REIDEMEISTER TORSION, THE THURSTON NORM AND HARVEY'S INVARIANTS

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ABSTRACT. Recently twisted and higher order Alexander polynomials were used by Cochran, Harvey, Friedl–Kim and Turaev to give lower bounds on the Thurston norm. We first show how Reidemeister torsion relates to these Alexander polynomials. We then give lower bounds on the Thurston norm in terms of the Reidemeister torsion which contain and extend all the above lower bounds and give an elegant reformulation of the bounds of Cochran, Harvey and Turaev. The Reidemeister torsion approach also gives a natural approach to proving and extending certain monotonicity results of Cochran and Harvey.

1. Introduction

The following algebraic setup allows us to define twisted non–commutative Alexander polynomials. First let $\mathbb{K}$ be a (skew) field and $\gamma : \mathbb{K} \to \mathbb{K}$ a ring homomorphism. Then denote by $\mathbb{K}_\gamma[t^{\pm 1}]$ the skew Laurent polynomial ring over $\mathbb{K}$. More precisely the elements in $\mathbb{K}_\gamma[t^{\pm 1}]$ are formal sums $\sum_{i=-r}^{s} a_i t^i$ with $a_i \in \mathbb{K}$. Addition is given by addition of the coefficients, and for multiplication one has to apply the rule $t^i a = \gamma(a)^i t^i$ for any $a \in \mathbb{K}$.

Let X be a finite connected CW–complex and let $\phi \in H^1(X;\mathbb{Z})$. We identify henceforth $H^1(X;\mathbb{Z})$ with $\mathrm{Hom}(H_1(X;\mathbb{Z}),\mathbb{Z})$. A representation $\alpha : \pi_1(X) \to \mathrm{GL}(\mathbb{K}_\gamma[t^{\pm 1}], d)$ is called *ϕ–compatible* if for any $g \in \pi_1(X)$ we have $\alpha(g) = At^{\phi(g)}$ for some $A \in \mathrm{GL}(\mathbb{K}, d)$. This generalizes a notion of Turaev [Tu02b].

Given a ϕ–compatible representation α we can consider the $\mathbb{K}_\gamma[t^{\pm 1}]$–modules $H_i^\alpha(X;\mathbb{K}_\gamma[t^{\pm 1}]^d)$ and we define twisted non–commutative Alexander polynomials $\Delta_i^\alpha(t) \in \mathbb{K}_\gamma[t^{\pm 1}]$ (cf. Section 3.1 for details), extending definitions in [Co04] and [KL99].

Furthermore we can consider $H_i^\alpha(X;\mathbb{K}_\gamma(t)^d)$, where $\mathbb{K}_\gamma(t)$ denotes the quotient field of $\mathbb{K}_\gamma[t^{\pm 1}]$. If these homology groups vanish, then we can define the Reidemeister torsion $\tau(M,\phi,\alpha) \in K_1(\mathbb{K}_\gamma(t))/\pm\alpha(\pi_1(X))$ (cf. Section 2.3 for details). The following result generalizes well–known commutative results of Turaev [Tu86] [Tu01]. Note that throughout the paper we will assume that all 3–manifolds are compact, orientable and connected.

Date: August 31, 2005.
2000 *Mathematics Subject Classification.* Primary 57M27; Secondary 57N10.
Key words and phrases. Thurston norm, Reidemeister torsion, 3-manifolds, Knot genus.

Theorem 1.1. *Let M be a 3–manifold with empty or toroidal boundary or let M be a 2–complex with $\chi(M) = 0$. Let $\phi \in H^1(M;\mathbb{Z})$ non–trivial and let $\alpha : \pi_1(M) \to GL(\mathbb{K}_\gamma[t^{\pm 1}], d)$ be a ϕ–compatible representation. Then $\tau(M, \phi, \alpha)$ is defined if and only if $\Delta_1^\alpha(t) \neq 0 \in \mathbb{K}_\gamma[t^{\pm 1}]$. Now assume that $\Delta_1^\alpha(t) \neq 0$. If M is a closed 3–manifold, then*

$$\tau(M, \phi, \alpha) = \Delta_0^\alpha(t)^{-1}\Delta_1^\alpha(t)\Delta_2^\alpha(t)^{-1} \in K_1(\mathbb{K}_\gamma(t))/K_1(\mathbb{K}_\gamma[t^{\pm 1}]).$$

If M has boundary or if M is a 2–complex, then

$$\tau(M, \phi, \alpha) = \Delta_0^\alpha(t)^{-1}\Delta_1^\alpha(t) \in K_1(\mathbb{K}_\gamma(t))/K_1(\mathbb{K}_\gamma[t^{\pm 1}]).$$

This theorem makes it possible to compute $\tau(M, \phi, \alpha)$ in terms of the Alexander polynomials, albeit with a larger indeterminacy, namely $K_1(\mathbb{K}_\gamma[t^{\pm 1}])$ instead of $\pm\alpha(\pi_1(M))$.

We quickly recall the definition of the Thurston norm of a 3–manifold M, we refer to [Th86] for details. Let $\phi \in H^1(M;\mathbb{Z})$. The *Thurston norm* of ϕ is defined as

$$||\phi||_T := \min\{\textstyle\sum_{i=1}^k \max\{-\chi(S_i), 0\} \mid \begin{array}{l} S_1 \cup \dots \cup S_k \subset M \text{ properly embedded,} \\ \text{dual to } \phi, S_i \text{ connected for } i = 1, \dots, k\}. \end{array}$$

As an example consider $X(K) := S^3 \setminus \nu K$, where $K \subset S^3$ is a knot and νK denotes an open tubular neighborhood of K in S^3. Let $\phi \in H^1(X(K);\mathbb{Z})$ be a generator, then $||\phi||_T = 2\,\text{genus}(K) - 1$ (cf. e.g. [FK05]).

We say (M, ϕ) *fibers over* S^1 if the homotopy class of maps $M \to S^1$ induced by $\phi : \pi_1(M) \to H_1(M;\mathbb{Z}) \to \mathbb{Z}$ contains a representative that is a fiber bundle over S^1.

The following theorem gives lower bounds on the Thurston norm using Reidemeister torsion and it gives obstructions to a manifold being fibered. It contains the lower bounds of McMullen [Mc02], Cochran [Co04], Harvey [Ha05], Turaev [Tu02b] and Friedl–Kim [FK05]. It also contains the fibering obstruction of Cha (cf. [Ch03] and [FK05]). To our knowledge this theorem is the strongest of its kind. We refer to Section 3.3 for the (non–trivial) definition of $\deg(\tau(M, \phi, \alpha))$.

Theorem 1.2. *Let M be a 3–manifold with empty or toroidal boundary. Let $\phi \in H^1(M;\mathbb{Z})$ and $\alpha : \pi_1(M) \to GL(\mathbb{K}_\gamma[t^{\pm 1}], d)$ a ϕ–compatible representation. If $\Delta_1^\alpha(t) \neq 0$, then*

$$||\phi||_T \geq \frac{1}{d}\,deg(\tau(M, \phi, \alpha)).$$

Furthermore if (M, ϕ) fibers over S^1, then

$$||\phi||_T = \max\{0, \frac{1}{d}\,deg(\tau(M, \phi, \alpha))\}.$$

Not only does this theorem contain the results in [Co04], [Ha05] and [Tu02b], the formulation in terms of the degrees of Reidemeister torsion also gives a particularly elegant way of putting their results. Reidemeister torsion also has the advantage over Alexander polynomials that it behaves well under ring homomorphisms. We already made use of this in [FK05b] and we will use it below to reprove and extend results of Cochran and Harvey.

The most commonly used skew fields are the quotient fields $\mathbb{K}(G)$ of group rings $\mathbb{Z}[G]$ for certain groups G, we refer to Section 4.1 for details. The following theorem says roughly that 'larger groups give better bounds on the Thurston norm'. We refer to Section 5 or to [Ha05b] for the definition of an admissible triple.

Theorem 1.3. *Let $\alpha : \pi_1(M) \to GL(\mathbb{F}, d)$, $\mathbb{F}$ a commutative field, be a representation and $(\varphi_G : \pi \to G, \varphi_H : \pi \to H, \phi)$ an admissible triple for $\pi_1(M)$, in particular we have an epimorphism $G \to H$. If $\tau(M, \phi, \varphi_H \otimes \alpha)$ is defined, then $\tau(M, \phi, \varphi_G \otimes \alpha)$ is defined. Furthermore*

$$deg(\tau(M, \phi, \varphi_G \otimes \alpha)) \geq deg(\tau(M, \phi, \varphi_H \otimes \alpha)).$$

A similar theorem holds for 2–complexes with Euler characteristic zero. As a special case consider the case that α is the trivial representation. Using Theorem 1.1 we can recover the monotonicity results of [Co04] and [Ha05b]. We hope that our alternative proof using Reidemeister torsion will contribute to the understanding of their results.

