TWISTED ALEXANDER NORMS GIVE LOWER BOUNDS ON THE THURSTON NORM

STEFAN FRIEDL AND TAEHEE KIM

Abstract. We introduce twisted Alexander norms of a compact connected orientable 3-manifold with first Betti number bigger than one generalizing norms of McMullen and Turaev. We show that twisted Alexander norms give lower bounds on the Thurston norm of a 3-manifold. Using these we completely determine the Thurston norm of many 3-manifolds which can not be determined by norms of McMullen and Turaev.

1. Introduction

Let M be a 3-manifold. Throughout the paper we will assume that all 3-manifolds are compact, connected and orientable. Let $\phi \in H^1(M;\mathbb{Z})$. There exists a (possibly disconnected) properly embedded surface S which represents a homology class which is dual to ϕ. (We also say that S is *dual* to ϕ.) The *Thurston norm* of ϕ is defined as

$$||\phi||_{T,M} := \min\{\textstyle\sum_{i=1}^k \max\{-\chi(S_i),0\} \mid S_1\cup\dots\cup S_k \subset M \text{ properly embedded, dual to } \phi,\ S_i \text{ connected for } i=1,\dots,k\}.$$

If the manifold M is clear, we will just write $||\phi||_T$.

Thurston [Th86] introduced $||-||_T$ in a preprint in 1976. He proved that the Thurston norm on $H^1(M;\mathbb{Z})$ is homogeneous and convex (that is, for $\phi,\phi_1,\phi_2 \in H^1(M;\mathbb{Z})$ and $k\in\mathbb{N}$, $||k\phi||_T = k||\phi||_T$ and $||\phi_1+\phi_2||_T \le ||\phi_1||_T + ||\phi_2||_T$). He also showed that the Thurston norm can be extended to a seminorm on $H^1(M;\mathbb{R})$ and that the Thurston norm ball (which is the set of $\phi\in H^1(M;\mathbb{R})$ with $||\phi||_T\le 1$) is a (possibly noncompact) finite convex polyhedron. A natural question arises; how do we determine the Thurston norm on $H^1(M;\mathbb{R})$?

To address this question McMullen [Mc02] used a homological approach. It is well-known that for a knot K in the 3-sphere

$$2\,\text{genus}(K) \ge \deg\left(\Delta_K(t)\right),$$

where $\Delta_K(t)\in\mathbb{Z}[t^{\pm1}]$ denotes the Alexander polynomial of K. Generalizing this McMullen [Mc02] considered the multivariable Alexander polynomial $\Delta_M \in \mathbb{Z}[FH_1(M;\mathbb{Z})]$ (cf. Section 2.2 for a definition) where $FH_1(M;\mathbb{Z}) := H_1(M;\mathbb{Z})/\text{Tor}_{\mathbb{Z}}(H_1(M;\mathbb{Z}))$ is

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the maximal free abelian quotient of $H_1(M;\mathbb{Z})$. Using the multivariable Alexander polynomial he defined another seminorm $||-||_A$ on $H^1(M;\mathbb{R})$ which is called *the Alexander norm* of M as follows. If $\Delta_M = 0$ then we set $||\phi||_A = 0$ for all $\phi \in H^1(M;\mathbb{R})$. Otherwise for $\Delta_M = \sum a_i f_i$ with $a_i \in \mathbb{Z}$ and $f_i \in FH_1(M;\mathbb{Z})$ and given $\phi \in H^1(M;\mathbb{R})$ we define

$$||\phi||_A := \sup \phi(f_i - f_j).$$

with the supremum over (f_i, f_j) such that $a_i a_j \neq 0$. Note that $\phi \in H^1(M;\mathbb{R})$ naturally induces a homomorphism $H_1(M;\mathbb{R}) \to \mathbb{R}$.

The Alexander norm ball is again a (possibly noncompact) finite convex polyhedron. McMullen showed that the Alexander norm gives a lower bound on the Thurston norm. More precisely he proved the following theorem.

Theorem 1.1. [Mc02, Theorem 1.1] *Let M be a 3-manifold whose boundary is empty or consists of tori. Then the Alexander and Thurston norms on $H^1(M;\mathbb{Z})$ satisfy*

$$||\phi||_T \geq ||\phi||_A - \begin{cases} 1 + b_3(M), & \text{if } b_1(M) = 1 \text{ and } H^1(M;\mathbb{Z}) \text{ is generated by } \phi, \\ 0, & \text{if } b_1(M) > 1. \end{cases}$$

Equality holds if $\phi : \pi_1(M) \to \mathbb{Z}$ is represented by a fibration $M \to S^1$ such that $M \neq S^1 \times D^2$ and $M \neq S^1 \times S^2$.

In [Mc02] using the Alexander norm McMullen completely determined the Thurston norm of many link complements. The computation was based on the following observation for the case $b_1(M) > 1$.

Observation: The Thurston norm ball lies inside the Alexander norm ball. If the Alexander norm ball and the Thurston norm ball agree on all extreme vertices of the Alexander norm ball, then they agree everywhere by convexity.

Note that Seiberg-Witten theory [KM97] and Heegard-Floer homology [OS04] can be used to completely determine the Thurston norm (cf. [Kr98, Kr99, Vi99, Vi03]), but computations are not combinatorial and therefore not very useful in practice. In this paper we will take a homology theoretic approach and find lower bounds on the Thurston norm which are easily computed in a combinatorial way.

McMullen's homological approach has been generalized by many authors. In [Co04, Ha02, Tu02b, FK05] much stronger lower bounds for $||\phi||_T$ for *specific* $\phi \in H^1(M;\mathbb{R})$ were found. In particular when $b_1(M) = 1$ these methods allow us to determine the Thurston norm ball in many cases. For the case $b_1(M) > 1$ Turaev introduced *the torsion norm* generalizing McMullen's Alexander norm using abelian representations [Tu02a, Chapter 4].

We will henceforth only consider 3-manifolds with $b_1(M) > 1$. In this case the above mentioned bounds except for bounds in [Mc02, Tu02a] can not completely determine the Thurston norm ball of M since they give bounds only for one $\phi \in H^1(M;\mathbb{Z})$ at a time. As McMullen observed we need to find a *norm* which gives a lower bound on the Thurston norm. For this purpose we define *twisted Alexander norms* and prove that they give lower bounds on the Thurston norm. This will greatly generalize the work of McMullen [Mc02] and Turaev [Tu02a].

In the following let $\mathbb{F}$ be a commutative field and $\alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{F}, k)$ a representation. Then we define *the twisted multivariable Alexander polynomial* $\Delta_M^\alpha \in \mathbb{F}[FH_1(M;\mathbb{Z})]$ associated to α and the natural surjection $\pi_1(M) \to FH_1(M;\mathbb{Z})$ (see Section 2.2). Similarly to the way the multivariable Alexander polynomial gives rise to the Alexander norm we use the twisted multivariable Alexander polynomial to define the twisted Alexander norm $||-||_A^\alpha$ on $H^1(M;\mathbb{R})$ associated to α (see Section 3.1).

Let $\phi \in H^1(M;\mathbb{Z})$. This defines a homomorphism $\phi : \pi_1(M) \to \mathbb{Z} \cong \langle t^{\pm 1}\rangle$. We now define $\Delta_\phi^{\alpha,i}(t) \in \mathbb{F}[t^{\pm 1}]$ to be the order of the i–th twisted homology module $H_i^\alpha(M; \mathbb{F}^k \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}])$ associated to α and ϕ. (See Section 2.2. We also refer to [KL99, FK05].) The notion of twisted Alexander polynomial originated from a preprint of Lin [Lin01] from 1990 and was developed by Wada [Wa94]. The homological definition of twisted Alexander polynomials, which we use in this paper, was first introduced by Kirk and Livingston [KL99]. We also refer to [Kit96, FK05] for more about twisted Alexander polynomials.

In [FK05, Theorem 3.1] the authors show that twisted one-variable Alexander polynomials give lower bounds on $||\phi||_T$ for *specific* $\phi \in H^1(M;\mathbb{Z})$. The following theorem allows us to translate bounds on $||\phi||_T$ for specific $\phi \in H^1(M;\mathbb{Z})$ from [FK05] to bounds on $||-||_T$ given by twisted Alexander norms. Note that ϕ induces a homomorphism $\phi : \mathbb{F}[FH_1(M;\mathbb{Z})] \to \mathbb{F}[t^{\pm 1}]$.

Theorem 3.4. *Let M be a 3-manifold with $b_1(M) > 1$ whose boundary is empty or consists of tori. Let $\alpha : \pi_1(M) \to GL(\mathbb{F}, k)$ be a representation. Let $\phi \in H^1(M;\mathbb{Z})$. Then*

$$\Delta_\phi^\alpha(t) = \phi(\Delta_M^\alpha)\Delta_\phi^{\alpha,0}(t)\Delta_\phi^{\alpha,2}(t).$$

Furthermore if $\phi(\Delta_M^\alpha) \neq 0$, then $\Delta_\phi^{\alpha,0}(t) \neq 0$ and $\Delta_\phi^{\alpha,2}(t) \neq 0$ and hence $\Delta_\phi^\alpha(t) \neq 0$.

The proof is based on the functoriality of Reidemeister torsion (see Section 6). The following two theorems are our main results.

Theorem 3.1 (**Main Theorem 1**)**.** *Let M be a 3-manifold with $b_1(M) > 1$ whose boundary is empty or consists of tori. Let $\alpha : \pi_1(M) \to GL(\mathbb{F}, k)$ be a representation. Then for the corresponding twisted Alexander norm $||-||_A^\alpha$, we have*

$$||\phi||_T \geq \frac{1}{k}||\phi||_A^\alpha$$

for all $\phi \in H^1(M;\mathbb{R})$.

Let M be a 3-manifold and $\phi \in H^1(M;\mathbb{Z})$. We say (M,ϕ) *fibers over* S^1 if the homotopy class of maps $M \to S^1$ induced by $\phi : \pi_1(M) \to H_1(M;\mathbb{Z}) \to \mathbb{Z}$ contains a representative that is a fiber bundle over S^1.

Theorem 3.2 (**Main Theorem 2**). *Let M be a 3-manifold with $b_1(M) > 1$ whose boundary is empty or consists of tori such that $M \neq S^1 \times D^2$ and $M \neq S^1 \times S^2$. Let $\alpha : \pi_1(M) \to GL(\mathbb{F}, k)$ be a representation. If $\phi \in H^1(M;\mathbb{Z})$ is such that (M,ϕ) fibers over S^1, then ϕ lies in the cone on a top-dimensional open face of the Thurston norm ball. Furthermore, if we denote this cone by C, then*

$$||\psi||_T = \frac{1}{k}||\psi||_A^\alpha$$

for all $\psi \in C$.

In Theorem 3.2 the first part of the conclusion is due to Thurston [Th86]. (See Theorem 3.5.) By Theorem 3.1 twisted Alexander norms give lower bounds on the Thurston norm. With the same reason as for the Alexander norm ball twisted Alexander norm balls are (possibly noncompact) finite convex polyhedra. Therefore we can use McMullen's observation in the above to determine the Thurston norm using twisted Alexander norms.

