A Lefschetz fixed point formula for elliptic complexes: I

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Introduction

In this paper, we give the proof of the main theorem announced in [1]. Interesting examples and applications will be developed in a subsequent paper [2].

Between the appearance of [1] and the writing of the present paper, we have experimented with a number of different variants of the proof (see § 8), all of which have some interest. The version presented here is technically and conceptually the simplest. In particular it is more elementary in nature than the proof outlined in [1].

As is well-known, the basic theorems on elliptic operators can be established either by use of a parametrix or by Hilbert-space methods. Although the second method is shorter, the first gives more information about the nature of the inverse or *Green's operator*; and for our purposes, this extra information is essential. Fortunately the recent development of pseudo-differential operator techniques [6] [7] [8] has led to a considerable streamlining of the classical parametrix method. Using these techniques therefore, we shall work in a C^∞-framework, avoiding Hilbert space entirely.

In § 1 we define elliptic complexes and formulate the *Lefschetz theorem* (Theorem A). The general idea of the proof is described in § 2. We show how the theorem will follow from a number of auxiliary lemmas and propositions, and the rest of the paper is devoted to the proofs of these auxiliary results. In § 3 we review the general facts about pseudo-differential operators, following [6]. Then in § 4 we consider more detailed properties which we shall need. These are put into suitable global form in § 5. In § 6 we apply pseudo-differential operators to the study of elliptic complexes. Finally in § 7 we study smooth endomorphisms. In § 8 we discuss briefly the alternative lines of proof mentioned above.

Although the local theory of pseudo-differential operators is very fully covered in [6], there is no adequate reference for a global theory. On the other hand the globalization is fairly routine, and so it did not seem desirable to obscure the main lines of proof by interspersing many minor technical points. For this reason we have added an appendix dealing with these questions. Results in the appendix are labelled (A1) (A2), etc.

It is possible to prove Theorem A, if one wants, without using pseudo-differential operators globally. In particular, one can avoid the proof of invariance under change of coordinates. However the details would be more cumbersome, and the general lines of the proof would not be so clear.

We are greatly indebted to L.Hörmander for instructing us in the techniques of pseudo-differential operators, and for much concrete advice.

1. Formulation of the theorem

Let X be a smooth (i.e., C^∞) compact manifold, and let E, F be smooth complex vector bundles over X. We denote by $\Gamma(E), \Gamma(F)$ the spaces of smooth sections of E, F. A differential operator

$$d\colon \Gamma(E) \longrightarrow \Gamma(F)$$

is then a linear map given locally by a matrix of partial differential operators with smooth coefficients. If d is of order k, then the terms of order k define in an invariant manner the *leading symbol* $\sigma_k(d)$ of d. This is a bundle homomorphism

$$\sigma_k(d)\colon \pi^*E \longrightarrow \pi^*F$$

over the cotangent space TX of X (π denotes the projection $TX \to X$).

The classical definition of ellipticity is that $\sigma_k(d)$ should be *an isomorphism* outside the zero section of TX. We shall generalize this in the following obvious way. Let $E_0, E_1, \cdots, E_N$ be a sequence of smooth vector bundles over X, and let

$$d_i\colon \Gamma(E_i) \longrightarrow \Gamma(E_{i+1})$$

be a sequence of differential operators. Following the usual terminology in homological algebra, the sequence is called a *complex* if $d^2=0$, i.e., if $d_{i+1}d_i=0$ for all i (we adopt the convention that $E_k = 0$ if $k < 0$ or $k > N$, and hence $d_k = 0$ if $k < 0$ or $k \geqq N$). For brevity we shall denote the complex simply by $\Gamma(E)$, the grading of E and the operator d being understood. The complex is *elliptic* if the sequence of leading symbols

$$\cdots \longrightarrow \pi^*E_i \xrightarrow{\sigma(d_i)} \pi^*E_{i+1} \longrightarrow \cdots$$

is *exact* outside the zero section. Note that we do not assume all d_i of the same order. Thus d_i has order k_i say, and

$$\sigma(d_i) = \sigma_{k_i}(d_i)$$

is given by the terms of order k_i. It is clear that, if $N = 1$, we just recover the usual definition of ellipticity for the operator d_0.

EXAMPLE. The simplest and most natural elliptic complex is the de Rham complex of X, where E_i is the bundle of (complex-valued) exterior differential forms of degree i, and d is the usual exterior derivative. The symbol sequence at $x \in X$ is just the exterior algebra on the cotangent space at x. More interesting examples are mentioned in [1] and will be discussed in detail in [2] (cf. also [3]).

The homology groups $H^i(\Gamma(E))$ of the complex

$$\longrightarrow \Gamma(E_i) \longrightarrow \Gamma(E_{i+1}) \longrightarrow$$

are defined as usual by

$$H^i(\Gamma(E)) = \operatorname{Ker} d_i/\operatorname{Im} d_{i-1} .$$

In § 6 we shall show that, *for an elliptic complex these homology groups are all finite-dimensional.* If $N = 1$, so that we are dealing with a single operator d_0, we have

$$H^0 = \operatorname{Ker} d_0$$
$$H^1 = \operatorname{Coker} d_0 ,$$

and the finite-dimensionality of these is classical. For the de Rham complex, the H^i are, by the theorems of de Rham, naturally isomorphic to the usual cohomology groups $H^i(X; \mathrm{C})$, and these are of course finite-dimensional.

By an *endomorphism* T of an elliptic complex $\Gamma(E)$, we mean a sequence of linear maps

$$T_i: \Gamma(E_i) \longrightarrow \Gamma(E_i)$$

such that $d_i T_i = T_{i+1} d_i$. Such an endomorphism induces endomorphisms $H^i T$ of $H^i(\Gamma(E))$. Since $H^i(\Gamma(E))$ is finite-dimensional, we can define Trace $H^i T$, and hence the *Lefschetz number*,

$$L(T) = \sum_{i=0}^{N} (-1)^i \operatorname{Trace} H^i T .$$

For example if $T = I$ is the identity, then

$$L(I) = \chi(\Gamma(E)) = \sum_{i=0}^{N} (-1)^i \dim H^i(\Gamma(E))$$

is just the Euler characteristic. In particular if $N = 1$, this is just the *index* of the elliptic operator d_0. The question of how to compute $L(T)$ is therefore a generalization of the index problem for elliptic operators (cf. [3]). In this paper, however, we shall be concerned with a situation which is, in a sense, at the opposite extreme from the case of the identity endomorphism. The case which we shall study is in fact much more elementary than the index problem, as will become evident later on.

Suppose then that $f: X \rightarrow X$ is a smooth map, and let $\varphi_i: f^*E_i \rightarrow E_i$ be smooth bundle homomorphisms. We can then define linear maps

$$T_i \colon \Gamma(E_i) \longrightarrow \Gamma(E_i)$$

as the composition

$$\Gamma(E_i) \xrightarrow{f^*} \Gamma(f^*E_i) \xrightarrow{\Gamma(\varphi_i)} \Gamma(E_i) \ .$$

Thus, if $s \in \Gamma(E_i)$, the section $T_i s$ is defined by

$$T_i s(x) = \varphi_i s(f(x)) \ .$$

The point to note is that $s(f(x)) \in (E_i)_{f(x)}$, but that φ_i takes us back to $(E_i)_x$. If further $d_i T_i = T_{i+1} d_i$, then the T_i defines an endomorphism of the elliptic complex $\Gamma(E)$. An endomorphism of this type we call a *geometric* endomorphism.

EXAMPLE. If $\Gamma(E)$ is the de Rham complex of X, then we have a natural choice for φ_i, namely the i^{th} exterior power of df. In this case any f defines a natural geometric endomorphism T and, in virtue of the de Rham theorems, the $H^i T$ are just the endomorphisms $H^i f$ induced by f on $H^i(X; \mathrm{C})$. Thus our Lefschetz number $L(T)$ coincides in this case with the classical Lefschetz number $L(f)$. In the general case, when the bundles E_i are not connected with the geometry of X, there is no natural construction for the φ_i, and their existence must be postulated. In many geometrically interesting examples, however, there is a natural choice for the φ_i.

Returning now to the map $f \colon X \to X$, we define a fixed point A of f to be *simple* if $\det(1 - df_A) \neq 0$ where df_A is the induced map on the tangent space at A. This is equivalent to requiring that the graph of f and the diagonal intersect transversally at (A, A) in $X \times X$. Thus a simple fixed point is an isolated fixed point. Hence if all fixed points of f are simple, it follows, since X is compact, that they are finite in number. We shall be concerned only with geometric endomorphisms arising from such maps.

The bundle homomorphism $\varphi_i \colon f^*E_i \to E_i$ is just a family of linear maps

$$\varphi_{i,A} \colon (E_i)_{f(A)} \longrightarrow (E_i)_A \ .$$

Hence at a fixed point A of f, we have

$$(E_i)_A = (E_i)_{f(A)} \ ,$$

and so $\varphi_{i,A}$ is an endomorphism of the vector space $(E_i)_A$. Thus Trace $\varphi_{i,A}$ is defined. We are now in a position to state our *Lefschetz fixed-point theorem.*

THEOREM A. *Let $\Gamma(E)$ be an elliptic complex on X, and let T be a geometric endomorphism of $\Gamma(E)$ defined by a map $f \colon X \to X$, with only simple fixed points, and bundle homomorphisms φ_i. Then the Lefschetz number $L(T)$ is given by the formula*

$$L(T) = \sum_A \nu(A)$$

where the summation is over the set of fixed points of f, and $\nu(A)$ is given by

$$\nu(A) = \frac{\sum (-1)^i \operatorname{Trace} \varphi_{i,A}}{|\det(1 - df_A)|} .$$

REMARKS. (1) Note that the formula $\nu(A)$ *does not explicitly involve the differential operators d_i*. Thus the formula for $L(T)$ is much simpler than the index formula [3; Th. 1]. Of course the d_i are implicitly involved by the condition $T_i d_i = T_{i+1} d_i$.

(2) If we take $\Gamma(E)$ to be the de Rham complex of X, with the natural choice for φ_i, we find

$$\begin{aligned}\sum (-1)^i \operatorname{Trace} \varphi_{i,A} &= \sum (-1)^i \operatorname{Trace} \Lambda^i(df_A) \\ &= \det(1 - df_A) ,\end{aligned}$$

so that $\nu(A) = \operatorname{sgn} \det(1 - df_A) = \pm 1$. Thus we recover the classical Lefschetz theorem (for simple fixed points).

(3) Note that in general $\nu(A)$ is a complex number, and not an integer. The classical Lefschetz formula, where $\nu(A) = \pm 1$, is highly special in this direction.

(4) Theorem A can be generalized by taking the φ_i to be differential operators. See Theorem B of § 2. The formula for $\nu(A)$ is then more complicated. See (5.4).

(5) If there are no fixed points, the proof of Theorem A is essentially a consequence of the fact that the Green's kernel $G(x, y)$ of an elliptic operator is smooth for $x \neq y$. If there are fixed points, we need more information about $G(x, y)$ near $x = y$. Roughly speaking, we need to know that the derivatives *parallel to the diagonal*:

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)^\alpha G(x, y)$$

have singularities at $x = y$ which are no worse than those of $G(x, y)$. This is trivially true for constant coefficient operators, and it is therefore plausible for operators with smooth coefficients. In the framework of pseudo-differential operators, these questions can be made quite precise and are easily answered.

