ON FORMULAE OF THOM AND WU

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1. Recently, in preparing a set of lectures on characteristic classes, I had occasion to consider the formulae of Thom and Wu (**10**) which relate Stiefel–Whitney classes to Steenrod squares. Briefly, they are as follows. Let M be a compact differentiable n-manifold, not necsssarily orientable, with fundamental class $\mu \in H_n(M; Z_2)$. Then there is a unique class $v_i \in H^i(M; Z_2)$ such that

$$\langle Sq^i x, \mu \rangle = \langle v_i x, \mu \rangle$$

for each $x \in H^{n-i}(M; Z_2)$; and the Stiefel–Whitney classes $w_k \in H^k(M; Z_2)$ satisfy

$$w_k = \sum_{i+j=k} Sq^j v_i.$$

Although these formulae are simple and attractive, I did not feel that they gave me that complete understanding which I sought. For example, they raise a problem first recorded by Thom (**9**); briefly, it is as follows. One may use these formulae to define Stiefel–Whitney classes w_k in the cohomology of a manifold which is not necessarily differentiable, or indeed, to define Stiefel–Whitney classes in any algebra over Z_2 which admits operations Sq^i and satisfies suitable axioms. Do these generalized Stiefel–Whitney classes satisfy every formula which holds in the differentiable case? In particular, in the differentiable case we have

$$Sq^i w_k = Q(w_1, w_2, \ldots, w_{i+k})$$

for a certain polynomial $Q = Q(i, k)$; does this formula hold for the generalized Stiefel–Whitney classes?

We shall prove that the answer to this question of Thom is in the affirmative (see Corollary 3 below). The paper is arranged as follows. In §§ 2, 3 we consider abstract algebras of the sort indicated above, and attempt to obtain a better understanding of their theory. § 2 contains sufficient to answer Thom's question, and § 3 contains the remainder of the work. In § 4 we make some supplementary remarks.

It is a pleasure to express my gratitude to R. Thom for a helpful letter. A similar acknowledgement to F. Hirzebruch appears in context in § 4.

2. We have said that generalized Stiefel–Whitney classes are defined in any algebra over Z_2 which admits operations Sq^i and satisfies suitable axioms. In this section, therefore, the initial object of study will be a graded, anticommutative algebra $H = \sum_{0 \leqslant i \leqslant n} H^i$ over the field Z_p which admits operations Sq^i (if $p = 2$) or P^k (if $p > 2$). More precisely, if $p = 2$ we define A to be the mod 2 Steenrod algebra (**3**, **6**); if $p > 2$ we define A to be that subalgebra of the mod p Steenrod algebra which is generated by the P^k; we now assume that H is a graded left module over the graded algebra A. We impose the following axioms.

(*a*) (Cartan formula.) Let $\Delta\colon A \to A \otimes A$ be the diagonal map (**6**); as a standard convention, we will write

$$\Delta a = \sum_r a'_r \otimes a''_r.$$

Then we have $$a(hk) = \sum_r (a'_r h)(a''_r k) \quad (h, k \in H).$$

(We need no signs in this formula, because either $p = 2$ or the elements of A are of even degree.)

(*b*) (Dimension axiom.) If $p = 2$, $h \in H^i$ and $i < j$, then $Sq^j h = 0$. If $p > 2$, $h \in H^i$, and $i < 2k$, then $P^k h = 0$.

(*c*) (Poincaré duality.) There is given an element μ in the vector-space dual of H^n. We write $\langle h, \mu \rangle$ for $\mu(h)$, to preserve the analogy with the topological case. The bilinear function $\langle hk, \mu \rangle$ of the variables $h \in H^i$, $k \in H^{n-i}$ gives a dual pairing from the finite-dimensional vector spaces H^i and H^{n-i} to Z_p.

For example, let M be a compact topological n-manifold, without boundary; and if $p > 2$, let M be oriented. Then the cohomology ring $H = H^*(M; Z_p)$ satisfies all these axioms, provided that we take μ to be the fundamental class.