The paper is organized as follows. In Section 2 we recall the definition of Reidemeister torsion and we show how to compute it for 3–manifolds and 2–complexes. In Section 3 we introduce the twisted non–commutative Alexander polynomials and show how they relate to the Reidemeister torsion. In Section 4 we give examples of ϕ–compatible representations. In Section 5 we prove Theorem 1.3 and in Section 6 we show that it implies Cochran's and Harvey's monotonicity results. We conclude with a few open questions and suggestions in Section 7.

2. Reidemeister torsion

2.1. **Definition of $K_1(R)$.** For the remainder of the paper we will only consider associative rings R with $1 \neq 0$ with the property that if $r \neq s$, then R^r is not isomorphic to R^s.

For such a ring R define $\mathrm{GL}(R) := \varinjlim \mathrm{GL}(R, n)$, where we have the following maps in the direct system: $\mathrm{GL}(R, n) \to \mathrm{GL}(R, n+1)$ given by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$. We define $K_1(R) = \mathrm{GL}(R)/[\mathrm{GL}(R), \mathrm{GL}(R)]$. In particular $K_1(R)$ is an abelian group. For details we refer to [Mi66] or [Tu01].

Note that for a commutative field $\mathbb{F}$ the determinant induces an isomorphism $K_1(\mathbb{F}) \cong \mathbb{F}^*$. If $\mathbb{K}$ is a skew field, then the Dieudonné determinant gives an isomorphism $K_1(\mathbb{K}) \to \mathbb{K}^*/[\mathbb{K}^*, \mathbb{K}^*]$.

There exists a canonical map $\mathrm{GL}(R, d) \to K_1(R)$ for every d. By abuse of notation we denote the image of $A \in \mathrm{GL}(R, d)$ in $K_1(R)$ by A as well. Often it is useful to view group multiplication as in the following lemma.

Lemma 2.1. *Let $A \in GL(R,k), B \in GL(R,l)$, then*

$$AB = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in K_1(R).$$

Proof. Clearly we can assume that $k = l$. Then

$$AB = \begin{pmatrix} AB & 0 \\ 0 & \mathrm{id} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix} \in K_1(R).$$

Furthermore

$$\begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix} = \begin{pmatrix} \mathrm{id} & B \\ 0 & \mathrm{id} \end{pmatrix} \begin{pmatrix} \mathrm{id} & 0 \\ \mathrm{id} - B^{-1} & \mathrm{id} \end{pmatrix} \begin{pmatrix} \mathrm{id} & -\mathrm{id} \\ 0 & \mathrm{id} \end{pmatrix} \begin{pmatrix} \mathrm{id} & 0 \\ \mathrm{id} - B & \mathrm{id} \end{pmatrix}.$$

But by [Mi66] upper and lower triangular matrices lie in $[\mathrm{GL}(R), \mathrm{GL}(R)]$. □

2.2. **Reidemeister torsion.** Let C_* be an acyclic complex of free R–modules. Pick bases $c_i \subset C_i$. Assume that $B_i := \mathrm{Im}(C_{i+1}) \subset C_i$ is free, pick a basis b_i of B_i and lifts $\tilde{b}_i$ of b_i to C_{i+1}. We write $b_i\tilde{b}_{i-1}$ for the collection of elements given by b_i and $\tilde{b}_{i-1}$. Since C_* is acyclic this is indeed a basis for C_i. Then we define the *Reidemeister torsion* of the based acyclic complex $(C, \{c_i\})$ to be

$$\tau(C, \{c_i\}) := \prod [b_i\tilde{b}_{i-1}/c_i]^{(-1)^{i+1}} \in K_1(R),$$

where $[d/e]$ denotes the matrix of a basis change, i.e. $[d/e] := (a_{ij})$ where $d_i := \sum_j a_{ij}e_j$. It is easy to see that $\tau(C, \{c_i\})$ is independent of the choice of b_i and of the choice of the lifts $\tilde{b}_i$. This is the definition used by Milnor [Mi66] except for a sign change. If the R–modules B_i are not free, then one can show that they are stably free and a stable basis will then make the definition work again (cf. [Mi66, p. 369] or [Tu01, p. 13]).

2.3. **Reidemeister torsion of a CW–complex.** Let X be a CW–complex, by this we will always mean a finite connected CW–complex. Denote the universal cover of X by $\tilde{X}$. We view $C_*(\tilde{X})$ as a right $\mathbb{Z}[\pi_1(X)]$–module via deck transformations.

Let R be a ring. Let $\alpha : \pi_1(X) \to \mathrm{GL}(R,d)$ be a representation, this equips R^d with a left $\mathbb{Z}[\pi_1(X)]$–module structure. We can therefore consider the right R–module chain complex $C_*^\alpha(X, R^d) := C_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} R^d$.

Assume that $H_i^\alpha(X; R^d) := H_i(C_*^\alpha(X; R^d)) = 0$ for all i (note that we will suppress the notation α if the representation is clear from the context). Denote the i–cells of X by $\sigma_i^1, \dots, \sigma_i^{r_i}$ and denote by $e_1, \dots, e_d$ the standard basis of R^d. Pick a lift $\tilde{\sigma}_i^j$ for each cell σ_i^j to the universal cover $\tilde{X}$. We get a basis $c_i := \{\tilde{\sigma}_i^j \otimes e_l\}_{j=1,\dots,r_i, l=1,\dots,d}$ for $C_i^\alpha(X, R^d)$. Then we can define

$$\tau(C_*^\alpha(X, R^d), \{c_i\}) \in K_1(R).$$

This element depends only on the ordering of the cells and on the choice of lifts of the cells to the universal cover. Therefore

$$\tau(X,\alpha) := \tau(C_*^\alpha(X,R^d),\{c_i\}) \in K_1(R)/\pm\alpha(\pi_1(X))$$

is a well–defined invariant of the CW–complex X. Chapman proved the following much deeper result.

Theorem 2.2 (Ch74)**.** *$\tau(X,\alpha) \in K_1(R)/\pm\alpha(\pi_1(X))$ only depends on the homeomorphism type of X.*

In particular this allows us to define $\tau(M,\alpha)$ for a manifold M by picking any CW–structure for M.

2.4. **Computation of Reidemeister torsion for 3–manifolds and 2–complexes.** Let M be a 3–manifold whose boundary consists of a (possibly empty) collection of tori. A duality argument shows that $\chi(M) = \frac{1}{2}\chi(\partial(M)) = 0$.

Choose a triangulation τ of M. Let T be a maximal tree in the 1-skeleton of τ and let T' be a maximal tree in the dual 1-skeleton. We collapse T to form a single 0-cell and join the 3-simplices of T' to form a single 3-cell. Since $\chi(M)=0$ the number n of 1–cells equals the number of 2–cells. Consider the chain complex of the universal cover $\tilde{M}$:

$$0 \to C_3(\tilde{M})^1 \xrightarrow{\partial_3} C_2(\tilde{M})^n \xrightarrow{\partial_2} C_1(\tilde{M})^n \xrightarrow{\partial_1} C_0(\tilde{M})^1 \to 0,$$

where the supscript indicates the rank over $\mathbb{Z}[\pi_1(M)]$.

Picking appropriate lifts of the cells of M to cells of $\tilde{M}$ we get bases $\tilde{\sigma}_i := \{\tilde{\sigma}_i^1,\dots,\tilde{\sigma}_i^{r_i}\}$ for the $\mathbb{Z}[\pi_1(M)]$–modules $C_i(\tilde{M})$, such that if A_i denotes the matrix corresponding to ∂_i, then A_1 and A_3 are of the form

$$\begin{array}{rcll} A_3 &=& (1-g_1,\dots,1-g_n)^t, & g_i \in \pi_1(M) \\ A_1 &=& (1-h_1,\dots,1-h_n), & h_i \in \pi_1(M). \end{array}$$

Clearly $\{h_1,\dots,h_n\}$ is a generating set for $\pi_1(M)$. If M is a closed 3–manifold then $\{g_1,\dots,g_n\}$ is a generating set for $\pi_1(M)$ as well.

Let R be a ring and let $\alpha : \pi_1(M) \to \mathrm{GL}(R,d)$ be a representation. If $A=(a_{ij})$ is an $r \times s$–matrix over $\mathbb{Z}[\pi_1(M)]$ then denote by $\alpha(A)$ the $rk \times sk$–matrix over R obtained by replacing each entry a_{ij} of A by the $d \times d$–matrix $\alpha(a_{ij})$.

Proposition 2.3. *Assume that there exist $k,l \in \{1,\dots,n\}$ such that $\alpha(1-g_k)$ and $\alpha(1-h_l)$ are invertible. Then denote by B_3 the k–th row of A_3, by B_2 the result of deleting the k–th column and the l–row of A_2 and denote by B_1 the l–th column of A_1. If $\tau(M,\alpha) \neq 0$, then $\alpha(B_2)$ is invertible and*

$$\tau(M,\alpha) = \alpha(B_3)^{-1}\alpha(B_2)\alpha(B_1)^{-1} \in K_1(R)/\pm\alpha(\pi_1(M)).$$

Conversely, if $\alpha(B_2)$ is invertible, then $\tau(M,\alpha) \neq 0$,

This proposition is the reason why Reidemeister torsion behaves well under ring homomorphisms.

