PROJECTIVE PLANES, SEVERI VARIETIES AND SPHERES

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Abstract. A classical result asserts that the complex projective plane modulo complex conjugation is the 4-dimensional sphere. We generalize this result in two directions by considering the projective planes over the normed real division algebras and by considering the complexifications of these four projective planes.

1. Introduction

There is an elementary but very striking result which asserts that the quotient of the complex projective plane CP^2 by complex conjugation is the 4-dimensional sphere. This result has attracted the attention of many geometers over the years, rediscovered afresh each time and with a variety of proofs. For some historical remarks about the origins of this theorem we refer to Arnold [3], where it is put in a more general context.

The purpose of the present paper is to give two parallel generalizations of this theorem. In the first we view CP^2 as the second member of the family of projective planes over the four normed real division algebras. This is close to Arnold's treatment and completes it by dealing with the octonions. The second generalization views CP^2 as the complex algebraic variety obtained by complexifying the real projective plane RP^2 and hence as the first member of the four algebraic varieties got by complexifying the four projective planes. This family of algebraic varieties has appeared in several contexts. First, in relation to Lie groups and the "magic square" of Freudenthal [13], somewhat independently in the characterization by Zak [26] of "Severi varieties", and also in the classification by Nakagawa and Takagi [19] of Kähler submanifolds with parallel second fundamental form in complex projective spaces. Since these varieties are not so widely known we shall give a brief account of them in an appendix. An extensive treatment based on the "magic square" can be found in [15].

Denoting by CP^2, HP^2 and OP^2 the projective planes over the complex numbers C, the quaternions H and the octonions O, and by S^d the sphere of dimension d, our first result may be formulated as follows.

There are natural diffeomorphisms

$$\begin{aligned} CP^2/O(1) &= S^4 \, , \\ HP^2/U(1) &= S^7 \, , \\ OP^2/Sp(1) &= S^{13} \, . \end{aligned} \tag{1.1}$$

Remarks. 1) The first equality is the "folklore" theorem which provides our starting point. The second one is proved in [3] and independently in [5].

2) The maps from the projective planes to the spheres in (1.1) are fibrations outside the "branch locus" given by the preceding projective plane. The sense in which the equations in (1.1) are diffeomorphisms is explained in the next section.

3) The embeddings of the branch loci

$$RP^2 \subset S^4 \ , \ CP^2 \subset S^7 \ , \ HP^2 \subset S^{13} \tag{1.2}$$

are well-known and will be elaborated on in Section 3.

4) We shall later in Section 4 formulate (1.1) more precisely as Theorem A. This will include an explicit and simple construction of the maps from the projective planes to the spheres. These maps will also be compatible with the relevant symmetry group $SO(3)$, $SU(3)$ and $Sp(3)$.

Before formulating our second result we need to introduce the *complexifications* of the four projective planes. The first case is clear, it leads from RP^2 to CP^2. Note that the action of $SO(3)$ on RP^2 extends to a complex action of $SO(3,C)$ on CP^2 and this leaves invariant the complex curve $z_1^2 + z_2^2 + z_3^2 = 0$ which, for reasons that will be clear later, we denote by $CP^2(\infty)$. All this generalizes to the other projective planes. If we denote the projective planes by P_n $(n = 0,1,2,3)$, so that $\dim P_n = 2^{n+1}$, then their complexifications[1] $P_n(C)$ are complex algebraic varieties of complex dimension 2^{n+1}. The isometry group of P_n extends to an action of its complexification on $P_n(C)$ leaving invariant a complex hypersurface $P_n(\infty)$. Moreover $P_n(C)$ has a "real structure", i.e. a complex conjugation. The real points are just P_n, and $P_n(\infty)$ inherits a real structure with no real points. We can now state our second result.

For each $n = 0,1,2,3$ we have a natural map

$$\varphi_n : P_n(C) \to S^{d(n)} \ , \ d(n) = 3 \cdot 2^n + 1 \ , \tag{1.3}$$

which is a fibration outside the branch locus P_n and the hypersurface $P_n(\infty)$. The fibres are the spheres S^0, S^1, S^3, S^7.

Remarks. 1) The case $n = 0$ of (1.3) is just the first case of (1.1).

2) The image of $P_n(\infty)$ under φ_n is the focal set of the branch locus P_n in $S^{d(n)}$ and turns out to be another embedding of P_n in $S^{d(n)}$: it is just the image under the antipodal map of the embedding $\varphi_n(P_n)$. For $n < 3$ this is just the embedding in (1.2), and for $n = 3$ it is the known fourth member of this sequence (as will be explained in Section 4).

3) The varieties $P_n(C)$ are in fact homogeneous spaces of a larger group as we shall explain later. For $n = 1$ the complexification of CP^2 is just the product $CP^2 \times CP^2$, while for $n = 2$ the complexification of HP^2 turns out to be the complex Grassmannian $Gr_2(C^6)$ of 2-planes in C^6, and for $n = 3$ the complexification of OP^2 is the exceptional Hermitian symmetric space $E_6/Spin(10)U(1)$.

[1] $P_n(C)$ should not be confused with the complex projective space of dimension n, for which we use the notation CP^n.

The paper is organized as follows. In Section 2 we review well-known elementary properties of the normed real division algebras and the associated geometries. In particular we explain the general notion of a branched fibration of which (1.3) is an illustration. In Section 3 we move on to projective planes and their automorphisms, where we pay special attention to the Cayley plane and its exceptional status. Section 4 examines the orbit structure for the projective planes and spheres in (1.1) with respect to the relevant symmetry groups. This enables us to establish (1.1). Section 5 is devoted to the formulation and proof of Theorem A, the more precise version of (1.1), while Section 6 deals similiarly with Theorem B, a more explicit version of (1.3). In Section 7 we establish projective versions of the two theorems, where the symmetry group is replaced by the larger group of projectivities of the relevant projective planes. In the appendix we elaborate more on the complexified projective planes $P_n(C)$.

We thank Friedrich Hirzebruch and Jean-Pierre Serre for drawing our attention to some of the relevant literature.

2. Projective Lines, Hopf Maps and Branching

We recall that there are four normed division algebras A_n over the reals, with $\dim A_n = 2^n$, namely

$n = 0$: R, the real field, of dimension 1;
$n = 1$: C, the complex field, of dimension 2;
$n = 2$: H, the non-commutative field of quaternions, of dimension 4;
$n = 3$: O, the non-associative algebra of octonions, of dimension 8.

To each of these we can associate the corresponding projective line A_nP^1 by adding a point ∞ to the algebra. This shows that the projective line is a sphere. The more classical definition of a projective line is to consider all lines (i.e. one-dimensional subspaces) in the 2-dimensional vector space A_n^2 over A_n. For $n = 3$ one has to be a little careful with the definition of a line because of the non-associativity of the octonions. A comprehensive introduction to the octonions can be found in the survey article [7].

Now consider the tautological line bundle over A_nP^1, where the fibre over each point in A_nP^1 is the corresponding line in A_n^2. It can also be obtained by cutting the sphere A_nP^1 into two closed discs and then gluing together the trivial line bundles over these two discs along the boundary by using the multiplication with elements of norm one in the normed division algebra A_n. We denote by $\mathfrak{L}_n$ the dual bundle of the tautological line bundle. The line bundle $\mathfrak{L}_n$ generates the reduced real K-theory of the sphere of dimension 2^n, and the octonionic line bundle $\mathfrak{L}_3$ induces a periodicity between the reduced real K-theories of higher-dimensional spheres, known as Bott periodicity.

Since each fibre of $\mathfrak{L}_n$ is equipped with a norm this line bundle naturally induces a sphere bundle $S(\mathfrak{L}_n)$ over A_nP^1. The manifolds $S(\mathfrak{L}_n)$ are again spheres. The fibrations $S(\mathfrak{L}_n) \to A_nP^1$ are usually called the Hopf fibrations (though for $n = 1$ it goes back originally to W.K. Clifford). In detail they are

$$\begin{aligned} S^1 &\xrightarrow{O(1)} RP^1 = S^1 \ , \\ S^3 &\xrightarrow{U(1)} CP^1 = S^2 \ , \\ S^7 &\xrightarrow{Sp(1)} HP^1 = S^4 \ , \\ S^{15} &\xrightarrow{S^7} OP^1 = S^8 \ . \end{aligned} \tag{2.1}$$

The first three fibrations are principal (i.e. group actions) while the last is not: S^7 is the set of elements of norm one in the octonions but is not a group since O is non-associative. The fact that every division algebra over R has dimension 2^n, $n = 0, 1, 2, 3$, can be proved topologically by showing that there are no further sphere fibrations beyond (2.1). More details about the construction of these fibrations can be found in §20 of [22]. The fibres in (2.1) will be denoted by Γ_n, so that

$$\begin{array}{lllll} \Gamma_0 & = & S^0 & = & O(1) \ , \\ \Gamma_1 & = & S^1 & = & U(1) \ , \\ \Gamma_2 & = & S^3 & = & Sp(1) \ , \\ \Gamma_3 & = & S^7 & . & \end{array} \tag{2.2}$$

We will discuss the Hopf fibrations again in Section 3 in relation with group actions.

We now discuss the notion of "branching" as we shall encounter it in this paper. The classical situation occurs in complex variable theory where one Riemann surface can appear as the branched covering of another. We shall restrict ourselves to the simple case of double coverings, where the local model is the equation $w = z^2$. Although the group of order 2, given by $z \mapsto -z$, has a fixed point at $z = 0$, the quotient is still a smooth surface. The underlying topological reason is that, on the small circles $|z| = \epsilon$ surrounding the fixed point, we get the double covering in the first line of (2.1) so that the quotient is still a circle and hence is the boundary of a small disc.

If we take the product with R^{n-2} we get the more general situation where a group of order 2 acting on an n-dimensional manifold, with fixed-point components all of codimension 2, has a manifold as quotient. Again we refer to the fixed-point set as the branch locus of the double covering.

The purpose of this lengthy analysis of a familiar situation was to point out that each of the equations in (2.1) gives rise to a similar story, except that the finite group $O(1)$ is replaced by a higher-dimensional group (or sphere) Γ_n so that, outside the fixed-point set, we have a fibration. We shall refer to such fibrations as *branched fibrations*.

Consider the case of the second equation in (2.1) involving $U(1)$. The local model here is the action of $U(1)$ on $C^2 = R^4$ via complex scalars. The quotient is R^3 with S^2 being the boundary of the branch point. The geometry of the branched $U(1)$-fibration $R^4 \to R^3$ is fundamental in physics where it describes the geometry of a magnetic monopole. It was a major discovery by Dirac that the quantization of electric charge could be explained (in modern terms) by the $U(1)$-bundle above, over the complement of the point magnetic source at the origin. R^4 in this situation is now referred to as the Kaluza-Klein model of the Dirac monopole.