In Section 5 we give examples which show how powerful twisted Alexander norms are. For example we determine the Thurston norm of the complement of the link L in Figure 1, which can not be determined by the (usual) Alexander norm. The components of L are K_1, the trefoil, and $K_2 = 11_{440}$ (here we use *knotscape* notation). Let $X(L)$ denote the complement of an open tubular neighborhood of L in the 3-

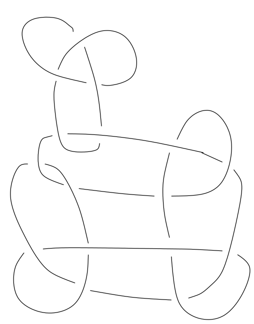

FIGURE 1. Link L

sphere. Then

$$\Delta_{X(L)}(x_1,x_2) = (x_1^2 - x_1 + 1)(x_2^4 - 2x_2^3 + 3x_2^2 - 2x_2 + 1) \in \mathbb{Q}[x_1^{\pm 1}, x_2^{\pm 1}].$$

The resulting Alexander norm ball is given in Figure 2 on the left. On the other hand using the program *KnotTwister* [F05] we found a representation $\alpha : \pi_1(X(L)) \to \mathrm{GL}(\mathbb{F}_{13}, 2)$ such that

$$\Delta^\alpha_{X(L)}(x_1,x_2) = \Delta_1(x_1)\Delta_2(x_2)$$

where $\deg(\Delta_1(x_1)) = 4$ and $\deg(\Delta_2(x_2)) = 12$. (Here $\mathbb{F}_n$ denotes the field of n elements.) Hence the twisted Alexander norm ball for $\frac{1}{2}||-||_A^\alpha$ is the shaded region given in Figure 2 on the right. By Theorem 3.1 we have $||\phi||_T \geq \frac{1}{2}||\phi||_A^\alpha$. It is clear

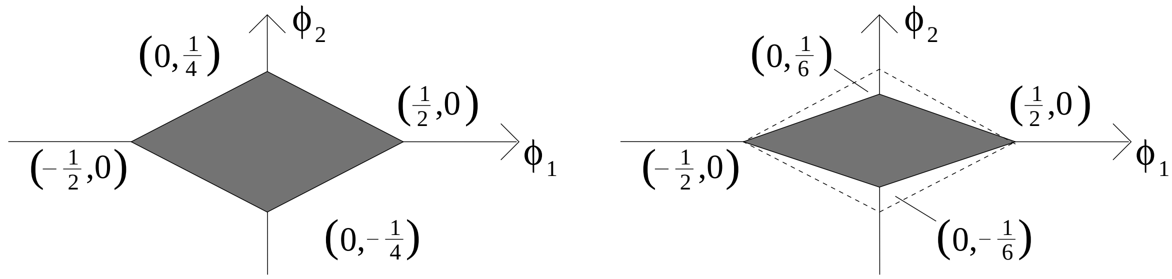

FIGURE 2. The untwisted and the twisted Alexander norm ball of L.

from Figure 2 that $\frac{1}{2}||-||_A^\alpha$ gives a strictly sharper bound on the Thurston norm than $||-||_A$ does. In Section 5.1 we will see that the norms $||-||_T$ and $\frac{1}{2}||-||_A^\alpha$ agree on the vertices of the norm ball of $\frac{1}{2}||-||_A^\alpha$. Therefore by McMullen's observation the norms agree everywhere. Hence the shaded region in Figure 2 on the right is in fact the Thurston norm ball of the link L. We point out that it follows immediately from Theorem 3.2 that $(X(L), \phi)$ does not fiber over S^1 for any $\phi \in H^1(M; \mathbb{Z})$. See Section 5 for more details.

Our approach works very well in many cases, but sometimes it is difficult to find an appropriate representation. Therefore it is sometimes convenient to find lower bounds on the Thurston norm of a finite cover $\tilde{M}$ of M. By a result of Gabai [Ga83, p. 484] (cf. also Theorem 5.5) the Thurston norm on $\tilde{M}$ determines the Thurston norm on M. In many cases it is easier to find representations of $\tilde{M}$. This approach allows us to determine the Thurston norm ball of Dunfield's link [Du01] (see Section 5.2).

Outline of the paper: In Section 2 we define twisted Alexander modules and twisted Alexander polynomials. In Section 3 we define twisted Alexander norms and prove the main theorems. We quickly discuss how to compute twisted Alexander polynomials in Section 4 and give examples in Section 5. In Section 6 we give a proof of Theorem 3.4 which shows the precise relationship between the twisted multivariable Alexander polynomials and the twisted one-variable Alexander polynomials.

Notations and conventions: For a link L in S^3, $X(L)$ denotes the exterior of L in S^3. (That is, $X(L) = S^3 \setminus \nu L$ where νL is an open tubular neighborhood of L in S^3.) An arbitrary (commutative) field is denoted by $\mathbb{F}$. $\mathbb{F}_n$ denotes the finite field of n elements. We identify the group ring $\mathbb{F}[\mathbb{Z}]$ with $\mathbb{F}[t^{\pm 1}]$. We denote the permutation group of order k by S_k. For a 3-manifold M we use the canonical isomorphisms to identify

$H^1(M;\mathbb{Z}) = \mathrm{Hom}(H_1(M;\mathbb{Z}),\mathbb{Z}) = \mathrm{Hom}(\pi_1(M),\mathbb{Z})$. Hence sometimes $\phi \in H^1(M;\mathbb{Z})$ is regarded as a homomorphism $\phi : \pi_1(M) \to \mathbb{Z}$ (or $\phi : H_1(M;\mathbb{Z}) \to \mathbb{Z}$) depending on the context.

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2. Twisted Alexander polynomials

In this section we give the definition of twisted Alexander polynomials.

2.1. **Torsion invariants.** Let R be a commutative Noetherian unique factorization domain (henceforth UFD). An example of R to keep in mind is $\mathbb{F}[t_1^{\pm}, t_2^{\pm}, \dots, t_n^{\pm}]$, a (multivariable) Laurent polynomial ring over a field $\mathbb{F}$. For a finitely generated R-module A, we can find a presentation

$$R^r \xrightarrow{P} R^s \to A \to 0$$

since R is Noetherian. Let $i \geq 0$ and suppose $s - i \leq r$. We define $E_i(A)$, *the i-th elementary ideal* of A, to be the ideal in R generated by all $(s-i) \times (s-i)$ minors of P if $s - i > 0$ and to be R if $s - i \leq 0$. If $s - i > r$, we define $E_i(A) = 0$. It is known that $E_i(A)$ does not depend on the choice of a presentation of A (cf. [CF77]).

Since R is a UFD there exists a unique smallest principal ideal of R that contains $E_0(A)$. A generator of this principal ideal is defined to be the *order of A* and denoted by $\mathrm{ord}(A) \in R$. The order is well-defined up to multiplication by a unit in R. Note that A is not R-torsion if and only if $\mathrm{ord}(A) = 0$. For more details, we refer to [Hi02].

2.2. **Twisted Alexander invariants.** Let M be a 3-manifold and $\psi : \pi_1(M) \to F$ a homomorphism to a free abelian group F. We do not demand that ψ is surjective. Note that $\Lambda := \mathbb{F}[F]$ is a commutative Noetherian UFD. Let $\alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{F}, k)$ be a representation.

Using α and ψ, we define a left $\mathbb{Z}[\pi_1(M)]$-module structure on $\mathbb{F}^k \otimes_{\mathbb{F}} \Lambda =: \Lambda^k$ as follows:

$$g \cdot (v \otimes p) := (\alpha(g) \cdot v) \otimes (\psi(g)p)$$

where $g \in \pi_1(M)$ and $v \otimes p \in \mathbb{F}^k \otimes_{\mathbb{F}} \Lambda = \Lambda^k$. Recall that the chain complex $C_*(\tilde{M})$ of the universal cover $\tilde{M}$ of M can be regarded as a right $\mathbb{Z}[\pi_1(M)]$-module. For $i \geq 0$, we define *the i-th twisted Alexander module of* (M, ψ, α) to be

$$H_i^\alpha(M; \Lambda^k) := H_i(C_*(\tilde{M}) \otimes_{\mathbb{Z}[\pi_1(M)]} \Lambda^k).$$

Since Λ^k is a right Λ-module twisted Alexander modules can be regarded as right Λ-modules. Since M is compact and Λ is Noetherian these modules are finitely generated over Λ.

Definition 2.1. The *i-th (twisted) Alexander polynomial of* (M, ψ, α) is defined to be $\operatorname{ord}(H_i^\alpha(M; \Lambda^k)) \in \Lambda$ and denoted by $\Delta^{\alpha,i}_{M,\psi}$. When $i = 1$, we drop the superscript i and abbreviate $\Delta^{\alpha,i}_{M,\psi}$ by $\Delta^{\alpha}_{M,\psi}$, and we call it the *(twisted) Alexander polynomial of* (M, ψ, α).

Twisted Alexander polynomials are well-defined up to multiplication by a unit in Λ. We drop the notation ψ when ψ is the natural surjection to $FH_1(M;\mathbb{Z})$. We also drop α when α is the trivial representation to $\mathrm{GL}(\mathbb{Q}, 1)$ and drop M in the case that M is clear from the context. If ψ is a homomorphism to $\mathbb{Z}$ then we identify $\mathbb{F}[\mathbb{Z}]$ with $\mathbb{F}[t^{\pm 1}]$ and we write $\Delta^{\alpha,i}_{M,\psi}(t) \in \mathbb{F}[t^{\pm 1}]$. The above homological definition of twisted Alexander polynomials was first introduced by Kirk and Livingston [KL99].

3. Twisted Alexander norms as lower bounds on the Thurston norm

In this section we define twisted Alexander norms, which generalize the Alexander norm of McMullen [Mc02] and the torsion norm of Turaev [Tu02a]. We show that twisted Alexander norms give lower bounds on the Thurston norm and that they give fibering obstructions of 3-manifolds.

3.1. Twisted Alexander norm.

Following an idea of McMullen's [Mc02] we now use the twisted multivariable Alexander polynomial corresponding to $\psi : \pi_1(M) \to FH_1(M;\mathbb{Z})$ to define a norm on $H^1(M;\mathbb{R})$. Let $\alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{F}, k)$ be a representation. If $\Delta^\alpha_M = 0$ then we set $||\phi||^\alpha_A = 0$ for all $\phi \in H^1(M;\mathbb{R})$. Otherwise we write $\Delta^\alpha_M = \sum a_i f_i$ for $a_i \in \mathbb{F}$ and $f_i \in FH_1(M;\mathbb{Z})$. Given $\phi \in H^1(M;\mathbb{R})$ we then define

$$||\phi||^\alpha_A := \sup \phi(f_i - f_j),$$

with the supremum over (f_i, f_j) such that $a_i a_j \neq 0$. Clearly this defines a seminorm on $H^1(M;\mathbb{R})$ which we call the *twisted Alexander norm of* (M, α). This is a generalization of the Alexander norm introduced by McMullen [Mc02]. Indeed, the Alexander norm is the same as the twisted Alexander norm corresponding to the trivial representation $\alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{Q}, 1)$. In this case we just write $|| - ||_A$. Twisted Alexander norms also generalize the torsion norm of Turaev [Tu02a].

