A Lefschetz fixed point formula for elliptic complexes: II. Applications*

By M. F. ATIYAH and R. BOTT

1. Introduction

Part I of this paper described an extension of the classical Lefschetz theorem in the framework of elliptic complexes. This second part is devoted to the applications and examples of this extension which for the most part were announced in [1].

Essentially these applications arise by specializing our formula to those elliptic complexes or operators which occur naturally in geometry as resolutions of certain sheaves associated to a G-structure on a manifold. Thus one has the de Rham complex giving a resolution of the locally constant sheaf; the d''-complex giving a resolution of the sheaf of germs of holomorphic functions on a complex manifold; the "signature operator" on a Riemann manifold, and the Dirac operator on a Riemann Spin manifold giving resolutions of the harmonic $*$ invariant forms and spinors.

Our formula applied to the de Rham complex of course yields only the classical theorems. However, in the other instances one obtains new invariants for mappings, which preserve the sheaves in question, and from which one may draw interesting conclusions.

To make this paper as self contained as possible we will start by reviewing our general fixed point theorem and also by discussing the de Rham complex once again because it plays such an essential role in all the cases. In § 4 we discuss the complex case and give some examples. One of these, dealing with induced representations, is so extensive that we have dealt with it in a section of its own, § 5. In § 6 we deal with the riemannian case deriving a formula (6.26), for the signature of an isometry. Some geometric applications of this formula are then given in § 7. The Dirac operator on Spin manifolds is dealt with in § 8, and applications to fixed-point-free involutions on spheres are given in § 9. We draw particular attention to the results in § 9 because they were not mentioned in [1].

2. The fixed point theorem reviewed

Underlying our whole discussion will be a smooth (i.e., C^∞) compact

* This research was partially supported by a grant from NSF-GP-6585.

manifold X. If E is a (smooth is always understood) vector bundle over X, $\Gamma(E)$ denotes the smooth sections of E.

To define the notion of an elliptic complex we need first of all to recall the definition of the leading symbol $\sigma(d)$ of a differential operator of order k,

$$d\colon \Gamma(E) \longrightarrow \Gamma(F)$$

between two bundles E and F. This symbol is constructed from the highest order terms of d and can be defined invariantly as follows.

Let $s \in \Gamma(E)$ be a section of E, let g be a smooth real valued function, and let λ be a real parameter. Then, $e^{-i\lambda g} d e^{i\lambda g} s \in \Gamma(F)$ will be a polynomial of order k in λ

$$e^{-i\lambda g} d e^{i\lambda g} s = \lambda^k p_k(g, s) + \cdots + p_0(g, s) \tag{2.1}$$

whose coefficients depend only on d, s, and g. In fact, the correspondence $s \mapsto p_j(g, s)$ is a differential operator of degree $k - j$, and $p_0(g,s) = ds$. Furthermore, the value of the leading coefficient $p_k(g, s)$ at a point $P \in M$ depends only on $s(P)$ and on the value of the differential of g at P. Finally, the assignment

$$s\,|_P \longmapsto i^{-k} p_k(g, s)_P$$

is linear in s and so defines, for each cotangent vector $\eta \in T_P X$, a linear function

$$\sigma_d(\eta)\colon E_P \longrightarrow F_P\ . \tag{2.2}$$

This is the *leading symbol map of d at the cotangent vector η*, and as η ranges over the cotangent bundle TX of X, these maps combine smoothly to yield a bundle homomorphism

$$\sigma(d)\colon \pi^* E \longrightarrow \pi^* F$$

of the pullbacks of E and F to TX under the natural projection $\pi\colon TX \to X$.

With this understood, the definition of an elliptic complex, $\Gamma(E)$, over X runs as follows.

DEFINITION 2.3. *We suppose given a sequence $E = \{E^k\}$ $k \in \mathbf{Z}$ of complex vector bundles over X, with E^k the zero bundle except for a finite number of indices, together with differential operators $d = \{d_k\}$,*

$$d_k\colon \Gamma(E^k) \longrightarrow \Gamma(E^{k+1})$$

of orders ρ_k, subject to the conditions

(2.4) $d_k \circ d_{k-1} = 0$, $k \in \mathbf{Z}$.

(2.5) *The leading symbol sequence of the d_k:*

$$\longrightarrow \pi^*E^k \xrightarrow{\sigma(d_k)} \pi^*E^{k+1} \xrightarrow{\sigma(d_{k+1})} \pi^*E^{k+2} \longrightarrow$$

is exact over the complement of the zero section in TX. *These data we then call an "elliptic complex"* $\Gamma(E)$ *over* X.

We also define the homology $H^k\{\Gamma(E)\}$ of such a complex in the customary manner by

$$H^k\{\Gamma(E)\} = \operatorname{Ker} d_k/\operatorname{Image} d_{k-1} . \tag{2.6}$$

The condition (2.3), which is a straightforward extension of the classical ellipticity concept to complexes, together with the compactness of X, then has the important consequence that $H^k\{\Gamma(E)\}$ *is finite-dimensional.*

It follows that any endomorphism T of an elliptic complex $\Gamma(E)$ over a compact manifold X has a well-defined Lefschetz number:

$$L(T) = \sum(-1)^k \operatorname{trace} H^k(T) . \tag{2.7}$$

Here of course $H^k(T)$ denotes the endomorphism induced by T on $H^k\{\Gamma(E)\}$.

The main result of Part I [2] is a simple formula for the Lefschetz number of T when T is derived from a smooth map $f\colon X \to X$ by means of a "lifting of f to E". By definition such a lifting consists of a family $\varphi = \{\varphi^k\}$ of bundle homomorphisms

$$\varphi^k\colon f^*E^k \longrightarrow E^k$$

such that the induced maps $T_k\colon \Gamma(E^k) \to \Gamma(E^k)$ defined by the composition

$$\Gamma(E^k) \xrightarrow{f^*} \Gamma(f^*E^k) \xrightarrow{\Gamma(\varphi^k)} \Gamma(E^k)$$

combine to yield an endomorphism $T = \{T_k\}$ of $\Gamma(E)$ as a complex, that is, satisfy the condition

$$T_{k+1}d_k = d_kT_k .$$

The resulting endomorphism $T = T(f, \varphi)$ of $\Gamma(E)$ is then called a geometric endomorphism of $\Gamma(E)$ and is said to be *derived from* f by the *lifting* φ.

Concerning these we now quote the following theorem from Part I.

THEOREM A. *Let* $\Gamma(E)$ *be an elliptic complex over the compact manifold* X. *Also let* $f\colon X \to X$ *be an endomorphism of* X *whose graph is transversal to the diagonal* Δ *in* $X \times X$, *and let* T *be a geometric endomorphism of* $\Gamma(E)$ *derived from* f *by a lifting* φ. *Then the Lefschetz number* $L(T)$ *of* T *is given by the formula*:

$$L(T) = \sum\nolimits_p \nu(P) \tag{2.8}$$

where P *ranges over the fixed points of* f *and* $\nu(P)$ *is defined by*

$$\nu(P) = \frac{\sum(-1)^k \operatorname{trace} \varphi_P^k}{|\det(1 - df_P)|} . \tag{2.9}$$

Remarks. First of all recall that the transversality condition for f is quite equivalent to the requirement that $\det(1 - df_P) \neq 0$ at all fixed points of f. Hence the absolute value in the denominator in (2.9) is never zero. Secondly, observe that a lifting φ amounts to a family of linear maps

$$\varphi_P^k : E_{f(P)}^k \longrightarrow E_P^k$$

parametrized by $P \in M$. Hence at fixed points f *and only at these* φ_P^k has a well defined trace, because it is an endomorphism of the fiber E_P^k.

3. The de Rham complex

The most natural elliptic complex in geometry is the de Rham complex of a smooth manifold. Its bundles are the complexified[1] exterior powers $\Lambda^k = \mathbf{C} \otimes_{\mathbf{R}} \lambda^k TX$ of the cotangent bundle TX of X and its operator is the usual exterior derivative $d: \Gamma(\Lambda^k) \longrightarrow \Gamma(\Lambda^{k+1})$.

The resulting complex

$$\Gamma(\Lambda^* X) : 0 \longrightarrow \Gamma(\Lambda^0) \xrightarrow{d} \Gamma(\Lambda^1) \xrightarrow{d} \cdots \longrightarrow \Gamma(\Lambda^n) \longrightarrow 0$$

is then seen to be elliptic. Indeed $d^2 = 0$ as is well known. Furthermore, the symbol sequence is easily computed from the derivation property of d: given a form $\omega \in \Gamma(\Lambda^k)$ one has

$$i^{-1} e^{-i\lambda g} d e^{i\lambda g} \omega = \lambda\, dg \wedge \omega + d\omega \,.$$

Hence $\sigma_d(\eta) =$ exterior multiplication by η. The resulting symbol sequence

$$0 \longrightarrow \lambda^0 T_P X \xrightarrow{\sigma_d(\eta)} \lambda^1 T_P X \longrightarrow \cdots \xrightarrow{\sigma_d(\eta)} \lambda^n T_P X \longrightarrow 0$$

is then well known to be exact at every non-zero cotangent vector $\eta \in T_P X$.

This complex also behaves naturally with respect to smooth maps. The differential df of f maps the tangent space T_P^* into $T_{f(P)}^*$, so that the exterior powers of the transpose of df furnish bundle maps

$$\lambda^k (df)^* : f^* \lambda^k TX \longrightarrow \lambda^k TX$$

which (on complexification) combine to furnish a natural lifting, traditionally denoted by f^*, of f to $\Gamma(\Lambda^* X)$. The induced $H^k(f^*)$ then coincides with the endomorphism in the k^{th} cohomology of X with values in the constant sheaf of complex numbers $\mathbf{C}$, by virtue of de Rham's theorem. Indeed this theorem asserts that, on the sheaf level, the operator d defines a fine resolution of the constant sheaf

$$0 \longrightarrow \mathbf{C} \longrightarrow \mathcal{E}(\Lambda^0) \xrightarrow{d} \mathcal{E}(\Lambda^1) \cdots \longrightarrow \cdots \mathcal{E}(\Lambda^n) \longrightarrow 0 \,,$$

[1] The de Rham complex is of course *real* but to fit in with our general formulation we consider the complexification.

($\mathcal{S}(\Lambda^i)$ denoting the sheaf of germs of C^∞ sections of Λ^i) from which our interpretation of $H^k(f^*)$ follows readily. In short, $L(f^*)$ is just the classical Lefschetz number of the map f.

Let us now compute the multiplicity $\nu(P)$ of a fixed point for this complex. According to (2.9) we have

$$\nu(P) = \sum (-1)^k \operatorname{trace} (\lambda^k df_P^*)/|\det (1 - df_P)| .$$

On the other hand if A is any endomorphism of a finite dimensional vector space and $\lambda^k A$ and A^* denote its k^{th} exterior power and transpose respectively, then it is well known that

$$\sum(-1)^k \operatorname{trace} (\lambda^k A) = \det (1 - A)$$

and that

$$\det (A^*) = \det (A) .$$

It follows that our multiplicity reduces to

$$\nu(P) = \frac{\det (1 - df_P)}{|\det (1 - df_P)|}$$

so that Theorem A specializes to the classical formula for $L(f^*)$

$$L(f^*) = \sum_{f(P)=P} \pm 1$$

with $\pm 1 = \operatorname{sign} \det (1 - df_P)$.

Note finally that the de Rham construction extends naturally to the tensor product of Λ^* with any *locally constant vector bundle* F. The operator $1 \otimes d$ is then well defined on $F \otimes \Lambda^*$ and again yields an elliptic complex $\Gamma(F \otimes \Lambda^*)$. Furthermore if f is a transversal map and

$$\varphi: f^*F \longrightarrow F$$

is a locally constant bundle map, then the tensor products $\varphi \otimes \lambda^k(df)^*$ define a lifting of f to $\Gamma(F \otimes \Lambda^*)$ and our previous computations show that the Lefschetz number of the resulting endomorphism is given by

$$L(f^*, \varphi) = \sum_{f(P)=P} \pm \operatorname{trace} \varphi_P$$

where $\pm 1 = \operatorname{sign} \det (1 - df_P)$ as before.

This formula is also generally known. We mention it here only to complete the analogy with the complex analytic case, to be considered next.

4. The complex analytic case

Suppose now that X is a complex analytic manifold. The complex cotangent bundle then splits naturally into a direct sum of complex sub-bundles

$$\mathbf{C} \otimes_{\mathbf{R}} TX = T'X \oplus T''X \tag{4.1}$$

where $T'X$ is spanned by the dz_k and $T''X$ by the $d\bar{z}_k$ of a local holomorphic coordinate system. The bundle $T'X$ therefore has a natural *holomorphic* structure and $T''X$ a natural *antiholomorphic* structure.

Corresponding to the decomposition (4.1) the bundles of the complexified de Rham complex decompose canonically into the tensor product

$$\lambda^*(\mathbf{C} \otimes_{\mathbf{R}} TX) = \lambda^* T'X \otimes_{\mathbf{C}} \lambda^* T''X$$

so that, in particular,

$$\Lambda^k = \sum_{p+q=k} \Lambda^{p,q}$$

with

$$\Lambda^{p,q} = \lambda^p(T'X) \otimes_{\mathbf{C}} \lambda^q(T''X) \,. \tag{4.2}$$

The exterior derivative d decomposes correspondingly into a direct sum

$$d = d' + d''$$

where $d'\colon \Gamma\Lambda^{p,q} \longrightarrow \Gamma\Lambda^{p+1,q}$ and $d''\colon \Gamma\Lambda^{p,q} \longrightarrow \Gamma\Lambda^{p,q+1}$. Furthermore these components satisfy the integrability conditions

$$(d'')^2 = (d')^2 = 0 \,.$$

Observe now that already the operator d'' acting on all of $\Gamma(\Lambda^*)$ is an elliptic operator. Indeed $\sigma_{d''}(\eta)$, $\eta \in T_P X$, is easily seen to be the exterior multiplication by the $T''_P X$ component, η'', of η in the decomposition (4.1). Because η is real, $\eta'' \neq 0$ whenever $\eta \neq 0$, whence $\sigma_{d''}(\eta)$ gives rise to an exact sequence. Thus under d'', the complex $\Gamma(\Lambda^*)$ breaks up into a direct sum of, $n = \dim_{\mathbf{C}} X$, elliptic complexes

$$\Gamma(\Lambda^{p,*}X)\colon 0 \longrightarrow \Gamma(\Lambda^{p,0}) \xrightarrow{d''} \Gamma(\Lambda^{p,1}) \longrightarrow \cdots \xrightarrow{d''} \Gamma(\Lambda^{p,n}) \longrightarrow 0$$

whose cohomology spaces are traditionally denoted by $H^{p,q}(X)$.