Proof. Without loss of generality we can assume that $k = 1$ and $l = n$. Denote the standard basis of R^d by $e_1, \dots, e_d$. In the following we write the elements of $C_i := C_i^\alpha(M, R^d)$ as column vectors with respect to the ordered bases

$$c_i := \{\tilde{\sigma}_i^1 \otimes e_1, \dots, \tilde{\sigma}_i^1 \otimes e_d, \dots, \tilde{\sigma}_i^{r_i} \otimes e_1, \dots, \tilde{\sigma}_i^{r_i} \otimes e_d\}.$$

Let b_2 be the columns of $\alpha(A_3)$ viewed as elements in C_2, $b_1 \subset C_1$ the last $d(n-1)$ columns of $\alpha(A_2)$. Furthermore let b_0 be the last d columns of $\alpha(A_1)$, i.e. the columns of $\alpha(1 - h_n)$.

Claim. b_i are bases for $\mathrm{Im}(C_{i+1})$ over R.

The claim is clear for b_2. Since $\alpha(1 - h_n)$ is invertible it follows that b_0 is a basis for $\mathrm{Im}(C_1) = C_0$. We now turn to b_1. Since $\partial_2 \circ \partial_3 = 0$ we have

$$\alpha(A_2)(\mathrm{id} - \alpha(g_1), \dots, \mathrm{id} - \alpha(g_n))^t = 0.$$

But since $\mathrm{id} - \alpha(g_1)$ is invertible this implies that

$$\mathrm{span}\{\text{first } d \text{ columns of } \alpha(A_2)\} \subset \mathrm{span}\{\text{last } d(n-1) \text{ columns of } \alpha(A_2)\}.$$

Hence b_1 generates $\mathrm{Im}(C_2)$. If the vectors in b_1 are linearly dependent, then there exists a non–zero vector w_2 in C_2 such that the first d entries are zero but such that $\partial_2(w_2) = 0$. In particular $w_2 = \partial_3(w_3)$ for some $w_3 \neq 0 \in C_3$. But this is not possible since $\alpha(1 - g_1)$ is invertible.

A similar argument, using the fact that $\partial_1 \circ \partial_2 = 0$ can be used to show that $\alpha(B_2)$ is invertible.

Now let $\tilde{b}_2 = c_3$, let $\tilde{b}_1$ be the last $d(n-1)$ vectors of c_2 and let $\tilde{b}_0$ be the last d vectors of c_1. Clearly $\tilde{b}_i$ are lifts of b_i to C_{i+1}. By definition

$$\tau(M, \alpha) = [\tilde{b}_2/c_3][b_2\tilde{b}_1/c_2]^{-1}[b_1\tilde{b}_0/c_1][b_0/c_0]^{-1}.$$

The equality in the proposition now follows easily from the following equalities in $K_1(R)$.

$$\begin{aligned}
[\tilde{b}_2/c_3] &= (\mathrm{id}),\\
[b_2\tilde{b}_1/c_2] &= \begin{pmatrix} \mathrm{id} - \alpha(g_1) & 0 & \dots & 0 \\ \mathrm{id} - \alpha(g_2) & \mathrm{id} & & 0 \\ \vdots & 0 & \ddots & \vdots \\ \mathrm{id} - \alpha(g_n) & 0 & 0 & \mathrm{id} \end{pmatrix} = \alpha(B_3),\\
[b_1\tilde{b}_0/c_1] &= \begin{pmatrix} 0 & \\ \vdots & \text{(d+1)–st to } dn\text{–th columns of } \alpha(A_2) \\ 0 & \\ \mathrm{id} & \end{pmatrix} = \alpha(B_2),\\
[b_0/c_0] &= (\mathrm{id} - \alpha(h_n)) = \alpha(B_1).
\end{aligned}$$

The last statement of the proposition is easy to show (cf. also [Tu01]). □

Now let X be a finite 2–complex with $\chi(X) = 0$. We can give X a CW–structure with one 0–cell. If n denotes the number n of 1–cells, then $n-1$ equals the number of 2–cells. Now consider the chain complex of the universal cover $\tilde{X}$:

$$0 \to C_2(\tilde{X})^{n-1} \xrightarrow{\partial_2} C_1(\tilde{X})^n \xrightarrow{\partial_1} C_0(\tilde{X})^1 \to 0.$$

As above we pick lifts of the cells of X to cells of $\tilde{X}$ to get bases such that if A_i denotes the matrix corresponding to ∂_i then

$$A_1 = (1 - h_1, \dots, 1 - h_n),$$

where $\{h_1, \dots, h_n\}$ is a generating set for $\pi_1(X)$. The proof of Proposition 2.3 can easily be modified to prove the following.

Proposition 2.4. *Let $\alpha : \pi_1(X) \to GL(R, d)$ be a representation. Assume that there exists $l \in \{1, \dots, n\}$ such that $\alpha(1 - h_l)$ is invertible. Then denote by B_2 the result of deleting the l–row of A_2 and denote by B_1 the l–th column of A_1. If $\tau(X, \alpha) \neq 0$, then $\alpha(B_2)$ is invertible and*

$$\tau(X, \alpha) = \alpha(B_2)\alpha(B_1)^{-1} \in K_1(R)/ \pm \alpha(\pi_1(X)).$$

Conversely, if $\alpha(B_2)$ is invertible, then $\tau(X, \alpha) \neq 0$,

Turaev proved these propositions in the commutative case (cf. e.g. [Tu86] and [Tu01, Theorem 2.2]). Our results can also be generalized to much more general situations along the lines of [Tu01, Theorem 2.2].

Now let M be again a 3–manifold whose boundary consists of a non–empty set of tori. Clearly M is homotopy equivalent to a 2–complex, which is often easier to work with. Since Reidemeister torsion is not a homotopy invariant we have to use a result of Turaev to make this approach work.

Lemma 2.5. [Tu01, p. 56 and Theorem 9.1] *Let M be a 3–manifold with boundary. Then there exists a 2–complex X and a* simple *homotopy equivalence $M \to X$. In particular, if $\alpha : \pi_1(X) \cong \pi_1(M) \to GL(R, d)$ is a representation, then*

$$\tau(M, \alpha) = \tau(X, \alpha) \in K_1(R)/ \pm \alpha(\pi_1(M)).$$

3. Reidemeister torsion and Alexander polynomials

3.1. **Alexander polynomials.** Let X be a CW–complex and let $\phi \in H^1(X; \mathbb{Z})$ non–trivial. For the remainder of this paper let $\mathbb{K}$ be a skew field and let $\mathbb{K}_\gamma[t^{\pm 1}]$ be a skew Laurent polynomial ring. Let $\alpha : \pi_1(X) \to \mathrm{GL}(\mathbb{K}_\gamma[t^{\pm 1}], d)$ be a ϕ–compatible representation.

The $\mathbb{K}_\gamma[t^{\pm 1}]$–modules $H_1(X; \mathbb{K}_\gamma[t^{\pm 1}]^d)$ are called twisted (non–commutative) Alexander modules. Similar modules were studied in [Co04] and [Ha05]. The rings $\mathbb{K}_\gamma[t^{\pm 1}]$

are principal ideal domains (PID) since $\mathbb{K}$ is a skew field (cf. [Co04, Proposition 4.5]). We can therefore decompose

$$H_i^\alpha(X;\mathbb{K}_\gamma[t^{\pm 1}]^d) \cong \mathbb{K}_\gamma[t^{\pm 1}]^f \oplus \bigoplus_{i=1}^{l} \mathbb{K}_\gamma[t^{\pm 1}]/(p_i(t))$$

for some $f \geq 0$ and $p_i(t) \in \mathbb{K}_\gamma[t^{\pm 1}]$ for $i = 1, \ldots, l$. If $f > 0$ then define $\Delta^\alpha_{\phi,i}(t) := 0$. Otherwise define $\Delta^\alpha_{\phi,i}(t) := \prod_{i=1}^l p_i(t) \in \mathbb{K}_\gamma[t^{\pm 1}]$. $\Delta^\alpha_{\phi,i}(t)$ is called the (twisted) *Alexander polynomial* of (X, ϕ, α). Note that $\Delta^\alpha_{\phi,i}(t) \in \mathbb{K}_\gamma[t^{\pm 1}]$ has a high degree of indeterminacy. For example writing the $p_i(t)$ in a different order will give a different Alexander polynomial. We refer to [Co04, p. 367] for a discussion of the indeterminacy of $\Delta^\alpha_{\phi,i}(t)$. We drop the subscript ϕ when ϕ is clear from the context.

In the case of one–dimensional representations we can determine $\Delta^\alpha_0(t)$. We call $\phi \in H^1(X;\mathbb{Z})$ *primitive* if the corresponding map $\phi : H_1(X;\mathbb{Z}) \to \mathbb{Z}$ is surjective.