3.2. Lower bounds on the Thurston norm.

Recall that McMullen showed that in the case $b_1(M) > 1$ the Alexander norm $|| - ||_A$ is a lower bound on the Thurston norm (see Theorem 1.1). We extend this result to twisted Alexander norms.

Theorem 3.1 (Main Theorem 1). *Let M be a 3-manifold with $b_1(M) > 1$ whose boundary is empty or consists of tori. Let $\alpha : \pi_1(M) \to GL(\mathbb{F}, k)$ be a representation. Then for the corresponding twisted Alexander norm $|| - ||^\alpha_A$ we have*

$$||\phi||_T \geq \frac{1}{k} ||\phi||^\alpha_A$$

for all $\phi \in H^1(M;\mathbb{R})$.

This theorem generalizes McMullen's theorem (Theorem 1.1). Turaev [Tu02a] proved this theorem in the special case of abelian representations.

Theorem 3.2 (**Main Theorem 2**). *Let M be a 3-manifold with $b_1(M) > 1$ whose boundary is empty or consists of tori such that $M \neq S^1 \times D^2$ and $M \neq S^1 \times S^2$. Let $\alpha : \pi_1(M) \to GL(\mathbb{F}, k)$ be a representation. If $\phi \in H^1(M;\mathbb{Z})$ is such that (M, ϕ) fibers over S^1, then ϕ lies in the cone on a top-dimensional open face of the Thurston norm ball. Furthermore, if we denote this cone by C, then*

$$||\psi||_T = \frac{1}{k}||\psi||_A^\alpha$$

for all $\psi \in C$.

The first part of the conclusion in Theorem 3.2 is due to Thurston [Th86] (see Theorem 3.5 below). The idea of the proofs of the main theorems is to combine the lower bounds for one-variable Alexander polynomials from [FK05] with Theorem 3.4. In [FK05] we proved the following theorem.

Theorem 3.3. [FK05, Theorem 3.1 and Theorem 6.1] *Let M be a 3-manifold whose boundary is empty or consists of tori. Let $\phi \in H^1(M)$ be nontrivial and $\alpha : \pi_1(M) \to GL(\mathbb{F}, k)$ a representation such that $\Delta_\phi^\alpha(t) \neq 0$. Then $\Delta_\phi^{\alpha,i}(t) \neq 0$ for $i = 0, 2$ and*

$$||\phi||_T \geq \frac{1}{k}\big(\deg\big(\Delta_\phi^\alpha(t)\big) - \deg\big(\Delta_\phi^{\alpha,0}(t)\big) - \deg\big(\Delta_\phi^{\alpha,2}(t)\big)\big).$$

Furthermore, if (M, ϕ) fibers over S^1 and if $M \neq S^1 \times D^2$ and $M \neq S^1 \times S^2$, then equality holds.

We also need the following theorem to prove the main theorems. This theorem clarifies the precise relationship between the twisted multivariable Alexander polynomial and the twisted one-variable Alexander polynomials of a 3-manifold.

Theorem 3.4. *Let M be a 3-manifold with $b_1(M) > 1$ whose boundary is empty or consists of tori. Let $\alpha : \pi_1(M) \to GL(\mathbb{F}, k)$ be a representation. Let $\phi \in H^1(M;\mathbb{Z})$ be nontrivial. Then*

$$\Delta_\phi^\alpha(t) = \phi(\Delta_M^\alpha)\Delta_\phi^{\alpha,0}(t)\Delta_\phi^{\alpha,2}(t).$$

Furthermore if $\phi(\Delta_M^\alpha) \neq 0$, then $\Delta_\phi^{\alpha,0}(t) \neq 0$ and $\Delta_\phi^{\alpha,2}(t) \neq 0$ and hence $\Delta_\phi^\alpha(t) \neq 0$.

The idea of the proof of Theorem 3.4 is to go from the twisted multivariable Alexander polynomials to Reidemeister torsion which is functorial, and then to go back to the twisted one-variable Alexander polynomials. The proof of Theorem 3.4 is postponed to Section 6.2. Now we give a proof of Theorem 3.1.

Proof of Theorem 3.1. If $\Delta_M^\alpha = 0$, then $||\phi||_A^\alpha = 0$ for all $\phi \in H^1(M;\mathbb{R})$, hence the theorem holds. We now consider the case $\Delta_M^\alpha \neq 0$.

First suppose that $\phi \in H^1(M;\mathbb{Z})$ is nontrivial and lies inside the cone on an open top-dimensional face of the twisted Alexander norm ball. Write $\Delta_M^\alpha = \sum a_i f_i$ where $a_i \in \mathbb{F} \setminus \{0\}$ and $f_i \in FH_1(M;\mathbb{Z})$. We have

$$\phi(\Delta_M^\alpha) = \sum a_i t^{\phi(f_i)}$$

in $\mathbb{F}[t^{\pm 1}]$. Since ϕ is inside the cone on an open top-dimensional face of the twisted Alexander norm ball, the highest and lowest values of $\phi(f_i)$ occur only once in the above equation. Therefore $\phi(\Delta_M^\alpha) \neq 0$ and

$$\deg\left(\phi(\Delta_M^\alpha)\right) = ||\phi||_A^\alpha.$$

By Theorem 3.4 we have $\Delta_\phi^\alpha(t) \neq 0$, $\Delta_\phi^{\alpha,0}(t) \neq 0$, $\Delta_\phi^{\alpha,2}(t) \neq 0$ and

$$\deg\left(\Delta_\phi^\alpha(t)\right) = ||\phi||_A^\alpha + \deg\left(\Delta_\phi^{\alpha,0}(t)\right) + \deg\left(\Delta_\phi^{\alpha,2}(t)\right). \tag{1}$$

Since $\Delta_\phi^\alpha(t) \neq 0$ we get by Theorem 3.3 that

$$||\phi||_T \geq \frac{1}{k}\left(\deg\left(\Delta_\phi^\alpha(t)\right) - \deg\left(\Delta_\phi^{\alpha,0}(t)\right) - \deg\left(\Delta_\phi^{\alpha,2}(t)\right)\right). \tag{2}$$

Combining the inequalities (1) and (2) we clearly get $||\phi||_T \geq \frac{1}{k}||\phi||_A^\alpha$. This proves Theorem 3.1 for all $\phi \in H^1(M;\mathbb{Z})$ inside the cone on an open top-dimensional face of the twisted Alexander norm ball. By homogeneity and continuity we get that in fact $||\phi||_T \geq \frac{1}{k}||\phi||_A^\alpha$ for all $\phi \in H^1(M;\mathbb{R})$. □

For the proof of Theorem 3.2 we need the following theorem proved by Thurston [Th86] and which can also be found in [Oe86, Theorem 9, p. 259].

Theorem 3.5. *Let M be a 3-manifold. If $\phi \in H^1(M;\mathbb{Z})$ is such that (M,ϕ) fibers over S^1, then ϕ lies in the cone on a top-dimensional open face of the Thurston norm ball. Furthermore, if we denote this cone by C, then (M,ψ) fibers over all $\psi \in C \cap H^1(M;\mathbb{Z})$.*

Proof of Theorem 3.2. Suppose $\phi \in H^1(M;\mathbb{Z})$. If ϕ is nontrivial and (M,ϕ) fibers over S^1 then the inequality in Theorem 3.3 and hence the inequality in Theorem 3.1 become equalities. Furthermore, by Theorem 3.5, ϕ lies in the cone on a top-dimensional open face C of the Thurston norm ball, and (M,ψ) fibers over S^1 for any $\psi \in C \cap H^1(M;\mathbb{Z})$. In particular we have

$$||\psi||_T = \frac{1}{k}||\psi||_A^\alpha$$

for every $\psi \in C \cap H^1(M;\mathbb{Z})$ which is nontrivial. By homogeneity and continuity it follows that

$$||\psi||_T = \frac{1}{k}||\psi||_A^\alpha$$

for all $\psi \in C$. □

4. Computation of twisted Alexander norms

Let M be a 3-manifold and $\psi : \pi_1(M) \to F$ a homomorphism to a free abelian group F such that $\psi : H_1(M;\mathbb{Q}) \to F \otimes_{\mathbb{Z}} \mathbb{Q}$ is surjective. (In this case we say ψ is *rationally surjective*.) Given a representation $\alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{F}, k)$ we quickly outline how to compute $\Delta^\alpha_{M,\psi}$ and hence the twisted Alexander norm.

Denote the universal cover of M by $\tilde{M}$. If p is a point in M, then denote the preimage of p under the map $\tilde{M} \to M$ by $\tilde{p}$. Then a presentation matrix for

$$H_i^\alpha(M, p; \mathbb{F}^k[F]) := H_i(C_*(\tilde{M}, \tilde{p}) \otimes_{\mathbb{Z}[\pi_1(M)]} \mathbb{F}^k[F]).$$

can be found using Fox calculus from a presentation of the group $\pi_1(M)$. We discuss this in detail in [FK05]. We also refer to the literature [Fo53, Fo54, CF77]), but we point out that we view $C_*(\tilde{M})$ as a *right* $\mathbb{Z}[\pi_1(M)]$-module, whereas the literature normally views $C_*(\tilde{M})$ as a *left* $\mathbb{Z}[\pi_1(M)]$-module (cf. also [Ha02, Section 6]).

By using the long exact sequence of the twisted homology modules of the pair of spaces (M, p), one can obtain the following short exact sequence of $\mathbb{F}[F]$-modules:

$$0 \to H_1^\alpha(M; \mathbb{F}^k[F]) \to H_1^\alpha(M, p; \mathbb{F}^k[F]) \to A \to 0$$

where $A = \mathrm{Ker}\{H_0^\alpha(p; \mathbb{F}^k[F]) \to H_0^\alpha(M; \mathbb{F}^k[F])\}$. Note that $H_0^\alpha(p; \mathbb{F}^k[F]) \cong \mathbb{F}^k[F]$ whereas $H_0^\alpha(M; \mathbb{F}^k[F])$ is a finite-dimensional $\mathbb{F}$-vector space by the following lemma.

Lemma 4.1. [FK05, Lemma 4.12] *Let X be a 3-manifold, $\psi : \pi_1(X) \to F$ a rationally surjective map with F a free abelian group, and $\alpha : \pi_1(X) \to GL(\mathbb{F}, k)$ a representation. Then*

$$H_i^\alpha(X; \mathbb{F}^k[F]) = H_i(Ker(\psi); \mathbb{F}^k)^n, i = 0, 1$$

where $n = |F/Im(\psi)|$.