More generally, in current physical theories where space-time is viewed as having higher dimension than 4, a $U(1)$-action with a fixed manifold of codimension 4 is viewed as providing a charge on the branch locus (which has codimension 3 in the quotient). Examples of such situations were, for example, studied in detail in [5] and provided some of the early motivation for this paper.

In a similar way an $Sp(1)$-action with a fixed-point set of codimension 8 (and the standard action on $H^2 = R^8$) gives a branched fibration carrying an $Sp(1) = SU(2)$-"charge" on the branch locus, which has codimension 5 in the quotient.

Finally the last equation in (2.1) gives a similar story for branched fibrations with fibre S^7 and branch locus having codimension 9 in the quotient. Notice that, in this case, the fibration is not a group action.

In all these cases the quotient manifold has the induced topology but *not* the induced differentiable structure. In other words, it is not true that a differentiable function above, which is invariant under the group action, is a differentiable function below. For example for the double cover $w = z^2$ the function x^2, where $x = \mathrm{Re}(z)$, is not a differentiable function of $\mathrm{Re}(w), \mathrm{Im}(w)$. However, there *is* a natural differentiable structure on the quotient and we shall always use this and refer to it as the quotient structure. Note that, for holomorphic functions the invariants are indeed the functions of w and so the induced holomorphic structure on the quotient agrees with our differentiable quotient structure.

In the examples of branched fibrations which we shall study there will be a further group action in addition to the actions of the type in (2.1). These will be actions of "cohomogeneity one", i.e. an action of a connected Lie group G on a connected smooth manifold M whose generic orbit has codimension one. If G and M are compact such an action has a simple global structure: either there are no exceptional orbits and we have a fibration over the circle, or else there are just two exceptional orbits and the quotient is the closed unit interval, see e.g. [17]. In this second case the exceptional orbits have isotropy groups K_1 and K_2 and the generic orbit has isotropy group $K \subset K_1 \cap K_2$ when we consider the isotropy groups along a suitable path which connects the two exceptional orbits. Moreover, the homogeneous spaces

$$K_1/K = S^p \quad \text{and} \quad K_2/K = S^q$$

must both be spheres, where $p+1$ and $q+1$ are the codimensions of the two exceptional orbits. The normal sphere bundles of these two orbits are the maps

$$G/K \to G/K_1 \quad \text{and} \quad G/K \to G/K_2 \ .$$

Finally the manifold M, with its G-action, is entirely determined by the conjugacy class of the triple of subgroups K_1, K_2 and K.

The prototype example, which will be analyzed carefully in Section 4, is when $M = CP^2$ and $G = SO(3)$ acting with two exceptional orbits of codimension 2, namely RP^2 and S^2 (the conic $z_1^2+z_2^2+z_3^2 = 0$). Here $K_1 = S(O(1)\times O(2)) \cong O(2)$ and $K_2 = SO(2)\times SO(1) \cong$

$SO(2)$ are embedded in $SO(3)$ so that $K = K_1 \cap K_2 = S(O(1) \times O(1)) \times SO(1) \cong Z_2$ and $p = q = 1$.

3. Projective Planes

In addition to the projective lines over the division algebras A_n we can consider higher-dimensional projective spaces. For $n = 0, 1, 2$, when A_n is associative, this gives us the classical projective spaces

$$RP^m \ , \ CP^m \ , \ HP^m \ (m \geq 2) \ .$$

For $n = 3$ however, $A_3 = O$ (the octonions) is not associative. In this case it is possible to construct a projective plane OP^2 (the Cayley plane), but not the projective spaces of higher dimension. In fact the non-associativity of O is related to the *non-Desarguesian* property of OP^2, and it is known (see e.g. [24]) that projective spaces of dimension ≥ 3 must be Desarguesian.

Just as with projective lines there are two ways of defining a projective plane over A_n. The first is to introduce the affine plane in the obvious way as pairs (x, y) of points x and y in A_n and then to compactify this by adding a projective line at infinity. This defines P_n as a manifold and the Hopf fibration appears naturally as the fibration of a spherical neighbourhood of the line at infinity. The fact that two lines meet in one point gets translated in this way to an assertion about the topology of the Hopf fibrations, namely that the Hopf invariant (or linking number of two fibres) is one. This is the fact which is used to show that the dimension of a division algebra over R must be 2^n, $n = 0, 1, 2, 3$: see [2] for a short proof.

This construction of P_n does not exhibit its homogeneity, and this is where an alternative construction is useful. For $n = 0, 1, 2$ the classical approach is to use the 3-dimensional vector space over A_n and to define P_n as the space of one-dimensional subspaces. This gives P_n as a homogeneous space of the relevant classical group

$$SL(3, R) \ , \ SL(3, C) \ , \ SL(3, H) \tag{3.1}$$

or of its compact form

$$SO(3) \ , \ SU(3) \ , \ Sp(3) \ . \tag{3.2}$$

The linear groups consist of projectivities, i.e. transformations preserving lines. The compact groups consist of isometries, where the projective plane is equipped with the Riemannian metric which is induced in the natural way from the Killing form of the group. For CP^2 the full isometry group is the extension of $SU(3)$ by Z_2 of complex conjugation. For the other two cases the isometry group is connected. In all cases the centre acts trivially so that it is really the adjoint group that acts effectively. The isotropy group for the actions of the three groups in (3.2) are

$$O(2) \cong S(O(1) \times O(2)) \ , \ U(2) \cong S(U(1) \times U(2)) \ , \ Sp(1) \times Sp(2) \ .$$

For $n = 3$ we cannot use this approach to construct the Cayley plane as there is no group $SL(3, O)$. However there is a substitute, both for the linear group and for its compact form, which plays the part of the fourth term of the sequences (3.1) and (3.2). For (3.2) we have the exceptional compact Lie group F_4 and for (3.1) we have the non-compact real form E_6^{-26} with character -26 of the exceptional complex Lie group $E_6(C)$. The group of projective transformations of the Cayley plane has been explicitly determined by Freudenthal in [14].

At this point it is easy to discuss the homogeneity of the Hopf fibrations in (2.1). If we fix a point o in P_n then the isotropy group at o of the connected isometry group of P_n acts transitively on the dual projective line o^* in P_n (the set of all antipodal points of o in P_n) and on the metric sphere bundle over o^* in P_n of sufficiently small radius. The projection of this sphere bundle from o onto o^* is just the Hopf fibration associated with A_n. The isotropy group of this action at a point in o^* acts transitively on the corresponding fibre.

The best way to unify all the projective planes P_n, and the associated groups of symmetries, is to introduce Jordan algebras. For a quite self-contained treatment in the octonionic case we refer to [14] and [18], the other cases work analogously and are easier to deal with. We summarize here the basic facts.

For each division algebra A_n we consider the real vector space H_n of 3×3 Hermitian matrices over A_n. Recall that in A_n we have a notion of conjugate $x \mapsto \bar{x}$ which fixes the "real" part and changes the sign of the "imaginary" part. Note that conjugation is an anti-involution of the algebra. Then, as usual, a matrix is Hermitian if its conjugate is equal to its transpose,

$$\bar{x}_{ij} = x_{ji} \quad (i, j = 1, 2, 3) \ .$$

We make H_n into a commutative (but non-associative) algebra by defining a multiplication

$$X \circ Y = \frac{1}{2}(XY + YX) \ , \tag{3.3}$$

where XY and YX denote usual matrix multiplication. Together with this multiplication H_n becomes a real Jordan algebra which we denote by J_n. The unit matrix I acts as an identity.

For $n = 0, 1, 2$ the groups in (3.2) act on H_n by

$$X \mapsto AXA^* \quad (A^* = \bar{A}^t)$$

and preserve the multiplication (3.3). Hence they are automorphisms of the Jordan algebra J_n. It can be shown that modulo their centres they are the full group $\mathrm{Aut}(J_n)$ of automorphisms of J_n, except for $n = 1$ when we get the identity component. For $n = 3$ the automorphism group $\mathrm{Aut}(J_3)$ provides an explicit model of the exceptional compact Lie group F_4.

For $n = 0, 1, 2$ we have a natural embedding of the projective plane P_n in H_n. We just associate to a one-dimensional subspace of the 3-dimensional vector space A_n^3 over A_n the Hermitian 3×3 matrix which represents orthogonal projection onto it. In terms of homogeneous coordinates (x_1, x_2, x_3), normalized so that $\|x_1\|^2 + \|x_2\|^2 + \|x_3\|^2 = 1$, this is given by

$$x \mapsto X = (x_i \bar{x}_j)_{i,j=1,2,3} \ .$$

Note that x and $x\lambda$ with λ in A_n, $\|\lambda\| = 1$, give the same matrix, and that X satisfies[2]

$$\begin{aligned} \operatorname{tr} X &= 1 \ , \\ \|X\|^2 &= 1 \ , \\ \det X &= 0 \ . \end{aligned} \tag{3.4}$$

Clearly the image of P_n in H_n is just the orbit of the diagonal matrix $\mathrm{Diag}(1,0,0)$ under the isometry group[3].

For $n = 3$ it can be shown that the orbit of $\mathrm{Diag}(1,0,0) \in J_3$ under the action of $F_4 = \mathrm{Aut}(J_3)$ provides a model for the Cayley plane P_3. The isotropy group is isomorphic to $Spin(9)$ and hence $OP^2 = P_3 = F_4/Spin(9)$ as a homogeneous space [8]. If we consider, as discussed above, the Hopf fibration $S^{15} \to S^8$ as a projection of a metric sphere bundle over o^* from a point o in $P_3 = OP^2$ onto the dual projective line o^*, and if $Spin(9)$ denotes the isotropy group of F_4 at o, then $Spin(9)$ acts transitively on S^{15} with isotropy group $Spin(7)$ and transitively on S^8 with isotropy group $Spin(8)$. Moreover, $Spin(8)$ acts transitively on the corresponding fibre S^7.

In all four cases J_n has three invariant polynomials of degrees $1, 2, 3$ as in (3.4), and we shall use the same notation. This follows from the fact that every element in J_n can be reduced by $\mathrm{Aut}(J_n)$ to real diagonal form. If the diagonal entries (the "eigenvalues") are $\lambda_1, \lambda_2, \lambda_3$ then the three invariant polynomials are

$$\begin{aligned} \operatorname{tr} &= \lambda_1 + \lambda_2 + \lambda_3 \ , \\ \| \ \|^2 &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \ , \\ \det &= \lambda_1 \lambda_2 \lambda_3 \ . \end{aligned} \tag{3.5}$$

In all cases only the cubic polynomial det is invariant under the group of projectivities of the projective plane P_n. The fact that it is actually given by a polynomial needs to be proved (see Section 7).