If H is an algebra satisfying the axioms we have given, then we can make H into a graded right module over A; in fact, if $h \in H^i$, $a \in A^j$ we define ha by the equation

$$\langle ha . k, \mu \rangle = \langle h . ak, \mu \rangle \quad (k \in H^{n-i-j}).$$

However, we cannot assert that these operations of A on the right commute with those on the left; nor can we assert that they satisfy the Cartan formula or the dimension axiom.

In particular, we shall have much to do with the classes $\mathscr{E}_H a$, where $\mathscr{E}_H$ is the unit in H. The characteristic property of $\mathscr{E}_H a$ is

$$\langle \mathscr{E}_H a . k, \mu \rangle = \langle ak, \mu \rangle.$$

If we take $p = 2$, then $\mathscr{E}_H Sq^i$ is the class v_i which appears in the formulae of Thom and Wu.

In any algebra H, we can define various classes by starting from the unit $\mathscr{E}_H$ and iterating the operations we have mentioned above. (For example,

$$w_k = \sum_{i+j=k} Sq^i(\mathscr{E}_H Sq^j)$$

is a class of this sort, and so is $Sq^l w_k$.) We wish to study how many classes we can obtain in this way, and what universal formulae they satisfy; that is, what formulae hold in every H. We therefore proceed as follows.

We first define a class of 'words' W, by laying down the following four inductive rules:

(i) The letter $\mathscr{E}$ is a word.

(We emphasize that here $\mathscr{E}$ is merely a formal symbol; in particular, it should not be confused with the unit of any particular algebra H.)

(ii) If W is a word and $a \in A$, then aW and Wa are words.
(iii) If W and W' are words, then the 'cup-product' WW' is a word.
(iv) If W and W' are words and λ, $\mu \in Z_p$, then $\lambda W + \mu W'$ is a word.

For example, if $p = 2$, the following formula is a word:

$$Sq^2\{[(\mathscr{E}Sq^1)(\mathscr{E}Sq^2)]+(\mathscr{E}Sq^3)\}.$$

And, in general, a formula W is a word if and only if it is shown to be such by a finite number of applications of the four given rules.

If H is an algebra, satisfying the axioms we have given above, then we can regard each word W as a formula defining a specific element of H. More formally, we can define a function θ_H which assigns to each word W an element of H, by giving the following four inductive rules.

(i) $\theta_H(\mathscr{E}) = \mathscr{E}_H$.
(ii) $\theta_H(aW) = a(\theta_H(W))$, $\qquad \theta_H(Wa) = (\theta_H(W))a$.
(iii) $\theta_H(WW') = (\theta_H(W))(\theta_H(W'))$.
(iv) $\theta_H(\lambda W + \mu W') = \lambda(\theta_H(W)) + \mu(\theta_H(W'))$.

We may refer to $\theta_H(W)$ as 'the value of W in H'.

We now divide the words W into equivalence classes, putting W and W' into the same class if we have $\theta_H(W) = \theta_H(W')$ for every H. We take these equivalence classes as the elements of a 'universal domain' U. The problem mentioned above is therefore equivalent to determining the structure of U.

It is trivial to check that U admits well-defined cup-products, linear combinations, and operations from A (both on the left and right). The operations from A are linear; the operations on the left satisfy the Cartan formula and the dimension axiom.

The ring-structure of U is given by Theorem 1 below; the remaining structure of U will be given in section 3.

THEOREM 1. *U is a polynomial algebra on the generators $u_1, u_2, \dots, u_i, \dots$ defined below.*

In order to define u_i, we write χ for the canonical anti-automorphism of A (**6**); this is defined, inductively, by the equations

$$\chi(1) = 1, \qquad \sum_r \chi(a'_r)a''_r = 0 \quad (\dim a > 0)$$

(where $\Delta a = \sum_r a'_r \otimes a''_r$, as always). We now define $u_i = \mathscr{E}(\chi Sq^i)$ if $p = 2$, $u_k = \mathscr{E}(\chi P^k)$ if $p > 2$.

The degree of u_i is thus i if $p = 2$, and $2i(p-1)$ if $p > 2$.