2. Outline of proof

We shall now describe in outline the proof of Theorem A. Let us begin by recalling the algebraic *alternating sum formula*.

PROPOSITION (2.1). *Let T be an endomorphism of finite rank of a complex of vector spaces*

$$0 \longrightarrow V^0 \longrightarrow V^1 \longrightarrow \cdots \longrightarrow V^N \longrightarrow 0 .$$

Then

$$\sum (-1)^i \operatorname{Trace} T^i = \sum (-1)^i \operatorname{Trace} H^i T .$$

The proof is quite elementary and will be given in § 7. The important thing of course is that each T^i is assumed to be of finite rank. If the V^i are the chain groups of a finite simplicial complex, Proposition 2.1 is the essential point in the proof of the classical Lefschetz fixed-point formula.

In our case, when $V^i = \Gamma(E_i)$, a geometric endomorphism T is, of course, not of finite rank. However we shall show that it can be approximated by endomorphisms of finite rank, so that Theorem A will follow by passing to the limit in Proposition (2.1). To put it another way, the definition of Trace will be extended by continuity (in a suitable sense) from operators of finite rank to a larger class which will include our geometric endomorphisms. The alternating sum formula of Proposition (2.1) will remain valid for operators of this extended class and for geometric T we shall find the explicit formula

$$\operatorname{Trace} T^i = \sum_{f(A)=A} \frac{\operatorname{Trace} \varphi_{i,A}}{|\det (1 - df_A)|} .$$

The extension of the notion of trace falls naturally into two steps, the first of which is quite classical. This is the extension from operators of finite rank to operators with *smooth kernel*. Let us briefly review what is involved.

Let E, F be smooth vector bundles over X. A linear operator $T: \Gamma(E) \longrightarrow \Gamma(F)$ will be called *smooth* if it is given by a smooth kernel K_T on $X \times X$. At a point (x, y) of $X \times X$, $K_T(x, y)$ is then a linear transformation $E_y \longrightarrow F_x \otimes \Omega_y$, where Ω is the volume bundle on X (see § 5). Hence if $E=F$, Trace $K_T(x, x)$ is a smooth volume on X and so can be integrated (X being compact). We *define*

$$\operatorname{Trace} T = \int \operatorname{Trace} K_T \tag{2.2}$$

as the value of this integral. If T is of finite rank (and smooth), it is easy to see (cf. § 7) that this definition agrees with the purely algebraic definition of Trace. If we give smooth operators their usual topology (the C^∞ topology of their kernels) we see that operators of finite rank are dense, and that our definition of trace is continuous. Thus (2.2) *is the unique continuous extension of the trace to smooth operators.*

For endomorphisms of an elliptic complex, we shall then need the following approximation lemma.

LEMMA (2.3). *The smooth endomorphisms of finite rank of an elliptic complex* $\Gamma(E)$ *are dense in the space of all smooth endomorphisms of* $\Gamma(E)$.

Lemma (2.3) is a fairly simple consequence of the basic facts about elliptic complexes, and will be proved in § 7. If now $T^{(t)} \to T$ is a family of endomorphisms of finite rank converging to a smooth endomorphisms T, we have also $H^i T^{(t)} \to H^i T$. Thus $\text{Trace } T_i^{(t)} \to \text{Trace } T_i$ and $\text{Trace } H^i T^{(t)} \to \text{Trace } H^i T$. Hence (2.1) and (2.3) imply the *alternating sum formula* for smooth endomorphisms.

PROPOSITION (2.4) *Let T be a smooth endomorphism of an elliptic complex. Then*

$$\sum (-1)^i \text{Trace } T_i = \sum (-1)^i \text{Trace } H^i T .$$

This completes the first stage of the extension of traces:

$$\text{finite rank} \longrightarrow \text{smooth} .$$

We come now to the second and more difficult stage of the extension. For this we fix, once and for all, a smooth map $f: X \to X$ with simple fixed points, If E is a smooth vector bundle over X, we have the natural homomorphism

$$f^*: \Gamma(E) \longrightarrow \Gamma(f^*E) .$$

Suppose that G is another smooth bundle over X, and that

$$P: \Gamma(G) \longrightarrow \Gamma(E) \qquad Q: \Gamma(f^*E) \longrightarrow \Gamma(G)$$

are continuous[1] linear operators. Then we can form the compositions

$$Qf^*P: \Gamma(G) \longrightarrow \Gamma(G)$$
$$f^*PQ: \Gamma(f^*E) \longrightarrow \Gamma(f^*E) .$$

If P is smooth, it extends to the distributional sections (cf. § 5) of G, and so therefore does Qf^*P. Thus Qf^*P is also smooth, and so Trace Qf^*P is defined. Since f here is fixed, we shall regard this as a function of P, Q and write it as[2] $\text{Trace}_f(Q, P)$. The extension we require is to take P, Q both *pseudo-differential operators* (§ 3). For such operators one has a definition of *bounded set*. Moreover the operators are continuous, and so we can take them with the *strong operator topology*. Then we have the following result about

[1] The topology of $\Gamma(E)$ may be defined by using coordinate patches. Equivalently it may be defined by the semi-norms $\sup |Ds|$ for all smooth differential operators

$$D: \Gamma(E) \longrightarrow \Gamma(1) .$$

[2] We prefer this notation because otherwise, when we extend the definition of traces, we would have to show that this depends only on Qf^*P and not just on (Q, P): note that $(Q, P) \to Qf^*P$ is not necessarily injective.

existence of traces.

PROPOSITION (2.5). *For Q pseudo-differential there is a unique extension of* $\mathrm{Trace}_f(Q, P)$ *from smooth P to pseudo-differential P, which is continuous in P (for the strong operator topology) on bounded sets (of pseudo-differential operators). If I denotes the identity endomorphism of f^*E, then*

$$\mathrm{Trace}_f(Q, P) = \mathrm{Trace}_f(I, PQ) .$$

If P is differential, then

$$\mathrm{Trace}_f(I, P) = \sum\nolimits_{f(A)=A} \nu_P(A)$$

where $\nu_P(A)$ can be computed locally near A in terms of P and f. In particular if P is of order zero, i.e. if it is induced by a bundle homomorphism φ, we have

$$\nu_P(A) = \frac{\mathrm{Trace}\,\varphi_A}{|\det(1 - df_A)|} .$$

This proposition is crucial for Theorem A. A local version of (2.5) is essentially given in [5; Th. (2.5)] but since the proof is quite simple, it will be given in detail in § 4. The globalisation which is straightforward is then given in § 5.

Passing now to elliptic complexes, we may consider endomorphisms T where $T_i = Q_i f^* P_i$ with P_i, Q_i pseudo-differential. We shall call these pseudo-differential endomorphisms. They include, as a special case, our geometric endomorphisms. We shall regard P, Q as part of the definition of T so that different sets of P, Q will define different endomorphisms even though the maps Qf^*P may be the same. Then we have the following approximation lemma.

LEMMA (2.6). *Let T be a pseudo-differential endomorphism of an elliptic complex $\Gamma(E)$. Then T can be approximated, in the sense of Proposition (2.5), by smooth endomorphisms. More precisely, if $T_i = Q_i f^* P_i$, then there exists a family of smooth operators*

$$S_i^t \colon \Gamma(f^*E_i) \longrightarrow \Gamma(E_i) \qquad t > 0$$

such that

(i) $(S_i^t f^*)d_i = d_{i+1}(S_i^t f^*)$,

(ii) $S_i^t \rightarrow P_i$ *in the strong operator topology.*

(iii) S_i^t *is a bounded set of pseudo-differential operators.*

Note that, if $T^t \rightarrow T$ in the strong operator topology, then $H^i(T^t) \rightarrow H^i(T)$. Hence applying (2.4) to the approximating family T^t of (2.6), letting $t \rightarrow 0$, and using (2.5), we get the *alternating sum formula for pseudo-*

differential endomorphisms.

THEOREM B. *Let $f\colon X \to X$ have simple fixed points, and let $T = Qf^*P$ be a pseudo-differential endomorphism of an elliptic complex. Then we have*

$$L(T) = \sum (-1)^i \operatorname{Trace} H^i T = \sum (-1)^i \operatorname{Trace}_f (Q_i, P_i).$$

In particular if P, Q are differential, we get the Lefschetz formula:

$$L(T) = \sum_{f(A)=A} \nu(A)$$

where

$$\nu(A) = \sum (-1)^i \nu_{P_i Q_i}(A) .$$

In view of the explicit formula at the end of (2.5), we see that Theorem A is just a special case of Theorem B. Explicit local formulas for $\nu_P(A)$, for differential operators P of order > 0, will be given in (5.4). Putting these into Theorem B, we then obtain the generalisation of Theorem A referred to in Remark 4 of the introduction.

The proof of (2.6) will be given in § 6. It is an easy consequence of the existence of a pseudo-differential parametrix (§ 5) and the following simple approximation lemma which is already implicit in (2.5) and will be proved in § 4.

LEMMA (2.7). *Let E be a vector bundle. Then the identity endomorphism of $\Gamma(E)$ can be approximated, in the sense of* (ii) *and* (iii) *in* (2.6), *by smooth endomorphisms.*

3. Review of pseudo-differential operators

In this section we shall review some of the basic facts concerning pseudo-differential operators. Our standard reference for the local theory will be [6], which contains complete proofs of all the results we need. As mentioned in the introduction, technical points concerning the globalisation of the results of [6] will be dealt with in the Appendix.

Consider first an open set U of $\mathbf{R}^n$, and let $x = (x_1, \cdots, x_n)$ be the standard coordinates. For any real number m we denote by $S^m(U)$ the set of all smooth functions $p(x, \xi)$ on $U \times \mathbf{R}^n$ such that for every compact $K \subset U$ and all multi-indices α, β we have

$$(3.1) \qquad | D_x^\beta D_\xi^\alpha p(x, \xi) | \leqq C_{\alpha,\beta,K}(1 + |\xi|)^{m-|\alpha|} , \qquad x \in K, \xi \in \mathbf{R}^n .$$

Here D_ξ^α stands for the partial derivative

$$\left(-i \frac{\partial}{\partial \xi_1}\right)^{\alpha_1}\left(-i \frac{\partial}{\partial \xi_2}\right)^{\alpha_2} \cdots \left(-i \frac{\partial}{\partial \xi_n}\right)^{\alpha_n} ,$$

$|\alpha| = \sum \alpha_i$ and $C_{\alpha,\beta,K}$ is a constant depending on α, β, K and p. For any such p, we define a linear operator

$$P\colon \mathfrak{D}(U) \longrightarrow \mathfrak{E}(U)$$

by the formula

$$Pu = (2\pi)^{-n}\int p(x, \xi)\hat{u}(\xi)e^{i\langle x,\xi\rangle}d\xi \ .$$

Here $\mathfrak{E}$ denotes the smooth functions on U, $\mathfrak{D}$ those with compact support, and $\hat{u}$ is the Fourier transform of u. If p is a polynomial in ξ of degree m with smooth coefficients, then $p \in S^m(U)$ and P is the differential operator associated to it in the usual way. For this reason, when we want to show the dependence of P on p, in the general case, we write

$$P = p(x, D)$$

where D stands formally for the vector with components $-i(\partial/\partial x_j)$.