4. Orbit Structures

As mentioned at the end of Section 2 the manifolds we are interested in have cohomogeneity one group actions compatible with the maps we need to construct to prove the identifications (1.1). We proceed to spell these out in detail beginning with the basic example of CP^2, which will be a model for the others.

We consider the action of $SO(3)$ on CP^2 via the natural embedding $SO(3) \subset SU(3)$. There are two special orbits, namely RP^2 with isotropy group $K_1 = S(O(1) \times O(2)) \cong O(2)$, and a 2-sphere S^2 (the conic $z_1^2 + z_2^2 + z_3^2 = 0$) with isotropy group $K_2 = SO(2) \times 1 \cong SO(2)$. The intersection

$$K = K_1 \cap K_2 = S(O(1) \times O(1)) \times 1 \cong Z_2 \tag{4.1}$$

consists of the two diagonal matrices $\mathrm{Diag}(\lambda, \lambda, 1)$ with $\lambda = \pm 1$. Each of the homogeneous spaces K_1/K and K_2/K is a circle. The generic orbit $SO(3)/K$ is 3-dimensional and it fibres over each of the two special orbits RP^2 and S^2 with S^1 as fibre.

[2]The definition of the determinant for $n = 2$ (the quaternionic case) requires a little care, see Section 7.
[3]This still works for $P_1 = CP^2$, where the isometry group is disconnected.

One way to establish the identity

$$CP^2/O(1) = S^4$$

of (1.1) is to analyze the $SO(3)$-orbit structure of S^4 and compare it with that of CP^2. Here we can view S^4 as the unit sphere in the vector space of symmetric 3×3 real matrices of trace zero equipped with its usual norm. The orbits are determined by the three real eigenvalues $\lambda_1 \leq \lambda_2 \leq \lambda_3$ with $\lambda_1 + \lambda_2 + \lambda_3 = 0$. There are two special orbits $\lambda_1 = \lambda_2$ and $\lambda_2 = \lambda_3$ each of which is an RP^2, while the generic orbit is the real flag manifold of all full flags in R^3. Thus the three isotropy groups are

$$K_1' = S(O(1) \times O(2)) \ , \ K_2' = S(O(2) \times O(1)) \ , \ K' = K_1' \cap K_2' = S(O(1)^3) \ . \tag{4.2}$$

Since the 4-manifold with its $SO(3)$-orbit structure is determined by the conjugacy class of the triples of isotropy groups, comparison of (4.1) and (4.2) shows that there is a natural map

$$CP^2 \to S^4$$

compatible with the two $SO(3)$-actions. Moreover this map identifies the special orbit RP^2 in CP^2 with one of the two RP^2 in S^4, while outside this we have a double covering, given by the action of $O(1)$ on CP^2 as complex conjugation.

The orbit structures of HP^2 and OP^2 are quite similar as are those of the corresponding spheres. We consider first the action of $SU(3)$ on HP^2 via the embedding $SU(3) \subset Sp(3)$, and on S^7 as the unit sphere in the vector space of Hermitian 3×3 complex matrices of trace zero equipped with its usual norm. For S^7 we find two copies of CP^2 as special orbits and the complex flag manifold of all full flags in C^3 as generic orbit, so that the isotropy groups are

$$K_1' = S(U(1) \times U(2)) \ , \ K_2' = S(U(2) \times U(1)) \ , \ K' = K_1' \cap K_2' = S(U(1)^3) \ . \tag{4.3}$$

For HP^2 the two special orbits are CP^2 and a circle bundle over the dual CP^2 which is a 5-dimensional sphere S^5 (as one sees by using all quaternion lines, i.e. 4-spheres, determined by complex lines). The isotropy groups are

$$K_1 = S(U(1) \times U(2)) \ , \ K_2 = SU(2) \times 1 \ , \ K = K_1 \cap K_2 = S(U(1)^2) \times 1 \ .$$

Comparison with (4.3) shows the existence of a map[4]

$$HP^2 \to S^7$$

compatible with the two $SU(3)$-actions. It identifies the CP^2 in HP^2 with one of the two CP^2 in S^7, and on the complement $HP^2 \setminus CP^2$ it is an S^1-bundle over the complement $S^7 \setminus CP^2$. The $SU(3)$ determines a maximal subgroup $U(3)$ of $Sp(3)$, and the central $U(1)$ in this $U(3)$ acts trivially on the CP^2 and gives the fibres on $HP^2 \setminus CP^2$.

Finally consider the action of $Sp(3)$ on the Cayley plane OP^2 and on S^{13}, the unit sphere in the space of Hermitian 3×3 quaternion matrices with trace zero equipped with its usual

[4] The proof outlined here is essentially that of [3], [5].

norm. Again we have two special orbits in S^{13}, both copies of HP^2, and the generic orbit is the quaternionic flag manifold of all full flags in H^3, so that the isotropy groups are

$$K_1' = Sp(1) \times Sp(2) \ , \ K_2' = Sp(2) \times Sp(1) \ , \ K' = K_1' \cap K_2' = Sp(1)^3 \ . \tag{4.4}$$

For the Cayley plane, HP^2 is clearly a special orbit. The generic orbit is just the normal sphere bundle of HP^2 in OP^2 with fibre S^7. This is the unit sphere in the normal R^8 which is a representation of the isotropy group $K_1 = Sp(1) \times Sp(2)$. Note that this representation is *not* the standard action on H^2 since $Sp(1)$ acts trivially and only $Sp(2)$ acts in the standard manner, so that the generic isotropy group is $K = Sp(1) \times Sp(1) \times 1$. By considering all the Cayley lines (8-spheres) determined by quaternion lines we see that the other special orbit is fibred over the dual HP^2 with 3-sphere fibres and therefore is an 11-dimensional sphere S^{11}. Hence the isotropy groups are

$$K_1 = Sp(1) \times Sp(2) \ , \ K_2 = Sp(2) \times 1 \ , \ K = K_1 \cap K_2 = Sp(1)^2 \times 1 \ .$$

Comparison with (4.4) then shows the existence of a map

$$OP^2 \to S^{13}$$

compatible with the action of $Sp(3)$. HP^2 is the branch locus and outside this we have a 3-sphere fibration. The $Sp(3)$ is contained in a maximal subgroup $Sp(3)Sp(1)$ in F_4, where the $Sp(1)$ centralizes the $Sp(3)$ in F_4, see e.g. [9]. This $Sp(1)$ fixes the HP^2 in OP^2 pointwise, and the orbits through the other points in OP^2 are just the 3-spheres of the fibration. This establishes the identity $OP^2/Sp(1) = S^{13}$ in (1.1).

We have thus established (1.1) from a complete description of the relevant orbit structures. In the next section we shall formulate and prove Theorem A, a more explicit version of (1.1) which does not rely on such a detailed knowledge of the orbit structure, and which provides such an explicit map.

The orbit structure of these group actions on the projective planes have also been studied in [20] in relation to homotopy theory.

5. An Explicit Map

In Section 3 we saw that in all four cases the projective plane P_n has a natural embedding in H_n, the vector space of 3×3 Hermitian matrices over the division algebra A_n. Moreover the formulae (3.4) show that P_n lies in the hyperplane $H_n(1)$ given by $\operatorname{tr} X = 1$ and on the sphere $\|X\|^2 = 1$. Note that the intersection of the hyperplane and the sphere is the sphere of one lower dimension with centre $I/3$ and radius ρ where I is the unit matrix $\operatorname{Diag}(1,1,1)$ and $\rho^2 = 2/3$. Since

$$\dim H_n = 3(2^n + 1)$$

we thus have an embedding

$$P_n \subset S^{d(n)} \ , \ d(n) = 3 \cdot 2^n + 1 \ . \tag{5.1}$$

For $n = 0, 1, 2$ these are the classical embeddings referred to in (1.2) and for $n = 3$ we have the corresponding one for the Cayley plane.

For $n \geq 1$ we have a natural inclusion of algebras

$$A_{n-1} \subset A_n$$

and hence, using the Euclidean metric given by the norm, an orthogonal projection $A_n \to A_{n-1}$ which induces a projection

$$\pi_n : H_n \to H_{n-1} \ .$$

Note that π_n commutes with the trace and hence it maps P_n into the affine hyperplane $H_{n-1}(1)$ in H_{n-1}. In this hyperplane we will choose the point $I/3$ as centre and use the shifted coordinate

$$\tilde{X} = X - I/3$$

for $H_{n-1}(0)$, the linear subspace of H_{n-1} given by $\operatorname{tr} X = 0$. Restricting π_n to $P_n \subset H_n$ we get a map

$$\pi_n : P_n \to H_{n-1}(1)$$

and a shifted map $\tilde{\pi}_n : P_n \to H_{n-1}(0)$. The following lemma will be crucial for our construction.

Lemma 1. *The matrix $I/3$ does not lie in the image $\pi_n(P_n)$, so that $\tilde{\pi}_n(X) \neq 0$ for all $X \in P_n$.*

We postpone the proof till later. Assuming that this lemma is true we can rescale the maps $\tilde{\pi}_n$ to define a map

$$f_n : P_n \to S^{d(n-1)} \tag{5.2}$$

given by

$$f_n(X) = \frac{1}{3}I + \rho \frac{\tilde{\pi}_n(X)}{\|\tilde{\pi}_n(X)\|} \ . \tag{5.3}$$

Thus $f_n(P_n)$ lies in the sphere $S^{d(n-1)}$ of radius ρ in $H_{n-1}(1)$ centred at $I/3$.

The map (5.2) will be the map inducing the diffeomorphism of (1.1), but before formulating Theorem A we shall need a few further properties of f_n. First we note that, when $X \in P_{n-1} \subset P_n$, $\pi_n(X) = X$ and $\|X\| = 1$, so that $\|\tilde{\pi}_n(X)\| = \rho$ and $f_n(X) = X$. Hence, when restricted to P_{n-1}, the map f_n is just the standard embedding (5.1) for $n-1$.

Let us denote by G_n the groups of isometries of P_n which we already discussed above, namely

$$\begin{aligned} G_0 &= SO(3) \ , \\ G_1 &= SU(3) \ , \\ G_2 &= Sp(3) \ , \\ G_3 &= F_4 \ . \end{aligned} \tag{5.4}$$

The space H_n of Hermitian matrices is a representation of G_n, which splits off a trivial factor (corresponding to the trace). When restricted to G_{n-1} ($n = 1, 2, 3$) it decomposes as

$$H_n = H_{n-1} \oplus H_{n-1}^{\perp} \ , \tag{5.5}$$

so that the projection $\pi_n : H_n \to H_{n-1}$ is compatible with the action of G_{n-1}. Thus, assuming Lemma 1, the map f_n of (5.2) also commutes with the action of G_{n-1}.