More generally if F is any *holomorphic* vector bundle over X the operator $1 \otimes d''$ is well defined on $\Gamma(F \otimes \Lambda^{p,*})$ and so determines an elliptic complex on X whose cohomology spaces are usually denoted by $H^{p,q}(X; F)$. In particular the complex

$$\Gamma(F \otimes_{\mathbf{C}} \Lambda^{0,*})\colon 0 \longrightarrow \Gamma(F) \xrightarrow{1 \otimes d''} \Gamma(F \otimes \Lambda^{0,1}) \longrightarrow \cdots \longrightarrow \Gamma(F \otimes \Lambda^{0,n}) \longrightarrow 0$$

will be referred to as the d'' complex of F. On the sheaf level it furnishes us with a fine resolution of the sheaf $\mathcal{O}(F)$ *consisting of the germs of holomorphic sections of* F.

$$\begin{aligned} 0 \longrightarrow \mathcal{O}(F) \longrightarrow \mathcal{E}(F \otimes \Lambda^{0,0}) \longrightarrow & \mathcal{E}(F \otimes \Lambda^{0,1}) \\ & \longrightarrow \cdots \longrightarrow \mathcal{E}(F \otimes \Lambda^{0,n}) \longrightarrow 0 \end{aligned} \tag{4.3}$$

so that $H^q(X; \mathcal{O}(F)) \cong H^{0,q}(X; F)$. Using (4.2) it is then also clear that $H^q(X; \mathcal{O}(F \otimes \Lambda^{p,0})) \cong H^{p,q}(X; F)$, so that the d'' complex of a holomorphic bundle is the basic concept.

Consider now a *holomorphic* map $f: X \to X$. The natural lifting of f to Λ^* is then compatible with d'' and therefore induces endomorphisms $f^{p,*}$ in each of the complexes $\Gamma(\Lambda^{p,*})$. The corresponding endomorphism in homology will be denoted by $H^{p,q}(f)$ so that the Lefschetz numbers of $f^{p,*}$ are given by

$$L(f^{p,*}) = \sum (-1)^q \operatorname{trace} H^{p,q}(f) . \tag{4.4}$$

We note that in view of (4.2) and (4.3), $L(f^{p,*})$ may also be interpreted as the Lefschetz number of the endomorphism induced by f in the sheaf cohomology $H^*(X; \mathcal{O}(\Lambda^{p,0}))$.

Let us now compute the multiplicity of a transversal fixed point P of f relative to the complex $\Lambda^{p,*}$.

Because f is holomorphic the complexification of df preserves the decomposition (4.1) so that

$$1 \otimes df_P = d'f_P \oplus d''f_P$$

with $d'f_P$ and $d''f_P$ endomorphisms of $T'_P X$ and $T''_P X$ respectively. It follows that

$$\lambda^k\{1 \otimes df_P\} = \sum\nolimits_{p+q=k} \lambda^p\{d'f_P\} \otimes \lambda^q\{d''f_P\} .$$

Hence the multiplicity $\nu(P)$ in question is given by:

$$|\det(1 - df_P)| \cdot \nu(P) = \sum\nolimits_q (-1)^q \operatorname{trace} \lambda^p(d'f_P^*) \cdot \operatorname{trace} \lambda^q(d''f_P^*) .$$

This now yields[2]

$$\nu(P) = \operatorname{Trace}_{\mathbf{C}}(\lambda^p d'f_P^*) \cdot \frac{\det_{\mathbf{C}}(1 - d''f_P^*)}{|\det_{\mathbf{R}}(1 - df_P)|} . \tag{4.5}$$

Finally observe that under the bar operation in $\mathbf{C} \otimes_{\mathbf{R}} T_P X$, $T'_P X$ is taken into $T''_P X$. Furthermore $1 \otimes df_P$ clearly commutes with this operation. It follows that

$$\det_{\mathbf{C}}(1 - d'f_P) = \overline{\det_{\mathbf{C}}(1 - d''f_P)}$$

whence in particular

$$\det_{\mathbf{R}}(1 - df_P) = \det_{\mathbf{C}}(1 - 1 \otimes df_P) = \det_{\mathbf{C}}(1 - d'f_P) \cdot \det_{\mathbf{C}}(1 - d''f_P) \tag{4.6}$$

is positive. The absolute value sign in (4.5) is therefore redundant and one obtains the formula:

$$\nu(P) = \operatorname{trace}_{\mathbf{C}} \lambda^p(d'f_P) \cdot \frac{1}{\det_{\mathbf{C}}(1 - d'f_P)} . \tag{4.7}$$

[2] To avoid confusion we write $\det_F$ for the determinant of an endomorphism of F-vector spaces ($F = \mathbf{R}$ or $\mathbf{C}$), and similarly for trace_F.

In this case then only half of the determinantal factor in the denominator of (2.9) is cancelled out by the numerator. Note also that $\nu(P)$ can be expressed without introducing the complexification of TX. Indeed, over $\mathbf{R}$, T_PM is isomorphic to T'_PM and so inherits a complex structure. Hence if A is any *complex linear endomorphism* of T_PM we can speak of its complex $\operatorname{trace}_{\mathbf{C}} A$, and determinant, $\det_{\mathbf{C}} A$. At a fixed point P of the holomorphic map f, df_P is of course such an endomorphism and considered as a $\mathbf{C}$-endomorphism agrees with $d'f_P$. Hence in terms of these notions one has

$$\nu(P) = \operatorname{trace}_{\mathbf{C}} \lambda^p(df_P)/\det_{\mathbf{C}}(1 - df_P) . \tag{4.8}$$

To recapitulate: *for a transversal endomorphism f of a complex analytic manifold X, our fixed point formula specializes to*

$$L(f^{p,*}) = \sum_{f(P)=P} \operatorname{trace}_{\mathbf{C}} (\lambda^p df_P)/\det_{\mathbf{C}} (1 - df_P) \tag{4.9}$$

where

$$L(f^{p,*}) = \sum (-1)^q \operatorname{trace} H^{p,q}(f) . \tag{4.10}$$

It is this formula which Shimura conjectured during a conference at Woods Hole in 1965, and which furnished the impetus for this work. For curves (4.9) had already been established by Eichler in [11]. Shimura and Eichler were of course thinking in the framework of algebraic geometry. There it turned out that the full duality theory of Serre and Grothendieck yields this result even over arbitrary characteristic.

The formula (4.9) now has an easy extension to the d''-complex of an arbitrary holomorphic bundle F. To lift a map f to this complex, one only needs a *holomorphic* bundle homomorphism

$$\varphi: f^*F \longrightarrow F .$$

In terms of it

$$\varphi \otimes \lambda^k(d''f)^*: f^*(F \otimes \Lambda^{0,k}) \longrightarrow F \otimes \Lambda^{0,k}$$

then serves to define the k^{th} lifting of f. We write $T(f, \varphi)$ for the induced endomorphism of $\Gamma(F \otimes \Lambda^{0,*})$ and $H^q(f, \varphi)$ for the induced homomorphism in $H^q(X; F)$. We also set

$$L(f, \varphi) = \sum (-1)^q \operatorname{trace} H^q(f, \varphi) \tag{4.11}$$

for its Lefschetz number. It is then clear that the multiplicity of P relative to this complex is given by

$$\nu(P) = \operatorname{trace}_{\mathbf{C}} \varphi_P/\det_{\mathbf{C}} (1 - df_P) .$$

Thus the holomorphic case of our general Lefschetz theorem takes the form

THEOREM 4.12. *Let X be a compact complex manifold, F a holomorphic*

vector bundle over X, $f\colon X \to X$ a holomorphic map with simple fixed points and $\varphi\colon f^*F \to F$ a holomorphic bundle homomorphism. Let $L(f, \varphi)$ denote the Lefschetz number given by the action of (f, φ) on $H^*(X; \mathcal{O}(F))$ as in (4.11). *Then*

$$L(f, \varphi) = \sum_P \frac{\mathrm{Trace}_{\mathbf{C}}\, \varphi_P}{\det_{\mathbf{C}}(1 - df_P)}$$

when P runs over the fixed points of f.

Taking F to be the trivial line bundle, φ the natural lift of f and recalling that the spaces $H^{0,q}(X)$ are birational invariants of X (when X is algebraic) we deduce

COROLLARY 4.13. *Let X be a connected compact complex manifold with $H^{0,q}(X) = 0$ for $q > 0$. Then any holomorphic map $f\colon X \to X$ has a fixed point. In particular this holds when X is a rational algebraic manifold.*

This corollary does not of course use the explicit nature of the fixed point contribution. We shall now illustrate Theorem 4.12 by more interesting special cases.

Example 1. Let X be a curve, and let f be a transversal endomorphism of X. Then (4.9) yields:

$$1 - \mathrm{trace}\, H^{0,1}(f) = \sum_{f(P)=P} \frac{1}{1 - f'(P)}\,. \tag{4.14}$$

Remarks. In this low dimensional case it is not difficult to extend (4.14) as follows. If u is a holomorphic coordinate centered at the fixed point z, then clearly

$$\frac{1}{1 - f'(P)} = \mathrm{Res}_P \frac{du}{u - f(u)}\,. \tag{4.15}$$

The right hand side of (4.15) makes sense also for non-transversal maps and turns out to be the correct multiplicity of a general fixed point in the sense that the formula

$$1 - \mathrm{trace}\, H^{0,1}(f) = \sum_{f(P)=P} \mathrm{Res}_P \frac{du}{u - f(u)} \tag{4.16}$$

is valid for all endomorphisms other than the constant one. On higher dimensional varieties the corresponding generalization involves the Grothendieck theory of residues [13]. One may further extend formula (4.16) to any self-correspondence of a curve X in a plausible manner. Applied to the Hecke transformations one then recaptures the formulas of Selberg and Eichler, see [11] and [16].

Note also that, combined with the usual Lefschetz theorem, (4.14) yields

a relation among the real parts of $1/(1 - f'(z))$ and the degree of f. Indeed let n be the degree of f, and let N denote the number of fixed points of f. Then the classical Lefschetz theorem yields

$$1 - \text{trace } H^{0,1}(f) - \text{trace } H^{1,0}(f) + n = N .$$

On the other hand $\overline{\text{trace } H^{0,1}(f)} = \text{trace } H^{1,0}(f)$. Hence one obtains the relation

$$\frac{N - n + 1}{2} = \text{Re} \sum_{f(P)=P} \frac{1}{1 - f'(P)} . \tag{4.17}$$

Example 2. Let P be the projective n-space over $\mathbf{C}$, with homogeneous coordinates $(x_0, \cdots, x_n)$. Let $f: P \to P$ be the linear map which sends x_i into $\gamma_i x_i$, $\gamma_i \neq 0$, $\gamma_i \neq \gamma_j$. There are then precisely $n + 1$ fixed points of f, namely the points $p_i = (0, \cdots, 1, \cdots 0)$ with 1 at the i^{th} place. Further $\det_{\mathbf{C}}(1 - df_P)$ at p_i is easily seen to be

$$\prod_{j \neq i} (1 - \gamma_j/\gamma_i) .$$

We have $H^{0,q}(P) = 0$ for $q > 0$, and $H^{0,0}(X) = \mathbf{C}$. Hence (4.9) yields the well-known interpolation formula

$$1 = \sum_{i=0}^{n} \frac{\gamma_i^n}{\prod_{j \neq i}(\gamma_i - \gamma_j)} . \tag{4.18}$$

Example 3. Consider n polynomials $g_\alpha(z)$ of degree d in the n complex variables $z_1, \cdots, z_n$. We study the hypersurfaces $g_\alpha = 0$, and make the following general position assumptions concerning them.

Let Σ denote the points common to all the hypersurfaces. We assume first of all, that the jacobian $\det \| \partial g_\alpha / \partial z_\beta \|$ is non-zero at all points of Σ. This implies that, in particular, Σ is a finite set. Secondly, we assume that the number of points in Σ is equal to d^n, in other words, that there are no common intersections of the g_α at ∞.

Under these circumstances one has the proposition

PROPOSITION 4.19. *If* $d > 1$, *then*

$$\sum_{p \in \Sigma} \{\det \| \partial g_\alpha / \partial z_\beta \|_p\}^{-1} = 0 , \qquad \Sigma = \{p \mid g_\alpha(p) = 0, \alpha = 1, \cdots, n\} . \tag{4.20}$$

This is then a residue type theorem. It may be derived from (4.9) by the following stratagem. Let $g_\alpha = \sum g_\alpha^q$ be the decomposition of the g_α into homogeneous constituents, and define

$$\tilde{g}_\alpha(z; z_0) = \sum g_\alpha^q \cdot z_0^{d-q} .$$

These are then homogeneous polynomials of degree d in the variables $(z_0, \cdots, z_n)$. Now define $\tilde{f}: \mathbf{C}^{n+1} \to \mathbf{C}^{n+1}$ by

$$\begin{aligned} \tilde{f}(z)_\alpha &= z_\alpha z_0^{d-1} - \tilde{g}_\alpha(z; z_0) \qquad \alpha > 0 \\ \tilde{f}(z)_0 &= z_0^d \end{aligned} \tag{4.21}$$

and let f be the transformation induced by $\tilde{f}$ on the projective space P with homogeneous coordinates $(z_0, \cdots, z_n)$. By our second assumption and the condition $d > 1$, f is well-defined. The set $z_0 \neq 0$, may then be identified with $\mathbf{C}^n$—the coordinates being z_α/z_0—and $P - \mathbf{C}^n$ is a projective space P' of dimension $n-1$. Now f clearly maps $\mathbf{C}^n$ and P' into themselves. Further it is clear that the fixed points of f on $\mathbf{C}^n$ coincide with the set Σ of our proposition, and that, for $p \in \Sigma$

$$\det_{\mathbf{C}}(1 - df_p) = \det \left\| \frac{\partial g_\alpha}{\partial z_\beta} \right\|_p . \tag{4.22}$$

Assume now that the fixed points of f are all non-degenerate, i.e., that f is transversal. On $\mathbf{C}^n$ this is already the case by our first assumption and on P' it can always be arranged by an arbitrarily small deformation of the terms of degree d in the g_α, so that this case easily implies the general one. With this understood, let Σ' be the fixed points of f on P'. Applying (4.9) with $p = 0$ and using the fact that $H^{0,q}(P) = 0$ for $q > 0$, $H^{0,0}(P) = \mathbf{C}$, we have the formula:

$$1 = \sum_{p \in \Sigma} 1/\det_{\mathbf{C}}(1 - df_p) + \sum_{p \in \Sigma'} 1/\det_{\mathbf{C}}(1 - df_p) . \tag{4.23}$$

Next let f' be the restriction of f to P'. Applying (4.9) to this endomorphism yields

$$1 = \sum_{p \in \Sigma'} 1/\det_{\mathbf{C}}(1 - df'_p) . \tag{4.24}$$

Finally, it is easily checked that if $d > 1$, then $\det_{\mathbf{C}}(1 - df_p) = \det_{\mathbf{C}}(1 - df'_p)$ and so (4.20) follows from (4.23) and (4.24).