A *pseudo-differential operator* is one which is locally of the above type. Precisely, we denote by $L^m(U)$ the set of all mappings $P\colon \mathfrak{D}(U) \to \mathfrak{E}(U)$ such that, for all $f \in \mathfrak{D}(U)$, there exists some $p_f \in S^m(U)$ with $P(fu) = p_f(x, D)u$ for all $u \in \mathfrak{D}(U)$. An equivalent definition [5, § 2] is that P is continuous, and that the commutator

$$p_f(x, \xi) = e^{-i\langle x,\xi\rangle}P(fe^{i\langle x,\xi\rangle}) \tag{3.2}$$

belongs to $S^m(U)$ for all $f \in \mathfrak{D}(U)$.

A set of functions $p \in S^m(U)$ is said to be *bounded* if the constants $C_{\alpha,\beta,K}$ in (3.1) can be chosen independent of p. A set of operators $P \in L^m(U)$ is said to be *bounded* if, for each $f \in \mathfrak{D}(U)$, the functions p_f given by (3.2) form a bounded set in $S^m(U)$. It is shown in[3] [5, § 2] that, if $p(x, \xi) \in S^m(U)$, then $p(x, D) \in L^m(U)$. Moreover, if $p(x, \xi)$ lies in a bounded set of $S^m(U)$, $p(x, D)$ lies in a bounded set of $L^m(U)$.

The first fundamental result proved in [6] is:

(3.3) *Invariance*. The space $L^m(U)$, and the bounded sets in $L^m(U)$, are invariant under change of coordinates.

REMARK. The invariance of bounded sets is not explicitly stated in [6] but is implicit in the proof for the invariance of $L^m(U)$. Alternatively, it can be deduced from the invariance of $L^m(U)$ by an application of the closed-graph theorem. Similar remarks apply later to the question of composition.

In view of this invariance property, it is clear how to globalise the definition to manifolds, and to sections of vector bundles. Thus let X be a manifold (not necessarily compact), and let E, F be two vector bundles over X. Let

$$P\colon \Gamma_c(E) \longrightarrow \Gamma(F)$$

[3] Actually this is a special case of facts about composition (see (3.4)).

be a continuous linear operator, where Γ_c denotes the sections with compact support. Observe first that, for any open set U of X, we have an operator (the restriction of P to U):

$$P\colon \Gamma_c(E \mid U) \longrightarrow \Gamma(F \mid U) ,$$

given by the composition

$$\Gamma_c(E \mid U) \xrightarrow{e} \Gamma_c(E) \xrightarrow{P} \Gamma(F) \xrightarrow{r} \Gamma(F \mid U) ,$$

where e denotes the natural extension by 0 and r is restriction. Then for any coordinate patch U of X over which E, F, are trivial, the restriction of P will be given (relative to bases in E and F) by a matrix $P_U^{i,j}$ of operators $\mathfrak{D}(U) \to \mathfrak{E}(U)$.

The space $L^m(X\colon E, F)$ is now defined as the set of all P as above so that, for all choices of coordinate patches U and bases of $E \mid U$, $F \mid U$, the operators $P_U^{ij} \in L^m(U)$. A set of P in $L^m(X; E, F)$ is said to be *bounded* if all the corresponding sets of P_U^{ij} are bounded in $L^m(U)$. It follows quite easily from the invariance property (cf. proof of (A1)) that, for $U \subset \mathbf{R}^n$, the new and old definitions of pseudo-differential operator (and of bounded sets) coincide, i.e., if 1_U denotes the trivial line-bundle over U, $L_U^m = L^m(U; 1_U, 1_U)$.

The second important result is:

(3.4) *Composition*. If $P \in L^m(X; E, F)$, $Q \in L^s(X; F, G)$ and $f \in \mathfrak{D}(X)$, then $QfP \in L^{m+s}(X; E, G)$. Moreover, if P or Q varies in a bounded set, so does QfP. This is proved in [6, Th. (2.10)] for $X = U \subset \mathbf{R}^n$. The general case will be discussed in the Appendix.

Note. If X is non-compact, we have to insert f in order to make the composition defined. If X is compact, we can take $f = 1$. If we put[4] $L^{-\infty}(E, F) = \bigcap_m L^m(E, F)$, it follows from [6, Th. (2.2)] that $L^{-\infty}$ consists precisely of the smooth operators (i.e., with C^∞ kernels). We then introduce an equivalence relation for elements of $L^m(E, F)$,

$$A \sim B \Longleftrightarrow A - B \text{ is smooth} .$$

The importance of psudo-differential operators of elliptic problems lies in the following result.

(3.5) *Existence of parametrix*. Let $d\colon \Gamma_c(E) \to \Gamma(F)$ be an elliptic differential operator of order m on a manifold X. Then there exists

$$P \in L^{-m}(F, E) \text{ with } dP \sim I_F , \qquad Pd \sim I_E ,$$

[4] When there is no possibility of confusion, we omit X and write $L^m(E, F)$ instead of $L^m(X; E, F)$.

where I_E, I_F denote the identity operators of E, F.

The construction of such a parametrix P is done locally and then patched together by a partition of unity. The local construction is done for example in [7] (in [6] the interest is in the more general hypo-elliptic case). If d_i is the restriction of d to U_i and, if P_i is a parametrix for d_i, then the required global parametrix P is given by

$$P = \sum \varphi_i P_i \psi_i$$

where φ_i is a partition of unity for the covering $\{U_i\}$, and $\psi_i \in \mathfrak{D}(U_i)$ equals 1 on the support of φ_i (see (A4)).

4. The transversal trace

In this section we shall discuss in more detail some local properties of pseudo-differential operators which play a key role in the proof of our theorem. First however, we shall digress in order to recall some basic facts about topological vector spaces. Our decision to avoid Hilbert spaces, and to work in a C^∞ framework, means that we need to consider more general locally convex spaces. In fact the spaces we mainly need are Montel spaces, and so we shall summarize briefly a few relevant facts concerning these spaces.

If E, F are topological vector spaces, we denote by $\mathfrak{L}(E, F)$ the vector space of all continuous linear maps $E \to F$. This space can be given three important topologies:

(a) pointwise or simple convergence, also called the strong operator topology;

(b) uniform convergence on compact sets (compact convergence);

(c) uniform convergence on bounded sets (bounded convergence).

The first general result concerning these topologies is [4; III, § 3, Prop. 5]:

(4.1) *If E, F are locally convex and Hausdorff and $H \subset \mathfrak{L}(E, F)$ is an equi-continuous subset, then the topologies of simple convergence and compact convergence coincide on H.*

An important and extensive class of locally convex spaces are the barrelled space (espaces tonnelés) [4; III, § 1]. For these one has [4; III, § 3, Prop. 7, and Th. 2]:

(4.2) *If E is barrelled, F locally convex, the following conditions on a subset $H \subset \mathfrak{L}(E, F)$ are equivalent*

(a) *H is equicontinuous;*

(b) *H is bounded for simple convergence;*

(c) *H is bounded for bounded convergence.*
In view of (b) and (c), one may just refer to bounded sets in $\mathfrak{L}(E, F)$ without ambiguity.

Finally let us recall that a *Montel space* is a barrelled space in which bounded sets are relatively compact. For such spaces (4.1) and (4.2) combine to yield

(4.3) *If E is a Montel space, F locally convex and Hausdorff, then on bounded sets of $\mathfrak{L}(E, F)$ the topologies of simple convergence and bounded convergence coincide.*

All the usual spaces occurring in the theory of distributions, namely $\mathfrak{D}$, $\mathfrak{D}'$, $\mathfrak{E}$, $\mathfrak{E}'$, $\mathfrak{S}$, $\mathfrak{S}'$-are Montel spaces [10]. We should perhaps warn the reader that, following Schwartz, we always take duals (e.g. $\mathfrak{D}'$) in the strong topology (i.e., bounded convergence), whereas Hörmander in [6] takes them in the weak topology (i.e., simple convergence). For many purposes these are equivalent, but we shall need (the elementary part of) the Schwartz kernel theorem [12; Prop. 22] where the strong topology is necessary. This theorem asserts that we have a topological isomorphism

$$\mathfrak{E}_{x,y} \longrightarrow \mathfrak{L}(\mathfrak{E}'_y, \mathfrak{E}_x)$$

where $\mathfrak{L}$ has the topology of bounded convergence, and the map is obtained by assigning to $k(x, y) \in \mathfrak{E}_{x,y}$ the linear mapping

$$\varphi(y) \longrightarrow \int h(x, y)\varphi(y)dy \ .$$

We refer to this topology briefly as the C^∞ topology of smooth operators.

Returning now to pseudo-differential operators, for $U \subset \mathbf{R}^n$, we observe first that bounded sets of $L^m(U)$, defined as in § 3, are certainly bounded in $\mathfrak{L}(\mathfrak{D}(U), \mathfrak{E}(U))$; we just put $\xi = 0$ in (3.1). Thus by (4.3), the topologies of simple convergence and bounded convergence coincide on such bounded sets. Since we shall be mainly concerned with bounded sets, there is thus no need to distinguish between the topologies. For definiteness, however, we agree to take $\mathfrak{L}(E, F)$ always with the topology of bounded convergence. For pseudo-differential operators, the topology is made more explicit by the following lemma.

LEMMA (4.4). *On bounded sets of $L^m(U)$, we have $P^t \to 0$ in $\mathfrak{L}(\mathfrak{D}(U), \mathfrak{E}(U))$, if and only if, for all $f \in \mathfrak{D}(U)$,*

$$p_f^t \longrightarrow 0 \qquad \text{in } \mathfrak{E}(U \times R^n) \ .$$

PROOF. Suppose $P^t \to 0$, and let $|\xi| < K$, $f \in \mathfrak{D}(U)$. Then $D_\xi^\alpha f(x) e^{i\langle x,\xi\rangle}$ is a bounded family in $\mathfrak{D}(U)$, and so (since P^t is continuous for each t)

$$D_\xi^\alpha P^t\big(f(x)e^{i\langle x,\xi\rangle}\big) = P^t D_\xi^\alpha\big(f(x)e^{i\langle x,\xi\rangle}\big) \longrightarrow 0 \qquad \text{in } \mathfrak{S}(U) .$$

This implies that

$$p_f^t(x, \xi) = e^{-i\langle x,\xi\rangle}p^t\big(f(x)e^{i\langle x,\xi\rangle}\big) \longrightarrow 0 \qquad \text{in } \mathfrak{S}(U \times \mathbf{R}^n) .$$

Conversely suppose $p_f^t \to 0$ in $\mathfrak{S}(U \times \mathbf{R}^n)$ for all $f \in \mathfrak{D}(U)$. Let $u \in \mathfrak{D}(U)$, and take $f = 1$ on $\operatorname{supp} u$. Then (cf. § 3)

$$P^t(u) = P^t(fu) = p_f^t(x, D)u = (2\pi)^{-n}\int e^{i\langle x,\xi\rangle}p_f^t(x, \xi)\hat{u}(\xi)d\xi ,$$

where $\hat{u}$ is the Fourier transform. Since p_f^t is bounded in $S^m(U)$, the integrand is bounded (uniformly in t and for x in a compact set) by a constant times $(1 + |\xi|)^m |\hat{u}(\xi)|$, and the integral of this converges since $u \in \mathfrak{D}(U)$. Since $p_f^t \to 0$ uniformly on compact sets, it then follows that $P^t u \to 0$ uniformly on compact sets. The same applies to all derivatives $D_x^\alpha P^t u$, and hence $P^t \to 0$ in $\mathfrak{L}(\mathfrak{D}(U), \mathfrak{S}(U))$ as required.