In addition we have an action of the group Γ_{n-1} of elements of norm one of the division algebra A_{n-1} on H_n which commutes with the action of G_{n-1} and preserves the fibres of π_n. Recall that G_n is, modulo its centre, the automorphism group $\mathrm{Aut}(J_n)$ of the Jordan algebra J_n except for $n = 1$ when $\mathrm{Aut}(J_1)$ has another connected component induced by complex conjugation. Note that this complex conjugation does not extend to an automorphism of J_2 but to an anti-automorphism. Then Γ_{n-1} can be viewed as the centralizer of the connected component of $\mathrm{Aut}(J_{n-1})$ in $\mathrm{Aut}(J_n)$. Explicitly we have

$n = 1$: $\Gamma_0 = O(1) \cong Z_2$ acts by complex conjugation on H_1;

$n = 2$: $\Gamma_1 = U(1)$ acts by conjugation on H_2 with respect to the elements of norm one of $C \subset H$;

$n = 3$: $\Gamma_2 = Sp(1)$. The action of $Sp(1)$ on H_3 cannot be described in a similar fashion because of the non-associativity of the octonions, but the construction of $Sp(1)$ as a subgroup of $\mathrm{Aut}(J_3) = F_4$ is quite simple. Consider a root space decomposition of the Lie algebra of F_4 such that the Lie algebra of $G_2 = Sp(3)$ is determined by the two short simple roots and the adjacent long simple root. Then the maximal root determines the Lie algebra of $\Gamma_2 = Sp(1)$.

Note that Γ_{n-1} acts trivially on the summand H_{n-1} in the decomposition (5.5) and hence trivially on the projective plane $P_{n-1} \subset H_{n-1}$. The action of $\Gamma_0 = O(1)$ on $H_0^\perp \cong R^3$ is just by scalar multiplication by ± 1, the one of $\Gamma_1 = U(1)$ on $H_1^\perp \cong C^3$ is the scalar action $(\lambda, z) \mapsto \lambda^2 z$ ($\lambda \in U(1) \cong S^1 \subset C$, $z \in C^3$), and the one of $\Gamma_2 = Sp(1)$ on $H_2^\perp \cong H^3$ is by right multiplication. The action of Γ_{n-1} on the normal bundle of P_{n-1} in P_n is obtained from the one on $H_{n-1}^\perp$ by suitable restriction.

Finally we are in a position to formulate our promised refinement of (1.1).

Theorem A. *For $n = 1, 2, 3$ the map*

$$f_n : P_n \to S^{d(n-1)}$$

defined by (5.3) induces a diffeomorphism

$$P_n/\Gamma_{n-1} \approx S^{d(n-1)} \ .$$

Moreover, this diffeomorphism is compatible with the natural action on P_n/Γ_{n-1} and $S^{d(n-1)}$ of the group G_{n-1} of (5.4).

Remark. We have already observed that the action of Γ_{n-1} on the normal bundle of P_{n-1} in P_n is just the scalar action of the relevant field (for $n = 2$ it is the square of the scalar action), so that the quotient P_n/Γ_{n-1} is indeed a manifold, branched along P_{n-1} in the sense of Section 2. Outside the branch locus the action of Γ_{n-1} is free (except for $n = 2$ where each orbit is a double covering of the circle, in which case we may consider $\Gamma_1/Z_2 \cong \Gamma_1$ to get a free action).

The case $n = 1$ of Theorem A gives the known diffeomorphism of $P_1 = CP^2$ modulo complex conjugation with S^4. Since this case is basic to the others we shall prove it first.

Consider, as a preliminary, the complex projective line CP^1 viewed as embedded by idempotents in the affine space R^3 of Hermitian 2×2 complex matrices of trace one. The projection onto the real symmetric matrices of trace one is easily seen to be the standard projection of S^2 onto the disc D^2 (with centre $I/2$ and radius σ with $\sigma^2 = 1/2$), which identifies conjugate points. Note that the $O(2)$-orbits on S^2, the "circles of latitude", go into the concentric circles in the disc.

We are now ready to look at CP^2 and the projection π_1. For every RP^1 in RP^2 its complexification is a CP^1 in CP^2 and these fill out CP^2, intersecting only at points of RP^2. Thus $CP^2 \setminus RP^2$ is fibred over the dual RP^2 with fibre $CP^1 \setminus RP^1 = S^2 \setminus S^1$ (two open discs). Alternatively it is fibred over S^2 with fibre an open disc. Moreover, $SO(3)$ acts transitively on RP^2 (or S^2), the base of this fibration, and the isotropy group $S(O(1) \times O(2)) \cong O(2)$ acts on each fibre. Because the projection π_1 is compatible with this action of $SO(3)$ it is entirely determined by its restriction to a single fibre. But such a fibre can be taken to be given by the equation $z_3 = 0$, and so we are reduced to studying the projection of CP^1 which we have just done above. The first implication of this is to establish Lemma 1 for $n = 1$, because the scalar 3×3 matrix $I/3$ cannot lie in the subspace of 2×2 matrices (with zeroes in the third row and column). Moreover, the orbit analysis of the 2-dimensional case shows that the map f_1 sends the $SO(3)$-orbits of CP^2 (modulo conjugation) diffeomorphically onto the $SO(3)$-orbits of S^4, thus proving Theorem A for $n = 1$.

To deal with the cases $n = 2, 3$ of Theorem A, i.e. with HP^2 and OP^2, we shall choose appropriate embeddings of CP^2 in the higher projective planes and then use Theorem A for CP^2. The embeddings we want are not the standard ones given by the original inclusions $C \subset H \subset O$, but are suitable conjugates of these. Thus, for H, we choose the embedding $C \to H$ which takes $i \in C$ into $j \in H$, which gives a second embedding of CP^2 into HP^2, which we will denote by $CP^2(j)$ to distinguish it from the original CP^2. Note that

$$CP^2(j) \cap CP^2 = RP^2 \ . \tag{5.6}$$

Similarly we will choose a third embedding of CP^2 into OP^2 (not coming from HP^2). Consider the sphere S^6 of imaginary elements of norm one in $O = R^8$. This contains the S^2 of imaginary quaternions of norm one. Choose an element $e \in S^6$ of norm one which lies in R^4 orthogonal to $H \subset O$, and embed C in O by sending i to e. Since the exceptional compact Lie group[5] G_2 of automorphisms of the octonions acts transitively on S^6 we can find $g \in G_2$ which takes i into e. Since G_2 acts naturally on the exceptional Jordan algebra J_3 (so that $G_2 \subset F_4$) we get a copy $g(CP^2) \subset OP^2$. Note that

$$g(CP^2) \cap CP^2 = RP^2 \ . \tag{5.7}$$

Because our elements j and e were chosen orthogonal to i it follows that the projections π of HP^2 and OP^2 restrict to the standard projection of $CP^2(j)$ and $g(CP^2)$. Because of (5.6) and (5.7) these copies of CP^2 are transversal to the orbits of the relevant groups $SU(3)$ and $Sp(3)$. More precisely, each orbit of the larger group intersects our CP^2 in just one $SO(3)$-orbit. This follows by examining the corresponding groups. Consider first the

[5]Note that this is not to be confused with the group $G_2 = Sp(3)$ of the sequence in (5.4).

embedding $CP^2(j) \subset HP^2$. This is an orbit of a copy of $SU(3)$ which we may denote by $SU(3)_j$. Clearly this intersects the original $SU(3) \subset Sp(3)$ precisely in $SO(3)$. Similarly $g(CP^2) \subset OP^2$ is an orbit of $g(SU(3))$ and we need to check that

$$SU(3) \cap g(SU(3)) = SO(3) .$$

But this is clear because the intersection must preserve

$$CP^2 \cap g(CP^2) = RP^2 .$$

This correspondence between the $SO(3)$-orbits on CP^2 and the orbits of $SU(3)$ on HP^2 and of $Sp(3)$ on OP^2 shows that Lemma 1 for $n = 2, 3$ follows from the case $n = 1$, which we have already proved. The correspondence also shows that the map f_n induces a diffeomorphism on the one-dimensional space (interval) of orbits. But each orbit in P_n is known and outside the branch locus it is just fibred over the corresponding fibre in $S^{d(n-1)}$. Together with the local behaviour near the branch locus this completes the proof of Theorem A.

6. The Complexified Projective Planes

As mentioned in Section 1 the four projective planes P_n have natural complexifications $P_n(C)$ as complex projective algebraic varieties. These have a "real structure", i.e. an anti-holomorphic involution τ, which has P_n as the real part (fixed by τ). We shall now examine these varieties in greater detail and study their symmetries.

Recall that $P_n \subset H_n$, the space of Hermitian 3×3 matrices over the division algebra A_n. It is the orbit of the diagonal matrix $\mathrm{Diag}(1, 0, 0)$ under the compact Lie group G_n (listed in (5.4)). Note that $P_n \subset H_n(1)$, the affine subspace of matrices of trace one. If we denote by $\mathfrak{P}_n$ the real projective space of the vector space H_n, so that $\dim \mathfrak{P}_n = 3 \cdot 2^n + 2$, then we can identify $\mathfrak{P}_n$ with the projective completion of the real affine space $H_n(1)$ and we have $P_n \subset \mathfrak{P}_n$.

The group G_n of isometries of P_n acts on the affine space $H_n(1)$. This action extends to an action of a larger group $\mathfrak{G}_n$ on the projective space $\mathfrak{P}_n$ which preserves P_n. This induces on P_n its group of projectivities. For $n = 0, 1, 2$ we have

$$\mathfrak{G}_n = SL(3, A_n)$$

as noted in (3.1), while $\mathfrak{G}_3$ is the non-compact real form E_6^{-26} of the exceptional complex Lie group $E_6(C)$. In all cases $\mathfrak{G}_n$ has a natural irreducible representation on the real vector space H_n, which splits off a trivial factor (corresponding to the trace) when restricted to the compact subgroup G_n.

For $n = 0$ the representation of $\mathfrak{G}_0 = SL(3, R)$ in $H_0 = R^6$ is via the symmetric square $S^2(R^3)$, and the embedding $P_0 \subset \mathfrak{P}_0$ is the embedding

$$RP^2 \subset RP^5$$

induced by the diagonal (squaring) map $R^3 \to S^2(R^3)$ given by $x \mapsto x^2$. It is the (real) Veronese embedding, and we shall use the same term for all n.

Note that, if $V = R^3$, then $SL(V)$ has two inequivalent irreducible representations of dimension 6, namely $S^2(V)$ and $S^2(V^*)$, where V^* denotes the dual vector space of V. These become equivalent when restricted to $SO(3)$. A similar story holds for all n, so that these Veronese embeddings come in dual pairs

$$P_n \subset \mathfrak{P}_n \ , \ P_n^* \subset \mathfrak{P}_n^* \ ,$$

where P_n^* is the dual projective plane of P_n, representing its projective lines. Reduction to the compact group gives an identification

$$P_n \to P_n^*$$

by associating to each point p of the plane P_n the “opposite” projective line, which may be defined as the set of all antipodal points q on a closed geodesic through p, the “polar” of p in P_n.