5. Induced representations. The Hermann Weyl formula

The multiplicity formulas (2.9) and (4.12) fit well into various branches of representation theory, and as our fourth example we will describe the connection of (4.12) with the *Hermann Weyl formula*, as well as comment on the general relation between the distributional trace of an induced representation and our Lefschetz formula.

We consider a Lie group G and let $\iota: H \to G$ be the inclusion of a closed Lie subgroup into G. The coset decomposition then gives G the structure of an H-bundle η over the quotient space G/H and we denote by π the natural projection $G \to G/H$. We assume further that there is given a finite dimensional (complex) left H-module F and denote the representation determined by the left action of H on F, by ρ_F:

(5.1) $$\rho_F: H \longrightarrow \mathrm{Aut}\,(F)\ .$$

The bundle associated to η by ρ_F is then a vector bundle over G/H which we denote by $\iota_* F$. The left action of G on G/H now lifts naturally into an action of G on $\Gamma(\iota_* F)$ by geometric endomorphisms, which we denote by $\iota_*(\rho_F)$:

$$\iota_*(\rho_F): G \longrightarrow \mathrm{Aut}\,\Gamma(\iota_* F)\ ,$$

and refer to as the induced representation of ρ_F.

The precise definition of these objects is as follows. The total space of $\iota_* F$ is the quotient space of $G \times F$ under the identification

$$(g \cdot h, f) \sim \big(g, \rho_F(h) f\big)\ .$$

The natural projection $G \times F \to G$ then induces the vector bundle projection $\iota_* F \to G/H$. Thus we have the commutative diagram

$$\begin{array}{ccc} G & \longleftarrow & G \times F \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \sigma} \\ G/H & \longleftarrow & \iota_* F \end{array}\ .$$

Furthermore, each $x \in G$ determines a linear isomorphism

$$j_x: F \longrightarrow (\iota_* F)_{\pi(x)}$$

by setting $j_x(f) = \sigma(x, f)$.

We come now to the induced action of G on $\Gamma(\iota_* F)$. Let L_g and l_g denote left translation in G and G/H respectively. Clearly

$$L_g \times 1 : G \times F \longrightarrow G \times F$$

preserves the fibers of σ and hence induces a map

$$L_g \times_H 1 : \iota_* F \longrightarrow \iota_* F$$

which maps the fiber over $l_{g^{-1}} \cdot \mathbf{x}$ linearly into the fiber over $\mathbf{x}$. Hence $\varphi_g = L_g \times_H 1$ *may be interpreted as a lifting of the map* $l_{g^{-1}}$. The resulting endomorphism $\Gamma(\varphi_g) \cdot l_{g^{-1}}^*$ of $\Gamma(\iota_* F)$ is denoted by T_g, and $g \mapsto T_g$ is the desired induced representation:

$$\iota_*(\rho)(g) = T_g\ .$$

Note. One has to use a lifting of $l_{g^{-1}}$ rather than of l_g to define T_g if one wishes $\iota_*(\rho)$ to be again a *left-representation.*

Our first aim now is to give a formula for the pertinent invariants of T_g at a fixed point, in terms of the adjoint actions of G and H. We write $\mathfrak{g}$ and $\mathfrak{h}$ for the Lie algebras in question, denote by Ad^G and Ad^H the adjoint action of G on $\mathfrak{g}$ and of H on $\mathfrak{h}$ respectively, and finally write

$$\mathrm{Ad}_H^G\colon H \longrightarrow \mathrm{Aut}\,(\mathfrak{g}/\mathfrak{h})$$

for the isotropy action of H on $\mathfrak{g}/\mathfrak{h}$.

Consider now a fixed point $\mathbf{x} \in G/H$ of $l_{g^{-1}}$. For any x in the coset $\mathbf{x}$ we must then have the relation

$$g^{-1}\cdot x = x\cdot h(g,x) \qquad h(g,x) \in H \tag{5.2}$$

and conversely if $g^{-1}x = x\cdot h$ holds for some $h \in H$, then πx is a fixed point of $l_{g^{-1}}$. Hence *$l_{g^{-1}}$ has a fixed point if and only if g is contained in the orbit of H under the adjoint action of G on G*; i.e., *if* $g \in \bigcup_{x\in G} xHx^{-1}$.

Observe also that as x varies over the coset of $\mathbf{x}$, $h(x)$ varies over a conjugacy class $h(g,\mathbf{x}) \subset H$. Thus to every fixed point $\mathbf{x}$ of $l_{g^{-1}}$ corresponds a definite conjugacy class $h(g,\mathbf{x}) \subset H$. With this understood we have the following proposition.

PROPOSITION 5.3. *Let x be a fixed point of $l_{g^{-1}}$, and let $h \in h(g,x)$ then*

$$\det\,(1 - dl_{g^{-1}})_{\mathbf{x}} = \det\,(1 - \mathrm{Ad}_H^G(h))\,. \tag{5.4}$$

Further if φ denotes the lifting of $l_{g^{-1}}$ to $\iota_* F$, then

$$\mathrm{trace}\,\varphi_g(x) = \mathrm{trace}\,\rho_F(h)^{-1}\,. \tag{5.5}$$

PROOF. Choose x in the coset $\mathbf{x}$ in such a manner that

$$g^{-1}x = xh\,.$$

The map $L_{g^{-1}} \circ R_{h^{-1}}$ (where R_g denotes right translation by g) then still induces $l_{g^{-1}}$, but also keeps x fixed. Consider the identification

$$dL_x \circ d\pi : \mathfrak{g}/\mathfrak{h} \longrightarrow (G/H)_{\mathbf{x}} \tag{5.6}$$

(where $(G/H)_{\mathbf{x}}$ denotes the tangent space to G/H at $\mathbf{x}$). The relation $L_{g^{-1}} \circ R_{h^{-1}} \circ L_x = L_x \circ L_h \circ R_{h^{-1}}$ then implies that under (5.6) $dl_{g^{-1}}$ goes over into $\mathrm{Ad}_H^G(h)$, and so establishes (5.4). To see (5.5) consider $j_x\colon F \to (\iota_* F)_{\mathbf{x}}$. We have $j_x(f) = \sigma(x,f)$. Hence $\varphi_g \circ j_x(f) = \sigma(gx,f) = \sigma(xx^{-1}gx,f) = \rho_F(h^{-1})j_x(f)$. q.e.d.

The expressions (5.4) and (5.5) occur in many branches of representation theory. As an example let us show how the "Hermann Weyl formula" fits into the framework of Theorem A and Proposition (5.3).

We assume then, that G is compact, and that $H = T$ is a maximal torus of G. In this case $\mathfrak{g}/\mathfrak{h}$ breaks into a direct sum of 2-planes, $\mathbf{e}_k$, $k = 1, \cdots, m$, on each of which Ad_H^G acts by rotations

$$\mathfrak{g}/\mathfrak{h} = \textstyle\sum_{k=1}^{m} \mathbf{e}_k\,. \tag{5.7}$$

Let ξ be a function which assigns to each 2-plane $\mathbf{e}_k$ an isomorphism with C compatible with the action of Ad_H^G. Such a function then determines m characters $\{\alpha_k^{\xi}\}$ on the torus H, as well as an almost complex structure on G/H.

(The complex structure determined by ξ on $\mathfrak{g}/\mathfrak{h}$ is compatible with the left action of G on G/H because it is invariant under Ad_H^G).

Now let ν be a character on H which is positive relative to ξ in the sense that $\nu^{n_0} = \prod(\alpha_i^\xi)^{n_i}$ for some $n_0 > 0$ and $n_i \geqq 0$; and consider the induced line bundle F_ν over G/H. The following facts concerning this situation (due to Borel and Weil) are then well known; see [7].

(5.8) The function ξ may be chosen so as to induce an integrable structure on G/H. Further, the bundle F_ν can then be given a corresponding holomorphic structure.

(5.9) The cohomology groups of the resulting complex $\Gamma(F_\nu \otimes \Lambda^{0,*})$ are zero in dimensions > 0:

$$H^{0,q}(G/H; F_\nu) = 0 \qquad \text{for } q > 0 ,$$

and the representation of G induced by $g \mapsto T_g$ on $H^{0,0}$ is irreducible.

The irreducible representation $\rho_\nu: G \to \mathrm{Aut}(V_\nu)$ obtained in this way is said to have maximal weight ν, relative to ξ, and a fundamental theorem asserts that all irreducible representations of G may be constructed in this manner.

The Hermann Weyl formula evaluates the trace of the representation ρ_ν on the maximal torus of G. Because traces depend only on conjugacy classes and T intersects each conjugacy class, this formula determines the trace on all of G.

Let us now apply our Lefschetz formula to the complex $\Gamma(F_\nu \otimes \Lambda^{0,*})$ and the geometric endomorphism T_g. Clearly, in view of (5.9) the Lefschetz number of T_g reduces to the trace of g acting on V_ν:

$$\mathrm{trace}\, \rho_\nu(g) = L(T_g) .$$

For transversal $l_{g^{-1}}$ we may therefore use Theorem A and Proposition 5.1 to compute trace $\rho_\nu(g)$. The resulting formula is precisely the Hermann Weyl formula, as we will now show.

Consider first the case when g is a generic element in T, that is one whose powers generate T. It follows that if $\mathbf{x}$ is fixed under g, and x is in the coset $\mathbf{x}$, then for all integers n,

$$x^{-1}g^{-n}x = h^n$$

where $h \in (g, \mathbf{x})$. Thus Ad x^{-1} keeps all of T invariant, so that the fixed points of g correspond precisely to the cosets of the normalizer of T, modulo T. The fixed points are therefore independent of the generic element in T, and naturally form a group $W(G, T) = N(T)/T$ called the Weyl group of G rel T. This group is well known to be finite, and its natural action on T permutes

the roots $\{\alpha_k\}$. From the formula (5.4) we have

$$\det_C (1 - dl_{g^{-1}}) |_X = \det_C \left(1 - \mathrm{Ad}_H^G(x^{-1}g^{-1}x)\right)$$

whence by (5.7) we obtain

$$\det_C(1 - dl_{g^{-1}}) |_X = \prod_{k=1}^{m} (1 - \alpha_k) |_{x^{-1}g^{-1}x} , \qquad \alpha_k = \alpha_k^{\xi}$$

which shows in particular, that a generic g gives rise to a transversal map $l_{g^{-1}}$.

Consider the action of $W(G, T)$ on the characters λ of T, defined by $\lambda^w(g) = \lambda(x^{-1} \cdot g \cdot x)$, with x in the coset of w. Also let $\beta_k = \alpha_k^{-1}$. Then the right hand side above takes the form

$$\prod_{k=1}^{m} (1 - \beta_k)^w(g) ,$$

so that by (5.5) the Lefschetz formula takes the form

$$L(T_g) = \sum_{w \in W(G,T)} \left[\frac{\nu}{\prod(1 - \beta_k)}\right]^w (g) . \tag{5.11}$$

Apart from some minor rewriting (5.11) is precisely the Hermann Weyl formula. Indeed to bring (5.11) into a more familiar form, assume that G is *simply connected*. Then the product of the positive roots turns out to have a square root a, whence the function Δ: $T \to C$ given by

$$\Delta = a \cdot \prod(1 - \beta_k)$$

is seen to be alternating under $W(G)$ that is, $w(\Delta) = \pm\Delta$ for all $w \in W(G)$. Using this fact, and putting Sign $(w) = w(\Delta)/\Delta$, (5.11) goes over into

$$\text{trace } T_g = \frac{\sum \text{Sign } (w)(\nu a)^w}{\Delta} \tag{5.12}$$

and this is the usual form. Note finally that because g is an arbitrary generic element in $H = T$, (5.12) implies a cancellation in the group ring of the character group of H and so determines the trace in question on all of H.

Some general remarks. One obtains a more thorough understanding of the relation between the Lefschetz and the Weyl formulas if one recalls some of the concepts needed in the proof of the Lefschetz formula, and applies these to the infinite dimensional induced representations. First of all recall that an operator Q: $\Gamma(E) \to \Gamma(F)$ is called a smooth operator if Q is given by

$$Qs(x) = \int K_Q(x, y)s(y)dy$$

where $K_Q(x, y)$ is a smooth kernel over the product of the respective base spaces of E and F. When $F = E$ such a Q has a natural trace because it can be approximated by endomorphisms of finite rank so that the natural notion of trace extends from these to Q by continuity. One then also finds that this

trace is given by the formula

$$\text{Trace } Q = \int \text{Trace } K_Q(x, x)dx \ .$$

Now the central point in Part I of this paper was the observation that this trace function on smooth operators can be extended by continuity to a larger class of operators. More precisely if T is a geometric endomorphism, obtained from a transversal map f by a lifting φ, $\varphi: f^*E \to E$, we defined

$$\text{Trace}_f T = \lim_{Q \to 1} \text{Trace } (T \circ Q)$$

when Q is a pseudo-differential operator tending to 1 in a suitable sense. In general we distinguished between the pair (f, φ) and the induced endomorphism T, and $\text{Trace}_f T$ was really a function of the pair (f, φ). However when f is an invertible map $X \to X$ this distinction need no longer be made (because T determines φ uniquely) and $\text{Trace}_f T$ is really a function of T and f. The dependence on f will be suppressed however and we shall write $\text{Trace}^\flat T$ instead of $\text{Trace}_f T$ and call it the *flat trace*.

In Part I we also obtained the explicit formula

$$\text{Trace}^\flat T = \sum_p \frac{\text{Trace } \varphi(p)}{|\det (1 - df_p)|}$$

summed over the fixed points of f.