Using (4.4) we can now proceed to

PROOF OF (2.7). First we consider the local situation for $U \subset \mathbf{R}^n$. Choose $\varphi \in \mathfrak{D}(\mathbf{R}^n)$ so that $\varphi(0) = 1$. Then the p^t defined (for $t > 0$) by

$$p^t(x, \xi) = \varphi(t\xi)$$

are a family of smooth functions, bounded in $S^0(U)$ and $p^t \to 1$ in $\mathfrak{S}(U \times \mathbf{R}^n)$ as $t \to 0$. Hence $P^t(x, D)$ is a family of smooth operators, bounded in $L^0(U)$ and $\to 1$ in $\mathfrak{L}(\mathfrak{D}(U), \mathfrak{S}(U))$. For systems of functions over U, we take p^t to be a scalar matrix. Finally for sections of a vector bundle E over a compact manifold X, we construct P_i^t over open sets U_i (of a finite coordinate covering with $E | U_i$ trivial) and put

$$P^t = \sum \varphi_i P_i^t \psi_i$$

where $\{\varphi_i\}$ is a partition of unity and $\psi_i = 1$ on $\operatorname{supp} \varphi_i$. Then (see (A1)) $P^t \to 1$ in $\mathfrak{L}(\Gamma(E), \Gamma(E))$, and is bounded in $L^0(\Gamma(E), \Gamma(E))$ as required.

If $P^t \to 1$ as above, and $Q: \Gamma(E) \to \Gamma(F)$ is pseudo-differential, then $QP^t \to Q$ in the same sense by (3.4). Thus we have the following corollary of (2.7).

COROLLARY (4.5). *Let $Q: \Gamma(E) \to \Gamma(F)$ be a pseudo-differential operator on a compact manifold. Then there is a family of smooth operators $Q^t: \Gamma(E) \to \Gamma(F)$ which are bounded in $L^m(\Gamma(E), \Gamma(F))$ and converge to Q in $\mathfrak{L}(\Gamma(E), \Gamma(F))$.*

We consider now the extension of pseudo-differential operators of distributions.

LEMMA (4.6). *If $P: \mathfrak{D}(U) \to \mathfrak{E}(U)$ belongs to $L^m(U)$, then it extends to a continuous linear operator $\mathfrak{E}'(U) \to \mathfrak{D}'(U)$. If P^t is bounded in $L^m(U)$ and $\to 0$ in $\mathfrak{L}(\mathfrak{D}(U), \mathfrak{E}(U))$, then its extension is bounded and $\to 0$ in $\mathfrak{L}(\mathfrak{E}'(U), \mathfrak{D}'(U))$.*

PROOF. This is essentially given in [5; (2.2)], but we recall the argument (particularly because we use the strong topology for distributions). Let K be a compact set in U, and let $f \in \mathfrak{D}(U)$ with $f = 1$ on K. Then, as above, for any $u \in \mathfrak{D}(K)$, we have

$$Pu = (2\pi)^{-n}\int e^{i\langle x,\xi\rangle}p_f(x, \xi)\hat{u}(\xi)d\xi .$$

Hence if $v \in \mathfrak{D}(U)$, we have

$$\begin{aligned}\langle Pu, v\rangle &= (2\pi)^{-n}\int v(x)dx\int e^{i\langle x,\xi\rangle}p_f(x, \xi)\hat{u}(\xi)d\xi \\ &= (2\pi)^{-n}\int p_{f,v}(\xi)\hat{u}(\xi)d\xi ,\end{aligned} \tag{4.7}$$

where we have interchanged the order of the absolutely convergent double integral and put

$$p_{f,v}(\xi) = \int v(x)p_f(x, \xi)e^{i\langle x,\xi\rangle}dx . \tag{4.8}$$

Differentiating with respect to ξ, and then integrating by parts, we obtain

$$\xi^\beta D_\xi^\alpha p_{f,v}(\xi) = \int\bigl(D_x^\beta(D_\xi - x)^\alpha v(x)p_f(x, \xi)\bigr)e^{-i\langle x,\xi\rangle}dx .$$

This leads to an estimate

$$| D_\xi^\alpha p_{f,v}(\xi) | < A(1 + \xi)^{m-|\beta|} , \tag{4.9}$$

where A is a constant depending on $L = \operatorname{supp} v$, on $\sup | D^\gamma v |$ (for $| \gamma | \leqq | \beta |$), and on the constants occurring in (3.1) (for $p = p_f$ and $K = L$). In particular this shows that, for fixed p_f, $v \mapsto p_{f,v}$ defines a continuous map $\mathfrak{D}(L) \to \mathfrak{S}$ (where $\mathfrak{S}$ is the Schwartz space of C^∞ functions on R^n decreasing rapidly at infinity). Since $\mathfrak{D}(U) = \operatorname{inj\,lim} \mathfrak{D}(L)$ with the direct limit topology [10, III, Th. II] it follows that $v \mapsto p_{f,v}$ defines a continuous map $\mathfrak{D}(U) \to \mathfrak{S}$. Now for any $u \in \mathfrak{E}'(K)$ we have a Fourier transform $\hat{u} \in \mathfrak{S}'$. Since $p_{f,v} \in \mathfrak{S}$, (4.7) defines Pu as a linear functional on $\mathfrak{D}(U)$. Since $p_{f,v}$ is continuous in v, Pu is a distribution. This gives the required extension of P to distributions. Since, on each compact K, it is defined as the composition of the two continuous maps,

$$\begin{aligned}&\mathfrak{E}'(K) \longmapsto \mathfrak{S}' \text{ given by } u \longmapsto (2\pi)^{-n}\hat{u} \\ &\mathfrak{S}' \quad \longrightarrow \mathfrak{D}'(U) \text{ dual to the map given by } v \longmapsto p_{f,v} ,\end{aligned}$$

it is continuous from $\mathfrak{E}'(K) \to \mathfrak{D}'(U)$. Since $\mathfrak{E}'(U) = \operatorname{inj\,lim} \mathfrak{E}'(K)$ [10, III, p. 90], it follows that (the extension of) P is continuous from $\mathcal{E}'(U) \to \mathcal{D}'(U)$.

If P^t is bounded in $L^m(U)$, then (4.9) shows that, for fixed u, v, $\langle P^t u, v\rangle$ is bounded. This means that (the extension of) P^t is bounded in $\mathcal{L}(\mathcal{E}'(U), \mathcal{D}'(U))$.

REMARKS. (1) It is shown in [6, (2.15)] that the dual of the distributional extension of P is in fact pseudo-differential, but we do not need this.

(2) Both (4.4) and (4.6) extend immediately to a global form, for vector bundles on a manifold. This is because the topology (and bounded sets) of $\mathcal{L}(\Gamma_c(E), \Gamma(F))$ may be defined by using coordinate patches.

We shall now examine the Schwartz kernels of pseudo-differential operators. Here we follow [6, § 2]. In the first place if $U \subset \mathbf{R}^n$ and $p \in S^m(U)$, the Schwartz kernel K_p of $p(x, D)$ is a distribution on $U \times U$. Off the diagonal Δ, it is a smooth function [6, (2.5)] given by

$$K_p(x, y) = \frac{(2\pi)^{-n}}{(x - y)^\alpha} \int e^{i\langle x-y, \eta\rangle}(-D_\eta)^\alpha p(x, \eta) d\eta ,$$

where α is arbitrarily large, and $(x - y)^\alpha \neq 0$. This expression shows the following

(4.10) *If p lies in a bounded set of $S^n(U)$ and $\to 0$ in $\mathcal{E}(U \times \mathbf{R}^n)$, then $K_p \to 0$ in $\mathcal{E}(U \times U - \Delta)$.*

We turn now to the more interesting question of the behaviour of K_p near the diagonal. Let U be open in $\mathbf{R}^n$, V open in U, and let $f\colon V \to U$ be a smooth map with a simple fixed point at A. We assume V chosen so small that $x \mapsto x - f(x)$ gives a diffeomorphism of V onto its image. Let $F\colon V \to U \times U$ be defined by $F(x) = (f(x), x)$, and consider the induced map

$$F^*\colon \mathcal{E}(U \times U) \longrightarrow \mathcal{E}(V)$$

of smooth functions. If $p(x, \xi) \in S_V^{-\infty}$, the operator $p(x, D)$ has a smooth Schwartz kernel K_p given by

$$K_p(x, y) = (2\pi)^{-n} \int e^{i\langle x-y, \xi\rangle} p(x, \xi) d\xi .$$

The induced function $F^* K_p$ is therefore

$$F^* K_p(x) = (2\pi)^{-n} \int e^{i\langle f(x)-x, \xi\rangle} p(f(x), \xi) d\xi .$$

The corresponding distribution in $\mathcal{D}'(V)$ may be written

$$\langle F^* K_p, v\rangle = (2\pi)^{-n} \int e^{i\langle f(x)-x, \xi\rangle} p(f(x), \xi) v(u) du ,$$

where the change in order of integration is permissible because $p \in S^{-\infty}(U)$.

Making the change of variable $y = x - f(x)$, we find

$$\langle F^*K_p, v\rangle = (2\pi)^{-n}\int d\xi \int e^{-i\langle y,\xi\rangle} \frac{p(f(x(y)), \xi)v(x(y))dy}{|\det(1 - f'(x(y)))|} . \tag{4.11}$$

But now applying (4.9) (with p instead of p_f, and $v/\det(1 - f')$ instead of v), we see that (4.11) *defines a distribution* for any $p \in S^m(U)$. Moreover if p, v remain in bounded sets (of $S^m(U)$ and $\mathfrak{D}(U)$ respectively), and $p \to 0$ in $\mathfrak{E}(U \times \mathbf{R}^n)$, then (4.9) shows that

$$\langle F^*K_p, v\rangle \longrightarrow 0 .$$

Hence we have

(4.12) *$p \mapsto F^*K_p$ is continuous from $\mathfrak{E}(U \times \mathbf{R}^n) \to \mathfrak{D}'(V)$ for p in a bounded set of $S^m(U)$.*

We propose next to calculate the distribution F^*K_p when $p = \sum a_\alpha(x)\cdot\xi^\alpha$ is a polynomial in ξ. We get[5]

$$\begin{aligned}\langle F^*K_p, v\rangle &= (2\pi)^{-n}\sum\nolimits_\alpha \int \xi^\alpha\Big(\frac{a_\alpha\cdot v}{|\det(1-f')|}\Big)^\wedge d\xi \\ &= (2\pi)^{-n}\sum\nolimits_\alpha \int\Big\{D_y^\alpha\Big(\frac{a_\alpha v}{|\det(1-f')|}\Big)\Big\}^\wedge d\xi \\ &= \sum\nolimits_\alpha D_y^\alpha\Big(\frac{a_\alpha v}{|\det(1-f')|}\Big)\Big|_{y=0}\end{aligned} \tag{4.13}$$

using the Fourier inversion formula. In particular, if $p = a(x)$ is a polynomial of degree zero in ξ, then

$$\langle F^*K_p, v\rangle = \frac{a(A)v(A)}{|\det(1 - f'(A))|}$$

or

$$F^*K_p = \frac{a(A)\delta_A}{|\det(1 - f'(A))|} , \tag{4.14}$$

where δ_A is the Dirac measure at A.