It follows that A is an $\mathbb{F}[F]$-module of rank k. (For the notion of rank over $\mathbb{F}[F]$ we refer to the first paragraph in Section 6.1.) If $H_0^\alpha(M; \mathbb{F}^k[F])$ is $\mathbb{F}[F]$-torsion, then by [Hi02, Theorem 3.4]

$$\Delta^\alpha_{M,\psi} = \mathrm{ord}(E_0(H_1^\alpha(M; \mathbb{F}^k[F]))) = \mathrm{ord}(E_k(H_1^\alpha(M, p; \mathbb{F}^k[F]))),$$

which can be computed using the presentation matrix for $H_1^\alpha(M, p; \mathbb{F}^k[F])$. If $H_1^\alpha(M; \mathbb{F}^k[F])$ is not $\mathbb{F}[F]$-torsion, $E_k(H_1^\alpha(M, p; \mathbb{F}^k[F])) = 0$ and $\Delta^\alpha_{M,\psi} = \mathrm{ord}(E_k(H_1^\alpha(M, p; \mathbb{F}^k[F]))) = 0$.

In the case that $\partial M \neq 0$ we can compute $\Delta^\alpha_{M,\psi}$ from Wada's invariant, which tends to be easier to compute. We refer to [Wa94, KL99] for more details.

5. Examples for twisted Alexander norms

In this section, using twisted Alexander norms, we completely determine the Thurston norm of two examples: certain Hopf-like links and Dunfield's link [Du01].

5.1. **Hopf-like links.** In this section, for a link L (possibly with one component), we write Δ_L^α for $\Delta_{X(L)}^\alpha$. Consider a link L as in Figure 3. We will call these links *Hopf-like*. Denote the meridian of K_1 by μ_1 and the meridian of K_2 by μ_2. Denote the corresponding elements in $H_1(X(L);\mathbb{Z})$ by x_1 and x_2. We then identify $\mathbb{Z}[H_1(X(L);\mathbb{Z})]$ with $\mathbb{Z}[x_1^{\pm 1}, x_2^{\pm 1}]$.

Let D_1 (respectively, D_2) be the 2-sphere cutting through L just below K_1 (respectively, above K_2). Denote the three components of $X(L)$ cut along $D_1 \cup D_2$ by P_1, P_0, P_2 (see Figure 3 below). Note that $P_i \cong X(K_i)$, $i = 1, 2$. In particular any representation $\alpha : \pi_1(X(L)) \to \mathrm{GL}(\mathbb{F}, k)$ induces representations $\pi_1(X(K_i)) \to \mathrm{GL}(\mathbb{F}, k), i = 1, 2$, which we also denote by α.

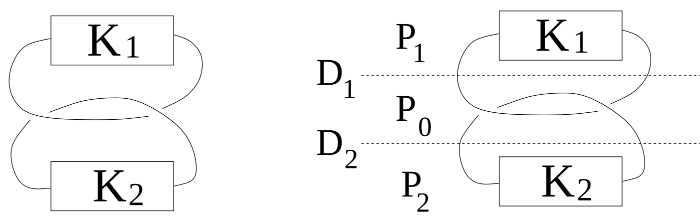

FIGURE 3. The link L and the link complement cut along spheres D_1 and D_2

Proposition 5.1. *Let $\alpha : \pi_1(X(L)) \to GL(\mathbb{F}, k)$ be a representation. Assume $\Delta_{K_i}^\alpha(x_i) \neq 0$ for $i = 1, 2$. Then*

$$\Delta_L^\alpha(x_1, x_2) = \Delta_1(x_1)\Delta_2(x_2) \in \mathbb{F}[x_1^{\pm 1}, x_2^{\pm 1}]$$

where

$$\Delta_i(x_i) = \Delta_{K_i}^\alpha(x_i)\frac{\det(\alpha(\mu_i)x_i - \mathrm{id})}{\Delta_{K_i}^{\alpha,0}} \in \mathbb{F}[x_i^{\pm 1}], \;\; i = 1, 2.$$

In particular

$$\deg(\Delta_i(x_i)) = \deg\left(\Delta_{K_i}^\alpha(x_i)\right) + k - \deg\left(\Delta_{K_i}^{\alpha,0}(x_i)\right), \;\; i = 1, 2.$$

Proof. First note that D_i is homotopy equivalent to the circle for $i = 1, 2$, hence it follows from Lemma 4.1 that $H_1^\alpha(D_i; \mathbb{F}^k[x_1^{\pm 1}, x_2^{\pm 1}]) = 0$. We now consider the Mayer-Vietoris sequence of $X(L) = P_1 \cup_{D_1} P_0 \cup_{D_2} P_2$.

$$0 \to \bigoplus_{i=0}^{2} H_1^\alpha(P_i; \mathbb{F}^k[x_1^{\pm 1}, x_2^{\pm 1}]) \to H_1^\alpha(X(L); \mathbb{F}^k[x_1^{\pm 1}, x_2^{\pm 1}]) \to$$
$$\bigoplus_{i=1}^{2} H_0^\alpha(D_i; \mathbb{F}^k[x_1^{\pm 1}, x_2^{\pm 1}]) \to \bigoplus_{i=0}^{2} H_0^\alpha(P_i; \mathbb{F}^k[x_1^{\pm 1}, x_2^{\pm 1}]) \to H_0^\alpha(X(L); \mathbb{F}^k[x_1^{\pm 1}, x_2^{\pm 1}]) \to 0.$$

By [Le67, Lemma 5, p. 76] for any exact sequence of $\mathbb{F}[x_1^{\pm 1}, x_2^{\pm 1}]$-modules the alternating product of the respective orders in $\mathbb{F}[x_1^{\pm 1}, x_2^{\pm 1}]$ equals one. The proposition now follows immediately from the following computations.

By Lemma 6.2 we have that $\operatorname{ord}(H_0^\alpha(X(L);\mathbb{F}^k[x_1^{\pm1},x_2^{\pm1}]))=1$. We compute the orders of the twisted Alexander modules of P_1 and P_2. Since $P_i\cong X(K_i)$, $i=1,2$, the natural surjection $\psi:\mathbb{Z}[\pi_1(X(L))]\to\mathbb{Z}[x_1^{\pm1},x_2^{\pm1}]$ restricted to P_i only has values in $\mathbb{Z}[x_i^{\pm1}]$. Thus we get

$$\begin{aligned} H_j^\alpha(P_1;\mathbb{F}^k[x_1^{\pm1},x_2^{\pm1}]) &\cong H_j^\alpha(X(K_1);\mathbb{F}^k[x_1^{\pm1}])\otimes_{\mathbb{F}}\mathbb{F}[x_2^{\pm1}] \text{ for all } j, \text{ and}\\ H_j^\alpha(P_2;\mathbb{F}^k[x_1^{\pm1},x_2^{\pm1}]) &\cong H_j^\alpha(X(K_2);\mathbb{F}^k[x_2^{\pm1}])\otimes_{\mathbb{F}}\mathbb{F}[x_1^{\pm1}] \text{ for all } j.\end{aligned}$$

Therefore

$$\operatorname{ord}\left(H_j^\alpha(P_i;\mathbb{F}^k[x_1^{\pm1},x_2^{\pm1}])\right)=\Delta_{K_i}^{\alpha,j}(x_i).$$

for all $j\geq 0$ and $i=1,2$.

Let us consider P_0. P_0 is homotopy equivalent to the torus and $\pi_1(P_0)$ is the free abelian group spanned by μ_1 and μ_2. By Lemma 4.1 we have $H_1^\alpha(P_0;\mathbb{F}^k[x_1^{\pm1},x_2^{\pm1}])=0$. Therefore $\operatorname{ord}(H_1^\alpha(P_0;\mathbb{F}^k[x_1^{\pm1},x_2^{\pm1}]))=1$. Furthermore the argument in the proof of Lemma 6.2 shows that $\operatorname{ord}(H_0^\alpha(P_0;\mathbb{F}^k[x_1^{\pm1},x_2^{\pm1}]))=1$.

Now consider D_1 and D_2. Using the cellular chain complex of the circle, one easily sees that

$$\operatorname{ord}(H_0^\alpha(D_i;\mathbb{F}^k[x_1^{\pm1},x_2^{\pm1}]))=\det(\alpha(\mu_i)x_i-\mathrm{id})$$

for $i=1,2$. □

Corollary 5.2. *For the trivial representation $\alpha:\pi_1(X(L))\to GL(\mathbb{F},1)$,*

$$\Delta_L^\alpha(x_1,x_2)=\Delta_{K_1}^\alpha(x_1)\Delta_{K_2}^\alpha(x_2).$$

Proof. Since α is a one-dimensional trivial representation,

$$H_0^\alpha(X(K_1);\mathbb{F}[x_1^{\pm1}])=\mathbb{F}[x_1^{\pm1}]/(x_1-1).$$

Hence $\Delta_{K_1}^{\alpha,0}(x_1)=x_1-1$. Also $\det(\alpha(\mu_1)x_1-\mathrm{id})=x_1-1=\Delta_{K_1}^{\alpha,0}(x_1)$. Similarly $\det(\alpha(\mu_2)x_2-\mathrm{id})=\Delta_{K_2}^{\alpha,0}(x_2)=x_2-1$. Now use Proposition 5.1. □

Corollary 5.3. *Let $d_i:=\deg(\Delta_i^\alpha(x_i))$, $i=1,2$ in Proposition 5.1. Then the norm ball of $\frac{1}{k}||-||_A^\alpha$ has exactly four extreme vertices namely $(\pm\frac{k}{d_1},0)$ and $(0,\pm\frac{k}{d_2})$.*

The above corollary easily follows from Proposition 5.1.

Now consider the Hopf-like link L in Figure 4. This consists of the knot K_1, the trefoil, and $K_2=11_{440}$ (here we use the *knotscape* notation). By Corollary 5.2 the usual multivariable Alexander polynomial with rational coefficients equals

$$\Delta_L(x_1,x_2)=\Delta_{K_1}(x_1)\Delta_{K_2}(x_2)=(x_1^2-x_1+1)(x_2^4-2x_2^3+3x_2^2-2x_2+1)\in\mathbb{Q}[x_1^{\pm1},x_2^{\pm1}].$$

Let $\{\phi_1,\phi_2\}\subset H^1(X(L);\mathbb{Z})=\operatorname{Hom}(H_1(X(L);\mathbb{Z}),\mathbb{Z})$ be the dual basis to $\{x_1,x_2\}$. It is known that $\operatorname{genus}(K_1)=1$ and $\operatorname{genus}(K_2)=3$. We can arrange the minimal Seifert surfaces such that they are punctured once by the other component. It follows that $||\phi_1||_T\leq 2\operatorname{genus}(K_1)=2$ and $||\phi_2||_T\leq 2\operatorname{genus}(K_2)=6$. In fact it is easy to see that the equality holds for each case since each surface dual to ϕ_1 (respectively ϕ_2) becomes a Seifert surface for K_1 (respectively K_2) after adding one or more disks.