In terms of this notion the Lefschetz formula is therefore equivalent to the compatibility statement

$$\sum (-1)^k \text{Trace}^\flat T_k = \sum (-1)^k \text{Trace } H^k(T) \ , \tag{5.18}$$

and it is in this framework that the proof was carried out in [2].

Let us return now to the homogeneous case where $X = G/H$, and $E = \iota_* F$ is the bundle induced by $\rho_F: H \to \text{Aut } F$.

The induced representation $\iota_*(\rho)$ then acts through geometric endomorphisms T_g of $\Gamma(\iota_* F)$. *Hence whenever*

(1) *G/H is compact and*

(2) *$l_{g^{-1}}: G/H \to G/H$ is transversal,*

then the map T_g will have a well defined flat trace $\text{Trace}^\flat (T_g)$.

On the other hand let μ be a smooth measure with compact support on G and define T_μ as the integral

$$T_\mu = \int_G T_g \mu(g) \ .$$

This endomorphism of $\Gamma(\iota_* F)$ then turns out to be a smooth operator, and hence has a well defined trace, at least when G/H is compact. Further it is seen that the function $\mu \mapsto \text{Trace } T_\mu$ defines a distribution on G. We refer to

this distribution as the trace of $\iota_*(\rho)$, and this is the distributional trace of representation theory.

One now has the plausible relation between trace $\iota_*(\rho)$ and our flat trace.

PROPOSITION 5.19. *Let $\iota_*\rho: G \to \mathrm{Aut}\,\Gamma(\iota_* F)$ be the induced representation from $\rho: H \to \mathrm{Aut}\,F$ and assume that G/H is compact. Also let $O(H) \subset G$ be the open subset of G consisting of those $g \in G$ for which $l_{g^{-1}}: G/H \to G/H$ is transversal.*

The function $\mathrm{Trace}^\flat: g \mapsto \mathrm{Trace}^\flat\, Tg$ *is then well defined on $O(H)$ and there represents the distributional trace of $\iota_*(\rho)$.*

The proof of this proposition hinges on two facts. First we observe that the convergence of the limit

$$\mathrm{Trace}^\flat\, T = \lim_{Q \to 1} \mathrm{Trace}\,(T \circ Q) \tag{5.20}$$

is uniform with respect to the map f underlying T provided f varies in a bounded set and that $\det(1 - df_p)$ is bounded away from zero at the fixed points of f. This assertion follows directly from (4.4) and (4.9) of [2].

Secondly we observe (cf. [2; § 5]) that the convergence $Q \to 1$ implies $T_\mu \circ Q \to T_\mu$ in the C^∞ topology of smooth operators and so

$$\mathrm{Trace}^\flat\, T_\mu = \lim_{Q \to 1} \mathrm{Trace}\; T_\mu \circ Q\ . \tag{5.21}$$

Now $T_\mu \circ Q = \int_G \mu(g) T_g \circ Q$ is the average over G of smooth operators. Since taking traces of smooth operators clearly commutes with averaging, we have

$$\mathrm{Trace}\,(T_\mu \circ Q) = \int_G \mu(g)\, \mathrm{Trace}\,(T_g \circ Q)\ . \tag{5.22}$$

If $\mathrm{Supp}\,\mu \subset O(H)$ the uniformity required in (5.20) is satisfied and so, as $Q \to 1$ in (5.22), we may pass to the limit under the integral. From this and (5.21) we obtain

$$\begin{aligned} \mathrm{Trace}\; T_\mu &= \int_G \mu(g)\big(\lim_{Q \to 1} \mathrm{Trace}\,(T_g \circ Q)\big) \\ &= \int_G \mu(g)\, \mathrm{Trace}^\flat\, T_g \end{aligned}$$

which establishes the proposition.

Perhaps we should make a few remarks on the reasons why T_μ is a smooth operator and $\mu \mapsto \mathrm{Trace}\; T_\mu$ is a distribution. These facts follow from quite general "integration over the fiber" arguments as we shall show.

Let $\pi: X \to Y$ be a smooth map of smooth manifolds and let E be a smooth bundle over Y. Then π induces a continuous map π_*, from the (compactly

supported) distributional sections of π^*E over X into the (compactly supported) distributional sections of E over Y, by simply setting

$$\pi_*\alpha(s) = \alpha \cdot (\pi^* s)$$

$s \in \Gamma(E)$, $\alpha \in \Gamma'_c(\pi^*E)$. Now if in addition the *differential* of π is *surjective at all points of* X, then a local integration over the fiber argument shows directly that π_* preserves *smoothness*.

Thus under this assumption

$$\pi_* : \Gamma_c\{\pi^*E^* \otimes \Omega(X)\} \longrightarrow \Gamma_c\{E^* \otimes \Omega(Y)\}$$

where $\Omega(X)$ and $\Omega(Y)$ denote the volume bundles of X and Y respectively and E^* denotes the dual bundle.

We apply this proposition to our situation, (with $\rho = 1$ for simplicity) by setting $M = G/H$ and defining

$$\pi : G \times M \longrightarrow M \times M$$

by

$$\pi(g, m) = (gm, m) .$$

Clearly our hypotheses are satisfied so that if we take for E the bundle $1 \times \Omega(M)$ on $M \times M$, π_* defines a map

$$\Gamma_c\{\Omega(G) \times 1\} \xrightarrow{\pi_*} \Gamma\{\Omega(M) \times 1\}$$

and one checks directly that for $u \in \Gamma_c\{\Omega(G)\}$

$$\pi_*(\mu \times 1) = \text{Kernel of } T_\mu .$$

Here of course G/H has to be compact in order for $(\mu \times 1)$ to have compact support.

Finally if $\Delta \in \Gamma'\{1 \times \Omega(M)\}$ is the kernel of the identity map then

$$\text{Trace } T_\mu = \pi_*(\mu \times 1) \cdot \Delta \tag{5.23}$$

from which our assertions concerning T_μ are evident.

In view of Proposition (5.19) we may say that our way of getting the Hermann Weyl formula is to express the character of a finite-dimensional representation of G as an alternating sum of characters of infinite-dimensional induced representations. The point of this is that the character of an induced representation is easily computed. In fact, as we have seen, on the open dense set $O(H) \subset G$ such a character is given by a sum over fixed points. Actually it is not difficult to show that the character $\chi i_*(\rho)$ is (as a distribution on G) equal to what one may call the *induced character*[3] $i_*(\chi_\rho)$. This is the direct

[3] Our understanding of these questions was greatly helped by some discussions with G. W. Mackey.

image under the projection $G \times M \to G$ of a distribution F_ρ sitting on the submanifold $Y \subset G \times M$ consisting of pairs (g, m) with $g(m) = m$. For example, if $\rho = 1$, and $F_\rho = \pi^*(\Delta)$ (5.23) asserts that

$$\iota_*(\chi_1) = \chi(\iota_* 1) .$$

The detailed formulation in the more general case is left to the reader.

6. The elliptic operators[4] associated to a riemannian structure

Every oriented riemannian manifold of even dimension has defined on it an elliptic operator

$$D^+ : \Gamma(\lambda^+ TX) \longrightarrow \Gamma(\lambda^- TX) \tag{6.1}$$

to which all isometries may be lifted, and which is closely related to the Dirac operator, the Hodge theory, and the Hirzebruch signature of the manifold.

This operator is treated in [6] but for the sake of completeness we will review the construction of (6.1) in some detail. Recall that the orientation and riemannian structure in TX single out a basic m-form $v \in \Gamma(\lambda^m TX)$, $m = \dim X$, which is characterized by the requirement that at every point P,

$$v_P = \theta^1 \wedge \cdots \wedge \theta^m \tag{6.2}$$

whenever $(\theta^1, \cdots, \theta^m)$ is any orthonormal frame for $T_P X$ in the orientation of X. Recall further that this m-form then serves to define an isomorphism

$$* : \lambda^q TX \longrightarrow \lambda^{m-q} TX \tag{6.3}$$

which is characterized by the identity

$$u \wedge *u' = \langle u, u' \rangle \cdot v_P \qquad u, u' \in \lambda^q T_P X \tag{6.4}$$

where $\langle u, u' \rangle$ denotes the inner product induced on $\lambda^* TX$ by the inner product in TX. In particular then

$$\langle u, u \rangle \geqq 0 . \tag{6.5}$$

Using the $*$ operator, one then defines the *global inner product* (u, v) for $u, v \in \Gamma(\lambda^* TX)$ by the formula

$$(u, v) = \int_{[X]} u \wedge *v . \tag{6.6}$$

In view of (6.5) this is a positive definite inner product, so that the operator d on $\Gamma(\lambda^* TX)$ has a well defined adjoint relative to it which we denote by δ. Thus

$$(du, u') = (u, \delta u') \qquad u, u' \in \Gamma(\Lambda^* TX) . \tag{6.7}$$

On an oriented riemannian manifold one is thus naturally led to the self-

[4] A single elliptic operator is of course a special case of an elliptic complex.

adjoint operator $D = d + \delta$. Note that $d^2 = \delta^2 = 0$ so that $D^2 = d\delta + \delta d$ is the usual laplacian Δ of the Hodge theory. This operator is well-known to be elliptic. On the other hand the highest order symbol of a composition is easily seen to be the composition of the highest order symbols. Hence D must also be elliptic. Taken by itself this operator is not of great interest for our purposes precisely because it is self adjoint, whence its Lefschetz numbers will turn out to be zero for quite trivial reasons.

However if X has an even dimension then one may fashion an interesting operator out of D in the following manner. First one observes that under this assumption

$$\delta = -*d* \,. \tag{6.8}$$

Next one checks that if $\alpha: \lambda^* TX \to \lambda^* TX$ is defined by the formula

$$\alpha u = (-1)^{q(q-1)/2} * u \,, \qquad u \in \lambda^q TX \tag{6.9}$$

then α satisfies the identities

$$\alpha^2 = (-1)^n \qquad \dim X = 2n \tag{6.10}$$

$$D\alpha = -\alpha D \,, \tag{6.11}$$

and has the multiplicative property that under the natural isomorphism $\lambda^* TX \otimes \lambda^* TY \cong \lambda^* T(X \times Y)$

$$\alpha_{X\times Y} = \alpha_X \otimes \alpha_Y \,. \tag{6.12}$$

Finally consider the action of $\alpha \otimes i^n$ on the complexification of $\lambda^* TX$. Under this action the total space breaks up into a direct sum

$$\lambda^* TX \otimes \mathbf{C} = \lambda^+ TX \oplus \lambda^- TX \tag{6.13}$$

where $\lambda^\pm TX$ are the eigenspaces of $\alpha \otimes i^n$ corresponding to the eigenvalues ± 1.

In view of (6.11) the operator $D \otimes 1$ then interchanges the $\Gamma(\lambda^\pm TX)$ and so induces operators $D^\pm: \Gamma(\lambda^\pm TX) \to \Gamma(\lambda^\mp TX)$ which are adjoints of each other.

The operator D^+ is of interest first of all because its index, that is, the Lefschetz number of the identity map, turns out to be a topological invariant of X. Precisely one has the

PROPOSITION 6.13. *If X is a compact oriented even dimensional riemannian manifold then*

$$\begin{aligned} \textit{index } D^+ &= \textit{Hirzebruch signature of } X, && \textit{if } \dim X \equiv 0 \bmod 4 \\ &= 0 && \textit{if } \dim X \equiv 2 \bmod 4 \,. \end{aligned} \tag{6.14}$$

This fact is an easy consequence of the Hodge theory. The first step is to identify the cohomology space of (6.1) with certain subspaces of the space of complex-valued harmonic forms $\mathcal{H}$ on X.

Since the adjoint of $D^{\pm}$ is $D^{\mp}$ we have the identifications

$$\begin{aligned} H^0 \text{ of the complex (6.1)} &= \operatorname{Ker} D^+ \\ H^1 \text{ of the complex (6.1)} &= \operatorname{Ker} D^- . \end{aligned} \tag{6.15}$$

Secondly recall that the harmonic forms $\mathcal{H}$ precisely constitute the kernel of $D \otimes 1$. This follows from $\{Du, Du\} = \{du, du\} + \{\delta u, \delta u\}$, whence $Du = 0 \Rightarrow du$ and $\delta u = 0$. Hence (6.15) implies that if $\mathcal{H}^{\pm}$ denotes the ± 1 eigenspaces of $\alpha \otimes i^n$ acting on $\mathfrak{H}$, then $H^0 \cong \mathcal{H}^+$ and $H^1 \cong \mathcal{H}^-$. In particular

$$\text{index } D^+ = \dim \mathcal{H}^+ - \dim \mathcal{H}^- . \tag{6.16}$$

To proceed further, let

$$\pi^{\pm} = \{1 \pm \alpha \otimes i^n\}/2 . \tag{6.17}$$

Then the $\pi^{\pm}$ project $\mathcal{H}$ onto $\mathcal{H}^{\pm}$ so that

$$\begin{aligned} \dim \mathcal{H}^+ - \dim \mathcal{H}^- &= \operatorname{trace} \pi^+ - \operatorname{trace} \pi^- \\ &= \operatorname{trace} (\alpha \otimes i^n) = i^n \operatorname{trace} \alpha . \end{aligned} \tag{6.18}$$

To prove Proposition 6.13 consider first the case n odd. Then $\alpha^2 = -1$, whence trace $\alpha = 0$ and therefore also trace $\alpha \otimes i^n = 0$.

Next consider the case $n = 2l$. Then $\alpha^2 = 1$ so that already the real harmonic forms, which we denote by $\mathcal{H}(\mathbf{R})$, decompose into a direct sum

$$\mathcal{H}(\mathbf{R}) = \mathcal{H}(\mathbf{R})^+ \oplus \mathcal{H}(\mathbf{R})^-$$

where $(-1)^l \alpha = \pm 1$ on $\mathcal{H}(\mathbf{R})^{\pm}$. It follows that

$$\text{index } D^+ = \dim \mathcal{H}(\mathbf{R})^+ - \dim \mathcal{H}(\mathbf{R})^- = \operatorname{trace} \{(-1)^l \alpha\} . \tag{6.19}$$

Observe now that $(-1)^l \alpha$ maps λ^q into λ^{2n-q} so that only the terms of dimension $2l$ enter into its trace. Thus we may also write

$$\text{index } D^+ = \dim \mathcal{H}(\mathbf{R})^+_{2l} - \dim \mathcal{H}(\mathbf{R})^-_{2l} . \tag{6.20}$$

Finally, consider the quadratic form $q(u)$ defined by

$$q(u) = \int_{[X]} u \wedge u , \qquad u \in \mathcal{H}(\mathbf{R})_{2l} .$$

The signature of this form, that is, the difference of dimensions of maximal subspaces on which it is positive and negative, is now seen to be precisely the right hand side of (6.20). Indeed when $u \in \mathcal{H}(\mathbf{R})^{\pm}_{2l}$ one checks that $q(u) = \pm\{u, u\}$. Thus index D^+ = signature of q. At this stage we appeal to the de Rham and Hodge theorems, which allow us to identify $\mathcal{H}(\mathbf{R})_{2l}$ with

$H^{2l}(X; \mathbf{R})$ in such a manner that q goes over into the topologically invariant form given by the cup product and the orientation on X. As the signature of X is by definition the signature of this form, Proposition 6.13 follows.