Lemma 3.1. *Let X be a CW–complex, $\phi \in H^1(X;\mathbb{Z})$ primitive. Let $\alpha : \pi_1(X) \to GL(\mathbb{K}_\gamma[t^{\pm 1}], 1)$ be a ϕ–compatible one–dimensional representation. If $Im(\alpha(\pi_1(X))) \subset \mathbb{K}_\gamma[t^{\pm 1}]$ is cyclic, then $\Delta^\alpha_0(t) = at - 1$ for some $a \in \mathbb{K}$. Otherwise $\Delta^\alpha_0(t) = 1$.*

Proof. This statement follows easily from considering the chain complex for X and from well–known properties of PID's. □

The following lemma follows from combining [Tu02b, Sections 4.3 and 4.4] with [FK05, Lemmas 4.7 and 4.9] (note that the results of [FK05] also hold in the non–commutative setting).

Lemma 3.2. *Let M be a 3–manifold, $\phi \in H^1(M;\mathbb{Z})$ non–trivial. Let $\alpha : \pi_1(X) \to GL(\mathbb{K}_\gamma[t^{\pm 1}], 1)$ be a ϕ–compatible one–dimensional representation. Assume that $\Delta^\alpha_1(t) \neq 0$. If M has boundary, then $\Delta^\alpha_2(t) = 1$, otherwise $\Delta^\alpha_2(t) = \Delta^\alpha_0(t)$.*

A slightly delicate duality argument shows that a similar results holds for unitary representations as well (cf. [FK05]).

3.2. **Proof of Theorem 1.1.** Let X be a CW–complex, $\phi \in H^1(X;\mathbb{Z})$ non–trivial and $\alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{K}_\gamma[t^{\pm 1}], d)$ a ϕ–compatible representation. Theorem 4.1 can easily be extended to show that $\mathbb{K}_\gamma[t^{\pm 1}]$ also has an Ore localization (cf. also [DLMSY03]) which is flat over $\mathbb{K}_\gamma[t^{\pm 1}]$. We denote the quotient field of $\mathbb{K}_\gamma[t^{\pm 1}]$ by $\mathbb{K}_\gamma(t)$.

If $H_i(X;\mathbb{K}_\gamma(t)^d) = 0$ for all i, then we can define $\tau(X, \phi, \alpha) \in K_1(\mathbb{K}_\gamma(t))/\pm \alpha(\pi_1(X))$, otherwise we set $\tau(X, \phi, \alpha) = 0$.

Proof of Theorem 1.1. We first show that $H_*(M;\mathbb{K}_\gamma(t)^d) = 0$ if and only if $\Delta^\alpha_1(t) \neq 0$. Recall that $\mathbb{K}_\gamma(t)$ is flat over $\mathbb{K}_\gamma[t^{\pm 1}]$. It follows that $H_i(M;\mathbb{K}_\gamma(t)^d) = 0$ if and only if $H_i(M;\mathbb{K}_\gamma[t^{\pm 1}]^d)$ is $\mathbb{K}_\gamma[t^{\pm 1}]$–torsion, which is equivalent to $\Delta^\alpha_i(t) \neq 0$.

It is easy to see that $H_i(M;\mathbb{K}_\gamma(t)^d) = 0$ for $i = 0,3$. If $\Delta_1^\alpha(t) \neq 0$, then $H_1(M;\mathbb{K}_\gamma(t)^d) = 0$. Since $\chi(H_i(M;\mathbb{K}_\gamma(t)^d)) = d\chi(M) = 0$ it follows that $H_2(M;\mathbb{K}_\gamma(t)^d) = 0$. We refer to [FK05] for details.

Now assume that $\Delta_1^\alpha(t) \neq 0$. We first consider the case that M is a closed 3–manifold. Let

$$0 \to C_3(\tilde{M})^1 \xrightarrow{\partial_3} C_2(\tilde{M})^n \xrightarrow{\partial_2} C_1(\tilde{M})^n \xrightarrow{\partial_1} C_0(\tilde{M})^1 \to 0$$

as in Section 2.4. Let $C_i := C_i(\tilde{M}) \otimes_{\mathbb{Z}[\pi_1(M)]} \mathbb{K}^d[t^{\pm 1}]$. Since $\partial_3 \otimes \mathrm{id} : C_3 \to C_2$ and $\partial_1 \otimes \mathrm{id} : C_1 \to C_0$ have full rank, and since $\mathbb{K}_\gamma[t^{\pm 1}]$ is a PID we can pick bases c_i over $\mathbb{K}_\gamma[t^{\pm 1}]$ for C_i such that $\partial_3 \otimes \mathrm{id}$ and $\partial_1 \otimes \mathrm{id}$ with respect to these bases is given by

$$A_3 = \begin{pmatrix} B_3 & 0 & \dots & 0 \end{pmatrix}^t, \ A_1 = \begin{pmatrix} 0 & \dots & 0 & B_1 \end{pmatrix},$$

where B_1, B_3 are $d \times d$ matrices over $\mathbb{K}_\gamma[t^{\pm 1}]$ of full rank. Since $\partial_2 \circ \partial_3 = 0$ and $\partial_1 \circ \partial_2 = 0$ we get

$$A_2 = \begin{pmatrix} 0 & & & \\ \vdots & & B_2 & \\ \vdots & & & \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

After another base change over $\mathbb{K}_\gamma[t^{\pm 1}]$ if necessary we can assume that B_1, B_2, B_3 are in fact diagonal. Since we did base changes over $\mathbb{K}_\gamma[t^{\pm 1}]$ it follows that

$$\tau(M,\phi,\alpha) = \tau(C_i, \{c_i\}) \in K_1(\mathbb{K}_\gamma(t))/K_1(\mathbb{K}_\gamma[t^{\pm 1}]).$$

It follows from the argument of Proposition 2.3 that

$$\tau(C_i, \{c_i\}) = B_3^{-1} B_2 B_1^{-1}.$$

It is clear that B_i presents the $\mathbb{K}_\gamma[t^{\pm 1}]$–torsion module $H_i(M;\mathbb{K}_\gamma[t^{\pm 1}]^d)$. The claim now follows immediately from Lemma 2.1.

The case that M has boundary, or that M is a 2–complex, follows from using Proposition 2.4 (together with Lemma 2.5) instead of Proposition 2.3. □

3.3. **Degrees of Alexander polynomials and Reidemeister torsion.** Let $\mathbb{K}_\gamma[t^{\pm 1}]$ be a skew Laurent polynomial ring. Let $f(t) \in \mathbb{K}_\gamma[t^{\pm 1}]$. If $f(t) = 0$ then we write $\deg(f(t)) = \infty$, otherwise, for $f(t) = \sum_{i=m}^n a_i t^i \in \mathbb{K}_\gamma[t^{\pm 1}]$ with $a_m \neq 0, a_n \neq 0$ we define $\deg(f(t)) := n - m$.

By [St75, Proposition I.2.3] and [Co85, p. 48] every right $\mathbb{K}$–module is free and has a well-defined dimension which is additive on short exact sequences. It is easy to see that

$$\deg(f(t)) = \dim_{\mathbb{K}}(\mathbb{K}_\gamma[t^{\pm 1}]/f(t)\mathbb{K}_\gamma[t^{\pm 1}]).$$

Since $\mathbb{K}[t^{\pm 1}]$ is a PID we immediately get the following result from the classification theorem of modules over PID's.

Lemma 3.3. [Co04, p. 368] *Let X be a CW–complex, $\phi \in H^1(X;\mathbb{Z})$ and let $\alpha : \pi_1(X) \to GL(\mathbb{K}_\gamma[t^{\pm 1}], d)$ be a ϕ–compatible representation. If $\Delta_i^\alpha(t) \neq 0$, then*

$$deg\big(\Delta_i^\alpha(t)\big) = \dim_{\mathbb{K}}(H_i(X;\mathbb{K}_\gamma[t^{\pm 1}]^d)).$$

The above lemma says in particular that $\deg\big(\Delta_i^\alpha(t)\big)$ is an invariant of (X,ϕ,α). As we will see, $\Delta_i^\alpha(t)$ is a convenient way to record $\dim_{\mathbb{K}}(H_i(X;\mathbb{K}_\gamma[t^{\pm 1}]^d))$.

Our goal is to extend the definition of degrees of Laurent polynomials to elements in $K_1(\mathbb{K}_\gamma(t))$. First let $B(t)$ be an $r \times r$–matrix over $\mathbb{K}_\gamma[t^{\pm 1}]$. We write $\deg(B(t)) = \infty$ if $B(t)$ is not invertible over $\mathbb{K}_\gamma(t)$. Otherwise $\mathbb{K}_\gamma[t^{\pm 1}]^r/B(t)\mathbb{K}_\gamma[t^{\pm 1}]^r$ is finite dimensional. This can be seen from diagonalizing $B(t)$ over the PID $\mathbb{K}_\gamma[t^{\pm 1}]$. We can therefore define

$$\deg(B(t)) := \dim_{\mathbb{K}}(\mathbb{K}_\gamma[t^{\pm 1}]^r/B(t)\mathbb{K}_\gamma[t^{\pm 1}]^r) \in \mathbb{N}.$$

Theorem 3.4. *There exists a homomorphism*

$$deg : K_1(\mathbb{K}_\gamma(t)) \to \mathbb{Z}$$

which sends an $r \times r$–matrix $B(t)$ over $\mathbb{K}_\gamma[t^{\pm 1}]$, which is invertible over $\mathbb{K}_\gamma(t)$, to $deg(B(t))$.