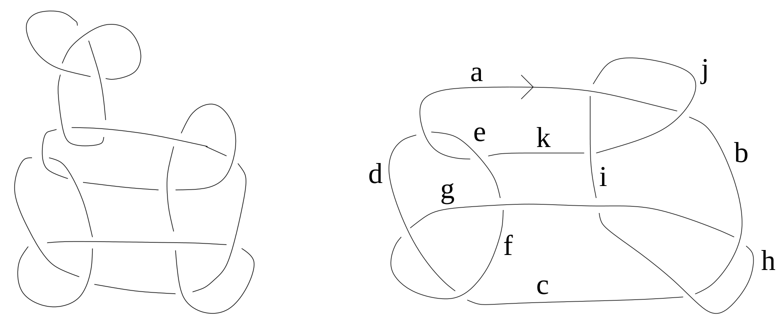

FIGURE 4. Link L and knot $K_2 = 11_{440}$ with meridians.

On the other hand it follows from the calculation of $\Delta_L(x_1, x_2)$ that $||\phi_1||_A = 2$ and $||\phi_2||_A = 4$. Therefore the Alexander norm and the Thurston norm do not agree for L. We also note that since $H_1(X(L); \mathbb{Z})$ is torsion-free, Turaev's torsion norm [Tu02a] agrees with the Alexander norm does.

The fundamental group of $\pi_1(X(K_2))$ is generated by the meridians $a, b, \dots, k$ of the segments in the knot diagram in Figure 4. Using the program *KnotTwister* [F05] we found the homomorphism $\varphi : \pi_1(X(K_2)) \to S_3$ given by

$$\begin{array}{cccccccccccccccccc} A & = & (23), & B & = & (12), & C & = & (13), & D & = & (23), & E & = & (23), & F & = & (12), \\ G & = & (13), & H & = & (23), & I & = & (12), & J & = & (13), & K & = & (23), & & & \end{array}$$

where we use the cycle notation. The generators of $\pi_1(X(K_2))$ are sent to the elements in S_3 given by the cycle with the corresponding capital letter. We then consider $\alpha := \alpha(\varphi) : \pi_1(X(K_2)) \xrightarrow{\varphi} S_3 \to \mathrm{GL}(V_2)$ where

$$V_2 := \{(v_1, v_2, v_3) \in \mathbb{F}_{13}^3 | \sum_{i=1}^{3} v_i = 0\}.$$

Clearly $\dim_{\mathbb{F}_{13}}(V_2) = 2$ and S_3 acts on it by permutation. With *KnotTwister* we compute

$$\Delta^\alpha_{K_2}(x_2) = 1 + 3x_2^2 + 12x_2^4 + x_2^6 + 10x_2^8 + 12x_2^{10} \in \mathbb{F}_{13}[x_2^{\pm 1}]$$

and $H_0^\alpha\left(X(K_2); \mathbb{F}_{13}^2[x_2^{\pm 1}]\right) = 0$. Hence $\Delta^{\alpha,0}_{K_2}(x_2) = 1$.

Denote the homomorphism $\alpha : \pi_1(X(L)) \to \pi_1(X(K_2)) \to \mathrm{GL}(V_2)$ by α as well. Here the map $\pi_1(X(L)) \to \pi_1(X(K_2))$ is induced from the inclusion. This induces a representation of $\pi_1(X(K_1))$ as in the proof of Proposition 5.1, and we also denote it by α. In fact, one easily sees that $\alpha : \pi_1(X(K_1)) \to \mathrm{GL}(V_2)$ is trivial. This implies that $\Delta^\alpha_{K_1}(x_1) = (\Delta_{K_1}(x_1))^2 = (1 - x_1 + x_1^2)^2$ and $\Delta^{\alpha,0}_{K_1}(x_1) = (x_1 - 1)^2$. By Proposition 5.1 we have

$$\Delta^\alpha_L(x_1, x_2) = \Delta^\alpha_1(x_1) \cdot \Delta^\alpha_2(x_2)$$

where

$$\deg(\Delta^\alpha_1(x_1)) = 2 \deg\left(\Delta_{K_1}(x_1)\right) + 2 - 2 = 4$$

and

$$\deg(\Delta_2^\alpha(x_2)) = \deg\left(\Delta_{K_2}^\alpha(x_2)\right) + 2 - 0 = 12.$$

Hence the twisted Alexander norm ball corresponding to $\frac{1}{2}||-||_A^\alpha$ has exactly four extreme vertices $(\pm\frac{1}{2}, 0)$ and $(0, \pm\frac{1}{6})$ by Corollary 5.3. Since $||\phi_1||_T = 2$ and $||\phi_2||_T = 6$, the norms $||\phi||_T$ and $\frac{1}{2}||\phi||_A^\alpha$ agree at the extreme vertices of the norm ball of $\frac{1}{2}||-||_A^\alpha$. Note that by Theorem 3.1 we have $||\phi||_T \geq \frac{1}{2}||\phi||_A^\alpha$. Since the norms $||\phi||_T$ and $\frac{1}{2}||\phi||_A^\alpha$ agree at all of the extreme vertices of the norm ball of $\frac{1}{2}||-||_A^\alpha$, they agree everywhere by convexity. Therefore the shaded region on the right in Figure 5 is the Thurston norm ball of the link L.

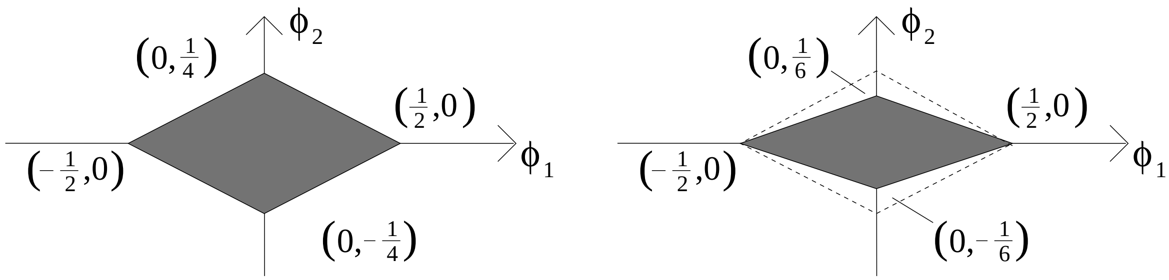

FIGURE 5. The untwisted and the twisted Alexander norm ball of L.

In Figure 5 on the right the closed region bounded by the dashed polygon is the Alexander norm ball. If $(X(L), \phi)$ fibers over S^1 for some $\phi \in H^1(X(L);\mathbb{Z})$ then it follows from Theorem 3.2 that the (usual) Alexander norm and the Thurston norm agree on the cone on a top-dimensional face of the Thurston norm ball. Figure 5 shows that the Alexander norm and the Thurston norm agree only for a multiple of ϕ_1. Hence $(X(L), \phi)$ does not fiber over S^1 for any $\phi \in H^1(X(L);\mathbb{Z})$. We state these results in the theorem below.

Theorem 5.4. *The Thurston norm ball of $X(L)$ is the shaded region on the right in Figure 5. Furthermore, $(X(L), \phi)$ does not fiber over S^1 for any $\phi \in H^1(X(L);\mathbb{Z})$.*

There exist 36 knots with 12 crossings or less such that $2\,\text{genus}(K) > \deg(\Delta_K(t))$. In all but three cases we found representations similar to the above such that the Thurston norm bound from Theorem 3.3 equals the Thurston norm of $X(K)$. Let L be the Hopf-like link as in Figure 3 with K_1 any knot such that $2\,\text{genus}(K_1) = \deg(\Delta_{K_1}(t))$ and K_2 any of the 33 knots mentioned above. In this case the argument above can be used to show that twisted Alexander norms completely determines the Thurston norm ball of $X(L)$ and it is always strictly smaller than the Alexander norm ball.

Now consider the case with K_1 the unknot and $K_2 = 11_{440}$. We use the same representation as above. In this case the norm ball for $\frac{1}{2}|| - ||_A^\alpha$ is given in Figure 6. The norm ball is a horizontal infinite strip, hence noncompact.

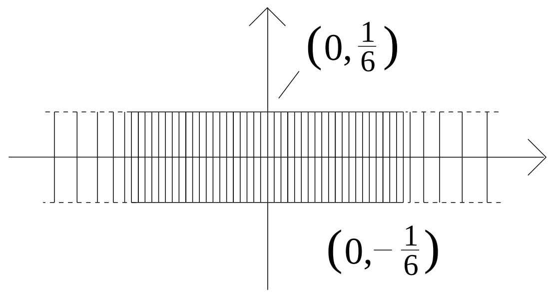

FIGURE 6. Thurston norm ball of L.

To show that $\frac{1}{2}|| - ||_A^\alpha = || - ||_T$ it is enough to show that for $\phi = (n, \pm 1), n \in \mathbb{Z}$ there exists a connected dual surface with $\chi(S) = -6$. Let S be a Seifert surface of genus 3 for K_2 which intersects K_1 just once. By deleting a disk from S we get a surface S' which is disjoint from K_1. The surface S' is dual to $\phi = (0, 1)$. We can make S' such that the two boundary components of S' are as close to each other as we wish. Now take a short path from one boundary component of S' to the other boundary component. Cut S' along that path and reglue the cut parts together by giving n full twists. The resulting surface is dual to $\phi = (n, 1)$ and has the Euler characteristic -6. Hence the Thurston norm ball in this case is the shaded (infinite) strip in Figure 6.

5.2. **Dunfield's example.** McMullen had asked whether for a fibered manifold the Thurston norm and the Alexander norm agree everywhere. To answer this question Dunfield [Du01] considers the link L in Figure 7.

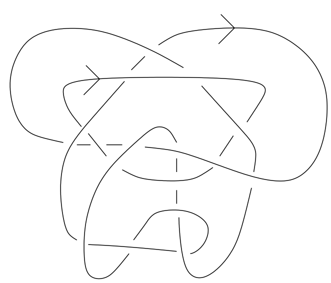

FIGURE 7. Dunfield's example.

Denote the knotted component by K_1 and the unknotted component by K_2. Let $x, y \in H_1(X(L); \mathbb{Z})$ be the elements represented by a meridian of K_1, respectively K_2.

Then the Alexander polynomial equals

$$\Delta_{X(L)} = xy - x - y + 1 \in \mathbb{Z}[H_1(X(L);\mathbb{Z})] = \mathbb{Z}[x^{\pm 1}, y^{\pm 1}].$$

We consider $H^1(X(L);\mathbb{Z})$ with the dual basis corresponding to $\{x, y\} \in H_1(X(L);\mathbb{Z})$. The Alexander norm ball is given in Figure 8.

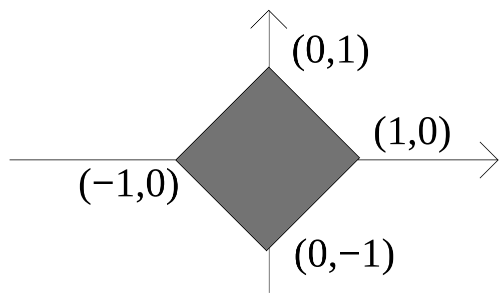

FIGURE 8. Alexander norm ball for Dunfield's link.