In view of (6.13) we shall refer to D^+ as the *signature operator* of X.

We turn next to the Lefschetz number of an isometry $f: X \to X$, relative to the signature operator (6.1). We have already seen that df induces a geometric endomorphism f^* in the de Rham complex. If f is an isometry this f^* necessarily commutes with the $*$ operator of X, and hence also with δ.

It follows that f^* induces endomorphisms $f^{\pm}$ in $\Gamma(\lambda^{\pm} TX)$ which commute with D^+ and so combine to define an automorphism of the complex (6.1).

We denote by $\mathcal{K}^{\pm}(f)$ the endomorphism in the cohomology of (6.1) so that the Lefschetz number of f is in this case given by

$$\text{Sign}\,(f, X) = \text{trace}\,\mathcal{K}^+(f) - \text{trace}\,\mathcal{K}^-(f)\,, \tag{6.21}$$

and we refer to $\text{Sign}\,(f, X)$ as the signature of the isometry f. Note that using the projections $\pi^{\pm}$ we also have

$$\text{Sign}\,(f, X) = \text{trace}\,(\pi^+ \circ f^* - \pi^- \circ f^*) = \iota^n \,\text{trace}\,(\alpha \cdot \mathcal{K}(f)) \tag{6.22}$$

and so by our previous argument, $\text{Sign}\,(f, X)$ is completely determined by the action of $\mathcal{K}(f)$ and α on $\mathcal{K}_n$ alone. In fact one can define $\text{Sign}\,(f, X)$ directly from the action f^* on $H^n(X; \mathbf{R})$ and the bilinear form given by the cup product. When n is even this form is symmetric and one proceeds much as in Proposition 6.13. For n odd the form is skew-symmetric and one proceeds in a different manner. Both cases are discussed in detail in [6; § 6].

Our next aim is to compute the multiplicity of a fixed point relative to this operator. Note first that, for an isometry, an isolated fixed point is necessarily transversal: in fact if df_P has a fixed tangent vector, the geodesic in that direction would consist of fixed points and P would not be isolated. Suppose therefore that $P \in X$ is an isolated fixed point of f and let f_P^* denote the automorphism of $\lambda^* T_P X$ induced by $\lambda^* df_P$. Then according to our general prescription the multiplicity of P relative to the operator (6.1) is given by

$$\nu(P) = \frac{\text{trace}_{\mathbf{C}}\,(f_P^* \otimes 1 \mid \lambda^+ T_P X) - \text{trace}_{\mathbf{C}}\,(f_P^* \otimes 1 \mid \lambda^- T_P X)}{|\det_{\mathbf{R}}\,(1 - df_P)|}\,. \tag{6.23}$$

Using projections $\pi^{\pm}$ just as we did above, only now acting on $\lambda^* TX$, the numerator of this expression reduces to $\text{trace}\,(f_P^* \circ \alpha \otimes \iota^n)$ so that

$$\nu(P) = \text{trace}_{\mathbf{C}}\,(\{f_P^* \circ \alpha\} \otimes \iota^n) \big/ |\det\,(1 - df_P)|\,.$$

A more explicit formula for $\nu(P)$ can be obtained in the following manner. Consider the differential

$$df_P: T_PX \longrightarrow T_PX .$$

Because f is an isometry of X, df_P will be an isometry of T_PX. Hence one may decompose T_PX into a direct sum of orthogonal 2-planes

$$T_PX = E_1 \oplus E_2 \oplus \cdots \oplus E_n$$

which are stable under df_P. Let (e_k, e'_k) be an orthogonal base of E_k, so chosen that

$$v_P(e_1 \wedge e'_1 \wedge e_2 \wedge e'_2 \cdots e_n \wedge e'_n) = 1 .$$

Relative to such a base df_P is then given by rotations through angles θ_k in E_k. That is,

$$\begin{cases} df_P \cdot e_k = \cos\theta_k e_k + \sin\theta_k e'_k \\ df_P \cdot e'_k = -\sin\theta_k e_k + \cos\theta_k e'_k . \end{cases} \tag{6.24}$$

and we call the resulting set of angles $\{\theta_k\}$ a *coherent system* for df_P. This understood the multiplicity formula we are seeking is given by

$$\nu(P) = i^{-n} \prod_1^n \cot(\theta_k/2) , \tag{6.25}$$

where $\{\theta_k\}$ is a system of coherent angles for df_P.

To prove (6.25) consider first the two-dimensional case. Then $*e_1 = e'_1$, $*e'_1 = -e_1$ whence $\lambda^+ T_P$ is spanned by $1 + ie_1 \wedge e'_1$ and $e_1 + ie'_1$, while $\lambda^- T_P$ is spanned by $1 - ie_1 \wedge e'_1$ and $e_1 - ie'_1$. Furthermore $df_P(e_1 + ie'_1) = e^{-i\theta}(e_1 + ie'_1)$ and $df_P(e_1 - ie'_1) = e^{i\theta}(e_1 - ie'_1)$ while the other elements remain fixed under $\lambda^* df_P$. It follows that trace $(f_P^* \circ \alpha \otimes i) = e^{-i\theta} - e^{+i\theta}$, so that

$$\nu(P) = \frac{e^{-i\theta} - e^{i\theta}}{(1 - e^{-i\theta})(1 - e^{i\theta})} = i^{-1} \cot(\theta/2) . \tag{6.26}$$

The proof of (6.25) is now completed by applying the multiplicative property (6.12) of α to the decomposition of T_PX into the 2-planes $\{E_i\}$.

To recapitulate, our Lefschetz theorem specializes in the following manner to the operator (6.1).

THEOREM 6.27. *Let $f: X \to X$ be an isometry of the compact oriented even dimensional Riemann manifold X. Assume further that f has only isolated fixed points $\{P\}$, and let $\{\theta_k^P\}$ be a system of coherent angles for df_P. Then the signature of f is given by*

$$\operatorname{Sign}(f, X) = \sum_P i^{-n} \prod_k \cot(\theta_k^P/2) , \qquad \dim X = 2n .$$

Remarks. Since we first noted this formula, the general problem of describing Sign (f, X) in terms of the fixed point sets of f has been completely solved. This solution proceeds *via* the general index theorem of Atiyah-Singer and the methods of K_G-theory (in particular the localization theorem

of Atiyah-Segal) and is described in [4], [5], and [6]. Thus this method of attack is considerably less elementary than the one under discussion here. Furthermore it is based essentially on the fact that the group of isometries of a Riemann structure is compact, and hence does not extend to cover other Lefschetz problems, even with only transversal fixed point sets.

7. Two applications

Theorem (6.27) imposes strong number-theoretic restrictions on the angles of an isometry at its fixed points. We give two illustrations of this fact in this section. As our first and simplest example we have

THEOREM 7.1. *Let X be a compact connected and oriented manifold (of positive dimension), and let $f: X \to X$ be an automorphism of X, of prime power order $n = p^l$ with p odd. Then f cannot have just one fixed point.*

PROOF. A compact group of diffeomorphisms can always be made into a group of isometries simply by averaging a given Riemann structure over the group. We may therefore assume that f acts as an isometry. Since p is odd (and X is connected) f must preserve the orientation. Assume now that f has just one fixed point P. Since $\det(1 - df_P) \neq 0$ and df_P is orthogonal it follows that $\dim X$ is even. We may therefore apply our Theorem (6.27) to obtain the expression

$$\text{Sign}(f, X) = i^{-m} \prod_1^m \cot(\theta_k/2) \qquad 2m = \dim X \tag{7.2}$$

for the signature of f, in terms of a system of coherent angles $\{\theta_k\}$ for df_P.

Setting $\xi_k = e^{-i\theta_k}$ we get

$$\text{Sign}(f, X) \prod (1 - \xi_k) = \prod (1 + \xi_k)\,. \tag{7.3}$$

The ξ_k are eigenvalues of df_P. Hence as $(df_P)^n = 1$, they must all be n^{th} roots of unity. Note further that $\text{Sign}(f, X)$, which by definition is $\text{trace}\,\mathcal{H}^+(f) - \text{trace}\,\mathcal{H}^-(f)$, must be a *linear combination of n^{th} roots with integer coefficients*. We may therefore interpret (7.3) as an equation in the ring $\mathbf{Z}[\xi]$, generated by a primitive n^{th} root of 1.

Let us now reduce mod p in this ring and raise both sides of (7.3) to the n^{th} power. Because $n = p^l$ and $m > 0$, this yields $0 \equiv 2^m \bmod p$, contradicting the fact that p is odd.

This theorem was originally conjectured by Conner and Floyd, and was also recently re-established by them using their own methods [10]. In their framework of bordism theory the Lefschetz formula leads to the following extension of Theorem 7.1.

Consider a representation ρ of the cyclic group Z_n on a complex vector

space V of dimension $m > 0$.

$$\rho\colon Z_n \longrightarrow \mathrm{Aut}(V)\ . \tag{7.4}$$

The character of ρ, evaluated on the generator 1 of Z_n then has the form

$$\chi_\rho(1) = \sum a_\lambda \xi^\lambda\ , \tag{7.5}$$

where $\lambda \in Z_n$, $a_\lambda \in \mathbf{Z}$, and $\xi = e^{2\pi i/n}$.

We call ρ primitive if $a_\lambda = 0$ unless $\lambda \in Z_n^*$, the set of residues prime to n. The restriction of such an action to the unit sphere $S(V)$ of V (relative to some invariant hermitian structure) then acts freely on $S(V)$ and so determines an element $[\rho]$ of the reduced bordism group $\tilde{\Omega}_*(Z_n)$. Theorem 7.1 clearly implies that when $n = p^l$, p an odd prime, we have $[\rho] \neq 0$. It is therefore natural to seek a lower bound for the order of $[\rho]$ by means of the Lefschetz formula and this is easily done.

In fact our previous argument leads directly to the following

THEOREM 7.6. *For any primitive representation ρ, let $\sigma(\rho) \in \mathbf{C}$ be determined in terms of $\chi_\rho(1) = \sum a_\lambda \xi^\lambda$ by the formula*

$$\sigma(\rho) = \prod \left(\frac{1 + \xi^\lambda}{1 - \xi^\lambda}\right)^{a_\lambda}\ . \tag{7.7}$$

Then if $\rho_1 \cdots \rho_k$ are primitive m-dimensional representations of Z_n, the relation

$$\sum [\rho_s] = 0 \qquad \text{in } \tilde{\Omega}_*(Z_n) \tag{7.8}$$

holds only if

$$\sum \sigma(\rho_s) \in \mathbf{Z}[\xi]\ . \tag{7.9}$$

To derive this criterion suppose (7.8) holds. Then there exists a manifold M, whose boundary ∂M consists of k $(2m-1)$-spheres on which Z_n acts freely, the action reducing to the actions $[\rho_s]$ on ∂M. If we fill in these spheres by unit discs $e_1, \cdots, e_k$ and extend the action linearly, there results a compact manifold $\tilde{M}$ on which Z_n acts freely except at the centers $P_1, \cdots, P_k$, of the attached discs. Theorem (6.27) applied to the generator of Z_n now yields the formula $\sum \nu(P_s) = \mathrm{Sign}\,(f, \tilde{M})$, with $\mathrm{Sign}\,(f, \tilde{M}) \in \mathbf{Z}[\xi]$. Since $\nu(P_s) = \sigma(\rho_s)$ the theorem follows.

As an example let us prove the following corollary to Theorem 7.6.

COROLLARY 7.10. *If $n = p^l$ with p an odd prime, then for any primitive m-dimensional representation ρ of Z_n the order of $[\rho]$ is divisible by p^{r+1}, where*

$$r = \left[\frac{m}{p^{l-1}(p-1)}\right] \tag{7.11}$$

PROOF. We recall a little elementary number theory. The minimal polynomial of a primitive n^{th} root of 1 is

$$\varphi(x) = (x^n - 1)/(x^{n/p} - 1) = 1 + x^{n/p} + \cdots + x^{n(p-1)/p} .$$

Hence $\varphi(x) = \prod (x - \xi^\lambda)$, $\lambda \in Z_n^*$; and so in particular

$$\prod_{(\lambda,n)=1} (1 - \xi^\lambda) = p . \tag{7.12}$$

It is clear that $1 - \xi$ divides $1 - \xi^\lambda$ in $\mathbf{Z}[\xi]$. But for any $\lambda \in Z_n^*$, ξ^λ is again primitive so that $(\xi^\lambda)^r = \xi$ for some r, and so $1 - \xi^\lambda$ divides $1 - \xi$. It follows that

$$(1 - \xi^\lambda) = (1 - \xi)\cdot\text{unit} . \tag{7.13}$$

In particular, (7.12) implies that

$$(1 - \xi)^{n(1-1/p)} = p\cdot\text{unit} , \qquad n = p^l . \tag{7.14}$$

Let us now set $\eta = 1 - \xi$. Applying (7.13) in the formula for $\sigma(\rho)$ we find

$$\sigma(\rho) = \frac{u}{\eta^m}$$

with $u \equiv 2^m \bmod \eta$ in $\mathbf{Z}[\xi]$. Applying Theorem (7.6) we see that $q\cdot[\rho] = 0 \Rightarrow q\cdot\sigma(\rho) \in \mathbf{Z}[\xi] \Rightarrow qu \in \eta^m\mathbf{Z}[\xi]$. Then (7.14) implies (by an argument used in (7.1)) that q must be divisible by p^{r+1} where

$$r = \left[\frac{m}{n(1 - 1/p)}\right] \qquad \text{q.e.d.}$$

So far in these applications we have only used the fact that Sign (f, X) is an algebraic integer. Its cohomological interpretation was not used, and so we did not exploit the full force of our Lefschetz formula. A more interesting application which really uses the full Lefschetz formula is the following theorem which confirms an old standing conjecture of P. A. Smith.