In (4.10)—(4.14) we have all the local information we need about the kernels of pseudo-differential operators. In the next section we put these on a global footing to obtain a proof of (2.5), the key step towards Theorem A.

5. Global formulation

This section is largely devoted to the notational questions necessary to formulate globally the results of the preceding section, and hence establish Proposition (2.5). All the analytical questions being essentially local, nothing

[5] Here ( )^ denotes the Fourier transform with respect to y.

really new is involved. First we need to introduce the *volume bundle* $\Omega(X)$ of a manifold X. This is the line-bundle associated to the tangent bundle by the representation $A \to |\det A|$ of $GL(n, \mathbf{R})$. The fibre Ω_x at $x \in X$ can be identified with the one-dimensional space of all Haar measures on the tangent space T_x at x. The smooth sections of Ω are the smooth measures on X, i.e., in a given coordinate patch they are of the form $f(x)\,|dx|$, where $|dx|$ denotes the Lebesgue measure of $\mathbf{R}^n$. Given an element $\omega \in \Gamma_c(\Omega)$, we can therefore, in an invariant manner, form the integral $\int_X \omega$.

For any bundle E, we have the dual bundle E^*, the fibre E_x^* being dual to E_x. We define $E' = E^* \otimes \Omega$. If $s \in \Gamma(E)$ and $t \in \Gamma(E')$, we can then define an element $\omega \in \Gamma(\Omega)$ by

$$\omega(x) = \big(s(x), t(x)\big)$$

where (,) denotes the natural pairing

$$E_x \times E'_x \longrightarrow \Omega_x \; .$$

By integrating $\int_X \omega$, we then obtain a (separately continuous) bilinear map

$$\Gamma_c(E) \times \Gamma(E') \longrightarrow \mathbf{C}$$

and hence a linear map

$$\Gamma(E') \longrightarrow \Gamma_c(E)' \; ,$$

where $\Gamma_c(E)'$ denotes the dual topological vector space of $\Gamma_c(E)$. Replacing E by E', and identifying E'' with E we thus obtain a map

$$\Gamma(E) \longrightarrow \Gamma_c(E')' \; .$$

It is easy to see that this is injective (with dense image), and we shall refer to an element of $\Gamma_c(E')'$ as a *distributional section* of E.

In a similar way a *distributional section* of E *with compact support* is an element of $\Gamma(E')'$. A more suggestive notation therefore is to follow Schwartz and write $\mathcal{D}'(E)$ for $\Gamma_c(E')'$ and $\mathcal{E}'(E)$ for $\Gamma(E')'$.

Suppose now that

$$P: \Gamma_c(E) \longrightarrow \Gamma(F)$$

is a continuous linear operator. Its Schwartz kernel K_P is then a distributional section on $X \times X$ of the bundle[6] $F \boxtimes E'$. If $\varphi \in \Gamma_c(F')$ and $\psi \in \Gamma_c(E)$, then the value of K_P on $\varphi \otimes \psi$ is given by

$$K_P \cdot \varphi \otimes \psi = \langle \varphi, P\psi \rangle \; .$$

[6] We use $E \boxtimes F$ for the *external tensor product*. It is a bundle on $X \times Y$ when E, F are bundles on X, Y respectively. If $Y = X$, the restriction of $E \boxtimes F$ to the diagonal gives the *internal tensor product* $E \otimes F$. This is a bundle on X.

If P is smooth, then K_P is a smooth function and, for $(x, y) \in X \times X$,

$$K_P(x, y) \in F_x \otimes E'_y .$$

Such a smooth operator extends by continuity to a continuous map $\mathcal{E}'(E) \to \Gamma(F)$. Moreover the Schwartz kernel theorem (cf. § 4) in its global form asserts that $P \mapsto K_P$ defines a topological isomorphism

$$\mathfrak{L}(\mathcal{E}'(E), \Gamma(F)) \longrightarrow \Gamma(F \boxtimes E') .$$

Suppose now that $f: X \to X$ is a smooth map with simple fixed points and that

$$P: \Gamma_c(f^*E) \longrightarrow \Gamma(E)$$

is a pseudo-differential operator. Put $F = f^*E$ for brevity. Then the kernel K_P of P is a distributional section of $E \boxtimes F'$ on $X \times X$. Then kernel K_{f^*P} of the composition

$$f^*P: \Gamma_c(F) \longrightarrow \Gamma(F)$$

is a distributional section of $F \boxtimes F'$. Outside the graph $y = f(x)$, it is a smooth function given by

$$K_{f^*P}(x, y) = K_P(f(x), y) . \tag{5.1}$$

If P is smooth, then K_{f^*P} is a smooth section given by (5.1) for all (x, y). If X is compact, the smooth trace of f^*P is then by definition

$$\operatorname{Trace} f^*P = \int_X \operatorname{Trace} K_P(f(x), x) . \tag{5.2}$$

Note that $K_P(f(x), x)$ is a section of $F \otimes F'$, and Trace $K_P(f(x), x)$ is therefore a section of $\Omega(X)$. Thus the integral in (5.2) is meaningful. As a main step towards (2.5) we shall now prove

PROPOSITION (5.3). *The map* $\Theta: \mathfrak{L}(\mathcal{E}'(F), \Gamma(E)) \to \Gamma(F \otimes F')$ *given by* $P \mapsto \Delta^* K_{f^*P} = K_P(f(x), x)$ *has a unique extension*

$$\Theta: L^m(F, E) \longrightarrow \mathfrak{D}'(F \otimes F') ,$$

which is continuous for the strong operator topology of P on bounded sets of $L^m(F, E)$. If P is a differential operator, then $\Theta(P) = \sum_{f(A)=A} \Theta(P)_A$ where $\Theta(P)_A$ is a distribution with support $\{A\}$ which may be calculated as follows. Let $e_1, \cdots, e_r$ be a local basis for E near A, let $(x_1, \cdots, x_n)$ be local coordinates near A, and suppose P is given there by

$$P(uf^*e_i) = \sum_{j=1}^r (P^{ij}u)e_j ,$$

where u is a scalar function and

$$P^{ij} = \sum_\alpha P^{ij}_\alpha(x) D^\alpha_x .$$

Then $\Theta(P)_A$ *is the distributional section of* $F \otimes F'$ *which, relative to the basis* $\{f^*e_i\}$ *of* F *and* $|dx|$ *of* Ω, *is given by the matrix* Q^{ij} *where* $Q^{ij} \in \mathfrak{D}'_x$ *is defined by*

$$\langle Q^{ij}, v\rangle = \sum_\alpha D_y^\alpha \Big(\frac{P_\alpha^{ij} v}{|\det(1 - df)|} \Big)_{y=0}$$

and $y = x - f(x)$.

PROOF. The uniqueness of the extension of Θ follows from (4.5). To show that Θ extends, it is enough to show that it extends locally near any point $x_0 \in X$, i.e., that

$$P \longrightarrow \Delta^* K_{f^*P} \mid U$$

can be extended for some neighbourhood U of x_0. We now choose U so that $\bar{U}$ and $f(\bar{U})$ are both contained in a coordinate patch W over which E and F are trivial. If $x_0 \neq f(x_0)$, then we take W disconnected. If $x_0 = A$ is a fixed point, we assume moreover that U contains no other fixed point, and that $\det(1 - df) \neq 0$ on U. Now let $\varphi \in \mathfrak{D}(W)$ with $\varphi = 1$ on $\bar{U}$. Then relative to the bases of E and F over W, the restriction P_U of any $P \in L^m(X, E, F)$ will be given by a matrix

$$P^{ij} = p_\varphi^{ij}(x, D) \qquad P_\varphi^{ij} \in S^m(W)\ .$$

In the case $x_0 \neq f(x_0)$, it follows from (4.10) that $K_P(f(x), x)$ is smooth for all $x \in U$ and that $P \mapsto \Delta^* K_{f^*P}$ defines a mapping

$$L^m(E, F) \longrightarrow \Gamma(F \otimes F' \mid U)\ ,$$

which is continuous in the sense of (5.3). It is continuous, *a fortiori*, when regarded as having values in the space of distributional sections. For the case $x_0 = A$ of a fixed point, we apply (4.11)—(4.14), and the results all follow.

The bundle homomorphism

$$\text{Trace}: F \otimes F' \longrightarrow \Omega$$

induces continuous linear maps (also called Trace) in the smooth and distributional sections of these bundles. Thus, by (5.3), for any $P \in L^m(E, F)$, we can form Trace $\Delta^* K_{f^*P}$. This will be a distributional section of $\Gamma(\Omega)$, i.e., an element of $\Gamma(\Omega')' = \Gamma(1)'$. It can therefore, for compact X, be evaluated on the constant function 1. For smooth P this value is, by 5.2, just Trace f^*P. Hence, as a corollary of (5.3), we deduce[7]

COROLLARY (5.4). *Let* X *be compact. Then the map* $P \mapsto$ Trace f^*P *has a unique extension (denoted by* Trace$_f$ P*) which is continuous in the sense*

[7] Strictly speaking (5.3) gives existence and (4.5) uniqueness.

of (5.3), *from smooth P to pseudo-differential P. If P is differential* $\mathrm{Trace}_f\, P$ *is given by*

$$\mathrm{Trace}_f\, P = \sum\nolimits_{f(A)=A} \nu_P(A)$$

where

$$\nu_P(A) = \sum\nolimits_\alpha D_y^\alpha\Big(\frac{P_\alpha^{ii}}{|\det(1-df)|}\Big)_{y=0}$$

in the notation of (5.3). *In particular, if P is induced by a bundle homomorphism φ, this can be written invariantly as*

$$\nu_P(A) = \frac{\mathrm{Trace}\,\varphi_A}{|\det(1-df_A)|}\ .$$

Corollary (5.4) is the special case of (2.5) when Q is the identity (and $f^*E = G$). We pass now to prove (2.5) in general. Thus we have to consider the composition Qf^*P where

$$Q\colon \Gamma(f^*E) \longrightarrow \Gamma(G) \qquad P\colon \Gamma(G) \longrightarrow \Gamma(E)$$

are pseudo-differential. Now we recall that smooth operators are just continuous operators from distributions to smooth sections. Thus if P is smooth so is Qf^*P, and so $\mathrm{Trace}\, Qf^*P$ is defined. We want to show that there is a unique continuous extension of this to all P. First we note that, when P and Q are both smooth, we have the commutation formula

$$\mathrm{Trace}\, Qf^*P = \mathrm{Trace}\, f^*PQ\ . \tag{5.5}$$

This follows at once from the integral expressions for these traces and the formula

$$\mathrm{Trace}\, AB = \mathrm{Trace}\, BA$$

for finite-dimensional spaces. We will now show that (5.5) continues to hold when P is smooth but Q is not. Note that, since Q extends to a continuous operator on distributions $\big((4.6)$ and Remark 2$\big)$, f^*PQ is continuous from distributions to smooth sections and so is still smooth. Now let Q^t be a smooth family converging to Q as in (4.5). Then by (4.6) (globalized) the extension of Q^t to distributions converges in $\mathfrak{L}(\mathfrak{D}'(f^*E), \mathfrak{D}'(G))$ to the extension of Q. Since composition of operators is separately continuous, it follows that

$$Q^t f^*P \longrightarrow Qf^*P \qquad\qquad \text{in } \mathfrak{L}\big(\mathfrak{D}'(G), \Gamma(G)\big)$$

and

$$f^*PQ^t \longrightarrow f^*PQ \qquad\qquad \text{in } \mathfrak{L}\big(\mathfrak{D}'(f^*E), \Gamma(f^*E)\big)\ .$$

