of these satellite knots equals.

Proposition 7.1. Let $K \subset S^3$ be a non-trivial knot, and $A \subset X(K)$ a simple closed curve such that $[A] = 0 \in H_1(X(K))$, which is unknotted if considered as a knot in S^3 . Let C be another knot. Now let S := S(K, C, A) be the satellite knot. Then

$$genus(S) = \frac{1}{2}(||\phi||_{T,X} + 1),$$

where $X := S^3 \setminus (\nu K \cup \nu A)$ and $\phi : H_1(X) \to \mathbb{Z}$ is given by sending the meridian of K to one, and the meridian of A to zero.

Proof. For convenience let us identify ∂X with $K \times S^1 \cup A \times S^1$. We also identify K with $K \times \{*\} \subset \partial X$. It follows from [Sc53], [BZ03, p. 21] that (F denotes a surface)

genus(S) = min{genus(F) | $F \subset X$ properly embedded and $\partial F = K$ }

since the linking number of A and K equals zero. This also implies that $\phi : H_1(A \times S^1) \to H_1(X) \xrightarrow{\phi} \mathbb{Z}$ is the zero map. Similar to the proof that for a knot K we have $||\phi||_{X(K)} = 2 \operatorname{genus}(K) - 1$ we can now show that

$$\begin{aligned} ||\phi||_T &= \min\{2\operatorname{genus}(F) - 1|F \subset X \text{ properly embedded and } K \subset \partial F, F \text{ dual to } \phi\} \\ &= \min\{2\operatorname{genus}(F) - 1|F \subset X \text{ properly embedded and } \partial F = K\} \\ &= 2\operatorname{genus}(S) - 1. \end{aligned}$$

Hence in order to determine the genus of S(K, C, A) for any knot C we have to determine the Thurston norm of $||\phi||_{T,X}$. For X, we compute that the untwisted

Alexander polynomial $\Delta_1(t) = 0$. Hence we need twisted coefficients to get non-trivial bounds.

Now consider the representation $\alpha : \pi_1(X) \to \operatorname{GL}(\mathbb{F}_{13}, 1)$ given by $\alpha(\mu_K) = 6$ and $\alpha(\mu_A) = 2$, where μ_K (respectively μ_A) denotes the meridian of K (respectively A). For X we compute $\Delta_1^{\alpha}(t) = 1 + 2t + 2t^2 + 4t^3 \in \mathbb{F}_{13}[t^{\pm 1}]$. It follows from Theorem 3.4 that $||\phi||_{T,X} \geq 3$.

Consider Figure 5. It shows the link $K \cup A$ and a Seifert surface of genus one for K. The knot A intersects this Seifert surface twice. Therefore adding a hollow 1-handle gives a Seifert surface of genus two for K which does not intersect A. Therefore $||\phi||_{T,X} \leq 3$. Hence $||\phi||_{T,X} = 3$ and by Proposition 7.1 we get genus(S) = 2.



FIGURE 5. Seifert surface for $K \subset S^3 \setminus K \cup A$.

7.5. **Ropelength.** Using work of Freedman and He [FH91] Cantarella, Kusner and Sullivan [CKS02, Corollary 22] showed that the Thurston norm can be used to give lower bounds on the ropelength of a link component. They formulated a conjecture for a certain link. This conjecture was proved by Harvey [Ha05, Section 8] using higher–order Alexander polynomials. Using one–dimensional representations together with Theorem 5.1 and Lemma 3.3 we can reprove this conjecture.

7.6. **Dunfield's link.** We will show that our invariants also detect subtle examples of pairs $(X(L), \phi)$ where L is a link in S^3 and $\phi \in H^1(X(L))$, which do not fiber over S^1 . Consider the link L in Figure 6 from [Du01]. Denote the knotted component by L_1 and the unknotted component by L_2 . Let $x, y \in H_1(X(L))$ be the elements represented by a meridian of L_1 respectively L_2 . Then the multivariable Alexander polynomial equals

$$\Delta_{X(L)} = xy - x - y + 1 \in \mathbb{Z}[H_1(X(L))] = \mathbb{Z}[x^{\pm 1}, y^{\pm 1}].$$

The Alexander norm ball (cf. [Mc02] for a definition) and the Thurston norm ball (which is determined in [FK05]) are given in Figure 7. Dunfield [Du01] showed that $(X(L), \phi)$ fibers over S^1 for all $\phi \in H^1(M)$ in the cone on the two open faces with vertices $(-\frac{1}{2}, \frac{1}{2}), (0, 1)$ respectively $(0, -1), (\frac{1}{2}, -\frac{1}{2})$. He also showed that $(X(L), \phi)$



FIGURE 6. Dunfield's link.



FIGURE 7. Alexander norm ball and Thurston norm ball for Dunfield's link.

does not fiber over S^1 for any $\phi \in H^1(X(L))$ lying outside the cone. Later in [FK05] the authors completely determined the Thurston norm of X(L).

Now let ϕ be the homomorphism given by $\phi(x) = 1$, $\phi(y) = -1$. In that case ϕ is inside the cone on an open face of the Alexander norm ball and $\Delta_1(t) = 1 - 3t + 3t^2 - t^3 \in \mathbb{Z}[t^{\pm 1}]$ is monic. Hence the abelian invariants are inconclusive whether $(X(L), \phi)$ is fibered or not. On the other hand we found a representation $\pi_1(X(L)) \to S_3 \to \operatorname{GL}(\mathbb{F}_2, 3)$ such that $\Delta_1^{\alpha}(t) = 0 \in \mathbb{F}_2[t^{\pm 1}]$. Therefore $(X(L), \phi)$ does not fiber over S^1 by Theorem 6.1.

Note that the fact that $(X(L), \phi)$ does not fiber over S^1 also follows from the fact that ϕ is not in the cone on a top-dimensional open face of the Thurston norm ball (cf. [Th86] and [Oe86]). But in this case, we do need to know the Thurston norm ball for Dunfield's link found in [FK05]. In general completely determining the Thurston norm ball is much harder than computing twisted Alexander polynomials.

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