Dunfield [Du01] showed that $(X(L), \phi)$ fibers over S^1 for all $\phi \in H^1(X(L);\mathbb{Z})$ in the cones on the two open faces of the Alexander norm ball with vertices $(-\frac{1}{2}, \frac{1}{2}), (0, 1)$ respectively $(0, -1), (\frac{1}{2}, -\frac{1}{2})$. Dunfield used the Bieri-Neumann-Strebel (BNS) invariant (see [BNS87]) to show that the Alexander norm and the Thurston norm do not agree for the 3-manifold $X(L)$. We will go one step further and completely determine the Thurston norm of $X(L)$.

We did not find a representation of $\pi_1(X(L))$ for which we can compute the twisted Alexander polynomial and which determines the Thurston norm. Therefore we study the Thurston norm of a 2-fold cover of $X(L)$ for which it is easier to find representation.

The following theorem by Gabai shows the relationship between the Thurston norm of $X(L)$ and that of a finite cover of $X(L)$.

Theorem 5.5. [Ga83, p. 484] *Let M be a 3-manifold and $\alpha : \pi_1(M) \to G$ a homomorphism to a finite group G. Denote the induced G-cover of M by M_G. Let $\phi \in H^1(M;\mathbb{Z})$ be nontrivial and denote the induced map $H_1(M_G;\mathbb{Z}) \to H_1(M;\mathbb{Z}) \to \mathbb{Z}$ by ϕ_G, which can be regarded as an element in $H^1(M_G;\mathbb{Z})$. Then ϕ_G is nontrivial and*

$$|G| \cdot ||\phi||_{T,M} = ||\phi_G||_{T,M_G}.$$

Thus to determine the Thurston norm of M, we only need to determine the Thurston norm of M_G. For this purpose, we generalize twisted Alexander norms and the main theorems a little bit further as follows.

Let M be a 3-manifold and $\psi : \pi_1(M) \to F$ a homomorphism to a free abelian group, we do not demand that ψ is surjective. We define a norm on $\text{Hom}(F, \mathbb{R})$. Note that if $F = FH_1(M;\mathbb{Z})$, then $\text{Hom}(F, \mathbb{R}) \cong H^1(M;\mathbb{R})$. Let $\alpha : \pi_1(M) \to \text{GL}(\mathbb{F}, k)$ be a representation. If $\Delta^\alpha_{M,\psi} = 0 \in \mathbb{F}[F]$ then we set $||\phi||^\alpha_{A,\psi} = 0$ for all $\phi \in \text{Hom}(F, \mathbb{R})$.

Otherwise we write $\Delta^{\alpha}_{M,\psi} = \sum a_i f_i$ for $a_i \in \mathbb{F}$ and $f_i \in F$. Given $\phi \in \mathrm{Hom}(F, \mathbb{R})$, we define *the (generalized) twisted Alexander norm of* (M, ψ, α) to be

$$||\phi||^{\alpha}_{A,\psi} := \sup \phi(f_i - f_j)$$

with the supremum over (f_i, f_j) such that $a_i a_j \neq 0$. If we consider the natural surjection $\psi : \pi_1(M) \to FH_1(M; \mathbb{Z})$, then clearly $|| - ||^{\alpha}_{A,\psi} = || - ||^{\alpha}_{A}$. Note that $|| - ||^{\alpha}_{A,\psi}$ is clearly a seminorm on $\mathrm{Hom}(F, \mathbb{R})$. The following theorem generalizes Theorem 3.1 and Theorem 3.2. The proof is almost identical.

Theorem 5.6. *Let M be a 3-manifold whose boundary is empty or consists of tori. Let $\alpha : \pi_1(M) \to GL(\mathbb{F}, k)$ be a representation. Let $\psi : \pi_1(M) \to F$ be a homomorphism to a free abelian group such that rank $F > 1$ and such that $H_1(M; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q} \to F \otimes_{\mathbb{Z}} \mathbb{Q}$ is surjective. Then*

$$||\phi \circ \psi||_T \geq \frac{1}{k} ||\phi||^{\alpha}_{A,\psi}$$

for all $\phi \in Hom(F, \mathbb{R})$.

Furthermore, if $M \neq S^1 \times D^2, M \neq S^1 \times S^2$ and if $\phi \in Hom(F, \mathbb{Z})$ is such that $(M, \phi \circ \psi)$ fibers over S^1, then $\phi \circ \psi$ lies in the cone on a top-dimensional open face of the Thurston norm ball (denoted by C) and for all $\phi' \in Hom(F, \mathbb{R})$ such that $\phi' \circ \psi \in C$ we have

$$||\phi' \circ \psi||_T = \frac{1}{k} ||\phi'||^{\alpha}_{A,\psi}.$$

We now return to the link L in Figure 7. Let $\varphi : H_1(X(L); \mathbb{Z}) \to \mathbb{Z}/2$ be the homomorphism given by $\varphi(x) = 1$, $\varphi(y) = 0$. Denote the induced two-fold cover by $X(L)_2$. Denote by ψ the homomorphism $\pi : \pi_1(X(L)_2) \to H_1(X(L)_2; \mathbb{Z}) \to H_1(X(L); \mathbb{Z})$ induced from the covering map $\pi : X(L)_2 \to X(L)$. We found a representation $\alpha : \pi_1(X(L)_2) \to \mathrm{GL}(\mathbb{F}_7, 1)$ such that

$$\Delta^{\alpha}_{X(L)_2,\psi} = 3x^6y^2 + 3x^4y^2 + 4x^4y + 2x^4 + x^2y^2 + 3x^2y - x^2 - 1 \in \mathbb{F}_7[H_1(X(L); \mathbb{Z})] = \mathbb{F}_7[x^{\pm 1}, y^{\pm 1}].$$

To use Theorem 5.6, we need the following well-known lemma.

Lemma 5.7. *Let M be a 3-manifold. Let $\pi_1(M) \to G$ be a homomorphism to a finite group. Denote the induced G-cover of M by M_G. Then $\pi_* : H_1(M_G) \otimes_{\mathbb{Z}} \mathbb{Q} \to H_1(M; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ induced from the covering map is surjective.*

Proof. Let $a \in \pi_1(M)$ represent an element in $H_1(M; \mathbb{Z})$. Then $a^{|G|}$ lifts to an element $a_G \in \pi_1(M_G)$. Therefore $|G|[a] = \pi_*([a_G]) \in H_1(M; \mathbb{Z})$ where π_* is the homomorphism induced from the covering map $\pi : M_G \to M$. □

Let $\phi \in H^1(X(L); \mathbb{Z})$. By Theorem 5.5 and Theorem 5.6, we have

$$||\phi||_{T,X(L)} = \frac{1}{2} ||\phi \circ \pi||_{T,X(L)_2} \geq \frac{1}{4} ||\phi||^{\alpha}_{A,\psi}.$$

The norm ball of $\frac{1}{4} || - ||^{\alpha}_{A,\psi}$ is drawn as the shaded region in Figure 9. We claim that this is exactly the Thurston norm ball.

By Theorem 5.6 the twisted Alexander norm ball in Figure 9 is an 'outer bound' for the Thurston norm ball of $X(L)$. But as we pointed out above, Dunfield showed that $(X(L), \phi)$ fibers over S^1 for all $\phi \in H^1(X(L); \mathbb{Z})$ which lie in the cones on the two open faces of the Alexander norm ball with vertices $(-\frac{1}{2}, \frac{1}{2}), (0, 1)$ respectively $(0, -1), (\frac{1}{2}, -\frac{1}{2})$. In particular, the Thurston norm ball and the twisted Alexander norm ball agree on these cones by the second part of Theorem 5.6. By continuity, the norms also agree on the vertices $(-\frac{1}{2}, \frac{1}{2}), (0, 1), (0, -1)$ and $(\frac{1}{2}, -\frac{1}{2})$. Now it follows from convexity that the Thurston norm ball coincides everywhere with the twisted Alexander norm ball given in Figure 9. Therefore the shaded region in Figure 9 is the Thurston norm ball of $X(L)$.

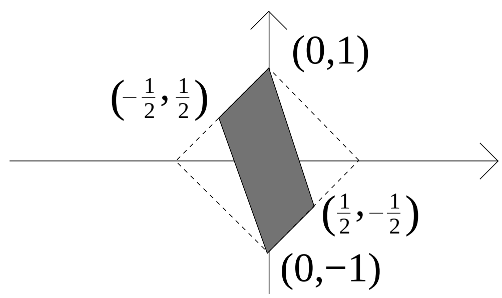

FIGURE 9. Twisted Alexander norm ball for Dunfield's link

Note that our calculation confirms Dunfield's result that $(X(L), \phi)$ does not fiber over S^1 for any ϕ outside the cones. We summarize these results in the following theorem.

Theorem 5.8. *The Thurston norm ball of $X(L)$ is the shaded region in Figure 9. Furthermore, $(X(L), \phi)$ fibers over S^1 exactly when ϕ lies inside the cones on the open faces of the two smaller faces of the Thurston norm ball of $X(L)$.*

6. Twisted multivariable Alexander polynomial and twisted one-variable Alexander polynomial

This section serves for proving Theorem 3.4. The main idea of the proof is to use the functoriality of Reidemeister torsion. To prove Theorem 3.4 we need some lemmas which show the nontriviality of certain twisted Alexander polynomials. Throughout this section we assume that M is a 3-manifold whose boundary is empty or consists of tori. Furthermore let $\alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{F}, k)$ be a representation.

6.1. **Computation of twisted Alexander polynomials.** We introduce the notion of *rank over a UFD*. Let Λ be a UFD and $Q(\Lambda)$ its quotient field. Let H be an Λ-module. Then we define $\mathrm{rank}_\Lambda(H) := \dim_{Q(\Lambda)}(H \otimes_\Lambda Q(\Lambda))$. We need the following well-known lemma. For the first part we refer to [Tu01, Remark 4.5], for the second part we refer to [FK05, Lemma 4.14]. The last statement follows from the fact that $Q(\Lambda)$ is flat over Λ.

Lemma 6.1. *Let Λ be a UFD.*

1. *Let H be a finitely generated Λ-module. Then the following are equivalent:*
   - (a) *H is Λ-torsion,*
   - (b) *$\operatorname{ord}_\Lambda(H) \neq 0$,*
   - (c) *$rank_\Lambda(H) = 0$,*
   - (d) *$Hom_\Lambda(H, \Lambda) = 0$.*
2. *Let N be an n-manifold and assume that Λ^k has a left $\mathbb{Z}[\pi_1(N)]$-module structure, then*
$$\sum_{i=0}^{n} (-1)^i rank_\Lambda(H_i(N; \Lambda^k)) = k\chi(N).$$
3. *$H_i(N; \Lambda^k \otimes_\Lambda Q(\Lambda)) = H_i(N; \Lambda^k) \otimes_\Lambda Q(\Lambda)$ for any i.*

Lemma 6.2. *Let M be a 3-manifold. Let $\varphi : \pi_1(M) \to H$ be a surjection to a free abelian group. Then $\Delta^{\alpha,3}_{M,\varphi} = 1$ and $\Delta^{\alpha,0}_{M,\varphi} \neq 0$. If furthermore $rank\, H > 1$, then $\Delta^{\alpha,0}_{M,\varphi} = 1 \in \mathbb{F}[H]$.*

Proof. We prove the lemma only in the case that M is closed. The proof for the case that ∂M consists of tori is very similar. Let $b := \operatorname{rank} H$. Pick a basis $t_1, \dots, t_b$ for H. We identify $\mathbb{F}^k[H]$ with $\mathbb{F}^k[t_1^{\pm 1}, \dots, t_b^{\pm 1}]$.