Since the topology in $\mathfrak{L}(\mathfrak{D}', \Gamma)$ is just the C^∞ topology of smooth kernels, and since the smooth trace is continuous for this topology, it follows that

$$\begin{aligned}\text{Trace } Q^t f^* P &\longrightarrow \text{Trace } Q f^* P \\ \text{Trace } f^* P Q^t &\longrightarrow \text{Trace } f^* P Q \ .\end{aligned}$$

Hence (5.5) follows by passing to the limit in

$$\text{Trace } Q^t f^* P = \text{Trace } f^* P Q^t \ .$$

Thus (5.5) holds for all smooth P. In view of (3.4) and (5.4), we can therefore define

(5.6) $$\text{Trace}_f\,(Q, P) = \text{Trace}_f\, PQ \ ,$$

and this will provide a continuous extension of Trace Qf^*P. The uniqueness follows by (4.5) as before. The proof of (2.5) is therefore complete.

6. Elliptic complexes

In this section, we shall use the results of § 3 to establish the basic facts about elliptic complexes. In particular, we shall give the proof of (2.3).

Let $\Gamma(E)$ be an elliptic complex over the compact[8] manifold X. By a *parametrix* P for $\Gamma(E)$, we shall mean a sequence of continuous linear operators

$$P_i\colon \Gamma(E_{i+1}) \longrightarrow \Gamma(E_i)$$

so that

$$d_{i-1}P_{i-1} + P_i d_i = 1 - S_i \ ,$$

where S_i is a *smooth* endomorphism of $\Gamma(E_i)$. Note that $S_{i+1}d_i = d_i P_i d_i = d_i S_i$ so that this defines a *smooth endomorphism* of the complex $\Gamma(E)$.

PROPOSITION (6.1). *For any elliptic complex* $\Gamma(E)$, *there is a pseudo-differential parametrix*, i.e., $P_i \in L^{-m_i}(E_{i+1}, E_i)$, *where* m_i *is the order of* $d_i\colon \Gamma(E_i) \to \Gamma(E_{i+1})$.

PROOF. Fix a riemannian metric on X. Then the leading symbol of a differential operator on X may be regarded as a function on the unit cotangent sphere bundle of X. For each i, we now take an elliptic differential operator

$$Q_i\colon \Gamma(E_i) \longrightarrow \Gamma(E_i)$$

of order $2(N - m_i)$ whose leading symbol is the identity (e.g., take Q_i to be the $(N - m_i)^{\text{th}}$ power of a *Laplace operator* for E_i). Here N is any integer so that $N - m_i \geqq 0$ for all i. We then define

$$\Delta_i\colon \Gamma(E_i) \longrightarrow \Gamma(E_i)$$

by

[8] X is compact throughout this section.

$$\Delta_i = Q_i d_i^* d_i + d_{i-1} Q_{i-1} d_{i-1}^*$$

where $d_i^*: \Gamma(E_{i+1}) \to \Gamma(E_i)$ is the adjoint of d_i relative to smooth hermitian metrics in the E_j and the riemannian measure on X. Clearly Δ_i is a differential operator of order $2N$, and its leading symbol $\sigma(\Delta_i)$ is given by

$$\sigma(\Delta_i) = \sigma(d_i)^* \sigma(d_i) + \sigma(d_{i-1})\sigma(d_{i-1})^* .$$

Now the definition of an elliptic complex is that, for any $x \in X$ and any non-zero ξ in the cotangent space at x, the sequence

$$\cdots \longrightarrow E_{i-1,x} \xrightarrow{\alpha_{i-1}} E_{i,x} \xrightarrow{\alpha_i} E_{i+1,x} \longrightarrow \cdots$$

is exact, where α_i is $\sigma(d_i)$ at the point (x, ξ). Elementary algebra then implies that

$$\sigma(\Delta_i)_{x,\xi}: E_{i,x} \longrightarrow E_{i,x}$$

is an isomorphism. In fact each $E_{i,x}$ is decomposed as a direct sum

$$E_{i,x} = \alpha_{i-1}(E_{i-1,x}) \oplus \alpha_i^*(E_{i+1,x}) ,$$

and $\alpha_{i-1}\alpha_{i-1}^*$ is an isomorphism on the first factor, zero on the second, with the reverse situation for $\alpha_i^*\alpha_i$. Hence Δ_i is an *elliptic differential operator* of order $2N$. Moreover we have (since $d_i d_{i-1} = 0$)

$$d_i \Delta_i = d_i Q_i d_i^* d_i = \Delta_{i+1} d_i , \tag{6.1}$$

so that the sequence of Δ_i defines an endomorphism of the complex $\Gamma(E)$. Now by (3.5), we can find a pseudo-differential parametrix R_i for Δ_i so that

$$R_i \Delta_i \sim 1 \quad \text{and} \quad \Delta_i R_i \sim 1 . \tag{6.2}$$

Multiplying (6.1) on the left by R_{i+1}, and on the right by R_i, we then obtain

$$R_{i+1} d_i \sim d_i R_i . \tag{6.3}$$

We now claim that

$$P_i = R_i Q_i d_i^*$$

is the required parametrix for $\Gamma(E)$. In fact we have

$$\begin{aligned} P_i d_i + d_{i-1} P_{i-1} &= R_i Q_i d_i^* d_i + d_{i-1} R_{i-1} Q_{i-1} d_{i-1}^* \\ &\sim R_i Q_i d_i^* d_i + R_i d_{i-1} Q_{i-1} d_{i-1}^* && \text{(by (6.3))} \\ &\sim R_i \Delta_i \\ &\sim 1 . \end{aligned}$$

and so $P_i d_i + d_{i-1} P_{i-1} = 1 - S_i$ with S_i smooth.

REMARK. If the m_i are all equal, we can take $N = m_i$ and all $Q_i = 1$. Then

$$\Delta_i = d_i^* d_i + d_{i-1} d_{i-1}^*$$

is an obvious generalisation of the Laplace operator of the Hodge theory.

Before proceeding further, let us review some facts about compact operators. A linear map $F\colon V \to W$ of topological vector spaces is called a *Fredholm operator* if

(i) $\dim \operatorname{Ker} F < \infty$;

(ii) $F(V)$ is closed;

(iii) $\dim \operatorname{Coker} F < \infty$.

The following result is usually proved for Banach spaces, but is actually true [5; § 9.6] for all locally convex Hausdorff topological vector spaces.

PROPOSITION (6.4). *If $K\colon V \to V$ is a compact linear operator, then $1 - K$ is a Fredholm operator.*

REMARK. The proof of properties (i) and (ii) is practically the same for general V as for Banach spaces. Property (iii) is proved by duality, using the dual V' with the topology of compact convergence, and applying (4.1), one verifies at once that K' is compact. We shall apply (6.4) when K is a smooth operator $\Gamma(E) \to \Gamma(E)$. A smooth operator (on a compact manifold) is compact because it factors through the compact inclusion

$$\Gamma(E) \longrightarrow \mathfrak{D}'(E)$$

of the smooth sections in the distributional sections.

Using (6.4) and (6.1), we can now establish the following basic properties of an elliptic complex

PROPOSITION (6.5). *Let*

$$\cdots \longrightarrow \Gamma(E_i) \xrightarrow{d_i} \Gamma(E_{i+1}) \longrightarrow \cdots$$

be an elliptic complex. Then

(i) $\dim H^i(\Gamma(E)) < \infty$ *for all* i;

(ii) $d_i\Gamma(E_i)$ *is closed in* $\Gamma(E_{i+1})$;

(iii) $\operatorname{Ker} d_i$ *is a topological direct factor of* $\Gamma(E_i)$.

PROOF. Let $Z_i = \operatorname{Ker} d_i$, $B_i = \operatorname{Im} d_{i-1}$. Let P be a parametrix for $\Gamma(E)$ so that

$$d_{i-1}P_{i-1} + P_i d_i = 1 - S_i \ ,$$

with S_i smooth and so compact. For $u \in Z_i$, we have

$$d_{i-1}P_{i-1}u = (1 - S_i)u \ .$$

Hence $B_i \supset (1 - S_i)Z_i$. But the restriction of S_i maps Z_i to Z_i and so, by (6.4), $(1 - S_i)Z_i$ is closed and of finite co-dimension in Z_i. The same is then true of B_i and, as Z_i is closed in $\Gamma(E_i)$, we deduce (i) and (ii). To prove (iii), we consider the map

$$F_i\colon Z_i \oplus B_{i+1} \longrightarrow \Gamma(E_i)$$

given by

$$F_i(z \oplus b) = z + P_i b \ .$$

Then $\operatorname{Im} F_i \supset \operatorname{Im}(1 - S_i)$ and so, by (6.4), it is closed and of finite co-dimension. Also we have

$$\begin{aligned}(1 - S_{i+1})b &= (d_i P_i + P_{i+1} d_{i+1})b = d_i P_i b \\ &= -d_i F_i(z \oplus b)\end{aligned}$$

which shows that

$$\dim \operatorname{Ker} F_i \leqq \dim \operatorname{Ker}(1 - S_{i+1}) < \infty \qquad \text{by (6.4).}$$

Thus F_i is a Fredholm operator, and it induces an isomorphism (algebraic and hence topological because all spaces are Frechet) of $Z_i \oplus (B_{i+1}/P_i^{-1}(Z_i) \cap B_{i+1})$ onto a closed subspace of finite codimension W_i of $\Gamma(E_i)$. If $W_i^{\perp}$ denotes a complement of W_i, it follows that

$$P_i B_{i+1}/P_i B_{i+1} \cap Z_i + W_i^{\perp}$$

is a complement of Z_i in $\Gamma(E_i)$. This proves (iii).

Using (6.1) and the results of § 4, it is now easy to give the proof of Lemma (2.6). Let P be a pseudo-differential parametrix for $\Gamma(E)$, and let R^t be a family of smooth operators converging to 1 as in (2.7). Put $P^t = (1-R^t)P$. Then, for $t > 0$, $R^t P$ is smooth, and so P^t is still a parametrix. Thus

$$P^t d + dP^t = 1 - K^t \ ,$$

where K^t is (for $t > 0$) a smooth endomorphism of $\Gamma(E)$. By (3.4) $K^t \longrightarrow 1$ in the sense of (ii) and (iii) in (2.6) as $t \longrightarrow 0$. Suppose now that $T_i = Q_i f^* P_i$ is a pseudo-differential endomorphism of $\Gamma(E)$. Then putting $S_i^t = P_i K_i^t$, we obtain a family $T^t = TK^t$ of smooth endomorphisms of the complex $\Gamma(E)$ which converge to T in the sense required for (2.6).