We are now in a position to complexify everything. We get a complex Lie group $\mathfrak{G}_n(C)$ acting on the complex projective space $\mathfrak{P}_n(C)$. Since P_n is an orbit of $\mathfrak{G}_n$ we get as complexification an orbit $P_n(C)$ of $\mathfrak{G}_n(C)$, which defines the complexified projective plane. It is a complex projective manifold with

$$\dim_C P_n(C) = \dim_R P_n = 2^n \ .$$

Since $\mathfrak{P}_n(C)$ has a natural Hermitian metric we can define the maximal compact subgroup $\hat{G}_n$ of $\mathfrak{G}_n(C)$. This preserves the induced Kähler metric on $P_n(C)$ and it clearly contains G_n. Note that $P_n(C)$ equipped with this induced Kähler metric is a Hermitian symmetric space.

Explicitly, the groups $\mathfrak{G}_n(C)$ and $\hat{G}_n$ are given by the following table, where for greater clarity we also list the groups G_n and $\mathfrak{G}_n$:

n	0	1	2	3
$\mathfrak{G}_n(C)$	$SL(3,C)$	$SL(3,C)\times SL(3,C)$	$SL(6,C)$	$E_6(C)$
$\mathfrak{G}_n$	$SL(3,R)$	$SL(3,C)$	$SL(3,H)$	E_6^{-26}
$\hat{G}_n$	$SU(3)$	$SU(3)\times SU(3)$	$SU(6)$	E_6
G_n	$SO(3)$	$SU(3)$	$Sp(3)$	F_4

. (6.1)

The compact complex manifolds $P_n(C)$ are necessarily homogeneous spaces also of the maximal compact subgroup $\hat{G}_n$ of the complex Lie group $\mathfrak{G}_n(C)$. Explicitly we have

$$\begin{aligned} P_0(C) &= SU(3)/S(U(1)\times U(2)) &&= CP^2 \ , \\ P_1(C) &= SU(3)^2/S(U(1)\times U(2))^2 &&= CP^2\times CP^2 \ , \\ P_2(C) &= SU(6)/S(U(2)\times U(4)) &&= Gr_2(C^6) \ , \\ P_3(C) &= E_6/Spin(10)U(1) \ . \end{aligned}$$

Here $Gr_2(C^6)$ denotes the complex Grassmannian of 2-planes in C^6. The identification of the isotropy groups is easy in the classical cases ($n = 0, 1, 2$) and follows from the representation theory of E_6 and F_4 for the last case [1]. The embeddings $P_n(C) \to \mathfrak{P}_n(C)$ are well-known classical embeddings for $n = 0, 1, 2$. For $n = 0$ it is the (complex) Veronese embedding $CP^2 \to CP^5$, for $n = 1$ it is the Segre embedding $CP^2 \times CP^2 \to CP^8$, and for $n = 2$ it is the Plücker embedding $Gr_2(C^6) \to CP^{14}$.

The homogeneous space P_n and $P_n(C)$, together with their isometry groups, are essentially given by the first two rows of Freudenthal's magic square

$\mathfrak{so}(3)$	$\mathfrak{su}(3)$	$\mathfrak{sp}(3)$	$\mathfrak{f}_4$
$\mathfrak{su}(3)$	$\mathfrak{su}(3) \oplus \mathfrak{su}(3)$	$\mathfrak{su}(6)$	$\mathfrak{e}_6$
$\mathfrak{sp}(3)$	$\mathfrak{su}(6)$	$\mathfrak{so}(12)$	$\mathfrak{e}_7$
$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

, (6.2)

see for example [13]. According to Freudenthal, the entries in the first row describe 2-dimensional elliptic geometries, in the second row 2-dimensional projective geometries, in the third row 5-dimensional symplectic geometries, and in the last row metasymplectic geometries.

We now want to look at the action of G_n on $P_n(C)$ and study its orbit structure. We already know that P_n is one orbit, say $P_n = G_n/M_n$ where M_n is given by

$$\begin{array}{lcl} M_0 &=& S(O(1) \times O(2)) \;\cong\; O(2)\ , \\ M_1 &=& S(U(1) \times U(2)) \;\cong\; U(2)\ , \\ M_2 &=& Sp(1) \times Sp(2)\ , \\ M_3 &=& Spin(9)\ . \end{array}$$

Since $P_n(C)$ is the complexification of P_n the normal bundle N_n is isomorphic to the tangent bundle and hence the action of M_n on N_n is just the representation of M_n on the quotient of the Lie algebras

$$L(G_n)/L(M_n)\ . \tag{6.3}$$

But for all these representations M_n is transitive on the unit sphere. It follows that the generic orbit of the action of G_n on $P_n(C)$ has codimension one. Moreover the generic isotropy groups are just the isotropy groups of M_n on the unit sphere in (6.3). Hence the generic orbits are

$$\begin{array}{lcl} n = 0 &:& SO(3)/O(1)\ , \\ n = 1 &:& SU(3)/U(1)\ , \\ n = 2 &:& Sp(3)/Sp(1) \times Sp(1)\ , \\ n = 3 &:& F_4/Spin(7)\ . \end{array}$$

For $n = 3$ this follows from the well-known fact that $S^{15} = Spin(9)/Spin(7)$, where the action of $Spin(9)$ on $S^{15} \subset R^{16}$ is via its irreducible spin representation [8].

In addition to the special orbit P_n in $P_n(C)$ there must be another special orbit. Since G_n acts on the affine space $H_n(1)$ it leaves invariant the hyperplane section of $P_n(C)$ at infinity, which we already denoted in Section 1 by $P_n(\infty)$.

After these preliminaries on the spaces $P_n(C)$ and the groups acting on them we now want to explicitly construct the maps referred to in (1.3). The method is very similar to that we used to construct the maps f_n of Theorem A, the essential point being a projection onto a linear space which misses the origin (Lemma 1) and hence can be normalized to map to the sphere. For P_n our projection was the orthogonal projection

$$\pi_n : H_n \to H_{n-1}$$

restricted to $P_n \subset H_n$. For such a method we need now to replace the projective embedding $P_n(C) \to \mathfrak{P}_n(C)$ by an embedding in a Euclidean space. Fortunately such an embedding exists for every complex projective algebraic variety which is a homogeneous space of a compact Lie group. We just have to pick an appropriate orbit in the Lie algebra. Thus we can find embeddings

$$P_n(C) \to L(\hat{G}_n)$$

compatible with the $\hat{G}_n$-action, where $\hat{G}_n$ is the group of isometries of $P_n(C)$ given by table (6.1).

We can be more precise. Note that $G_n \subset \hat{G}_n$ and that $\hat{G}_n/G_n$ is the compact dual of the non-compact symmetric space $\mathfrak{G}_n/G_n$, since $\hat{G}_n$ and $\mathfrak{G}_n$ are both real forms of the same complex Lie group $\mathfrak{G}_n(C)$. The compact symmetric space $\hat{G}_n/G_n$ is just the space that parametrizes all real P_n in $P_n(C)$, whereas the non-compact symmetric space $\mathfrak{G}_n/G_n$ is the space that parametrizes the standard metrics on P_n (up to overall scale) and so can be identified with the open set

$$H_n^+(1) \subset H_n(1)$$

consisting of matrices which are positive-definite (i.e. all X in $H_n(1)$ for which the eigenvalues $\lambda_1, \lambda_2, \lambda_3$ as in (3.5) are positive). Here we choose the diagonal matrix $I/3$ to define our base metric (base point of $\mathfrak{G}_n/G_n$). It follows that there is a natural isomorphism between the tangent spaces

$$L(\hat{G}_n)/L(G_n) \cong L(\mathfrak{G}_n)/L(G_n) \cong H_n(0) \ , \tag{6.4}$$

where $H_n(0)$ is the vector space obtained from the affine space $H_n(1)$ with $I/3$ as its origin.

P_n is the G_n-orbit of the matrix $\mathrm{Diag}(1,0,0)$ in $H_n(1)$ or equivalently the G_n-orbit of $\mathrm{Diag}(2/3,-1/3,-1/3)$ in $H_n(0)$. From (6.4) we get an orthogonal decomposition

$$L(\hat{G}_n) = L(G_n) \oplus H_n(0) \ . \tag{6.5}$$

The G_n-orbit P_n in $H_n(0)$ then generates the $\hat{G}_n$-orbit $P_n(C)$ in $L(\hat{G}_n)$. The projection

$$\sigma_n : L(\hat{G}_n) \to H_n(0)$$

defined by (6.5) restricts to give a map

$$\sigma_n : P_n(C) \to H_n(0) \ . \tag{6.6}$$

Parallel to Lemma 1 we now have

LEMMA 2. *The image of the map σ_n in (6.6) does not contain* 0.

Assuming for the moment the truth of Lemma 2 we can now define a map

$$\varphi_n : P_n(C) \to S^{d(n)} \ , \ d(n) = 3 \cdot 2^n + 1 \ ,$$

by the normalization

$$\varphi_n(Z) = \rho \frac{\sigma_n(Z)}{\|\sigma_n(Z)\|} \ , \tag{6.7}$$

where ρ with $\rho^2 = 2/3$ is inserted to ensure that, on restriction to $P_n \subset P_n(C)$, the map φ_n coincides with the standard inclusion $P_n \to H_n(0)$.

We are now in a position to formulate the main result of this section, refining (1.3),

Theorem B. *The map $\varphi_n : P_n(C) \to S^{d(n)}$ defined by (6.7) for $n = 0, 1, 2, 3$ is a fibration outside the branch locus P_n and the hypersurface $P_n(\infty)$ which gets mapped to the antipodal P_n. The fibres are the norm one elements Γ_n of A_n (namely the spheres S^0, S^1, S^3, S^7) and φ_n commutes with the action of G_n.*

Remark. The case $n = 0$ of Theorem B coincides with the case $n = 1$ of Theorem A, since $P_0(C) = P_1 = CP^2$ is the complexification of RP^2 and

$$L(\hat{G}_1) \cong \mathfrak{su}(3) \cong iH_1(0) \ .$$

Note that Lemma 2 for $n = 0$ reduces to Lemma 1 for $n = 1$. We can therefore use the case $n = 0$ of Lemma 2 and Theorem B for the other three cases. We will simply use the natural inclusions

$$P_0(C) \subset P_1(C) \subset P_2(C) \subset P_3(C) \tag{6.8}$$

and the corresponding inclusions

$$H_0(0) \subset H_1(0) \subset H_2(0) \subset H_3(0) \ .$$

The maps σ_n and φ_n are compatible with these inclusions. Moreover, for $n = 1, 2, 3$, the G_n-orbits in $P_n(C)$ intersect $P_{n-1}(C)$ in the G_{n-1}-orbits. Thus each inclusion in (6.8) induces naturally a diffeomorphism from the space of G_n-orbits onto the space of G_{n-1}-orbits (which are both closed intervals). Since the property in Lemma 2 of not containing 0 is G_n-invariant it is a property of orbits. Lemma 2 for $n \geq 1$ therefore follows from the case $n = 0$ (since the σ_n-image of any G_n-orbit of $P_n(C)$ contains the corresponding σ_0-image). Moreover, for the same reasons we see that φ_n maps the generic G_n-orbit in $P_n(C)$ onto a generic G_n-orbit in $S^{d(n)}$ in a smooth manner (i.e. by a diffeomorphism of the orbit parameter). The identification of the fibres as Γ_n follows from their explicit description. Finally, the branch locus P_n being preserved by φ_n, the other exceptional orbit $P_n(\infty)$ in $P_n(C)$ must go to the other (dual or antipodal) P_n in $S^{d(n)}$. This completes the proof of Theorem B.