THEOREM 7.15. *Let p be an odd prime and consider a smooth action of Z_p on a homology sphere which has precisely two fixed points. Then the induced representations of Z_p on the tangent spaces of the two fixed points are isomorphic.*

PROOF. Let f generate the action and choose a riemannian structure for S^{2n} on which f acts as an isometry. Also let P and Q be the fixed points of f. We have to show that df_P and df_Q have the same set of eigenvalues. Because $(df_P)^p = 1$ these eigenvalues will all be p^{th} roots of unity. Hence if we set $\xi = e^{2\pi i/p}$, and let a_λ^P denote the number of eigenvalues of df_P which are equal to ξ^λ, then we have to show that $a_\lambda^P = a_\lambda^Q$.

Now according to our fixed point theorem:

$$\text{Sign}\,(f, S^{2n}) = \nu(P) + \nu(Q)\ . \tag{7.16}$$

On the other hand $\text{Sign}\,(f, S^{2n}) = 0$ because a homology sphere has no harmonic forms in its middle dimension. Thus under our assumptions we must have

$$\nu(P) = -\nu(Q)\ . \tag{7.17}$$

Now according to (7.2),

$$\nu(P)\cdot\overline{\nu(P)} = \prod \frac{1+\xi_k}{1-\xi_k}\ ,$$

where ξ_k range over all the eigenvalues of df_P. In terms of our multiplicity function a^P we therefore have

$$|\,\nu(P)\,|^2 = \prod \{(1+\xi^\lambda)/(1-\xi^\lambda)\}^{a_\lambda^P} \tag{7.18}$$

where λ ranges over the set Z_p^* of congruence classes mod p which are prime to p.

Hence if we set $a_\lambda = a_\lambda^P - a_\lambda^Q$ then (7.17) implies that

$$\prod_{(\lambda,p)=1} \left\{\frac{1+\xi^\lambda}{1-\xi^\lambda}\right\}^{a_\lambda} = 1\ . \tag{7.19}$$

At this stage one is therefore reduced to showing that (7.19) $\Rightarrow a_\lambda \equiv 0$. Now a theorem of Kummer implies an assertion of this type, which for our purposes may be stated as follows.

THEOREM. (Kummer). *If* $\{a_\lambda\}$, $\lambda \in Z_p^*$, *is any set of integers subject to*

$$a_\lambda = a_{-\lambda} \tag{7.20}$$

$$\sum a_\lambda = 0 \tag{7.21}$$

and,

$$\prod (1-\xi^\lambda)^{a_\lambda} = 1\ , \tag{7.22}$$

then $a_\lambda \equiv 0$.

From this theorem one deduces ours in the following fashion. The conditions (7.20) and (7.21) are trivially met by our a_λ. Using the relation

$$(1+\xi)/(1-\xi) = (1-\xi^2)/(1-\xi)^2 \tag{7.23}$$

the condition (7.19) is transformed into

$$\prod (1-\xi^{2\lambda})^{a_\lambda}\cdot\prod (1-\xi^\lambda)^{-2a_\lambda} = 1\ . \tag{7.24}$$

Now $\lambda \mapsto 2\lambda$ defines a bijection on Z_p^* whose inverse we denote by $\lambda \mapsto \lambda/2$. Thus (7.24) is also given by

$$\prod (1-\xi^\lambda)^{\{a_{\lambda/2}-2a_\lambda\}} = 1\ . \tag{7.25}$$

Applying the Kummer theorem now yields

$$a_{\lambda/2} = 2a_\lambda \,. \tag{7.26}$$

However on each orbit of the transformation $\lambda \mapsto 2\lambda$, the function a_λ must take a maximum. This is compatible with (7.26) only if $a_\lambda \equiv 0$, as was to be shown.

This argument was extended by Milnor to yield the following theorem.

THEOREM 7.27. *Let G be a compact group of diffeomorphisms of a homology sphere with fixed points P, Q, the action being free except at P and Q. Then the induced representations of G on the tangent spaces T_P and T_Q are isomorphic.*

Milnor's proof runs as follows. G is a closed subgroup of isometries (for some riemannian metric) and so is a Lie group. The elements of finite order are therefore dense in G. Hence it is sufficient to establish the theorem for an arbitrary cyclic group Z_n. The case $n = 2$ is trivially valid and so we may assume $n \geqq 3$. What is needed therefore first of all, is the following generalization of the Kummer theorem due to Franz. [See [12].]

Let Z_n^* denote the residue classes λ mod n, with $(\lambda, n) = 1$, $n \geqq 3$, and set $\xi = e^{2\pi i/n}$. Also let d range over the positive divisors of n, other than n itself. With this understood one has the following.

THEOREM 7.28 (Franz). *If $\{a_\lambda\}$, $\lambda \in Z_n^*$ is a set of integers subject to*

$$a_\lambda = a_{-\lambda} \tag{7.29}$$

$$\sum a_\lambda = 0 \tag{7.30}$$

and if for every divisor d of n, $d \neq n$,

$$\prod (1 - \xi^{d\lambda})^{a_\lambda} = 1 \,, \tag{7.31}$$

then $a_\lambda \equiv 0$.

Consider now a generator f for our group Z_n and apply Theorem 6.27 to the powers f^d, where d ranges over the divisors of n not equal to n, or $n/2$. By assumption all these f^d act freely on $X - \{P \cup Q\}$, whence, as none of them has order 2, the multiplicities ν_P and ν_Q relative to f^d will be non-zero. Thus, if a_λ^P and a_λ^Q are the multiplicity functions of df_P and df_Q respectively and a_λ is their difference, our earlier argument leads to the condition

$$\prod \left\{\frac{(1 + \xi^{d\lambda})}{(1 - \xi^{d\lambda})}\right\}^{a_\lambda} = 1 \,, \qquad \lambda \in Z_n^* \tag{7.32}$$

for all divisors d of n not equal to n or $n/2$.

Our aim is therefore to deduce that $a_\lambda \equiv 0$ from (7.32). We consider several cases.

Case 1. *n odd.* Here $n/2$ is not a divisor. Further the map $a \mapsto 2\lambda$ is a

bijection on Z_n^*. Hence by our earlier argument (7.32) $\Rightarrow \prod (1 - \xi^{d\lambda})^{a_\lambda/2 - 2a_\lambda} = 1$, from which we conclude by Franz's theorem that $a_\lambda \equiv 0$. q.e.d.

Case 2. $n = 2m$, m *odd*. In this case the projection $Z_{2m} \to Z_m$ induces a bijection of Z_{2m}^* onto Z_m^*. Thus in (7.32) λ may be thought of as varying over Z_m^*. Taking $d = 2d'$ where $d' \mid m$, $d' \neq m$ we are reduced to *Case* 1 and again $a_\lambda \equiv 0$.

Case 3. $n = 2m$, m *even*. This case is treated by induction. We assume that (7.29), (7.30), and (7.32) $\Rightarrow a_\lambda \equiv 0$ for the case $n = m$, then prove it for $n = 2m$. Note that in this case $Z_{2m} \to Z_m$ induces a surjection $Z_{2m}^* \to Z_m^*$ with kernel Z_2. Now for every $d' \mid m$, $d' \neq m$, we may apply (7.32) with $d = 2d'$, to obtain

$$\prod \{(1 + \xi^{2\lambda d'})/(1 - \xi^{2\lambda d'})\}^{a_\lambda} = 1 \qquad \lambda \in Z_{2m}^* .$$

On the other hand here the terms λ and $\lambda + m$ may be lumped together whence one obtains

$$\prod \{(1 + \xi^{2\lambda d'})/(1 - \xi^{2\lambda d'})\}^{a_\lambda + a_{\lambda+m}} = 1 , \qquad \lambda \in Z_m^* .$$

Hence, by the inductive hypothesis, $a_\lambda + a_{(\lambda+m)} \equiv 0$.

To complete the proof we will now show that (7.32) implies the relation (7.31) i.e., that (7.32) $\Rightarrow \prod (1 - \xi^{\lambda d})^{a_\lambda} = 1$ for all $d \mid 2m$, $d \neq 2m$. If $d = 2d'$ with $d' \mid m$, then we may again combine the λ and $(\lambda + m)^{\text{th}}$ terms. Hence, by the relation $a_\lambda + a_{\lambda+m} = 0$, the product equals 1. Consider now a divisor $d \neq 2d'$. Then $\xi^{dm} = -1$, so that (7.32) may be written

$$\begin{aligned} 1 &= \prod \{(1 - \xi^{(\lambda+m)d})/(1 - \xi^{\lambda d})\}^{a_\lambda} \qquad \lambda \in Z_{2m}^* \\ &= \prod \{1 - \xi^{\lambda d}\}^{a_{\lambda+m} - a_\lambda} \\ &= \prod \{1 - \xi^{\lambda d}\}^{-2a_\lambda} . \end{aligned}$$

Taking the reciprocal of the square root one gets

$$\prod (1 - \xi^{\lambda d})^{a_\lambda} = \pm 1 .$$

Finally the $+1$ must hold since $a_\lambda = a_{-\lambda}$. q.e.d.

Remark. When G is a finite cyclic group Theorem 7.27 can be reformulated as a result about lens spaces. It asserts that two lens spaces which are h-cobordant are isometric. For further information about these questions we refer the reader to [15].

8. The Dirac operator on Spin manifolds

Notice that if $f: X \to X$ is an isometry of order two, then at every isolated fixed point P of f, $df_P = -$ identity. Hence the multiplicity $\nu(P)$ of P, relative to the signature of f, is identically zero. Theorem 6.26 therefore yields the proposition that an involution of an oriented $4k$-dimensional compact manifold of odd Euler number must have a fixed point set of dim > 0. Indeed under the assumptions Sign (f, X) cannot possibly be zero as $\mathcal{H}(f)$ has

eigenvalues ± 1, and $\dim \mathcal{H}$ is odd.

In this section we will describe a more interesting operator, closely related to D^+, which attaches multiplicities $\pm(i/2)^m$ (where $2m = \dim X$) to the isolated fixed points of orientation preserving involutions and which can therefore be applied to involutions in the same manner that we applied the operator D^+ to transformations of odd period. This operator which we call the Dirac operator exists only under certain topological conditions on X, and we will start by reviewing these.

First recall that if Y is a connected CW-complex then the set of isomorphism classes of double coverings of Y is in one to one correspondence with the set $H^1(Y; Z_2)$, and so in particular, inherits a group structure. This comes about because a double covering $\hat{Y} \to Y$ corresponds to a homomorphism $\pi_1(Y) \to Z_2$.

Suppose now that X is a connected and oriented riemannian manifold. We then denote by $\mathbf{F} = \mathbf{F}TX$, the (oriented) orthogonal frame bundle of the *cotangent bundle* to X. Thus the fiber $\mathbf{F}_P$ of $\mathbf{F}$ is isomorphic to $SO(n)$, $n = \dim X$.

By definition, *a Spin structure on X, will be a double covering $\hat{\mathbf{F}}$ of $\mathbf{F}$ whose restriction to a fiber $\mathbf{F}_P$ induces the non-trivial double covering* Spin (n) *of* $SO(n)$.

Two such coverings will be called isomorphic if they are isomorphic *qua* double coverings, so that the *isomorphism classes* of *Spin structures on X* may be identified with the set of elements in $H^1(\mathbf{F}; Z_2)$ which have a non-trivial restiction to $H^1(\mathbf{F}_P; Z_2)$ This set is therefore either vacuous, or a coset, in $H^1(\mathbf{F}_P; Z_2)$ of the kernel of the restriction map $H^1(\mathbf{F}; Z_2) \to H^1(\mathbf{F}_P; Z_2)$.

From the exact sequence

$$(8.1) \qquad \longrightarrow H^1(X; Z_2) \xrightarrow{\pi^*} H^1(\mathbf{F}; Z_2) \xrightarrow{i^*} H^1(\mathbf{F}_P; Z_2) \xrightarrow{\delta} H^2(X; Z_2) \longrightarrow ,$$

where δ denotes the transgression in the fibering $\mathbf{F} \xrightarrow{\pi} X$, we conclude first of all that X admits a Spin structure if and only if $\delta \equiv 0$. The value of δ on the generator of $H^1(\mathbf{F}_P; Z_2)$ is by definition the second Stiefel-Whitney class $w_2(X)$ of X. Hence this condition is equivalent to $w_2(X) = 0$.

Secondly one sees that Ker $(i^*) = \pi^* H^1(X; Z_2)$. Thus the difference of two Spin structures is measured by an element of $H^1(X; Z_2)$.

Suppose now that $\hat{\mathbf{F}}$ is a given Spin structure on X and let $\dim X = 2m$. The Dirac operator relative to $\hat{\mathbf{F}}$ is then constructed as follows. Let

$$(8.2) \qquad \sigma\colon \text{Spin}\,(2m) \longrightarrow SO(2m)$$

be the projection and let ε generate the kernel of σ. Also denote by α a fixed

element in $\sigma^{-1}(-1)$.