Our next theorem will show that U is faithfully represented in the cohomology of differentiable manifolds.

THEOREM 2. *Suppose given an integer N; let* M *run over those monomials in the u_i $(i > 0)$ whose degree is N or less. Then there is a differentiable manifold D (orientable if $p > 2$) such that the values in $H^*(D; Z_p)$ of the monomials* M *are linearly independent.*

Theorems 1 and 2 lead immediately to the following corollary.

COROLLARY 3. *Let W be a word of the sort considered above. Suppose that the value of W in $H^*(D; Z_p)$ is zero for every differentiable manifold D; then the value of W in H is zero for any H.*

It is clear that this corollary answers Thom's question in the affirmative. In fact, let us take $p = 2$; then

$$w_k = \sum_{i+j=k} Sq^i(\mathscr{E}Sq^j)$$

is a word of the sort considered; hence so is

$$W = Sq^i w_k - Q(w_1, w_2, \dots, w_{i+k})$$

for any polynomial Q. If we choose Q so that $W = 0$ in every differentiable manifold, then the corollary shows that $W = 0$ in every H.

The remainder of this section will be devoted to proving Theorems 1 and 2. The manifolds D which we exhibit to prove Theorem 2 are cartesian products of projective spaces. We begin work as follows.

LEMMA 4. *Take $p = 2$; let x be the cohomology generator in $H^1(RP^\infty; Z_2)$. If $a \in A^i$ and $i+j+k = 2^s-1$, then*

$$x^j . ax^k = x^k . (\chi a)x^j.$$

Take $p > 2$; let y be the cohomology generator in $H^2(CP^\infty; Z_p)$. If $a \in A^{2i}$ and $i+j+k = p^s-1$, then

$$y^j . ay^k = y^k . (\chi a)y^j.$$

In proving this lemma, and later, it will be convenient to make a convention concerning the expansion

$$\Delta a = \sum_r a'_r \otimes a''_r.$$

If $\dim a > 0$, we may assume that this expansion contains the term $a \otimes 1$ (for $r = \alpha$, say) and the term $1 \otimes a$ (for $r = \omega$, say), while the remaining terms have $\dim a'_r > 0$, $\dim a''_r > 0$.

We give the proof of Lemma 4 for the case $p > 2$; the case $p = 2$ is closely similar. We proceed by induction over $\dim a$. The result is trivial if $\dim a = 0$; as an inductive hypothesis, we suppose it true if $\dim a < l$ ($l > 0$); we must deduce it when $\dim a = l$.

It is easy to see that if $N = p^s - 1$ and $m > 0$, then the operation

$$P^m\colon H^{2N-2m(p-1)}(CP^\infty;\, Z_p) \to H^{2N}(CP^\infty;\, Z_p)$$

is zero. Hence any operation a taking values in $H^{2N}(CP^\infty; Z_p)$ is zero, at least if $\dim a > 0$. In particular, if $a \in A^{2i}$ and $i+j+k = N$, we have $a(y^k . y^j) = 0$. That is,

$$ay^k . y^j + \sum_{r \neq \alpha} a'_r y^k . a''_r y^j = 0.$$

Using the inductive hypothesis, we have

$$ay^k . y^j + \sum_{r \neq \alpha} y^k . \chi(a'_r) a''_r y^j = 0.$$

Using the characteristic property of χ, we have

$$ay^k . y^j = y^k . \chi(a) y^j.$$

This completes the induction.

From this point onwards, we shall permit ourselves to write $\mathscr{E}$ instead of $\mathscr{E}_H$ for the unit in any algebra H under consideration.

LEMMA 5. *Take $p = 2$, $n = 2^s - 2$ ($s \geqslant 2$), $M = RP^n$, and let x be the cohomology generator in $H^1(M; Z_2)$. Then in $H^*(M; Z_2)$ we have $\mathscr{E}(\chi Sq^1) = x$ and $\mathscr{E}(\chi Sq^i) = 0$ for $i > 1$.*

Take $p > 2$, $m = p^s - 2$ ($s \geqslant 2$), $M = CP^m$, and let y be the cohomology generator in $H^2(M; Z_p)$. Then in $H^*(M; Z_p)$ we have $\mathscr{E}(\chi P^1) = y^{p-1}$ and $\mathscr{E}(\chi P^k) = 0$ for $k > 1$.