We use an idea in the proof of [Mc02, Theorem 5.1]. Choose a triangulation τ of M. Let T be a maximal tree in the 1-skeleton of τ and let T' be a maximal tree in the dual 1-skeleton. We collapse T to form a single 0-cell and join the 3-simplices of T' to form a single 3-cell. Denote the 1-cells by $h_1, \dots, h_n$. Denote the corresponding elements in $\pi_1(M)$ by $h_1, \dots, h_n$ as well. If $\operatorname{rank} H > 1$ then we can arrange that $\varphi(h_i) = t_i$ for $i = 1, 2$.

Since M is closed it follows that $\chi(M) = 0$, hence there are n 2-cells. Write $\pi := \pi_1(M)$. From the CW structure we obtain a chain complex $C_* := C_*(\tilde{M})$ (where $\tilde{M}$ denotes the universal cover of M):

$$0 \to C_3^1 \xrightarrow{\partial_3} C_2^n \xrightarrow{\partial_2} C_1^n \xrightarrow{\partial_1} C_0^1 \to 0$$

for M, where the C_i are free $\mathbb{Z}[\pi]$-right modules. In fact $C_i^k \cong \mathbb{Z}[\pi]^k$. Consider the chain complex $C_* \otimes_{\mathbb{Z}[\pi]} \mathbb{F}^k[H]$:

$$0 \to C_3^1 \otimes_{\mathbb{Z}[\pi]} \mathbb{F}^k[H] \xrightarrow{\partial_3 \otimes \mathrm{id}} C_2^n \otimes_{\mathbb{Z}[\pi]} \mathbb{F}^k[H] \xrightarrow{\partial_2 \otimes \mathrm{id}} C_1^n \otimes_{\mathbb{Z}[\pi]} \mathbb{F}^k[H] \xrightarrow{\partial_1 \otimes \mathrm{id}} C_0^1 \otimes_{\mathbb{Z}[\pi]} \mathbb{F}^k[H] \to 0.$$

Let A_i, $i = 0, \dots, 3$, be the matrices with entries in $\mathbb{Z}[\pi]$ corresponding to the boundary maps $\partial_i : C_i \to C_{i-1}$ with respect to the bases given by the lifts of the cells of M to $\tilde{M}$. Then we can arrange the lifts such that

$$\begin{aligned} A_3 &= (1 - g_1, 1 - g_2, \dots, 1 - g_n)^t, \\ A_1 &= (1 - h_1, 1 - h_2, \dots, 1 - h_n), \end{aligned}$$

where $\{g_1, \dots, g_n\}$ and $\{h_1, \dots, h_n\}$ are generating sets for $\pi_1(M)$.

Let $B = (b_{rs})$ be a $p \times q$ matrix with entries in $\mathbb{Z}[\pi]$. We write $b_{rs} = \sum b^g_{rs} g$ for $b^g_{rs} \in \mathbb{Z}, g \in \pi$. We define $(\alpha \otimes \varphi)(B)$ to be the $p \times q$ matrix with entries $\sum b^g_{rs}\alpha(g)\varphi(g)$. Since each $\sum b^g_{rs}\alpha(g)\varphi(g)$ is a $k \times k$ matrix with entries in $\mathbb{F}[H]$ we can think of $(\alpha \otimes \varphi)(B)$ as a $pk \times qk$ matrix with entries in $\mathbb{F}[H]$.

Since φ is nontrivial there exist k, l such that $\varphi(g_k) \neq 0$ and $\varphi(h_l) \neq 0$. It follows that $(\alpha \otimes \varphi)(A_1)$ and $(\alpha \otimes \varphi)(A_3)$ have full rank over $\mathbb{F}[H]$. The first part of the lemma now follows immediately.

Now assume that rank $H > 1$. Then ord $\left(H_0^\alpha(M; \mathbb{F}^k[t_1^{\pm 1}, \dots, t_b^{\pm 1}])\right)$ divides $\det(\alpha(g_1)t_1 - \mathrm{id}) \in \mathbb{F}[t_1^{\pm 1}]$ and $\det(\alpha(g_2)t_2 - \mathrm{id}) \in \mathbb{F}[t_2^{\pm 1}]$. These two polynomials are clearly relatively prime. This implies that ord $\left(H_0^\alpha(M; \mathbb{F}^k[t_1^{\pm 1}, \dots, t_b^{\pm 1}])\right) = 1$.

□

Lemma 6.3. *Let $\varphi : \pi_1(M) \to H$ be a surjection to a free abelian group H. If $\Delta^{\alpha,1}_{M,\varphi} \neq 0$ then $\Delta^{\alpha,2}_{M,\varphi} \neq 0$.*

Proof. Note that by assumption and by Lemma 6.2 we have $\Delta^{\alpha,i}_{M,\varphi} \neq 0$ for $i = 0, 1, 3$. Let $\Lambda := \mathbb{F}[H]$. It follows from the long exact homology sequence for $(M, \partial M)$ and from duality that $\chi(M) = \frac{1}{2}\chi(\partial M)$. So $\chi(M) = 0$ in our case. It follows from Lemma 6.1 that

$$\sum_{i=0}^{3}(-1)^i \dim_{Q(\Lambda)} \left(H_i^\alpha(M; \Lambda^k \otimes_\Lambda Q(\Lambda))\right) = k\chi(M) = 0.$$

Note that $H_i^\alpha(M; \Lambda^k \otimes_\Lambda Q(\Lambda)) \cong H_i^\alpha(M; \Lambda^k) \otimes_\Lambda Q(\Lambda)$ by Lemma 6.1. By assumption $H_i^\alpha(M; \Lambda^k) \otimes_\Lambda Q(\Lambda) = 0$ for $i \neq 2$, hence $H_2^\alpha(M; \Lambda^k) \otimes_\Lambda Q(\Lambda) = 0$. □

The following corollary is now immediate.

Corollary 6.4. *Let $\varphi : \pi_1(M) \to H$ be a surjection to a free abelian group H. If $\Delta^{\alpha,1}_{M,\varphi} \neq 0$ then $\Delta^{\alpha,i}_{M,\varphi} \neq 0$ for all i.*

Lemma 6.5. *Let $\varphi : \pi_1(M) \to H$ be a surjection to a free abelian group with rank $H > 1$. If $\Delta^{\alpha,1}_{M,\varphi} \neq 0$ then*

$$\Delta^{\alpha,2}_{M,\varphi} = 1 \in \mathbb{F}[H].$$

Proof. Let $\Lambda := \mathbb{F}[H]$ and $\pi := \pi_1(M)$. By Poincaré duality,

$$H_2^\alpha(M; \Lambda^k) \cong H^1_\alpha(M, \partial M; \Lambda^k) = H^1(\mathrm{Hom}_{\mathbb{Z}[\pi]}(C_*(\tilde{M}, \partial\tilde{M}), \Lambda^k))$$

where $\tilde{M}$ is the universal cover of M. On the right we view Λ^k as a right $\mathbb{Z}[\pi]$-module by taking $f \cdot g = g^{-1} \cdot f = \varphi(g^{-1})\alpha(g^{-1})f$ for $f \in \Lambda^k$ and $g \in \pi$.

We use an argument in [KL99, p. 638]. Let $\langle\, , \rangle : \mathbb{F}^k \times \mathbb{F}^k \to \mathbb{F}$ be the canonical inner product on $\mathbb{F}^k$. Then there exists a unique representation $\overline{\alpha} : \pi_1(M) \to \mathrm{GL}(\mathbb{F}, k)$ such that

$$\langle \alpha(g^{-1})v, w\rangle = \langle v, \overline{\alpha}(g)w\rangle$$

for all $g \in \pi_1(M)$ and $v, w \in \mathbb{F}^k$. We denote by $\overline{\Lambda^k}$ the left $\mathbb{Z}[\pi]$-module with underlying Λ-module Λ^k and $\mathbb{Z}[\pi]$-module structure given by $\overline{\alpha} \otimes (-\phi)$.

Using the inner product we get a map

$$\begin{array}{rcl} \operatorname{Hom}_{\mathbb{Z}[\pi]}(C_*(\tilde{M}, \partial\tilde{M}), \Lambda^k) & \to & \operatorname{Hom}_\Lambda\big(C_*(\tilde{M}, \partial\tilde{M}) \otimes_{\mathbb{Z}[\pi]} \overline{\Lambda^k}, \Lambda\big) \\ f & \mapsto & (c \otimes w) \mapsto \langle f(c), w\rangle. \end{array}$$

Using $\langle \alpha(g^{-1})v, w\rangle = \langle v, \overline{\alpha}(g)w\rangle$ it is now easy to see that this map is well-defined and that it defines in fact an isomorphism of Λ-module chain complexes.

Now we can apply the universal coefficient spectral sequence to the Λ-module chain complex $\operatorname{Hom}_\Lambda\big(C_*(\tilde{M}, \partial\tilde{M}) \otimes_{\mathbb{Z}[\pi]} \overline{\Lambda^k}, \Lambda\big)$ to conclude that there exists a short exact sequence

$$0 \to \operatorname{Ext}^1_\Lambda(H_0^{\overline{\alpha}}(M, \partial M; \overline{\Lambda^k})) \to H^1_\alpha(M, \partial M; \Lambda^k) \to \operatorname{Hom}_\Lambda(H_1^{\overline{\alpha}}(M, \partial M; \overline{\Lambda^k})).$$

Since $\Delta^{\alpha,2}_{M,\varphi} \neq 0$ by Lemma 6.3 it follows that $H^1_\alpha(M, \partial M; \Lambda^k)$ is Λ-torsion. Hence

$$H^1_\alpha(M, \partial M; \Lambda^k) \cong \operatorname{Ext}^1_\Lambda(H_0^{\overline{\alpha}}(M, \partial M; \overline{\Lambda^k}).$$

First assume that ∂M is nonempty. Note that $\pi_1(\partial M) \to \mathrm{GL}(\mathbb{F}, k)$ factors through $\pi_1(M)$. It follows from

$$H_0^{\overline{\alpha}}(X; \overline{\Lambda^k}) \cong \overline{\Lambda^k}/\{gv - v | g \in \pi_1(X), v \in \overline{\Lambda^k}\}$$

that $H_0^{\overline{\alpha}}(\partial M; \overline{\Lambda^k})$ surjects onto $H_0^{\overline{\alpha}}(M; \overline{\Lambda^k}) = 0$, hence $H_0^{\overline{\alpha}}(M, \partial M; \overline{\Lambda^k}) = 0$ (cf. [FK05, Lemma 4.13]).