The group Spin $(2m)$ has a complex representation Δ of dimension 2^m called the Spin representation. This is the direct sum of two irreducible inequivalent representations Δ^+ and Δ^- of dimension 2^{m-1} (the half-Spin-representations). For these we have

$$\Delta^{\pm}(\varepsilon) = -1 \tag{8.3}$$

$$\Delta^{\pm}(\alpha) = \pm i^{-m} . \tag{8.4}$$

Now let $\Delta^{\pm}\hat{\mathbf{F}}$ denote the vector bundles associated to $\hat{\mathbf{F}}$ by these representations. Then the Dirac operator δ^+ relative to $\hat{\mathbf{F}}$ will be a first order elliptic differential operator

$$\delta^+ : \Gamma(\Delta^+\hat{\mathbf{F}}) \longrightarrow \Gamma(\Delta^-\hat{\mathbf{F}}) . \tag{8.5}$$

To define it one must recall that if

$$\rho : SO(2m) \longrightarrow \mathrm{Aut}\,(\mathbf{R}^{2m}) \tag{8.6}$$

is the standard, or identity, representation then there is a natural pairing

$$(\rho \circ \sigma) \otimes \Delta^{\pm} \longrightarrow \Delta^{\mp} . \tag{8.7}$$

Now TX is clearly the bundle associated to $\hat{\mathbf{F}}$ *via* the representation $\sigma^*\rho = \rho \circ \sigma$. Hence (8.7) induces a pairing

$$\mu^{\pm} : TX \otimes \Delta^{\pm}\hat{\mathbf{F}} \longrightarrow \Delta^{\mp}\hat{\mathbf{F}} . \tag{8.8}$$

Composed with the covariant derivative

$$\nabla^{\pm} : \Gamma(\Delta^{\pm}\hat{\mathbf{F}}) \longrightarrow \Gamma(TX \otimes \Delta^{\mp}\hat{\mathbf{F}}) \tag{8.9}$$

which these bundles inherit from the canonical connection on $\mathbf{F}$, μ therefore gives rise to operators

$$\delta^{\pm} : \Gamma(\Delta^{\pm}\hat{\mathbf{F}}) \longrightarrow \Gamma(\Delta^{\mp}\hat{\mathbf{F}}) \tag{8.10}$$

and these are by definition the Dirac operators we were seeking. The operators are elliptic for the following reason. The pairing μ^+ induces a map

$$TX \xrightarrow{\mu_+^t} \mathrm{Hom}\,(\Delta^+\mathbf{F}, \Delta^-\mathbf{F}) , \tag{8.11}$$

which is non-singular in the sense that $\mu_+^t(\xi)$ is an isomorphism for $\xi \neq 0$. Indeed one has

$$\mu_+^t(\xi) \cdot \mu_-^t(\xi) = -\,\mathrm{Identity}\,(\xi, \xi) . \tag{8.12}$$

On the other hand observe that the symbol of the covariant derivative ∇^+ is induced by the identity map

$$TX \otimes \Delta^+ \longrightarrow TX \otimes \Delta^+ , \tag{8.13}$$

so that the symbol of δ^+ is simply given by $\mu^t_+(\xi)$. Thus (8.12) implies that δ^+ is non-singular. Similary for δ^-. The pertinent facts concerning spinors we have used here can be found in [3], however, for completeness, we will now very briefly review the construction of some of these objects.

If E is a real vector space with positive definite inner product, cE shall denote the Clifford algebra of E. Thus cE is defined as the quotient of the full tensor algebra over E modulo the ideal I generated by elements of the form $e \otimes e + (e, e)\cdot 1$. Alternatively cE is the free algebra generated over $\mathbf{R}$ by a unit 1, and an orthogonal base $\{e_1, \cdots, e_n\}$ for E, subject to the defining relations

$$e_j^2 = -1\,, \qquad e_j e_k + e_k e_j = 0 \qquad j \neq k\,. \tag{8.14}$$

As a module over $\mathbf{R}$, cE is then seen to be spanned by the products $e_{j_1} \cdots e_{j_k}$, $j_1 < j_2 \cdots < j_k, 0 \leqq k \leqq n$ so that, *qua* $\mathbf{R}$ module, cE is isomorphic to the exterior algebra $\lambda^* E$. Multiplicatively this is not so, but cE does inherit the structure of a Z_2-graded module $c(E) = c_+E + c_-E$ where c_+E and c_-E are additively generated by the even and odd products respectively. Furthermore E is naturally included in c_-E.

The group Spin (n) exists naturally as a subgroup of the group of invertible elements c^*E of cE. Indeed let $x \mapsto \bar{x}$ be the anti-automorphism which, acting on our $\mathbf{R}$-basis, sends $e_{j_1} \cdots e_{j_k}$ into $(-1)^k e_{j_k} \cdots e_{j_1}$. Then Spin (n) is the subgroup of c^*E characterized by

$$x \in c_+E \tag{8.15}$$

$$xex^{-1} \in E \qquad \text{for all } e \in E \tag{8.16}$$

$$\bar{x}\cdot x = 1\,. \tag{8.17}$$

If follows from (8.15) and (8.16) that for all $x \in$ Spin (n), the transformation $\sigma(x)$: $E \to E$ defined by $e \mapsto xex^{-1}$ is an orientation preserving isometry of E. Hence σ maps Spin(n) into $SO(n)$ and it is not difficult to show that σ is onto. In fact if[5] $n = 2m$ and $(f_1, f_1', \cdots, f_m, f_m')$ is an orthonormal frame for E, then one checks directly that the element

$$x(\theta_1, \cdots, \theta_m) = \prod_j (\cos\theta_j - f_j f_j' \sin\theta_j) \tag{8.18}$$

is contained in Spin $(2m)$, and projects under σ onto the rotation of E which rotates the plane E_j spanned by (f_j, f_j') through the angle $2\theta_j$. Because every rotation can be brought into this form the assertion follows. It is clear that ± 1 are in Spin$(2m)$ and in the kernel of σ, and then in view of (8.18) it is not hard to see that this is the entire kernel σ. Thus the element ε of Spin $(2m)$

[5] From now on we restrict to the even case because this is all we need.

is simply $-1 \in c^*(E)$. Note finally that if

$$\alpha = f_1 f_1' \cdots f_m f_m' \tag{8.19}$$

then

$$\alpha^2 = (-1)^m , \tag{8.20}$$

$$\sigma(\alpha) = -1 \in SO(2m) , \tag{8.21}$$

and

$$\alpha x = -x\alpha \qquad \text{for all } x \in c_-E \tag{8.22}$$

while,

$$\alpha x = x\alpha \qquad \text{for } x \in c_+E . \tag{8.23}$$

This model of Spin $(2m) \subset c_+E$ furnishes us with a natural representation

$$c_+ : \text{Spin } (2m) \longrightarrow \text{Aut } (c_+E) , \tag{8.24}$$

given by the left action of c_+E on c_+E. This representation is far from irreducible. Rather its complexification breaks up into 2^{m-1} copies of Δ^+ and Δ^-

$$c_+ \otimes_{\mathbf{C}} 1 = 2^{m-1}(\Delta^+ \oplus \Delta^-) . \tag{8.25}$$

To see this let Q_j be given by right multiplication with the elements $f_j f_j' \otimes i$ of our basis $(f_1, f_1', \cdots, f_m, f_m')$. Then the Q_j commute with each other and also, of course, with the action of Spin $(2m)$ because Spin $(2m)$ acts from the left. Further $Q_j^2 = +1$. The simultaneous eigenspaces of the Q_j therefore decompose $c_+E \otimes \mathbf{C}$ into 2^m Spin $(2m)$-invariant subspaces. Furthermore the representations arising in this manner are of only two types, depending on the value α takes on them. This follows from the following construction. Let

$$a = (f_j + f_j')(f_k + f_k')/2 \tag{8.26}$$

then one checks that $a \in$ Spin $(2m)$ and that $Q_s a = aQ_s$ for $s \neq j, k$ while

$$aQ_j = Q_k a . \tag{8.27}$$

Hence right multiplication with a permutes the ± 1 eigenspaces of Q_j with those of Q_k, and it then follows easily that the isomorphism class of a simultaneous eigenspace as a Spin $(2m)$-module depends precisely on the parity of the number of Q_j's with value -1 on it. As $Q_1 \cdots Q_m = \alpha \otimes i^m$, and as $\Delta^\pm$ are distinguished by (8.4), the assertion and (8.25) follow.

An immediate corollary of these remarks is the following

PROPOSITION 8.28. *For any* $x \in$ Spin $(2m)$

$$\text{trace } \Delta^+(x) - \text{trace } \Delta^-(x) = \frac{i^m}{2^{m-1}} \text{trace } c_+(\alpha x) \,. \tag{8.29}$$

In particular, if $x = x(\theta)$ *is the element* (8.18), *then*

$$\text{trace } \Delta^+(x) - \text{trace } \Delta^-(x) = i^m 2^m \prod_1^m \sin \theta_j \,. \tag{8.30}$$

PROOF. The operators $(1 \pm \alpha \otimes i^m)/2$ project $c_+E \otimes \mathbf{C}$ onto the $\Delta^\pm$ types of that module and (8.29) then follows by an argument we have already encountered in § 6. To see (8.30) observe first that left multiplication by an element $e = e_1 \cdots e_{2s}$, $s > 0$, of c_+E induces a transformation of c_+E with trace 0. This follows, for instance, from the fact that this transformation maps the basis elements into multiples of each other, but clearly maps no element into a multiple of itself. Hence the only term in $\alpha x(\theta_1, \cdots, \theta_m)$ which contributes to the trace is

$$\prod_{j=1}^m \sin \theta_j \,.$$

Thereafter (8.30) follows directly from (8.29) and the fact that $\dim c_+E = 2^{2m-1}$.

So much for a quick review of the Spin construction. We leave to the reader the fact that the multiplication $E \otimes c_+(E) \to c_-(E)$ induces the desired pairing (8.7), that this pairing satisfies the condition (8.12) etc. Note finally that the element $\alpha = e_1 \cdot e_2 \cdots e_{2m}$ is not canonically defined in c_+E, but depends on the orientation in which the frame $e_1, e_2, \cdots, e_{2m}$ is taken. The distinction between Δ^+ and Δ^- in the last analysis, therefore depends on the orientation of E.

We turn now to the Lefschetz formula of the Dirac operator

$$\delta^+ \colon \Gamma(\Delta^+\hat{\mathbf{F}}) \longrightarrow \Gamma(\Delta^-\hat{\mathbf{F}}) \,. \tag{8.31}$$

Assume then that X admits a Spin-structure $\hat{\mathbf{F}}$, and let $f \colon X \to X$ be an isometry. The differential of f then induces a lifting

$$df^* \colon f^*\mathbf{F} \longrightarrow \mathbf{F} \tag{8.32}$$

and it is clear that df^* lifts to a bundle isomorphism

$$\hat{f} \colon f^*\hat{\mathbf{F}} \longrightarrow \hat{\mathbf{F}} \,, \tag{8.33}$$

(*qua* bundles over X) if and only if df^* preserves the characteristic class of $\hat{\mathbf{F}}$ in $H^1(\mathbf{F}; Z_2)$. There are then two possible choices of $\hat{f}$ over each component of X and they can be distinguished by their values at a given point. In particular, if X is 2-connected, then X has a unique Spin-structure $\hat{\mathbf{F}}$, and every isometry f has precisely two liftings $\hat{f} \colon f^*\hat{\mathbf{F}} \to \hat{\mathbf{F}}$.

A lifting $\hat{f}$ now induces liftings $\Delta^\pm(\hat{f}) \colon f^*\Delta^\pm(\hat{\mathbf{F}}) \to \Delta^\pm(\hat{\mathbf{F}})$ and so induces a geometric automorphism $\hat{f}_*$ of the Dirac complex

$$\delta^+ : \Gamma(\Delta^+\hat{\mathbf{F}}) \longrightarrow \Gamma(\Delta^-\hat{\mathbf{F}}) .$$

We write $\text{Spin}(\hat{f}, X)$ for the corresponding Lefschetz number

$$\text{Spin}(\hat{f}, X) = \text{trace } H^0(\hat{f}_*) - \text{trace } H^1(\hat{f}_*) . \tag{8.34}$$

Concerning this "Spin-number" of f we now have the following theorem.

THEOREM 8.35. *Suppose that $f: X \to X$ is an isometry of the $2m$-dimensional compact oriented manifold X, with only isolated fixed points $\{P\}$. Suppose further that X admits a* Spin *structure $\hat{\mathbf{F}}$, and that f has a lifting $\hat{f}$ to this* Spin *structure. The* Spin-*number* $\text{Spin}(\hat{f}, X)$ *is then given by the expression*

$$\text{Spin}(\hat{f}, X) = \sum \nu(P) \tag{8.36}$$

where P ranges over the fixed points of f and

$$\nu(P) = \varepsilon(P, \hat{f}) i^m 2^{-m} \prod_1^m \text{cosec}\,(\theta_j/2) \tag{8.37}$$

where $\theta_1, \cdots, \theta_m$ is a coherent system of angles for df_P, and $\varepsilon(P, \hat{f}) = \pm 1$.

PROOF. Only the formula (8.37) has to be derived from the general multiplicity expression

$$\nu(P) = \frac{\text{trace } \varphi_P^0 - \text{trace } \varphi_P^1}{|\det(1 - df_P)|} . \tag{8.38}$$

of § 2. Therefore let P be a fixed point of f. Then $\hat{f}_P: \hat{\mathbf{F}}_P \to \hat{\mathbf{F}}_P$ is a well defined map which commutes with the right action of Spin $(2m)$ on this fiber. For any $y \in \hat{\mathbf{F}}_P$, $\hat{f}_P(y)$ is therefore equal to some right multiple of y.

$$\hat{f}_P(y) = y \cdot x(P; y) , \qquad x(P; y) \in \text{Spin}\,(2m)$$

and the conjugacy class of $x(P, y)$ is independent of $y \in \mathbf{F}_P$. Now just as in the homogeneous case (see § 5) one concludes that trace φ_P^0 and trace φ_P^1 are determined by the element x according to

$$\text{trace } \varphi_P^0 = \text{trace } \Delta^+ x(P; y) \qquad y \in \mathbf{F}_P \tag{8.39}$$

$$\text{trace } \varphi_P^- = \text{trace } \Delta^- x(P; y) \qquad y \in \mathbf{F}_P . \tag{8.40}$$

On the other hand $\sigma x(P, y) \in SO(2m)$ clearly represents the matrix of df_P relative to the frame determined by y at P. It follows that in this frame $x(P, y)$ has the form $\pm x(\theta_1/2, \cdots, \theta_m/2)$ as given by (8.18) for some system $(\theta_1, \cdots, \theta_m)$ of coherent angles for df_P. Applying (8.30) to (8.39) and (8.40), the expression (8.38) goes over into

$$\nu(P) = \pm i^m 2^m \prod_1^m \sin(\theta_j/2) \Big/ |\det(1 - df_P)| . \tag{8.41}$$

However the denominator is given by

$$\prod_{j=1}^m (1 - e^{i\theta_j})(1 - e^{-i\theta_j})$$

and hence equals $2^{2m}\prod\{\sin(\theta_j/2)\}^2$. The result follows.

The sign $\pm$ in the formula for $\nu(P)$ depends on P and $\hat{f}$, as is made clear in the notation of (8.37) where it appears as $\varepsilon(P, \hat{f})$. Clearly

$$\varepsilon(P, -\hat{f}) = -\varepsilon(P, \hat{f})$$

where $-\hat{f}$ is the opposite lifting. Thus we cannot specify the sign in terms of f alone. What we can do however is to compare two different fixed points P, Q: the product

$$\varepsilon(P, Q; f) = \varepsilon(P, \hat{f})\cdot\varepsilon(Q, \hat{f})$$

will depend only on f and not on the choice of lifting $\hat{f}$. We shall show how to compute $\varepsilon(P, Q; f)$ in the special case when f is an involution and X is 2-connected.