Together with Theorem 1.1 we get the following corollary.

Corollary 3.5. *Assume that we are in the situation of Theorem 1.1 and assume that $\Delta_\alpha^1(t) \neq 0$. If M is a closed 3–manifold, then*

$$deg(\tau(M,\phi,\alpha)) = deg(\Delta_1^\alpha(t)) - deg(\Delta_2^\alpha(t)) - deg(\Delta_0^\alpha(t)).$$

If M has boundary or if M is a 2–complex, then

$$deg(\tau(M,\phi,\alpha)) = deg(\Delta_1^\alpha(t)) - deg(\Delta_0^\alpha(t)).$$

Proof. The equality follows immediately from noticing that $\deg : K_1(\mathbb{K}_\gamma[t^{\pm 1}]) \to K_1(\mathbb{K}_\gamma(t)) \to \mathbb{Z}$ is the zero map. □

We now turn to the proof of Theorem 3.4. Denote by $\mathbb{H}(\mathbb{K}_\gamma[t^{\pm 1}], \mathbb{K}_\gamma(t))$ the exact category of $\mathbb{K}_\gamma[t^{\pm 1}]$–torsion modules. More precisely, it is the category of all $\mathbb{K}[t^{\pm 1}]$–modules P with a resolution

$$0 \to \mathbb{K}_\gamma[t^{\pm 1}]^r \xrightarrow{d} \mathbb{K}_\gamma[t^{\pm 1}]^r \to P \to 0$$

such that $\mathbb{K}_\gamma[t^{\pm 1}]^r \otimes_{\mathbb{K}_\gamma[t^{\pm 1}]} \mathbb{K}_\gamma(t) \xrightarrow{d \otimes id} \mathbb{K}_\gamma[t^{\pm 1}]^r \otimes_{\mathbb{K}_\gamma[t^{\pm 1}]} \mathbb{K}_\gamma(t)$ is an isomorphism. We now consider the corresponding K_0–group $K_0(\mathbb{H}(\mathbb{K}[t^{\pm 1}], \mathbb{K}(t)))$ (cf. [Ra98] for details). Since the dimension over a skew field $\mathbb{K}$ is additive there exists a well–defined map

$$\deg : K_0(\mathbb{H}(\mathbb{K}[t^{\pm 1}], \mathbb{K}(t))) \to \mathbb{Z}$$

which sends $[P]$, P a $\mathbb{K}[t^{\pm 1}]$–torsion module, to $\dim_{\mathbb{K}}(P)$. Theorem 3.4 now follows immediately from the following theorem.

Theorem 3.6. *There exists a homomorphism*

$$K_1(\mathbb{K}_\gamma(t)) \to K_0(\mathbb{H}(\mathbb{K}_\gamma[t^{\pm 1}], \mathbb{K}_\gamma(t)))$$

which sends an $r \times r$*–matrix* $B(t)$ *over* $\mathbb{K}_\gamma[t^{\pm 1}]$ *to* $\mathbb{K}_\gamma[t^{\pm 1}]^r / B(t)\mathbb{K}_\gamma[t^{\pm 1}]^r$.

Proof. The theorem follows from [Ra98, Proposition 9.8] since $\mathbb{K}_\gamma[t^{\pm 1}] \to \mathbb{K}_\gamma(t)$ is in fact the Cohn localization of $\mathbb{K}_\gamma[t^{\pm 1}]$ with respect to $\Sigma = \{(f) | f \in \mathbb{K}_\gamma[t^{\pm 1}] \setminus \{0\}\}$, as can easily be checked.

□

Remark. Sakasai [Sa05] gives an alternative definition of $\deg : K_1(\mathbb{K}_\gamma(t)) \to \mathbb{Z}$ using the Dieudonné determinant and using the extension of the degree function on $\mathbb{K}_\gamma[t^{\pm 1}]$ to $\mathbb{K}_\gamma(t)$.

3.4. **Proof of Theorem 1.2.**

Proof of Theorem 1.2. Let M be a 3–manifold whose boundary is empty or consists of tori. Let $\phi \in H^1(M;\mathbb{Z})$ be non–trivial, and $\alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{K}_\gamma[t^{\pm 1}], d)$ a ϕ–compatible representation.

Claim.

$$||\phi||_T \geq \frac{1}{d}\left(\dim_{\mathbb{K}}\left(H_1^\alpha(M;\mathbb{K}_\gamma[t^{\pm 1}]^d)\right) - \dim_{\mathbb{K}}\left(H_0^\alpha(M;\mathbb{K}_\gamma[t^{\pm 1}]^d)\right) - \dim_{\mathbb{K}}\left(H_2^\alpha(M;\mathbb{K}_\gamma[t^{\pm 1}]^d)\right)\right).$$

Furthermore this inequality becomes an equality if (M, ϕ) fibers over S^1 and if $M \neq S^1 \times D^2, M \neq S^1 \times S^2$.

First note that if ϕ vanishes on $X \subset M$ then α restricted to $\pi_1(X)$ lies in $\mathrm{GL}(\mathbb{K}, d) \subset \mathrm{GL}(\mathbb{K}_\gamma[t^{\pm 1}], d)$ since α is ϕ–compatible. Therefore $H_i^\alpha(X;\mathbb{K}_\gamma[t^{\pm 1}]^d) \cong H_i^\alpha(X;\mathbb{K}^d) \otimes_{\mathbb{K}} \mathbb{K}_\gamma[t^{\pm 1}]$. The proofs of [FK05, Theorem 3.1] and [FK05, Theorem 6.1] can now easily be translated to this non–commutative setting. This proves the claim.
Combining the results of the claim with Corollary 3.5 and Lemma 3.3 we immediately get a proof for Theorem 1.2. □

Remark. It follows immediately from Lemma 3.1 and 3.2 and the discussion in Section 4 that Theorem 1.2 contains the results of [Mc02], [Co04], [Ha05], [Tu02b] and [FK05].

Remark. Given a 2–complex X Turaev [Tu02a] defined a norm $||-||_X : H^1(X;\mathbf{R}) \to \mathbf{R}$, modelled on the definition of the Thurston norm of a 3–manifold. In [Tu02a] and [Tu02b] Turaev gives lower bounds for the Turaev norm which have the same form as certain lower bounds for the Thurston norm. Going through the proofs in [FK05] it is not hard to see that the obvious version of Theorem 1.2 for 2–complexes also holds.

If M is a 3–manifold with boundary, then it is homotopy equivalent to a 2–complex X. It is not known whether the Thurston norm of M equals the Turaev norm on X, but the fact that Theorem 1.2 holds in both cases suggests that they do in fact agree.

4. Examples for skew fields and ϕ–compatible representations

4.1. **Skew fields of group rings.** A group G is called locally indicable if for every finitely generated subgroup $U \subset G$ there exists a non–trivial homomorphism $U \to \mathbb{Z}$.

Theorem 4.1. *Let G be a locally indicable torsion–free amenable (LITFA for short) group and let R be a subring of $\mathbb{C}$.*

(1) *$R[G]$ is an Ore domain, in particular it embeds in its classical right ring of quotients $\mathbb{K}(G)$.*
(2) *$\mathbb{K}(G)$ is flat over $R[G]$.*

It follows from [Hi40] that $R[G]$ has no zero divisors. The first part now follows from [Ta57] or [DLMSY03, Corollary 6.3]. The second part is a well–known property of Ore localizations (cf. e.g. [Ra98, p. 99]).

A group G is called poly–torsion–free–abelian (PTFA) if there exists a filtration

$$1 = G_0 \subset G_1 \subset \cdots \subset G_{n-1} \subset G_n = G$$

such that G_i/G_{i-1} is torsion free abelian. It is well–known that PTFA groups are amenable and locally indicable (cf. [Lu02, p. 256] and [St74]). The group rings of PTFA groups played an important role in [COT03], [Co04] and [Ha05].

4.2. **Examples for ϕ–compatible representations.** Let M be a 3–manifold and $\phi \in H^1(M;\mathbb{Z})$. We give examples of ϕ–compatible representations.

Let $\mathbb{F}$ be a commutative field. Note that $\phi \in H^1(M;\mathbb{Z}) \cong \mathrm{Hom}(H_1(M;\mathbb{Z}), \langle t\rangle)$ induces a ϕ–compatible representation $\phi : \mathbb{Z}[\pi_1(M)] \to \mathbb{F}[t^{\pm 1}]$. Furthermore if $\beta : \pi_1(M) \to \mathrm{GL}(\mathbb{F}, d)$ is a representation, then $\beta \otimes \phi : \pi_1(M) \to \mathrm{GL}(\mathbb{F}^d \otimes_{\mathbb{F}} \mathbb{F}[t^{\pm 1}]) \cong \mathrm{GL}(\mathbb{F}[t^{\pm 1}], d)$ is clearly ϕ–compatible as well. In this particular case Theorem 1.2 was proved in [FK05].