In the appendix we will give more information about the map from $P_n(\infty)$ onto this dual P_n.

Remark. As can be seen Theorems A and B are very similar. However, in one sense Theorem B was easier to prove by induction on n because we could use the natural inclusions (6.8), whereas the natural inclusions of the real projective plane were not transversal to the G_n-action and had to be replaced by a second set of embeddings. On the other hand, Theorem B has an exceptional fibre $P_n(\infty)$ for $n \geq 1$, whereas in Theorem A, f_n is a fibration outside the branch locus. Notice that, in each case, our branch locus P_n is embedded in a manifold of twice its dimension, namely

$$\begin{array}{ll} P_n \subset P_{n+1} & \text{(Theorem A)} \ , \\ P_n \subset P_n(C) & \text{(Theorem B)} \ . \end{array}$$

These coincide only for $n = 0$. For $n \geq 1$ even the normal bundles are different.

The four spheres $S^{d(n)}$ together with their cohomogeneity one actions by G_n appear in several contexts in differential geometry. We discuss briefly two of them.

a) *Isoparametric hypersurfaces.* A real-valued function f on a Riemannian manifold is isoparametric if the first and second differential parameter of f (i.e. $\|\text{grad}\, f\|^2$ and Δf) are constant along the level sets of f. The interest in such functions originated from geometrical optics. Any regular level set of an isoparametric function is called an isoparametric hypersurface. E. Cartan proved that a hypersurface in a space of constant curvature is isoparametric if and only if it has constant principal curvatures. The number of distinct principal curvatures of an isoparametric hypersurface in a sphere is $1, 2, 3, 4$ or 6. It is easy to show that the isoparametric hypersurfaces in spheres with 1 or 2 distinct principal curvatures are the distance spheres or Clifford tori, respectively. In [12] E. Cartan proved that the isoparametric hypersurfaces with 3 distinct principal curvatures exist only in $S^{d(n)}$, $n = 0, 1, 2, 3$, and moreover they are the regular orbits of the cohomogeneity one action of G_n on $S^{d(n)}$. For a survey about this topic see [23].

b) *Positive curvature.* A classical problem is to classify all simply connected closed smooth manifolds which admit a Riemannian metric with positive sectional curvature. The standard examples of such manifolds are the spheres S^m and the projective spaces CP^m, HP^m and OP^2 ($m \geq 2$). Wallach [25] proved that the only simply connected closed smooth manifolds admitting a *homogeneous* Riemannian metric with positive sectional curvature are, apart from the even-dimensional spheres and the above projective spaces, precisely the regular orbits of the cohomogeneity one actions of G_n on $S^{d(n)}$, $n = 1, 2, 3$. Explicitly these are the flag manifolds $SU(3)/U(1)^2$, $Sp(3)/Sp(1)^3$ and $F_4/Spin(8)$ of all full flags in CP^2, HP^2 and OP^2. However, we should point out that the positive curvature metrics on these flag manifolds are not the induced metrics from the spheres with their standard metrics.

7. The Projective Version

So far we have concentrated on constructing maps compatible with the compact isometry groups G_n of the projective planes P_n. Now we shall extend these results appropriately to the non-compact groups $\mathfrak{G}_n$ of projectivities of P_n.

In Section 6 we saw that the embeddings $P_n \subset H_n(1)$, which are compatible with the action of G_n, extend naturally to embeddings

$$P_n \subset \mathfrak{P}_n$$

compatible with the action of $\mathfrak{G}_n$. Here $\mathfrak{P}_n$ is the real projective space associated with H_n, and so is the projective completion of the affine space $H_n(1)$. The vector space H_n is an irreducible representation of $\mathfrak{G}_n$ which splits off a one-dimensional trivial factor (corresponding to the trace) on restriction to G_n, so that $\mathfrak{G}_n$ acts on $\mathfrak{P}_n$ with G_n preserving the hyperplane at infinity and the central point given by the scalar multiples of $I/3$.

The projection $\pi_n : H_n \to H_{n-1}$ induces a corresponding projection

$$\pi_n : \mathfrak{P}_n \setminus \mathfrak{P}(H_{n-1}^\perp) \to \mathfrak{P}_{n-1} \tag{7.1}$$

which is defined in the complement of the "axis" of projection arising from $H_{n-1}^\perp$. The map (7.1) is compatible with the action of $\mathfrak{G}_{n-1}$. Note that P_n is contained in $H_n(1)$ and so does not intersect the axis of the projection (7.1). Hence we get a well-defined map

$$\pi_n : P_n \to \mathfrak{P}_{n-1} \tag{7.2}$$

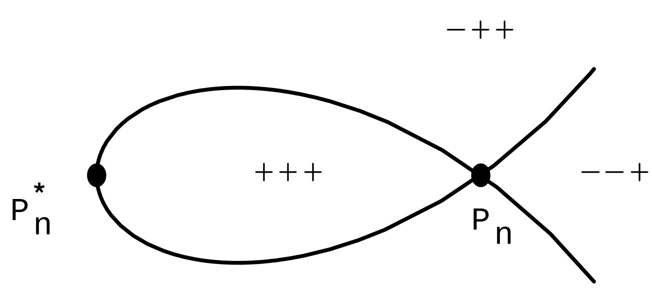
compatible with the action of $\mathfrak{G}_{n-1}$. We propose to examine (7.2) with respect to the orbits of $\mathfrak{G}_{n-1}$.

We begin by looking at the action of $\mathfrak{G}_n$ on $\mathfrak{P}_n$ (and we shall then replace n by $n-1$). We recall that we have a cubic polynomial det in H_n whose vanishing defines a hypersurface $Z_n \subset \mathfrak{P}_n$. In the appendix which follows, the complexification $Z_n(C) \subset \mathfrak{P}_n(C)$ is discussed in detail. The group $\mathfrak{G}_n$ leaves Z_n invariant: this is clear for $n = 0, 1$, requires a little verification for $n = 2$, and is a classical result of E. Cartan [11] when $n = 3$ and $\mathfrak{G}_3 = E_6^{-26}$ is a real form of $E_6(C)$. Moreover Z_n contains P_n as a $\mathfrak{G}_n$-orbit and P_n is the singular locus of Z_n.

The group G_n acts on the affine part $H_n(1)$ of $\mathfrak{P}_n$ with $I/3$ as fixed point and its orbits are parametrized by three real eigenvalues

$$\lambda_1 \le \lambda_2 \le \lambda_3 \quad , \quad \lambda_1 + \lambda_2 + \lambda_3 = 1 \ .$$

Z_n is given by $\lambda_1\lambda_2\lambda_3 = 0$, while P_n is given by $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 1$. We can indicate the affine part of Z_n schematically by the following picture (which is actually what a 2-dimensional (affine) slice would look like: a real cubic curve):



where the complement of Z_n in $H_n(1)$ is divided into regions, depending on the signs of the three eigenvalues as indicated. We shall focus attention on the bounded region of positive-definite matrices, which we denote by Λ_n^+, and its boundary which will be denoted by Σ_n. Notice that Σ_n is a semi-algebraic set (given by polynomial equations and inequalities), since it is only part of the real algebraic variety Z_n.

Note that Σ_n contains not only P_n, as a singular locus, but also another copy P_n^* given by

$$\lambda_1 = 0 \ , \ \lambda_2 = \lambda_3 = \frac{1}{2} \ .$$

Unlike P_n the copy P_n^* consists of smooth points of Z_n.

Our first key lemma is

LEMMA 3. *The set Λ_n^+ is convex and its boundary Σ_n is homeomorphic to a sphere of dimension $d(n) = 3 \cdot 2^n + 1$.*

The convexity follows from the fact that Z_n is a *cubic* hypersurface so that any line meets it in at most three points. Thus a chord of Σ_n cannot exit and then reenter Λ_n^+. Projection from any interior point, say $I/3$, then gives the required homeomorphism with the sphere.

REMARK. For the classical cases $n = 0, 1, 2$ this lemma is directly evident from the properties of eigenvalues of Hermitian matrices: in fact, as we see, it also holds for the octonionic case $n = 3$. The classical cases were used by Arnold [3] to establish Theorem A for $n = 1, 2$. He also had results for larger matrices. By contrast we stick to 3×3 matrices but handle also the Cayley case.

Our next lemma describes the $\mathfrak{G}_n$-orbit structure of $\mathfrak{P}_n$.

LEMMA 4. *The $\mathfrak{G}_n$-orbits on $\mathfrak{P}_n$ are as follows:*

(1) *Two open orbits, given by matrices in Λ_n^+ and by matrices with $\lambda_1 < 0 < \lambda_2 \leq \lambda_3$;*
(2) *Two orbits of codimension one, namely $\Sigma_n \setminus P_n$ (which is given by $\lambda_1 = 0 < \lambda_2 \leq \lambda_3$) and $Z_n \setminus \Sigma_n$ (which is given by $\lambda_1 < 0 = \lambda_2 < \lambda_3$);*
(3) *One orbit of codimension $2^n + 2$, namely P_n, given by $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = 1$.*

Proof. This follows from the fact that the $\mathfrak{G}_n$-orbits on H_n are characterized by the rank of the matrix and the sign of the eigenvalues (in the projective space $\mathfrak{P}_n$ a matrix X is also equivalent to $-X$).