Now assume that M is closed. Let $H_0 := H_0^{\overline{\alpha}}(M; \overline{\Lambda^k})$. We define a finitely generated Λ-module A to be *pseudonull* if $A_\wp = 0$ for every height 1 prime ideal $\wp$ of Λ where $A_\wp$ is the localization of A at $\wp$. (See p. 51 in [Hi02].) By [Hi02, Theorem 3.1], $E_0(H_0) \subset \operatorname{Ann}(H_0)$. Since $\Delta^{\alpha,0}_{M,\varphi} = 1$ by Lemma 6.2, $\widetilde{\operatorname{Ann}}(H_0) = \Lambda$ where $\widetilde{\operatorname{Ann}}(H_0)$ is the smallest principal ideal of Λ which contains $\operatorname{Ann}(H_0)$. Thus by [Hi02, Theorem 3.5], H_0 is pseudonull. Finally, by [Hi02, Theorem 3.9], $\operatorname{Ext}^1_\Lambda(H_0, \Lambda) = 0$. Hence $H_2^\alpha(M; \Lambda^k) \cong H^1_\alpha(M, \partial M; \Lambda^k) = 0$. □

6.2. **Functoriality of Reidemeister torsion.** Define F to be the free abelian group $FH_1(M; \mathbb{Z})$. Let $\psi : \pi_1(M) \to F$ be the natural surjection and $\phi \in H^1(M; \mathbb{Z})$ nontrivial. Note that ϕ induces a homomorphism $\phi : \mathbb{F}[F] \to \mathbb{F}[t^{\pm 1}]$. In this section we go back to the notation $\Delta^{\alpha,i}_M = \Delta^{\alpha,i}_{M,\psi}$ and $\Delta^{\alpha,i}_\phi = \Delta^{\alpha,i}_{M,\phi}$.

Theorem 6.6. *Suppose* $b_1(M) > 1$.

(1) *If* $\phi\left(\Delta^{\alpha,1}_{M,\psi}\right) \neq 0$ *then* $\Delta^{\alpha,1}_{M,\phi} \neq 0$ *and*

$$\phi\left(\Delta^{\alpha,1}_{M,\psi}\right) = \prod_{i=0}^{3} \phi\left(\Delta^{\alpha,i}_{M,\psi}\right)^{(-1)^{i+1}} = \prod_{i=0}^{3} \Delta^{\alpha,i}_{M,\phi}(t)^{(-1)^{i+1}} \in \mathbb{F}[t^{\pm 1}].$$

(2) *If* $\phi\left(\Delta^{\alpha,1}_{M,\psi}\right) = 0$ *then* $\Delta^{\alpha,1}_{M,\phi} = 0$.

Note that if $\Delta^{\alpha,1}_{M,\psi}$, then by Lemmas 6.2 and 6.5 $\prod_{i=0}^{3} \phi\left(\Delta^{\alpha,i}_{M,\psi}\right)^{(-1)^{i+1}}$ is defined and the first equality in the first part is obvious. Also if $\Delta^{\alpha,1}_{M,\phi} \neq 0$ then by Lemmas 6.2 and 6.3, $\prod_{i=0}^{3} \Delta^{\alpha,i}_{M,\phi}(t)^{(-1)^{i+1}}$ is defined.

Proof. We will only consider the case that M is a closed 3-manifold. The proof for the case that $\partial M \neq \emptyset$ is similar.

Let us prove (1). Write $\pi := \pi_1(M)$. As in the proof of Lemma 6.2 we can find a CW-structure for M such that the chain complex $C_*(\tilde{M})$ of the universal cover is of the form

$$0 \to C_3^1 \xrightarrow{\partial_3} C_2^n \xrightarrow{\partial_2} C_1^n \xrightarrow{\partial_1} C_0^1 \to 0$$

for M, where the C_i are free $\mathbb{Z}[\pi]$-right modules. In fact $C_i^k \cong \mathbb{Z}[\pi]^k$. Let $\varphi : \pi_1(M) \to H$ be an epimorphism to a free abelian group H. Consider the chain complex $C_* \otimes_{\mathbb{Z}[\pi]} \mathbb{F}^k[H]$:

$$0 \to C_3^1 \otimes_{\mathbb{Z}[\pi]} \mathbb{F}^k[H] \xrightarrow{\partial_3 \otimes \mathrm{id}} C_2^n \otimes_{\mathbb{Z}[\pi]} \mathbb{F}^k[H] \xrightarrow{\partial_2 \otimes \mathrm{id}} C_1^n \otimes_{\mathbb{Z}[\pi]} \mathbb{F}^k[H] \xrightarrow{\partial_1 \otimes \mathrm{id}} C_0^1 \otimes_{\mathbb{Z}[\pi]} \mathbb{F}^k[H] \to 0.$$

Let A_i, $i = 0, \dots, 3$, be the matrices with entries in $\mathbb{Z}[\pi]$ corresponding to the boundary maps $\partial_i : C_i \to C_{i-1}$ with respect to the bases given by the lifts of the cells of M to $\tilde{M}$. Then we can arrange the lifts such that

$$\begin{aligned} A_3 &= (1-g_1, 1-g_2, \dots, 1-g_n)^t, \\ A_1 &= (1-h_1, 1-h_2, \dots, 1-h_n), \end{aligned}$$

where $\{g_1, \dots, g_n\}$ and $\{h_1, \dots, h_n\}$ are generating sets for $\pi_1(M)$. Since ϕ is non-trivial there exist k, l such that $\phi(g_k) \neq 0$ and $\phi(h_l) \neq 0$. Let B_3 be the k-th row of A_3. Let B_2 be the result of deleting the k-th column and the l-th row. Let B_1 be the l-th column of A_1.

Note that

$$\det((\alpha \otimes \phi)(B_3)) = \det(\mathrm{id} - (\alpha \otimes \phi)(g_k)) = \det(\mathrm{id} - \phi(g_k)\alpha(g_k)) \neq 0 \in \mathbb{F}[t^{\pm 1}]$$

since $\phi(g_k) \neq 0$. Similarly $\det((\alpha \otimes \phi)(B_1)) \neq 0$ and $\det((\alpha \otimes \psi)(B_i)) \neq 0, i = 1, 3$.

Denote the quotient field of $\mathbb{F}[H]$ by $Q(H)$. Note that $C_* \otimes_{\mathbb{Z}[\pi]} Q(H)$ is acyclic if and only if $\Delta^{\alpha,1}_{M,\varphi} \neq 0$ by Corollary 6.4. We now need the following theorem.

Theorem 6.7. [Tu01, Theorem 2.2, Lemma 2.5 and Theorem 4.7] *Let $\varphi : \pi \to H$ be a homomorphism to a free abelian group. Suppose* $\det((\alpha \otimes \varphi)(B_i)) \neq 0$, $i = 1, 3$.

(1) $C_* \otimes_{\mathbb{Z}[\pi]} Q(H)^k$ *is acyclic* $\Leftrightarrow \det((\alpha \otimes \varphi)(B_2)) \neq 0 \Leftrightarrow \Delta^{\alpha,1}_{M,\varphi} \neq 0$.
(2) *If* $C_* \otimes_{\mathbb{Z}[\pi]} Q(H)^k$ *is acyclic then*

$$\tau(M, \alpha, \varphi) := \prod_{i=1}^{3} \det((\alpha \otimes \varphi)(B_i))^{(-1)^{i+1}} = \prod_{i=0}^{3} \left(\Delta^{\alpha,i}_{M,\varphi}\right)^{(-1)^{i+1}}.$$

Remark. Lifting the cells of M to $\tilde{M}$ makes C_* a based complex. If

$$C_* \otimes_{\mathbb{Z}[\pi]} Q(H)^k := C_* \otimes_{\mathbb{Z}[\pi]} \mathbb{F}^k[H] \otimes_{\mathbb{F}[H]} Q(H)$$

is acyclic, then we can define its Reidemeister torsion, by [Tu01] it equals $\tau(M,\alpha,\varphi)$ in Theorem 6.7. Theorem 6.6 can therefore be summarized as saying that Reidemeister torsion is functorial. We refer to [Tu01] for an excellent introduction to Reidemeister torsion.

By Theorem 6.7 we only need to prove that $C_* \otimes_{\mathbb{Z}[\pi]} Q(F)^k$ and $C_* \otimes_{\mathbb{Z}[\pi]} \mathbb{F}(t)^k$ are acyclic and $\tau(M,\alpha,\phi) = \phi(\tau(M,\alpha,\psi))$. (We define $\phi(f/g) := \phi(f)/\phi(g)$ for $f, g \in \mathbb{F}[F]$.)

Since $\phi\left(\Delta^{\alpha,1}_{M,\psi}\right) \neq 0$ by our assumption, $\Delta^{\alpha,1}_{M,\psi} \neq 0$. Therefore $C_* \otimes_{\mathbb{Z}[\pi]} Q(F)^k$ is acyclic by Corollary 6.4. Since $\det((\alpha\otimes\psi)(B_i)) \neq 0, i = 1,3$, it follows from Theorem 6.7 that $\det((\alpha\otimes\psi)(B_2)) \neq 0$ and

$$\tau(M,\alpha,\psi) = \prod_{i=1}^{3} \det((\alpha\otimes\psi)(B_i))^{(-1)^{i+1}}.$$

Note that

$$\begin{aligned}\prod_{i=1}^{3} \det((\alpha\otimes\phi)(B_i))^{(-1)^{i+1}} &= \prod_{i=1}^{3} \phi\Big(\det((\alpha\otimes\psi)(B_i))\Big)^{(-1)^{i+1}} \\ &= \prod_{i=0}^{3} \phi\left(\Delta^{\alpha,i}_{M,\psi}\right)^{(-1)^{i+1}} \\ &= \phi(\tau(M,\alpha,\psi)).\end{aligned}$$

In the above the second equality follows from Theorem 6.7. Since $\phi(\Delta^{\alpha,1}_{M,\psi}) \neq 0$ and $\det((\alpha\otimes\phi)(B_i)) \neq 0$ for $i = 1,3$, it follows that $\det((\alpha\otimes\phi)(B_2)) \neq 0$. It follows from Theorem 6.7 that $C_* \otimes_{\mathbb{Z}[\pi]} \mathbb{F}(t)^k$ is acyclic and

$$\tau(M,\alpha,\phi) = \prod_{i=1}^{3} \det((\alpha\otimes\phi)(B_i))^{(-1)^{i+1}}.$$

Therefore $\tau(M,\alpha,\phi) = \phi(\tau(M,\alpha,\psi))$.

For the part (2), using similar arguments as above one can easily show that if $\Delta^{\alpha,1}_{M,\phi} \neq 0$ then $\phi\left(\Delta^{\alpha,1}_{M,\psi}\right) \neq 0$.

□

Proof of Theorem 3.4. Clearly Theorem 3.4 follows from Theorem 6.6 and Lemmas 6.2, 6.3 (applied to $\psi : \pi_1(M) \to FH_1(M)$ and $\phi : \pi_1(M) \to \mathbb{Z}$) and from Lemma 6.5. □

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