To describe the ϕ–compatible representations of Cochran [Co04] and Harvey [Ha05][Ha05b] we need the following definition.

Definition. Let π be a group, $\phi : \pi \to \mathbb{Z}$ an epimorphism and $\varphi : \pi \to G$ an epimorphism to a LITFA group G such that there exists a map $\phi_G : G \to \mathbb{Z}$ (which is necessarily unique) such that

$$\begin{tikzcd} \pi \arrow[r, "\varphi"] \arrow[dr, "\phi"'] & G \arrow[d, "\phi_G"] \\ & \mathbb{Z} \end{tikzcd}$$

commutes. Following [Ha05b, Definition 1.4] we call (φ, ϕ) an *admissible pair* for π. If ϕ_G is an isomorphism, then (φ, ϕ) is called *initial*.

Now let $(\varphi : \pi_1(M) \to G, \phi)$ be an admissible pair for $\pi_1(M)$. Consider $G' := \mathrm{Ker}\{\phi : G \to \mathbb{Z}\}$. Clearly G' is still a LITFA group. Pick an element $\mu \in G$ such that

$\phi(\mu) = 1$. Let $\gamma : \mathbb{K}(G') \to \mathbb{K}(G')$ be the homomorphism given by $\gamma(a) := \mu a \mu^{-1}$. Then we get a homomorphism

$$\begin{array}{rcl} G & \to & \mathbb{K}(G')_\gamma[t^{\pm 1}] \\ g & \mapsto & g\mu^{-\phi(g)}t^{\phi(g)}. \end{array}$$

It is clear that $\alpha : \pi_1(M) \to G \to \mathbb{K}(G')_\gamma[t^{\pm 1}]$ is ϕ–compatible. Note that this map depends on the choice of μ. We will nonetheless suppress γ in the notation since different choices of splittings give isomorphic rings. We often make use of the fact that $f(t)g(t)^{-1} \to f(\mu)g(\mu)^{-1}$ defines an isomorphism $\mathbb{K}(G')(t) \to \mathbb{K}(G)$ (cf. [Ha05, Proposition 4.5]). Similarly $\mathbb{Z}[G'][t^{\pm 1}] \xrightarrow{\cong} \mathbb{Z}[G]$.

An important example of admissible pairs is provided by Harvey's rational derived series of a group G (cf. [Ha05, Section 3]). Let $G_r^{(0)} := G$ and define inductively

$$G_r^{(n)} := \left\{ g \in G_r^{(n-1)} | \, g^d \in \left[G_r^{(n-1)}, G_r^{(n-1)}\right] \text{ for some } d \in \mathbb{Z} \setminus \{0\} \right\}.$$

Note that $G_r^{(n-1)}/G_r^{(n)} \cong \left(G_r^{(n-1)}/\left[G_r^{(n-1)}, G_r^{(n-1)}\right]\right)/\mathbb{Z}$–torsion. By [Ha05, Corollary 3.6] the quotients $G/G_r^{(n)}$ are PTFA groups for any G and any n. If $\phi : G \to \mathbb{Z}$ is a homomorphism, then $(G \to G/G_r^{(n)}, \phi)$ is an admissible pair for (G, ϕ) for any $r > 0$.

For example if K is a knot, $G := \pi_1(S^3 \setminus K)$, then it follows from [St74] that $G_r^{(n)} = G^{(n)}$, i.e. the rational derived series equals the ordinary derived series (cf. also [Co04] and [Ha05]).

Remark. For a knot K denote the knot complement by $X(K)$. Let $\pi := \pi_1(X(K))$ and let $\phi \in H^1(X(K); \mathbb{Z})$ primitive. Then

$$\overline{\delta}_n(K) := \dim_{\mathbb{K}(\pi'/(\pi')_r^{(n)})}(H_1(X(K), \mathbb{K}(\pi'/(\pi')_r^{(n)})[t^{\pm 1}])$$

is a knot invariant. Cochran [Co04, p. 395, Question 5] asked whether $\overline{\delta}_n(K)$ is of finite type.

Eisermann [Ei00, Lemma 7] shows that the genus is not a finite type knot invariant. Recall that $\overline{\delta}_n(K) \le 2\,\text{genus}(K)$ (cf. [Co04]), this follows also from Theorem 1.2 together with Corollary 3.5 and Lemmas 3.1 and 3.2. Eisermann's argument can now be used to show that $\overline{\delta}_n(K)$ is not of finite type either.

Remark. In [FK05] we showed that twisted Alexander polynomials over the rings $\mathbb{F}_p[t^{\pm 1}]$, p a prime number, (instead of $\mathbb{Q}[t^{\pm 1}]$) give powerful fibering obstructions. Similarly Alexander polynomials over rings of the form $(\mathbb{Z}[G']/\mathfrak{m})[t^{\pm 1}]$, $\mathfrak{m}$ a maximal ideal, should provide interesting fibering obstructions.

The two types of ϕ–compatible representations given above can be combined as follows. Let $\alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{F}, d)$ be a representation and let $\varphi : \pi_1(M) \to G$ be an admissible homomorphism to a LITFA group G. Denote the Ore localization of $\mathbb{F}[G']$ by $\mathbb{K}(G')$. Then $\varphi \otimes \alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{K}(G')[t^{\pm 1}] \otimes_{\mathbb{F}} \mathbb{F}^d) \cong \mathrm{GL}(\mathbb{K}(G')[t^{\pm 1}]^d) \cong \mathrm{GL}(\mathbb{K}(G')[t^{\pm 1}], d)$ defines a ϕ–compatible representation.

5. Comparing different ϕ–compatible maps

We now recall a definition from [Ha05b].

Definition. Let π be a group and $\phi : \pi \to \mathbb{Z}$. Furthermore let $\varphi_1 : \pi \to G_1$ and $\varphi_2 : \pi \to G_2$ be epimorphisms to LITFA groups G_1 and G_2. We call $(\varphi_1, \varphi_2, \phi)$ an *admissible triple* for π if there exist epimorphisms $\varphi_2^1 : G_1 \to G_2$ (which is not an isomorphism) and $\phi_2 : G_2 \to \mathbb{Z}$ such that $\varphi_2 = \varphi_2^1 \circ \varphi_1$, and $\phi = \phi_2 \circ \varphi_2$.

The situation can be summarized in the following diagram

$$\begin{tikzcd}
& G_1 \arrow[d, "\varphi_2^1"] \\
\pi \arrow[ur, "\varphi_1"] \arrow[r, "\varphi_2"] \arrow[dr, "\phi"'] & G_2 \arrow[d, "\phi_2"] \\
& \mathbb{Z}.
\end{tikzcd}$$

Note that in particular $(\varphi_i, \phi), i = 1, 2$ are admissible pairs for π. Given an admissible triple we can pick splittings $\mathbb{Z} \to G_i$ of $\varphi_i, i = 1, 2$ which make the following diagram commute:

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[dr] & G_1 \arrow[d, "\varphi_2^1"] \\
& G_2.
\end{tikzcd}$$

We therefore get an induced commutative diagram of ring homomorphisms

$$\begin{tikzcd}
\mathbb{Z}[\pi] \arrow[r] \arrow[dr] & \mathbb{Z}[G_1'][t^{\pm 1}] \arrow[d, "\varphi_2^1"] \\
& \mathbb{Z}[G_2'][t^{\pm 1}].
\end{tikzcd}$$

Note that we are suppressing the notation for the twisting in the skew Laurent polynomial rings. Denote the ϕ–compatible maps $\mathbb{Z}[\pi] \to \mathbb{K}(G_i')[t^{\pm 1}], i = 1, 2$ by φ_i as well. For convenience we recall Theorem 1.3.

Theorem 1.3. *Let M be a 3–manifold whose boundary is a (possibly empty) collection of tori or let M be a 2–complex with $\chi(M) = 0$. Let $\alpha : \pi_1(M) \to GL(\mathbb{F}, d)$ be a representation and $(\varphi_1, \varphi_2, \phi)$ an admissible triple for $\pi_1(M)$. If $\tau(M, \phi, \varphi_2 \otimes \alpha) \neq 0$, then $\tau(M, \phi, \varphi_1 \otimes \alpha) \neq 0$. Furthermore*

$$deg(\tau(M, \phi, \varphi_1 \otimes \alpha)) \geq deg(\tau(M, \phi, \varphi_2 \otimes \alpha)).$$

We only treat the case that M is a closed 3–manifold. The other cases follows from a very similar argument using Proposition 2.4 (together with Lemma 2.5) instead of Proposition 2.3.