Finally we shall need to know the $\mathfrak{G}_{n-1}$-orbits on P_n:

LEMMA 5. *The action of $\mathfrak{G}_{n-1}$ on P_n has two orbits, namely P_{n-1} and its complement.*

Proof. First we prove this for $n = 1$. We have to show that $SL(3, R)$ acts transitively on $CP^2 \setminus RP^2$. Let ξ be in this open set. Then $\xi \neq \bar{\xi}$, so we have a unique CP^1 joining ξ and $\bar{\xi}$. This CP^1 is the complexification of a suitable line RP^1 in RP^2. The subgroup of $SL(3, R)$ that leaves this RP^1 invariant induces on CP^1 an action of $GL(2, R)$. Now $SL(2, R)$ acting on $CP^1 = S^2$ acts transitively on each hemisphere (the model of the hyperbolic plane), and an element of determinant -1 switches the two hemispheres. Together with the fact that $SL(3,R)$ is transitive on lines in RP^2 this establishes the lemma for $n=1$. To prove it for $n=2,3$ we use the second embeddings introduced in Section 5,

$$CP^2(j) \subset HP^2 \ \text{ and } \ g(CP^2) \subset OP^2 \ ,$$

which according to (5.6) and (5.7) intersect the original CP^2 in RP^2. We recall that the orbits of the compact groups $Sp(3)$ and F_4 cut out on $CP^2(j)$ and $g(CP^2)$ the orbits of $SO(3)$. Since we have shown that $SL(3,R)$ acts transitively on $CP^2 \setminus RP^2$, and since

$$SL(3,R) = \mathfrak{G}_0 \subset \mathfrak{G}_1 \subset \mathfrak{G}_2 \ ,$$

Lemma 5 follows for $n=2,3$.

By definition the map $\pi_n : P_n \to \mathfrak{P}_{n-1}$ of (7.2) is the identity on P_{n-1}, and from Lemma 4 and Lemma 5 we see that π_n must map P_n into Σ_{n-1}, since this is the only compact union of two $\mathfrak{G}_{n-1}$-orbits which lies in the affine part $H_{n-1}(1)$ of $\mathfrak{P}_{n-1}$. It is easy to check that we cannot have $\pi_n(P_n) = P_{n-1}$: it is enough to check this for $n=1$, when it cannot happen for dimension reasons ($\pi_1(P_1)$ being of dimension 4). Thus we deduce the following projective refinement of Theorem A:

Theorem A′ *The map $\pi_n : P_n \to \mathfrak{P}_{n-1}$ has image Σ_{n-1} and $P_n \setminus P_{n-1} \to \Sigma_{n-1} \setminus P_{n-1}$ is a homogeneous fibration for the group $\mathfrak{G}_{n-1}$.*

Remarks. 1) Note that Σ_{n-1} does not have a natural smooth structure compatible with the action of $\mathfrak{G}_{n-1}$, since P_{n-1} is a singular locus. However, if we restrict to the compact subgroup $G_{n-1} \subset \mathfrak{G}_{n-1}$, which fixes $I/3$, then projection from $I/3$ maps Σ_{n-1} to the sphere $S^{d(n-1)}$ (as shown by Lemma 3) and now the map

$$P_n \to S^{d(n-1)}$$

identifies the smooth structure of $S^{d(n-1)}$ with the quotient smooth structure of P_n as explained in Section 2. Thus Theorem A follows from Theorem A′ and Lemma 3. The smoothing of Σ_{n-1} by the radial projection effectively "rounds off the corners" as exemplified by the projection of a square from its centre onto a circle.

2) By considering the projective lines of P_n we get the following more precise picture of the geometry of Z_n in relation to P_n. Consider any line in P_n. This is a sphere of dimension 2^n embedded in the standard way as a real quadric in $\mathfrak{P}_n$ (lying in a linear subspace of dimension 2^n+1). Take its interior, an open ball, and its closure. This lies inside Σ_n, and Σ_n is filled up by the union of the closed balls. If we fix a standard metric on P_n, and hence a compact subgroup $G_n \subset \mathfrak{G}_n$, then the sphere becomes a round sphere in $H_n(1)$ and its centre lies on P_n^*, which is the locus of such centres under the action of G_n. This shows that P_n^* is the closest part of Σ_n to the centre $I/3$. For a fixed line in P_n the geometry of its interior is just hyperbolic geometry and the subgroup of $\mathfrak{G}_n$ preserving the line is the conformal group of the sphere or the isometry group of its interior. The corresponding complex picture is described in the appendix.

3) Since π_n is a linear projection it takes the interiors of real quadrics to the interiors of their projections. Remark 2 then shows that π_n maps $\Sigma_n \setminus P_n$ into the interior of Σ_{n-1}.

Finally, since this latter is an open orbit of $\mathfrak{G}_{n-1}$, it follows that we get the whole of the interior.

We shall now consider to what extent there is a Theorem B′ analogous to Theorem A′. It is easy to see that there can be no version compatible with $\mathfrak{G}_{n-1}$, since this moves the projective plane P^*_{n-1} while, for Theorem B (for $n > 0$), P^*_{n-1} is a distinguished subspace of the sphere, being the image of the exceptional fibre. However the first part of Theorem A′ extends to give

Theorem B′. *The image of the map $\sigma_n : P_n(C) \to H_n(0)$ is Σ_n.*

Proof. For $n = 0$ we have $\sigma_0 = \pi_1$ (of Theorem A′) and hence Theorem B′ for $n = 0$ follows from Theorem A′ for $n = 1$. For $n \geq 1$ we have a commutative diagram

$$\begin{array}{ccc} P_{n-1}(C) & \xrightarrow{\sigma_{n-1}} & H_{n-1}(0) \\ \downarrow & & \downarrow \\ P_n(C) & \xrightarrow[\sigma_n]{} & H_n(0) \end{array} .$$

Moreover the codimension one orbits of G_n in $P_n(C)$ cut out the codimension one orbits of G_{n-1} on $P_{n-1}(C)$. Hence the image of σ_n is the union of the G_n-orbits of the image of σ_{n-1}. But, from the characterization of the G_n-orbits in H_n by their eigenvalues, it then follows that

$$G_n(\Sigma_{n-1}) = \Sigma_n \ .$$

Thus Theorem B′ follows by induction on n.

8. Appendix

In Section 6 we studied the differential geometry of the complexified projective varieties $P_n(C)$. In this appendix we will review (without complete proofs) some of the algebraic geometry which is of independent interest and provides further background. All dimensions in this section are complex dimensions. For further details see [15].

The algebraic variety

$$P_n(C) \subset \mathfrak{P}_n(C) \quad (n = 0, 1, 2, 3)$$

of dimension 2^{n+1} has appeared as an orbit of the complex Lie group $\mathfrak{G}_n(C)$ in the complex projective space $\mathfrak{P}_n(C)$ of dimension $3 \cdot 2^n + 2$. The group $\mathfrak{G}_n(C)$ is the complexification of the compact Lie group $\hat{G}_n$ given in table (6.1). Explicitly we have the sequence of groups $\mathfrak{G}_n(C)$ $(n = 0, 1, 2, 3)$

$$SL(3, C) \ , \ SL(3, C) \times SL(3, C) \ , \ SL(6, C) \ , \ E_6(C) \ .$$

These have irreducible representations on the vector spaces $H_n(C) = H_n \otimes C$, which are the complexifications of the real vector spaces H_n of all 3×3 Hermitian matrices over the division algebra A_n. These representations have dimension $3(2^n + 1)$, explicitly (for $n = 0, 1, 2, 3$)

$$6 \ , \ 9 \ , \ 15 \ , \ 27 \ ,$$

and they projectivize to give the spaces $\mathfrak{P}_n(C)$.

We can also consider the dual representations on $H_n(C)^*$ giving dual projective spaces $\mathfrak{P}_n(C)^*$, and in these there is a unique compact orbit $P_n(C)^*$ of $\mathfrak{G}_n(C)$.

On $H_n(C)$ there is a unique (up to scalars) cubic polynomial invariant under $\mathfrak{G}_n(C)$ which we have denoted by det, for reasons explained in Section 3. For $n = 0, 1$ it is just the usual determinant. For $n = 2$ it is the $SL(6, C)$-invariant cubic on $\Lambda^2(C^6)$ given by the exterior cube into $\Lambda^6(C^6) \cong C$. For $n = 3$ with $\mathfrak{G}_3(C) = E_6(C) \subset SL(27, C)$ this invariant cubic was discovered by E. Cartan [11]. In all cases it defines a cubic hypersurface in $\mathfrak{P}_n(C)$ which we denote by $Z_n(C)$, and which contains $P_n(C)$.

The action of $\mathfrak{G}_n(C)$ on $\mathfrak{P}_n(C)$ has just three orbits, namely $P_n(C)$, $Z_n(C) \setminus P_n(C)$ and $\mathfrak{P}_n(C) \setminus Z_n(C)$. When $n = 0$ these just correspond to symmetric matrices of ranks $1, 2, 3$, and we could use the same terminology in the general case. In fact $Z_n(C) \setminus P_n(C)$ is the $\mathfrak{G}_n(C)$-orbit of the diagonal matrix $\mathrm{Diag}(1, 1, 0)$ of rank 2, and as we have already observed $P_n(C)$ is the $\mathfrak{G}_n(C)$-orbit of the diagonal matrix $\mathrm{Diag}(1, 0, 0)$. Points of rank 3 constitute the $\mathfrak{G}_n(C)$-orbit of the unit matrix. The complex Lie group $\mathfrak{G}_n(C)$ is just the identity component of the group of holomorphic transformations of $P_n(C)$.

The natural embedding of 2×2 matrices into 3×3 matrices by adding zeroes in the third row and column gives a linear subspace

$$\mathfrak{L}_n(C) \subset \mathfrak{P}_n(C) \ , \ \dim \mathfrak{L}_n(C) = 2^n + 1 \ ,$$

and its orbit under $\mathfrak{G}_n(C)$ fills out the whole of $Z_n(C)$. The intersection

$$\mathfrak{L}_n(C) \cap P_n(C) = L_n(C)$$

is a complex quadric of dimension 2^n. In fact, $L_n(C)$ inside $\mathfrak{L}_n(C)$ plays the same role (for 2×2 matrices) that $Z_n(C)$ does inside $\mathfrak{P}_n(C)$: it is the hypersurface given by the invariant *quadratic* $\det_2$ for 2×2 matrices.

The family of all transforms of $\mathfrak{L}_n(C)$ under $\mathfrak{G}_n(C)$ therefore cuts out on $P_n(C)$ a corresponding family of quadrics $L_n(C)$ with

$$\dim L_n(C) = \frac{1}{2} \dim P_n(C) \ .$$

The dual $\mathfrak{L}_n(C)^*$ of $\mathfrak{L}_n(C)$ is a linear subspace of $\mathfrak{P}_n(C)^*$ of dimension

$$(3 \cdot 2^n + 2) - (2^n + 1) - 1 = 2^{n+1} = \dim P_n(C)^* \ .$$

In fact, $\mathfrak{L}_n(C)^*$ is the tangent space to $P_n(C)^*$ at a point ℓ_n. The correspondence

$$L_n(C) \longleftrightarrow \ell_n$$

represents $P_n(C)^*$ as the parameter family of the quadrics $L_n(C)$ on $P_n(C)$, and the situation is symmetrical (or dual): points of $P_n(C)$ parametrize quadrics (of half the dimension) on $P_n(C)^*$.