Notice first that if P is a fixed point of the involution f and $\hat{f}$ is a lifting to the Spin-structure $\hat{\mathbf{F}}$, then for any $y \in \hat{\mathbf{F}}_P$,

$$\hat{f}_P(y) = yx(\hat{f}, P) \tag{8.42}$$

with $x(\hat{f}, P) = \pm\alpha \in \mathrm{Spin}(2m)$. Hence

$$\nu(P) = \pm i^m 2^{-m}, \tag{8.43}$$

correspondingly.

We will call two fixed points P and Q, f-equivalent if they have equal multiplicity, that is if $\varepsilon(P, Q; f) = 1$. Then we have the following criterion cencerning equivalence of fixed points.

PROPOSITION 8.44. *Let $t \mapsto s(t)$ be a curve in $\mathbf{F}X$ starting at $y \in \mathbf{F}_P$ and ending in $\mathbf{F}_Q$. Let $df^*: \mathbf{F}X \to \mathbf{F}X$ denote the map induced by the differential and consider the curve $r = -df^*s$. This curve then has the same endpoints as s so that the composition $r^{-1}*s$ is a well defined loop $c: S^1 \to \mathbf{F}$, with $c(1) = P$ and $c(-1) = Q$. Then*

P and Q are f-equivalent if and only if $c = 0$ in $\pi_1(\mathbf{F}; y)$.

PROOF. By assumption $\pi_0(X) = \pi_1(X) = 0$. Hence $\pi_1(\mathbf{F}, y) = Z_2$. Hence the pullback $c^*\hat{\mathbf{F}}$ of $\hat{\mathbf{F}}$ to S^1 is the non-trivial or trivial double covering of the circle, depending on whether c is non-trivial or trivial in $\pi_1(\mathbf{F}, y)$. Furthermore note that, by construction, the composition $\hat{f}\circ\alpha$ of $\hat{f}$ with right multiplication by α induces an automorphism of $c^*\hat{\mathbf{F}}$ over the reflection map $z \mapsto \bar{z}$ of $S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$. The result now follows by inspection: if $c^*\mathbf{F}$ is the trivial covering and $\hat{f}\cdot\alpha$ is the identity over $z = 1$, it will also have to be the identity over $z = -1$; on the other hand, if $c^*\mathbf{F}$ is the non-trivial covering $e^{i\theta} \mapsto e^{2i\theta}$ and $\hat{f}\cdot\alpha$ is the identity over 1 then $\hat{f}\cdot\alpha$ is given by $\theta \mapsto -\theta$ and so is minus the identity at $\theta = \pi/2$, that is at $z = -1$. q.e.d.

In general, i.e., when X can have several Spin structures, this argument generalizes in a straightforward manner to yield the following criterion.

PROPOSITION 8.45. *Suppose $\hat{f}$ is a lifting of an involution to the* Spin-*structure $\hat{\mathbf{F}}$ and that P and Q are fixed points of f in the same component of X. Also suppose that c is constructed as above. Then P and Q are f-equivalent if and only if the characteristic class $\omega_1(\hat{\mathbf{F}}) \in H^1(\mathbf{F}; Z_2)$ vanishes on c.*

We conclude this section with a few further remarks concerning involutions of Spin-manifolds. These remarks are independent of our Lefschetz theorem but they throw some light on the formula (8.37) for involutions.

Thus let X be a Spin-manifold and let $f: X \to X$ be an involution preserving the orientation and the Spin-structure, so that we get two liftings $\pm\hat{f}$ to $\hat{\mathbf{F}}X$. As we have pointed out $\hat{f}$ is at most of order 4. Let us call an involution of *even type* if $\hat{f}^2 = 1$ and of *odd type* if $\hat{f}$ has order 4 (note that this does not depend on which lifting we choose). If f has no fixed points, so that the orbit space Y is a manifold, then one sees at once that

f is of even type $\Leftrightarrow$ Y is a Spin-*manifold.*

Thus the antipodal map on the sphere S^{2n-1} is of even type if and only if the real projective space P^{2n-1} has second Stiefel-Whitney class zero, and this is known to be true precisely for *n even*. Hence $S^3 \times S^5$ provides an example of a manifold having involutions of both types, the anti-podal maps on the factors.

Suppose now that f has fixed points. Since $f^2 = 1$ each connected component of the fixed point set is a submanifold and df acts as -1 in the normal planes. Restricting f to the normal sphere at a fixed point we therefore get the antipodal map, and the type of this restriction is the same as the type of f. From the results about projective spaces mentioned above we therefore deduce[6]

PROPOSITION 8.46. *Let X be a* Spin-*manifold, $f: X \to X$ an involution preserving the orientation and* Spin-*structure, and let Y_j be the connected components of the fixed point set of f. Then*

$$\begin{aligned} \operatorname{codim} Y_j &\equiv 0 \mod 4 && \text{if } f \text{ is of even type} \\ &\equiv 2 \mod 4 && \text{if } f \text{ is of odd type}. \end{aligned}$$

In particular we always have

$$\dim Y_j \equiv \dim Y_k \mod 4$$

[6] One can of course verify (8.45) by working directly with Clifford algebras and spinors. In fact this is one way to determine which projective spaces are Spin-manifolds.

for any two components Y_j, Y_k.

The final part of (8.46) is not true if X is not a Spin-manifold as is shown by the example of the complex projective plane with the involution $(x_0, x_1, x_2) \mapsto (-x_0, x_1, x_2)$: this has a fixed point and a fixed 2-sphere.

From (8.45) we see that the existence of an *isolated* fixed point implies

$$\begin{aligned} \dim X &\equiv 0 \bmod 4 && \text{for } f \text{ of even type} \\ &\equiv 2 \bmod 4 && \text{for } f \text{ of odd type} . \end{aligned}$$

Hence the number $\nu(P)$ in (8.37), for involutions, is *real* when f has even type and *imaginary* when f has odd type. This fits in with the formula (8.36) because Spin $(\hat{f}, X)$ is clearly real when f has even type. One can also show directly from the analysis that, if $m \equiv 2 \bmod 4$ and f has even type, then Spin $(\hat{f}, X) = 0$. This again fits with (8.36) because, as we have just seen, there cannot be any isolated fixed points in this case.

9. Exotic involutions

Consider the hypersurface $V(a)$ in $\mathbf{C}^n$ given by the equation

$$z_1^{a_1} + \cdots + z_n^{a_n} = 0 . \tag{9.1}$$

When all the a_i are greater than 1, $V(a)$ has an isolated singularity at $z = 0$ whose topological and differentiable nature is by now well understood from several points of view thanks to the work of Brieskorn, Pham, Milnor, Hirzebruch, and others. We will need a few special instances of these results, all of which can be found in [9].

Let $\Sigma(a)$ denote the intersection of $V(a)$ with the unit sphere, S^{2n-1} in $\mathbf{C}^n$, and let D^{2n} be the unit disk $|z| \leqq 1$ in $\mathbf{C}^n$. Now for $t \in \mathbf{C}$ let

$$V_t(a) = \{z \mid z_1^{a_1} + \cdots + z_n^{a_n} = t\} \tag{9.2}$$

$$\Sigma_t(a) = V_t(a) \cap S^{2n-1} \tag{9.3}$$

$$M_t(a) = V_t(a) \cap D^{2n} . \tag{9.4}$$

It is then easy to see that, for small t, $\Sigma_t(a)$ is the smooth boundary of $M_t(a)$ and that $\Sigma_t(a)$ is diffeomorphic to $\Sigma(a)$.

Furthermore it can be shown that, for $n \geqq 4$, $\Sigma(a_1, \cdots, a_n)$ is $(n-3)$-connected while $M_t(a_1, \cdots, a_n)$ is $(n-2)$-connected.

In [9] one finds a numerical algorithm on the a_i which decides whether $\Sigma(a)$ is a topological sphere.

In particular this criterion implies that $\Sigma(a_1, \cdots, a_n)$ is a topological S^{2n-3} for the a_i we are interested in. These are of the form

$$a^k = (a_1, \cdots, a_{2m}) = (2, 2, \cdots, 2, k)\,, \qquad k \text{ odd}\,. \tag{9.5}$$

Furthermore these $\Sigma(a^k)$ are also classified there according to diffeomorphism type

(9.6) *$\Sigma(a^k)$ is the standard $(4m-3)$ sphere unless m is $\geqq 3$ and $k \equiv \pm 3 \bmod 8$.*

Our concern will be with the following involution on $\Sigma = \Sigma(a^k)$, a^k as in (9.5).

Let $T\colon \mathbf{C}^n \to \mathbf{C}^n$ be defined by

$$Tz_i = \begin{cases} -z_i & 1 \leqq i \leqq n-1 \\ z_i & i = n\,. \end{cases} \tag{9.7}$$

Clearly T preserves Σ and has no fixed points there. The resulting involutions occur in the work of Bredon [8] and Hirsch-Milnor [14]. In fact for $m = 2$, $k = 3$ one obtains precisely the exotic involution of [8].

Our aim is the following generalization of this example.

THEOREM 9.8. *If the actions of T on the topological spheres $\Sigma = \Sigma(a^k)$ and $\Sigma = \Sigma(a^l)$ are isomorphic, then*

$$k \equiv \pm l \mod 2^{2m}\,. \tag{9.9}$$

In particular the involution T acting on $\Sigma(a^3) = S^{4m-3}$ is not isomorphic to the standard antipodal map whenever $m \geqq 2$.

PROOF. Clearly T preserves the set $M_t(a^k)$ and its boundary $\Sigma_t(a^k)$. Furthermore for small t the action of T on $\Sigma(a^k)$ will be isomorphic to its action on $\Sigma(a^k)$. Hence, if the conditions of the theorem are met, there exists a diffeomorphism

$$\mu\colon \Sigma_t(a^k) \longrightarrow \Sigma_t(a^l) \tag{9.10}$$

which commutes with the action of T. Consider now the compact manifold X obtained from $M = M_t(a^k)$ and $M' = M_t(a^l)$ by gluing their boundaries together with μ

$$X = M \cup_\mu M'\,. \tag{9.11}$$

The involution T acts naturally on X, and its fixed points will coincide with the union of those of $T \mid M$ and $T \mid M'$. For $m \geqq 2$, X will furthermore be 2-connected and so has a natural Spin structure. Let $\hat{T}$ be a lifting of T to $\hat{\mathbf{F}}X$ and consider the Spin number Spin $(\hat{T}, X)$. Clearly $\hat{T}$ has period at most 4. Hence Spin $(\hat{T}, X) \in \mathbf{Z}[i]$. On the other hand by Theorem 8.35 we have the relation

$$\Sigma_P \varepsilon(P, \hat{T})(i/2)^{2m-1} = \text{Spin}\,(\hat{T}, X) \tag{9.12}$$

where P runs over the fixed points of T.

Now consider $T \mid M$ first. Clearly the fixed points of T in M have the form:

$$(0, \cdots, 0, u) \qquad \text{with } u^k = t \ .$$

Hence there are k of them. Note further that the transformation $S\colon \mathbf{C}^n \to \mathbf{C}^n$ given by

$$S(z_i) = \begin{cases} z_i & 1 \leqq i \leqq n-1 \\ \xi z_n & \xi = e^{2\pi i/k},\ i = n \end{cases}$$

maps M into itself, commutes with T, and cyclically permutes the fixed points of T on M. Since S has *odd* order k its action on M can be lifted to a transformation $\hat{S}$ of order k on $\hat{F}M$ which commutes with $\hat{T}$. This implies that, if P is any one of the fixed points of T in M,

$$\varepsilon(P, \hat{T}) = \varepsilon(SP, \hat{T}) \ .$$

Since S permutes these fixed points cyclically it follows that the signs $\varepsilon\,(P, \hat{T})$ for $P \in M$ are all the same. A similar result holds for the fixed points of $T \mid M'$ and so (9.12) takes the form

$$\left(\frac{i}{2}\right)^{2m-1} \{\pm k \pm l\} = \text{Spin}\,(\hat{T}, X) \ .$$

Since $\text{Spin}\,(\hat{T}, X) \in \mathbf{Z}[i]$, this implies that $k \equiv \pm l \bmod 2^{2m-1}$. q.e.d.

OXFORD UNIVERSITY
HARVARD UNIVERSITY

BIBLIOGRAPHY

[1] M. ATIYAH and R. BOTT, *A Lefschetz fixed point formula for elliptic differential operators*, Bull. Amer. Math. Soc. **72** (1966), 245-250.

[2] ———, *A Lefschetz fixed point formula for elliptic complexes*: I. Ann. of Math. **86** (1967), 374-407.

[3] ——— and A. SHAPIRO, *Clifford modules*, Topology **3** (1964), 3-38.

[4] M. ATIYAH and G. SEGAL, *The index of elliptic operators*: II, Ann. of Math. **87** (1968), 531-545.

[5] ——— and I. M. SINGER, *The index of elliptic operators*: I, Ann. of Math. **87** (1968), 484-530.

[6] ———, *The index of elliptic operators*: III Ann. of Math. **87** (1968), 546-604.

[7] A. BOREL and A. WEIL, Representations lineaires et espaces homogenes Kählerians des groupes de Lie compactes, Seminaire Bourbaki, May 1954 (Exposé by J.P. Serre).

[8] G. E. BREDON, *Examples of differentiable group actions*, Topology **3** (1965), 115-122.

[9] E. BRIESKORN, *Beispiele zur Differentialtopologie von Singularitäten*. Invent. Math. **2** (1966), 1-14.

[10] P. E. CONNER and E. E. FLOYD, *Maps of odd period*, Ann. of Math. **84** (1966), 132-156.

[11] M. EICHLER, *Eine Verallgemeinerung der Abelschen Integrale*, Math. Z. **67** (1957), 267-298.

[12] W. FRANZ, *Über die Torsion einer Überdeckung*, J. Reine Angew. Math. **173** (1935), 245-254.

[13] R. HARTSHORNE, Residues and duality, Lecture notes in Mathematics, Springer-Verlag 1966.

[14] M. W. HIRSCH and J. MILNOR, *Some curious involutions of spheres*, Bull. Amer. Math. Soc. **70** (1964), 372-377.

[15] J. MILNOR, *Whitehead torsion*, Bull. Amer. Math. Soc. **3** (1966), 358-426.

[16] A. SELBERG, *Harmonic analysis and discontinuous groups in weakly symmetric spaces with applications to Dirichlet series*, J. Indian Math. Soc. **20** (1956), 47-48.

(Received March 1, 1968)