We write $\alpha_i := \varphi_i \otimes \alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{K}(G_i')[t^{\pm 1}] \otimes \mathbb{F}^d) \to \mathrm{GL}(\mathbb{K}(G_i')[t^{\pm 1}], d)$, $i = 1, 2$ and we write $\varphi := \varphi_2^1$. Now pick a cell decomposition of M as in Section 2.4. Picking appropriate lifts of the cells of M to cells of $\tilde{M}$ we get bases c_i for the $\mathbb{Z}[\pi_1(M)]$–modules $C_*(\tilde{M})$, such that if A_i denotes the matrix corresponding to ∂_i, A_1 and A_3 are of the form

$$\begin{array}{rcll} A_3 & = & (1-g_1, \dots, 1-g_n)^t, & g_i \in \pi_1(M), \\ A_1 & = & (1-h_1, \dots, 1-h_n), & h_i \in \pi_1(M). \end{array}$$

Recall that $\{h_1, \dots, h_n\}$ and $\{g_1, \dots, g_n\}$ are generating sets for $\pi_1(M)$. Therefore there exist $k, l \in \{1, \dots, n\}$ such that $\phi(g_k) \neq 0, \phi(h_l) \neq 0$. Denote by B_3 the k–th row of A_3, by B_2 the result of deleting the k–th column and the l–row of A_2 and by B_1 the l–th column of A_1.

Lemma 5.1.

$$\begin{array}{rcccl} deg(\alpha_1(B_1)) & = & deg(\alpha_2(B_1)) & = & d|\phi(h_l)| \\ deg(\alpha_1(B_3)) & = & deg(\alpha_2(B_3)) & = & d|\phi(g_k)|. \end{array}$$

In particular the matrices $\alpha_i(B_1), \alpha_i(B_3)$ are invertible over $\mathbb{K}(G_i')(t)$ for $i = 1, 2$.

Proof. Recall that $\alpha_i(B_1) = \mathrm{id} - \alpha_i(h_l)$, $\alpha_i(B_3) = \mathrm{id} - \alpha_i(g_k)$ and that $\phi(h_l) \neq 0$, $\phi(g_k) \neq 0$. The lemma now follows immediately from the fact that $\alpha_i, i = 1, 2$ is ϕ–compatible and from the following claim.

Claim. Let $\mathbb{K}_\gamma[t^{\pm 1}]$ be a skew Laurent polynomial ring and let A, B be invertible $d \times d$–matrices over $\mathbb{K}$ and $r \neq 0$. Then $\deg(A + Bt^r) = kr$.

We can clearly assume that $r > 0$. Let $\{e_1, \dots, e_d\}$ be a basis for $\mathbb{K}^d$. Consider the map $p : \mathbb{K}_\gamma[t^{\pm 1}]^d \to P := \mathbb{K}_\gamma[t^{\pm 1}]^d / (A + Bt^r)\mathbb{K}_\gamma[t^{\pm 1}]^d$. We claim that $p(e_i t^j), i \in \{1, \dots, d\}, j \in \{0, \dots, r-1\}$ form a basis for P.

It follows easily from A, B invertible that this is indeed a generating set. Let $v = \sum_{i=n}^m v_i t^i, v_i \in \mathbb{K}^d$ with $v_n \neq 0, v_m \neq 0$. Since A, B are invertible it follows that $(A + Bt^r)v$ has terms with t–exponent n and terms with t–exponent $m + r$. This observation can be used to show that the above vectors are linearly independent in P. □

Now assume that $\tau(M, \phi, \alpha_2) \neq 0$. Then $\alpha_2(B_2)$ is invertible over $\mathbb{K}(G_2')(t)$ by Proposition 2.3 and Lemma 5.1. Note that $\alpha_i(B_2)$ is defined over $\mathbb{Z}[G_i'][t^{\pm 1}] \subset \mathbb{K}(G_i')(t)$. In particular $\alpha_2(B_2) = \varphi(\alpha_1(B_2))$. It follows from the following lemma that $\alpha_1(B_2)$ is invertible as well.

Lemma 5.2. *Let $B(t)$ be an $r \times s$–matrix over $\mathbb{Z}[G_1'][t^{\pm 1}]$. If $\varphi(B(t)) : \mathbb{Z}[G_2]^s \to \mathbb{Z}[G_2]^r$ is invertible (injective) over $\mathbb{K}(G_2')(t)$, then $B(t)$ is invertible (injective) over $\mathbb{K}(G_1')(t)$.*

Proof. Assume that $\varphi(B(t))$ is injective over $\mathbb{K}(G_2')(t)$. Since $\mathbb{Z}[G_2] \to \mathbb{K}(G_2')(t) = \mathbb{K}(G_2)$ is injective it follows that $\varphi(B(t)) : \mathbb{Z}[G_2]^s \to \mathbb{Z}[G_2]^r$ is injective. By Proposition 5.3 the map $B(t) : \mathbb{Z}[G_1]^s \to \mathbb{Z}[G_1]^r$ is injective as well. Since $\mathbb{K}(G_1')(t) = \mathbb{K}(G_1)$ is flat over $\mathbb{Z}[G_1]$ it follows that $B(t) : \mathbb{K}(G_1')(t)^s \to \mathbb{K}(G_1')(t)^r$ is injective.

If $\varphi(B(t))$ is invertible over the skew field $\mathbb{K}(G_2')(t)$, then $r = s$. But an injective homomorphism between vector spaces of the same dimension is in fact an isomorphism. This shows that $B(t)$ is invertible over $\mathbb{K}(G_1')(t)$. □

Proposition 5.3. *If G_1 is locally indicable, and if $\mathbb{Z}[G_1]^s \to \mathbb{Z}[G_1]^r$ is a map such that $\mathbb{Z}[G_1]^s \otimes_{\mathbb{Z}[G_1]} \mathbb{Z}[G_2] \to \mathbb{Z}[G_1]^r \otimes_{\mathbb{Z}[G_1]} \mathbb{Z}[G_2]$ is injective, then $\mathbb{Z}[G_1]^s \to \mathbb{Z}[G_1]^r$ is injective as well.*

Proof. Let $K := \operatorname{Ker}\{\varphi : G_1 \to G_2\}$. Clearly K is again locally indicable. Note that $\mathbb{Z}[G_1]^s \to \mathbb{Z}[G_1]^r$ can also be viewed as a map between free $\mathbb{Z}[K]$–modules. Pick any right inverse $\lambda : G_2 \to G_1$ of φ. It is easy to see that $g \otimes h \mapsto g\lambda(h) \otimes 1, g \in G_1, h \in G_2$ induces an isomorphism

$$\mathbb{Z}[G_1] \otimes_{\mathbb{Z}[G_1]} \mathbb{Z}[G_2] \to \mathbb{Z}[G_1] \otimes_{\mathbb{Z}[K]} \mathbb{Z}.$$

By assumption $\mathbb{Z}[G_1]^s \otimes_{\mathbb{Z}[K]} \mathbb{Z} \to \mathbb{Z}[G_2]^r \otimes_{\mathbb{Z}[K]} \mathbb{Z}$ is injective. Since K is locally indicable it follows immediately from [Ge83] or [HS83] (cf. also [St74] for the case of PTFA groups) that $\mathbb{Z}[G_1]^s \to \mathbb{Z}[G_1]^r$ is injective. □

By Proposition 2.3 we now showed that if $\tau(M, \phi, \alpha_2) \neq 0$, then $\tau(M, \phi, \alpha_1) \neq 0$. Furthermore

$$\deg(\tau(M, \phi, \alpha_i)) = \deg(\alpha_i(B_2)) - \deg(\alpha_i(B_3)) - \deg(\alpha_i(B_1)), i = 1, 2.$$

Theorem 1.3 now follows immediately from Lemma 5.1 and from the following proposition.

Proposition 5.4. *Let $B(t)$ be an $r \times r$–matrix over $\mathbb{Z}[G_1'][t^{\pm 1}]$. If $\varphi(B(t))$ is invertible over $\mathbb{K}(G_2')(t)$, then*

$$deg(B(t)) \geq deg(\varphi(B)(t))$$

Remark.

(1) If $\varphi : R \to S$ is a homomorphism of *commutative* rings, and if $B(t)$ is a matrix over $R[t^{\pm 1}]$, then clearly

$$\begin{array}{rcl} \deg(B(t)) = \deg(\det(B(t))) & \geq & \deg(\varphi(\det(B(t)))) = \deg(\det(\varphi(B(t)))) \\ & = & \deg(\varphi(B(t))). \end{array}$$

Similarly, several other results in this paper, e.g. Theorem 3.4 and Lemma 5.1 are clear in the commutative world, but require more effort in our non-commutative setting.

(2) If

$$(\mathbb{Z}[G_1'], \{f \in \mathbb{Z}[G_1'] | \varphi(f) \neq 0 \in \mathbb{Z}[G_2']\})$$

has the Ore property, then one can give an elementary proof of the proposition by first diagonalizing over $\mathbb{K}(G_2')[t^{\pm 1}]$ and then over $\mathbb{K}(G_1')[t^{\pm 1}]$. Since this is not known to be the case, we have to give a more indirect proof.

The following proof is based on arguments in [Co04] and [Ha05b].