REMARK. Note that our method of constructing the family T^t does a little more than required. In fact, the induced action on homology is independent of t.

7. Trace of smooth endomorphisms

This section is devoted to the proof of (2.4), the *alternating sum formula* for smooth endomorphisms. As pointed out in § 2, this follows from (2.1), a purely algebraic result, and (2.3), an approximation lemma.

We begin with the general algebraic situation. An endomorphism of finite rank of a vector space V may be identified with an element of $V \otimes V^*$, where V^* is the algebraic dual. The trace is then just the natural linear form on $V \otimes V^*$ given by the pairing. Note in particular when $V = \Gamma(E)$, we have the subspace

$$\Gamma(E) \otimes \Gamma(E') \qquad \text{of } V \otimes V^*$$

consisting of smooth endomorphisms of finite rank. The trace of an element

$$s \otimes t \in \Gamma(E) \otimes \Gamma(E')$$

is therefore given by $\int \operatorname{Trace} s \otimes t$. On the other hand, the Schwartz kernel of this endomorphism is just

$$K(x, y) = s(x) \otimes t(y) .$$

This therefore justifies the formula (2.2) for such endomorphisms.

To prove (2.1), we first establish the additivity of traces for short exact sequences. Namely, if T is an endomorphism of finite rank of the exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 ,$$

then $b = a + c$, where we write $a = \operatorname{Trace} T \mid A$, etc. If $\dim C < \infty$, we can split the sequence so that $B = A \oplus C$, and the result is then easy. For the general case, we observe that, since T has finite rank, it factors through a finite-dimensional space. Thus

$$C \xrightarrow{T} C_0 \subset C$$

and the trace on C clearly coincides with the trace of the restriction to C_0. Replacing C by C_0 (and B by the inverse image) we are reduced to the previous case, and the additivity is therefore proved.

For a general complex V, we then put $Z_i = \operatorname{Ker} d_i$, $B_i = \operatorname{Im} d_{i-1}$, so that we have short exact sequences

$$(7.1) \quad 0 \longrightarrow Z_i \longrightarrow V_i \longrightarrow B_{i+1} \longrightarrow 0 \qquad 0 \longrightarrow B_i \longrightarrow Z_i \longrightarrow H_i \longrightarrow 0 .$$

If we write $z_i = \operatorname{Trace} T \mid Z_i$, etc., then the additivity of traces implies that

$$v_i = z_i + b_{i+1} , \qquad h_i = z_i - b_i .$$

From these, we deduce at once

$$\sum (-1)^i v_i = \sum (-1)^i z_i - \sum (-1)^i b_i = \sum (-1)^i h_i ,$$

which proves (2.1).

Suppose now that V is a complex of topological vector spaces, with the

d_i continuous. We shall say that V is a *split complex* if both exact sequences (7.1) have topological splittings. This means we have topological direct sum decompositions

$$V_i \simeq Z_i \oplus B_{i+1} \qquad Z_i \simeq B_i \oplus H_i ,$$

and hence $V_i = B_i \oplus H_i \oplus B_{i+1}$.

In particular, this implies that B is a closed subspace of V_i. Note that the dual of a split complex is split.

If the V_i are Frechet spaces, then the following conditions together imply that V is split.

(i) $\dim H_i < \infty$,

(ii) B_i is closed in V_i,

(iii) Z_i is a topological direct factor of V_i.

In fact, the decomposition

$$Z_i \cong B_i \oplus F_i$$

follows at once from (i) and (ii). Using (iii), we can decompose $V_i = Z_i \oplus W_i$, and the projection $V_i \rightarrow B_{i+1}$ then induces an algebraic isomorphism $W_i \rightarrow B_{i+1}$. Since both of these are Frechet spaces (being closed subspaces of Frechet spaces), it follows that $W_i \cong B_{i+1}$ topologically.

Hence Proposition (6.5) implies that *an elliptic complex* $\Gamma(E)$, *and hence also its dual* $\Gamma(E)'$, *is split.*

If V, W are topological vector spaces, then as before we denote by $\mathfrak{L}(V, W)$ the space of continuous linear map $V \rightarrow W$ with the topology of bounded convergence. The subspace of operators of finite rank, we denote by $\mathfrak{L}_f(V, W)$. We employ a similar notation for homomorphisms of complexes. Thus $\mathfrak{L}(V, W)$ is the subspace of $\prod \mathfrak{L}(V_i, W_i)$ consisting of all $T = \prod T_i$ such that $d_i T_i = T_{i+1} d_i$, and $\mathfrak{L}_f(V, W)$ is the subspace for which each $T_i \in \mathfrak{L}_f(V_i, W_i)$.

We then have the following easy lemma.

LEMMA (7.2). *Let V, W be two split complexes of topological vector spaces. Assume that, for all i, $\mathfrak{L}_f(V_i, W_i)$ is dense in $\mathfrak{L}(V_i, W_i)$. Then $\mathfrak{L}_f(V, W)$ is dense in $\mathfrak{L}(V, W)$.*

PROOF. The hypotheses imply that, for any topological direct factors $X_i \subset V_i$, $Y_i \subset W_i$, $\mathfrak{L}_f(X_i, Y_i)$ is dense in $\mathfrak{L}(X_i, Y_i)$. This can be applied to the various spaces Z_i, B_i, H_i. Now an endomorphism of a split exact sequence is given by a *triangular* 2×2 matrix. Hence we can approximate endomorphisms of the exact sequences (7.1) by ones of finite rank. Moreover any approximation of the first term can be extended to an approximation of the whole sequence. To construct a finite rank approximation S to a given endomorphism T of V, we now proceed inductively over i. Assume S_j constructed for $j<i$. Then the induced endomorphism of B_i is given, and may be extended to one of Z_i and then to one of V_i.

We can now proceed to the proof of (2.3). Let $\Gamma(E)$ be an elliptic complex. We can then define a transpose complex $\Gamma(E')$ where

$$d'_i\colon \Gamma(E'_{i+1}) \longrightarrow \Gamma(E'_i)$$

is the transpose differential operator to d_i, i.e.,

$$\langle d_i u, v\rangle = \langle u, d'_i v\rangle \qquad\qquad u \in \Gamma(E_i),\ v \in \Gamma(E'_{i+1}) .$$

It is trivial to verify that $\Gamma(E')$ is also elliptic. Now as remarked above, $\Gamma(E)$ and $\Gamma(E')$ are split complexes, and so is the dual $\Gamma(E')'$. Since $\mathfrak{L}(\Gamma(E'_i)', \Gamma(E_i))$ is just the space of smooth endomorphisms of $\Gamma(E_i)$, the subspace $\mathfrak{L}_f$ is dense. Hence applying (7.2) with

$$V = \Gamma(E')' , \qquad W = \Gamma(E) ,$$

we deduce (2.3) as required.

8. Alternative methods

We shall now discuss two variations of the proof of Theorem A.

Homological method. The first variation, although essentially equivalent in analytical content, involves a different algebraic point of view. It is well known in various *homology theories* in topology and algebraic geometry that a Lefschetz fixed-point formula is a purely formal consequence of three things:

(i) Künneth formula: giving the cohomology of a product,

(ii) Poincaré duality: the isomorphism between homology and cohomology,

(iii) compatibility between *intersection of cycles* and cup-product of cohomology classes.

The precise nature of these statements depends of course on the context. For those familiar with this homological point of view, it may therefore be helpful to explain what form (i)—(iii) take in the context of elliptic complexes. Once these have been established, our Lefschetz theorem will follow in the routine manner.

Let $\Gamma(E)$ be an elliptic complex on X, $\Gamma(F)$ an elliptic complex on Y. Then[9] we can define in a natural way an elliptic complex $\Gamma(E \boxtimes F)$ on $X \times Y$: in fact $\Gamma(E \boxtimes F)$ will be a completion in the usual way of the complex $\Gamma(E) \otimes \Gamma(F)$. Then we have

[9] At least when all differentials d_i are of the same order. In the general case it would be necessary to enlarge the definition of ellipticity. We could take (6.1) as the definition.

Künneth formula. $H(\Gamma(E \boxtimes F)) \cong H(\Gamma(E)) \otimes H(\Gamma(F))$. This is in fact a special theorem of a more general theorem of Grothendieck about topological tensor products. It can also be proved by Hilbert space methods.

Next we have

Poincaré duality. The inclusion of complexes

$$\Gamma(E) \longrightarrow \Gamma(E')'$$

induces an isomorphism of homology.

This is an immediate consequence of the existence (6.1) of a parametrix P. In fact if

$$dP + Pd = 1 - S ,$$

then S is chain homotopic to 1, and (since it is smooth) retracts $\Gamma(E')'$ onto $\Gamma(E)$.

Finally we come to the formulation of (iii). Let $\Gamma(E)$ be an elliptic complex on X, and let Y be a closed submanifold of X. Suppose $a \in \Gamma(\mathrm{Hom}\,(E' \mid Y, \Omega(Y))$. Then a defines an element $\alpha \in \Gamma(E')'$ as follows

$$\alpha(s) = \int_Y a(s \mid Y) \qquad s \in \Gamma(E') .$$

If $d\alpha = 0$, we shall say that α is a geometric cycle of $\Gamma(E)$ carried by Y. Suppose now that Z is another closed submanifold of X, with $\dim Z + \dim Y = \dim X$, and that Y and Z intersect transversally in a finite set of points $\{A\}$. Let β be a geometric cycle of $\Gamma(E')$ carried by Z, and defined by an element $b \in \Gamma(\mathrm{Hom}\,(E \mid Z, \Omega(Z))$. Then at any point $A \in Y \cap Z$, we have

$$a_A \in E_A \otimes \Omega(X)^*_A \otimes \Omega(Y)_A$$
$$b_A \in E^*_A \otimes \Omega(Z)_A ,$$

so that

$$a_A \otimes b_A \in E^*_A \otimes E_A .$$

Thus we can form

$$\mathrm{Trace}\, a_A \otimes b_A \in \mathbf{C} .$$

The sum $\sum_A \mathrm{Trace}\, a_A \otimes b_A$ may be called the *intersection number* of the two geometric cycles α, β.

Now the pairing

$$\Gamma(E) \otimes \Gamma(E') \longrightarrow \mathbf{C}$$

is compatible with the differentials, and so it induces a pairing in homology

$$H(\Gamma(E)) \otimes H(\Gamma(E')) \longrightarrow \mathbf{C} .$$

In view of the Poincaré duality, we may therefore form the *cup-product* $[\alpha]\,[\beta]$

of the homology classes of α, β.

With these explanations, we can now formulate property (iii): *the intersection number of two geometric cycles α, β is equal to the cup product $[\alpha]\,[\beta]$ of their cohomology classes.*

The proof of this involves the same local analysis as the proof of (2.5). If one wants to prove Theorem B, and not just Theorem A, then it is necessary to generalize the notion of *geometric cycle carried by Y*. One has to allow α to involve not only restrictions to Y, but also normal derivatives. In classical terminology α would be a *multiple layer*, smooth along Y.