Note that the quadrics $L_n(C)$ are all non-singular (since all points are of "rank 1"), whereas $Z_n(C)$ has $P_n(C)$ as a *singular locus*: a generic point of $Z_n(C)$ has rank 2, whereas points of $P_n(C)$ are of rank 1. Thus $Z_n(C)$ determines $P_n(C)$ (as its singular locus), and conversely $P_n(C)$ determines $Z_n(C)$, as the space generated by the linear spaces $\mathfrak{L}_n(C)$ spanned by the quadrics $L_n(C)$.

It may be helpful at this stage if we looked in detail at the special case $n = 0$, so that we are dealing with the classical embedding of CP^2 as the Veronese surface V in CP^5. The lines of CP^2 become conics on V and these lie in planes. The lines of CP^2 are parametrized by the dual CP^2 which can be identified with the dual Veronese surface V^* in $(CP^5)^*$. Thus the planes spanned by the conics in V form a 2-parameter family and they fill out a (cubic) hypersurface Z. Since every pair of distinct points on CP^2 lies on a unique line, every pair of distinct points of V lies on a unique conic. In particular it follows that the chordal variety of V (i.e. the closed subspace generated by all chords) is also the space generated by all the planes spanned by the conics and hence is the hypersurface Z. This is a very unusual situation for a surface in CP^5. On dimension grounds one could expect the chordal variety to be the whole ambient space. Equivalently, when the chordal variety is only a hypersurface, the projection from a generic point gives an *embedding*[6] (without singularities) in CP^4. In fact, it is a classical result of Severi [21] that the Veronese surface is the only surface (not contained in a hyperplane) in CP^5 with this property.

Zak [26] (see also [16]) has investigated this "Severi property" for higher dimensions when $V_d \subset CP^N$. The critical case is when $d = \frac{2}{3}(N-2)$. Zak proved the remarkable result that, firstly

$$d = 2^{n+1}\ ,\ n = 0, 1, 2, 3\ ,$$

and secondly that the only such varieties in these dimensions are the complexified projective planes $P_n(C)$ in their standard projective embeddings in $\mathfrak{P}_n(C)$. For this reason these varieties have been named Severi varieties [15].

To see how this fits into the picture we have described we have to note that in all cases *the chordal variety of $P_n(C)$ is the cubic hypersurface $Z_n(C)$*. The proof is very similar to the case $n = 0$, but with a caveat. Given any two distinct points x and y in $P_n(C)$ there are just two possibilities, either

(i) the projective line in $\mathfrak{P}_n(C)$ containing x and y lies entirely on $P_n(C)$, or
(ii) there is a unique quadric $L_n(C)$ on $P_n(C)$ containing both x and y.

For $n = 0$ case (i) never happens: the classical Veronese variety contains no lines. Clearly (ii) is the generic situation and as before this implies that the *hypersurface $Z_n(C)$* (generated by the planes $\mathfrak{L}_n(C)$ spanned by the $L_n(C)$) *is precisely the chordal variety of $P_n(C)$*. This shows that $P_n(C)$ does indeed have the Severi property in $\mathfrak{P}_n(C)$. The power of Zak's Theorem is that these are the only ones.

There is a striking resemblance between Zak's theorem in complex algebraic geometry and the classical results about division algebras and projective planes. It would be interesting to see if a purely topological proof of Zak's theorem could be found. We recall that the use of Steenrod squares enables one to prove that a projective plane must have dimension a power of 2 (analogous to the first part of Zak's Theorem), while K-theory is needed for the final part [2]. One is therefore tempted to expect a K-theory proof of Zak's

[6]Over the reals this gives an embedding $RP^2 \subset RP^4$. Since this is of degree 2 it lifts to the double cover S^4 giving the embedding $RP^2 \subset S^4$ of (1.2).

theorem, particularly in view of the role that K-theory, as developed by Grothendieck, plays in algebraic geometry.

It is at this point that we should perhaps pass from the purely complex approach to the varieties $P_n(C)$ and introduce real structures. Recall that $P_n(C)$ has a complex conjugation, preserved only by the real subgroup $\mathfrak{G}_n$ of $\mathfrak{G}_n(C)$, and that the set of real points just recover the original projective planes P_n. The complex quadrics $L_n(C)$ on $P_n(C)$, parametrized by $P_n(C)^*$, include those preserved by conjugation and parametrized by P_n^*. These are just the complexifications of the projective lines in P_n (i.e. spheres of dimension 2^n). The incidence properties of these projective lines in P_n imply generically the corresponding properties of the complex quadrics $L_n(C)$ in $P_n(C)$, ensuring (ii) above. However, for $n \geq 1$, the exceptional case (i) does occur. For example, when $n = 1$, we have $P_1(C) = CP^2 \times CP^2$, and two distinct point pairs whose first components agree give case (i).

The manifolds $P_n(C)$, with their family of submanifolds $L_n(C)$ of middle dimension, are examples of generalized projective planes in the sense of Atsuyama [6], who studied these from the point of view of differential geometry.

Recall that we have the following subgroups of $\mathfrak{G}_n(C)$:

$$\begin{array}{ccc} G_n & \longrightarrow & \mathfrak{G}_n \\ \big\downarrow & & \big\downarrow \\ \hat{G}_n & \longrightarrow & \mathfrak{G}_n(C) \end{array} ,$$

where the groups in the first column are the maximal compact subgroups of those in the second column and preserve the metrics on P_n and $P_n(C)$ respectively. We can also introduce the groups $G_n(C)$, the complexification of G_n. Since the representation $H_n(C)$ splits off a trivial factor of dimension one (given by the trace) when restricted to $G_n(C)$, it follows that there is a hyperplane section $P_n(\infty)$ of P_n invariant under $G_n(C)$. In fact, $G_n(C)$ acts on $P_n(C)$ with just two orbits, $P_n(\infty)$ and its complement. We propose to examine the geometry of $P_n(\infty)$ which is a homogeneous space of $G_n(C)$ and so also of G_n. In particular we will see that we can reconstruct $P_n(C)$ canonically from $P_n(\infty)$.

It is instructive to consider first the simple case $n = 0$. Then $P_0(\infty)$ is a rational normal quartic curve, the image under the Veronese embedding of a conic in CP^2. This is the conic $z_1^2 + z_2^2 + z_3^2 = 0$ invariant under $G_0(C) = SO(3, C)$. Consider now the conics on $V = P_0(C)$ (the Veronese surface) which are complexifications of projective lines in $P_0 = RP^2$. In the "abstract" CP^2 which maps to V these just correspond to projective lines with real equations. Each such line meets the conic $z_1^2 + z_2^2 + z_3^2 = 0$ in a pair of conjugate points. The quotient of the conic by this involution is naturally identified with the *dual* RP^2. This is part of the content of Theorem B for $n = 0$. If we now consider all conics on V, they come from all projective lines on CP^2, and they meet the conic $z_1^2 + z_2^2 + z_3^2 = 0$ in any pair of points. Thus if we start with our rational normal curve $P_0(\infty)$ on $P_0(C)$ and consider the variety of all (unordered) pairs of points on $P_0(\infty)$ (i.e. its symmetric square) we get the dual CP^2. From this we can (by duality) recover the original CP^2 and also its Veronese embedding. Thus $P_0(\infty)$ determines the whole picture.

We want to show that this is typical of the general case (i.e. for all n).

We consider the real family of complex quadrics $L_n(C)$ parametrized by points of $P_n^* \subset P_n(C)^*$. These are just the complexifications of the projective lines of the projective plane P_n. Since any two distinct points of P_n are joined by a unique line they are never in the special position of (i). Dually this means that no two quadrics of the real family of $L_n(C)$ meet in more than one point. Since they already have one common point on P_n they meet nowhere else. In particular the family of quadrics of one lower dimension cut out on $P_n(\infty)$,

$$L_n(\infty) = L_n(C) \cap P_n(\infty) ,$$

are all disjoint. A dimension count shows that they must fill out the whole of $P_n(\infty)$, and since they are by construction parametrized by the dual space P_n^* we get a fibration

$$P_n(\infty) \to P_n^* \tag{8.1}$$

with fibre $L_n(\infty)$. This gives a more explicit description of the behaviour at infinity of the map of Theorem B. Note that, for $n = 0$, $L_0(\infty)$ has dimension zero and is a point-pair as we have already seen.

Motivated by the case $n = 0$ we now consider the full complex family of quadrics $L_n(C)$. These intersect $P_n(\infty)$ in the full complex family of quadrics (of one lower dimension) whose real members are the fibres of (8.1). This full family is therefore parametrized by the complexification $P_n(C)^*$ of P_n^*. Since we have cut down the symmetry from $\mathfrak{G}_n(C)$ to $G_n(C)$ there is a distinguished subset $P_n(\infty)^*$. These quadrics in $P_n(\infty)$ are *singular*, they arise from quadrics $L_n(C)$ which touch $P_n(\infty)$: note that these are never the real members (the fibres of (8.1)) since P_n^* and $P_n(\infty)^*$ are disjoint. All of this checks with what we saw for $n = 0$. Thus $P_n(\infty)$ contains a family of complex quadrics $L_n(\infty)$, parametrized by $P_n(C)^*$. This enables us to recover $P_n(C)^*$ and hence $P_n(C)$ from $P_n(\infty)$.

The fibration (8.1) can be viewed as a *twistor fibration*. For example when $n = 1$, $P_1(C) = CP^2 \times (CP^2)^*$ and $P_2(\infty)$ is the incidence locus. It is therefore the flag manifold of $SU(3)$ and (8.1) is the twistor fibration for CP^2 regarded as a 4-manifold with self-dual metric [4]. For $n = 2, 3$ the fibration (8.1) is a partial twistor fibration in the sense of Bryant [10], and it is given as an interesting example of a general theory. Explicitly, the two fibrations are

$$P_2(\infty) = \frac{Sp(3)}{Sp(1)U(2)} \longrightarrow \frac{Sp(3)}{Sp(1)Sp(2)} = HP^2 \cong P_2^*$$

and

$$P_3(\infty) = \frac{F_4}{Spin(7)U(1)} \longrightarrow \frac{F_4}{Spin(9)} = OP^2 \cong P_3^* ,$$

where the fibres are $L_2(\infty) = Sp(2)/U(2)$ and $L_3(\infty) = Spin(9)/Spin(7)U(1)$ respectively. Note that $Sp(2)/U(2)$ is isomorphic to the 3-dimensional quadric $SO(5)/SO(3)SO(2)$ and that $Spin(9)/Spin(7)U(1)$ is isomorphic to the 7-dimensional quadric $SO(9)/SO(7)SO(2)$.

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