Proof of Proposition 5.4. Let $s := \deg(\varphi(B(t)))$. Pick a map $f : \mathbb{Z}[G_1']^s \to \mathbb{Z}[G_1'][t^{\pm 1}]^r$ such that the induced map

$$\mathbb{K}(G_2')^s \to \mathbb{K}(G_2')[t^{\pm 1}]^r \to \mathbb{K}(G_2')[t^{\pm 1}]^r / \varphi(B(t))\mathbb{K}(G_2')[t^{\pm 1}]^r$$

is an isomorphism. Denote by $0 \to C_1 \xrightarrow{B(t)} C_0 \to 0$ the complex

$$0 \to \mathbb{Z}[G_1'][t^{\pm 1}]^r \xrightarrow{B(t)} \mathbb{Z}[G_1'][t^{\pm 1}]^r \to 0,$$

and denote by $0 \to D_0 \to 0$ the complex with $D_0 = \mathbb{Z}[G_1']^s$. We have a chain map $D_* \to C_*$ given by $f : D_0 \to C_0$. Denote by $\operatorname{Cyl}(D_* \xrightarrow{f} C_*)$ the mapping cylinder of the complexes. We then get a short exact sequence of complexes

$$0 \to D_* \to \operatorname{Cyl}(D_* \xrightarrow{f} C_*) \to \operatorname{Cyl}(D_* \xrightarrow{f} C_*)/D_* \to 0.$$

More explicitly we get the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & C_1 \oplus D_0 \arrow[r, "\begin{pmatrix} \mathrm{id} & \mathrm{id} \end{pmatrix}"] \arrow[d, "\begin{pmatrix} B(t) & -f \\ 0 & \mathrm{id} \end{pmatrix}"] & C_1 \oplus D_0 \arrow[r] \arrow[d, "\begin{pmatrix} B(t) & -f \end{pmatrix}"] & 0 \arrow[d] \\
0 \arrow[r] & D_0 \arrow[r, "\begin{pmatrix} 0 & \mathrm{id} \end{pmatrix}"] & C_0 \oplus D_0 \arrow[r] & C_0 \arrow[r] & 0
\end{tikzcd}$$

Recall that $\operatorname{Cyl}(D_* \xrightarrow{f} C_*)$ and C_* are chain homotopic. Using the definition of f we therefore see that

$$f : H_0(D_*; \mathbb{K}(G_2')) \to H_0(\operatorname{Cyl}(D_* \xrightarrow{f} C_*), \mathbb{K}(G_2'))$$

is an isomorphism. Since $B(t)$ is invertible over $\mathbb{K}(G_2')(t)$ it follows that $H_1(\operatorname{Cyl}(D_* \xrightarrow{f} C_*); \mathbb{K}(G_2')) = 0$. It follows from the long exact homology sequence corresponding to the above short exact sequence of chain complexes that $H_1(\operatorname{Cyl}(D_* \xrightarrow{f} C_*)/D_*; \mathbb{K}(G_2')) = 0$, i.e. $\begin{pmatrix} B(t) & -f \end{pmatrix}$ is injective over $\mathbb{K}(G_2')$. It follows from Lemma 5.2 that $H_1(\operatorname{Cyl}(D_* \xrightarrow{f} C_*)/D_*; \mathbb{K}(G_1')) = 0$ as well. Again looking at the long exact homology sequence we get that

$$f : H_0(D_*; \mathbb{K}(G_1')) \to H_0(\operatorname{Cyl}(D_* \xrightarrow{f} C_*); \mathbb{K}(G_1')) = H_0(C_*; \mathbb{K}(G_1'))$$

is an injection. Hence

$$\begin{array}{rcl} \deg(\varphi(B(t)) = s & = & \dim_{\mathbb{K}(G_2')}(H_0(D_*; \mathbb{K}(G_2'))) \\ & = & \dim_{\mathbb{K}(G_1')}(H_0(D_*; \mathbb{K}(G_1'))) \\ & \leq & \dim_{\mathbb{K}(G_1')}(H_0(C_*; \mathbb{K}(G_1'))) \\ & = & \deg(B(t)). \end{array}$$

$\square$

6. Harvey's monotonicity theorem for groups

Let π be a group and let $(\varphi : \pi \to G, \phi : \pi \to \mathbb{Z})$ be an admissible pair for π. Consider $G' := \text{Ker}\{\phi_G : G \to \mathbb{Z}\}$ and pick a splitting $\mathbb{Z} \to G$ of ϕ_G. As in Section 4.2 we can consider the skew Laurent polynomial ring $\mathbb{K}(G')[t^{\pm 1}]$ together with the ϕ–compatible map $\pi \to \mathbb{K}(G')[t^{\pm 1}]$.

Following [Ha05b, Definition 1.6] we define $\overline{\delta}_G(\phi)$ to be zero if $H_1(\pi, \mathbb{K}(G')[t^{\pm 1}])$ is not $\mathbb{K}(G')[t^{\pm 1}]$–torsion and

$$\overline{\delta}_G(\phi) := \dim_{\mathbb{K}(G')}(H_1(\pi, \mathbb{K}(G')[t^{\pm 1}]))$$

otherwise. We give an alternative proof for the following result of Harvey [Ha05b, Theorem 2.9].

Theorem 6.1. *If $\pi = \pi_1(M)$, M a closed 3–manifold, and if $(\varphi_1 : \pi \to G_1, \varphi_2 : \pi \to G_2, \phi)$ is an admissible triple for π, then*

$$\begin{array}{lcll} \overline{\delta}_{G_1}(\phi) & \geq & \overline{\delta}_{G_2}(\phi), & \textit{if } (\varphi_1, \varphi_2, \phi) \textit{ is not initial}, \\ \overline{\delta}_{G_1}(\phi) & \geq & \overline{\delta}_{G_2}(\phi) - 2, & \textit{otherwise.} \end{array}$$

Proof. We clearly only have to consider the case that $\overline{\delta}_{G_2}(\phi) > 0$. We can build $K(\pi, 1)$ by adding i–handles to M with $i \geq 3$. It therefore follows that for the admissible pairs $(\varphi_i : \pi \to G_i, \phi)$ we have

$$\overline{\delta}_{G_i}(\phi) = \dim_{\mathbb{K}(G_i')}(H_1(M; \mathbb{K}(G_i')[t^{\pm 1}])).$$

We combine this equality with Theorem 1.3, Corollary 3.5 and Lemmas 3.1, 3.2, 3.3. The theorem follows now immediately from the observation that $\text{Im}\{\pi_1(M) \to G_i \to \mathbb{K}(G_i')[t^{\pm 1}]\}$ is cyclic if and only if $\phi : G_i \to \mathbb{Z}$ is an isomorphism. $\square$

This monotonicity result gives in particular an obstruction for a group π to be the fundamental group of a closed 3–manifold. For example Harvey [Ha05b, Example 3.2] shows that as an immediate consequence we get the (well–known) fact that $\mathbb{Z}^m, m \geq 4$ is not a 3–manifold group.

Let π be a finitely presented group of deficiency at least one, for example $\pi = \pi_1(M)$ where M is a 3–manifold with boundary. Using a presentation of deficiency one we can build a 2–complex X with $\chi(X) = 0$ and $\pi_1(X) = \pi$. The same proof as the proof of Theorem 6.1 now gives the following theorem of Harvey [Ha05b, Theorem 2.2]. In the case that $\pi = \pi_1(S^3 \setminus K)$, K a knot, this was first proved by Cochran [Co04].

Theorem 6.2. *If π is a finitely presented group of deficiency one, and if $(\varphi_1, \varphi_2, \phi)$ is an admissible triple for π, then*

$$\begin{array}{lcll} \overline{\delta}_{G_1}(\phi) & \geq & \overline{\delta}_{G_2}(\phi), & \textit{if } (\varphi_1, \varphi_2, \phi) \textit{ is not initial}, \\ \overline{\delta}_{G_1}(\phi) & \geq & \overline{\delta}_{G_2}(\phi) - 1, & \textit{otherwise.} \end{array}$$

7. Questions and conjectures

Let M be a 3–manifold and $\phi \in H^1(M;\mathbb{Z})$. We propose the following two questions.

(1) If $(\varphi : \pi_1(M) \to G, \phi)$ is an admissible pair for $\pi_1(M)$ and if $\alpha : \pi_1(M) \to \mathrm{GL}(\mathbb{F}, d)$ factors through φ, does it follow that

$$\frac{1}{d}\deg(\tau(M,\phi,\alpha)) \le \deg(\tau(M,\phi,\mathbb{Z}[\pi_1(M)] \to \mathbb{K}(G')(t)))?$$

(2) It is well–known that in many cases $\deg(\tau(M,\phi,\mathbb{Z}[\pi_1(M)] \to \mathbb{K}(G')(t))) < ||\phi||_T$ for any admissible pair $(\varphi : \pi_1(M) \to G, \phi)$. For example this is the case if K is a knot with $\Delta_K(t) = 1$ and $M = X(K)$. It is an interesting question whether invariants can be defined for any map $\pi_1(M) \to G$, G a (locally indicable) torsion–free group. For example it might be possible to work with $\mathcal{U}(G)$ the algebra of affiliated operators (cf. e.g. [Re98]) instead of $\mathbb{K}(G)$. If such an extension is possible, then it is a natural question whether the Thurston norm is determined by such more general bounds. This might be too optimistic in the general case, but it could be true in the case of a knot complement.

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