Zeta-function method. The second variation is based on the *Zeta-functions* of [9] suitably generalized. As we shall explain, this method requires more subtle analysis but, in principle, it leads to more general Lefschetz formulas than the ones considered here. In particular, one can obtain an index formula, although not the topological formula of [3].

Suppose first that Δ is an elliptic differential operator on a compact manifold X which is (strictly) positive and self-adjoint, relative to suitable metrics. Then one can show (cf. [10]):

(i) Δ^{-s} is an operator of trace class (in the Hilbert space sense) for $\mathrm{Re}(s)$ large,

(ii) $\zeta(s) = \mathrm{Trace}\,\Delta^{-s}$ is an analytic function of s having a meromorphic continuation to the whole s-plane,

(iii) the value of $\zeta(s)$ for $\zeta = 0, 1, 2, \cdots$ can be computed as the integral over X of an expression explicitly constructed from Δ.

Suppose now that $\Gamma(E)$ is an elliptic complex, and let

$$\Delta_i = 1 + d_i^* d_i + d_{i-1} d_{i-1}^* ,$$

where d^* is the adjoint of d. Assuming for simplicity that all d_i have the same order, it follows that Δ_i is elliptic, self-adjoint, and strictly positive. Thus $\zeta_i(s) = \mathrm{Trace}\,\Delta_i^{-s}$ is defined. Now it is not difficult to see that

$$\sum (-1)^i \zeta_i(s) = \sum (-1)^i \dim H^i\bigl(\Gamma(E)\bigr) \tag{8.1}$$

for $\mathrm{Re}(s)$ large. Hence by analytic continuation this holds for all s. In particular taking $s = 0$, we obtain by (iii) an explicit integral formula for the Euler characteristic (or index) of $\Gamma(E)$. Unfortunately this explicit expression is, in general, very complicated. For example it involves the n^{th} derivatives of the coefficients of d where $n = \dim X$. Except in low dimensions, it seems very hard to reduce this formula to that given in [3]. Note that the formula of [3] can also be written in integral form using the representation of characteristic classes by curvature forms. This involves, however, only one or two derivatives whatever the dimension of X.

For maps $f\colon X \to X$ with simple fixed points, however, the situation is much simpler. In this case, if T is a geometric endomorphism of $\Gamma(E)$, defined by f and bundle maps φ_i, we define new zeta-functions (depending on T) by

$$\zeta_i(s) = \operatorname{Trace}(\Delta_i^{-s} T_i) .$$

Because f has simple fixed points, it turns out that this zeta-function is *holomorphic* for all s and that

$$\zeta_i(0) = \sum_{f(A)=A} \frac{\operatorname{Trace} \varphi_{i,A}}{|\det(1 - df_A)|} .$$

This follows by applying (2.5). We need to know of course that Δ_i^{-s} is a pseudo-differential operator depending analytically on s. This is one of the main results of [10]. Now just as in (8.1) one can prove

$$\sum (-1)^i \zeta_i(s) = L(T) \tag{8.2}$$

for Re (s) large. Putting $s = 0$, one then obtains our Lefschetz formula.

As a proof of Theorem A, this method is unnecessarily sophisticated. We need (2.5), which is the main step in the proof given in detail in this paper; but, in addition, we need the results of [10] concerning the fractional powers Δ^{-s}. It is only for the index formula, or more generally for intermediate maps f with higher-dimensional fixed-point sets, that this more delicate approach is needed.

It has been pointed out to us by Hörmander that the *heat operator* $e^{-\Delta t}$ can be used instead of the fractional powers Δ^{-s}, and gives essentially identical results. This has technical advantages because all that one needs for this treatment is the pseudo-differential operator calculus for parabolic operators, and this is included in [6].

Appendix

We shall now discuss in a little more detail some of the points involved in passing from pseudo-differential operators on an open set $U \subset R^n$ to operators on a manifold. For simplicity, we shall consider only functions and not sections of vector bundles. The extension to vector bundles is quite straightforward, using matrices instead of scalars. The effect of a change of basis in the fibres is covered by the local form of (3.4).

LEMMA (A.1). *Let U be a coordinate patch on X, $P \in L^m(U)$ and φ, $\psi \in \mathfrak{D}(U)$. Then $\varphi P \psi$ is a pseudo-differential operator on X of order m and lies in a bounded set if P does.*

PROOF. Let V be another coordinate patch on X, and let $L \subset V$ be compact. Then we have to show that the estimate (3.1) holds for $f \in \mathcal{D}(L)$, K compact in V and

$$p = e^{-i\langle x,\xi\rangle}\varphi(x)P\{\psi(x)f(x)e^{i\langle x,\xi\rangle}\}$$

where (x) are the coordinates of V. Moreover we want (3.1) to hold uniformly for a bounded set of P. But since $P \in L^m(U)$, its restriction to $U \cap V$ belongs to $L^m(U \cap V)$, and here we can use either coordinate system by the basic invariance established in [6; (2.16)]. Since $\psi f \in \mathcal{D}(U \cap V)$, the required estimates then follow.

Before proceeding to examine composition globally, we need a simple lemma about compact spaces.

LEMMA (A.2). *Let K_i $(i = 1, \cdots, n)$ be compact spaces. Then any open covering of $\prod_{i=1}^n K_i$ can be refined by a product of finite open coverings of the factors,* i.e., *by a covering of the form $\{U_I\}$, where $I = (i_1, \cdots, i_n)$,*

$$U_I = U^1_{i_1} \times U^2_{i_2} \times \cdots \times U^n_{i_n}$$

and $\{U^j_i\}$ is a finite covering of K_j.

PROOF. By induction on n we see that we need only treat the case $n=2$. Let $\{U_i\}$ be a given covering of $K_1 \times K_2$. Since $x \times K_2$ is compact, for each $x \in K_1$, we can find an open set $U^1_x \subset K_1$ and a finite open covering $\{V^2_{x,j}\}$ of K_2 so that

$$U^1_x \times V^2_{x,k} \subset U_i \qquad \text{for some } i = i(x, k) .$$

Since K_1 is compact, we can then choose a finite set $x_1, \cdots, x_n$ so that the sets $U^1_j = U^1_{x_j}$ cover K_1. Now let $\{U^2_i\}$ be a common refinement of the finite set of coverings $\{V^2_{x_j,k}\}$. Then the product covering $\{U^1_j \times U^2_i\}$ of $K_1 \times K_2$ refines $\{U_i\}$ as required.

Next let us observe again that a coordinate patch on a manifold X need not be connected. Thus if $x, y \in X$ with $x \neq y$, we can always find a disconnected coordinate patch containing both x and y. We can therefore find a family $\{U_i\}$ of coordinate patches so that the family $\{U_i \times U_i\}$ cover $X \times X$. Similarly we could choose one so that $\{U_i \times U_i \times U_i\}$ covers $X \times X \times X$.

We can now establish

PROPOSITION (A.3). *Let P, Q be pseudo-differential operators on X of orders m, n respectively and let $g \in \mathcal{D}(X)$. Then PgQ is pseudo-differential of order $m + n$ and varies in a bounded set if P or Q does (the other being fixed).*

PROOF. Let $\{U_i\}$ be a coordinate covering so that $\{U_i \times U_i \times U_i\}$ covers $X \times X \times X$. Let V be any other coordinate patch of X, $K_1, K_3 \subset V$ compact sets and put $K_2 = \operatorname{supp} g$. By (A2) we can find finite collections of open sets $\{U_j^i\}$, $(i = 1, 2, 3; j = 1, \cdots, n)$ of X so that

(1) $U_j^i \cap K_i$ cover K_i,

(2) for each triple j, k, l, there exists an i so that $U_j^1 \subset U_i$, $U_k^2 \subset U_i$, $U_l^3 \subset U_i$.

Now let φ_j^i be a partition of unity on K_i $(i = 1, 2, 3)$. Thus $\varphi_j^i \in \mathfrak{D}(U_j^i)$, $\varphi_j^i \geqq 0$ and

$$\sum_j \varphi_j^i = 1 \qquad \text{on } K_i\,.$$

We want now to establish the estimates (3.1) for $x \in K_3$, $f \in \mathfrak{D}(K_1)$ and

$$p = e^{-i\langle x,\xi\rangle} PgQfe^{i\langle x,\xi\rangle}\,.$$

Using the partitions of unity, we can write this as a sum $p = \sum p_{jkl}$ where

$$p_{jkl} = e^{-i\langle x,\xi\rangle}\varphi_l^3 P\varphi_k^2 gQf\varphi_j^1 e^{i\langle x,\xi\rangle}\,.$$

By our choice of the U_j^i we have $\operatorname{supp}\varphi_l^3$, $\operatorname{supp}\varphi_k^2 g$, $\operatorname{supp} f\varphi_j^1$, all contained in one coordinate patch U_i. By the basic formula [6; (2.10)′] for composition of operators defined in U_i, we then obtain estimates for P_{jkl} with x replaced by the coordinates y of U_i. However, since the composite operator is from $\mathfrak{D}(U_i \cap V) \to \mathfrak{E}(U_i \cap V)$ the estimates with y imply those with x, by the invariance under change of coordinates.

Finally we come to the question of a global parametrix. If $P_i \in L^m(U_i)$ and $\varphi_i, \psi_i \in \mathfrak{D}(U_i)$, then by (A.1), $\sum \varphi_i P_i \psi_i$ is a pseudo-differential operator of order m on X.

PROPOSITION (A.4). *Let d be an elliptic differential operator on X. Let $\{U_i\}$ be a coordinate covering, P_i a parametrix for $d_i = d_{U_i}$, φ_i a partition of unity with $\varphi_i \in \mathfrak{D}(U_i)$, and $\psi_i \in \mathfrak{D}(U_i)$ with $\psi_i = 1$ on* $\operatorname{supp}\varphi_i$. *Then $P = \sum \varphi_i P_i \psi_i$ is a parametrix for d.*

PROOF. By hypothesis, we have

$$(1) \qquad P_i d = 1 + S_i\,, \qquad dP_i = 1 + T_i$$

where S_i, T_i are smooth operators on U_i. We have to prove that $Pd - 1$ and $dP - 1$ are both smooth. Since P is a pseudo-differential (A.1), so are Pd and dP (A.3), and all operators are smooth outside the diagonal of $X \times X$. Thus it is sufficient to prove that the restrictions to each U_i are smooth. Since d is differential, and so local, we can compute the restriction of Pd and dP by restricting each factor.

Now in $U_i \cap U_j$, we have $P_i - P_j = Q_{ij}$ where Q_{ij} is smooth. The symbol of a parametrix is unique (see for example [6]). Hence in U_i we have

$$
\begin{aligned}
P &= \sum_j \varphi_j P_j \psi_j = \sum_j \varphi_j (P_i + Q_{ij}) \psi_j \\
&= \sum_j \varphi_j P_i + \sum_j \varphi_i P_i(\psi_j - 1) + \sum_j \varphi_j Q_{ij} \psi_j \qquad (2) \\
&= P_i + A_i + B_i \, .
\end{aligned}
$$

Now B_i is smooth because Q_{ij} is, and A_i is smooth because φ_j and $(\psi_j - 1)$ have disjoint support. Composing with d these remain smooth, and the required results then follow from (1) and (2).

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