Proof. According to Lemma 4, the calculation of an operation χa which maps into the top dimension of M reduces to the calculation of ax or ay, as the case may be. The values of $Sq^i x$ and $P^k y$ are well known, and lead to the result stated.

LEMMA 6. *In a product manifold $M = M' \times M''$, the values of the classes $\mathscr{E}a$ are given by*

$$\mathscr{E}a = \sum_r \mathscr{E}' a'_r \otimes \mathscr{E}'' a''_r,$$

where $\mathscr{E}'b$ and $\mathscr{E}''c$ denote the corresponding classes in M' and M''.

The verification is obvious.

We will now prove Theorem 2. We take the manifold D to be a cartesian product of N factors, where each factor is a projective space RP^n (if $p = 2$) or CP^m (if $p > 2$). We suppose, of course, that $n = 2^s-2 \geqslant N$, or that $m = p^s-2 \geqslant \frac{1}{2}N$, according to the case considered. If $p = 2$, we write $x_1, x_2,..., x_N$ for the cohomology generators in the separate factors; if $p > 2$, we call them $y_1, y_2,..., y_N$.

LEMMA 7. *If* $p = 2$, $\mathscr{E}(\chi Sq^i)$ *is the i-th elementary symmetric function in* $x_1, x_2,..., x_N$.

If $p > 2$, $\mathscr{E}(\chi p^k)$ *is the k-th elementary symmetric function in* $(y_1)^{p-1}$, $(y_2)^{p-1},..., (y_N)^{p-1}$.

This lemma follows immediately from Lemmas 5 and 6; and it completes the proof of Theorem 2.

In order to prove Theorem 1, it is now sufficient to show that U is multiplicatively generated by the elements u_i, $i > 0$. We begin work as follows:

LEMMA 8 (i). *If* $\dim a > 0$ *and* $h \in H$, *then*

$$(\mathscr{E}a).h = ah + \sum_{r \neq \alpha,\omega} (a'_r h)a''_r + ha.$$

(ii) *If* $\dim a > 0$ *and* $u \in U$, *then*

$$(\mathscr{E}a).u = au + \sum_{r \neq \alpha,\omega} (a'_r u)a''_r + ua.$$

(iii) *If* $\dim a > 0$, *then*

$$(\mathscr{E}a).(\mathscr{E}b) = a(\mathscr{E}b) + \sum_{r \neq \alpha,\omega} (a'_r(\mathscr{E}b))a''_r + \mathscr{E}ba.$$

Proof. We begin with (i). If k is also in H and of the appropriate dimension, we have

$$\begin{aligned}\langle(\mathscr{E}a).h.k, \mu\rangle &= \langle a(hk), \mu\rangle \\ &= \langle ah.k, \mu\rangle + \sum_{r \neq \alpha,\omega} \langle a'_r h.a''_r k, \mu\rangle + \langle h.ak, \mu\rangle \\ &= \langle ah.k, \mu\rangle + \sum_{r \neq \alpha,\omega} \langle (a'_r h)a''_r.k, \mu\rangle + \langle ha.k, \mu\rangle.\end{aligned}$$

Since this holds for each k, it establishes (i).

Now take an element $u \in U$. The equation

$$(\mathscr{E}a).u = au + \sum_{r \neq \alpha,\omega} (a'_r u)a''_r + ua$$

holds in every H; therefore it holds in U. This proves part (ii). Part (iii) follows by substituting $u = \mathscr{E}b$.

LEMMA 9. *U is multiplicatively generated by the elements* $\mathscr{E}a$, $a \in A$.

Proof. It is sufficient to prove that if $a \in A$, and W is a polynomial in the elements $\mathscr{E}b$, then aW, Wa may also be written as polynomials in the $\mathscr{E}b$. We will prove this proposition by induction over $\dim a$. The proposition is trivial if $\dim a = 0$; as an inductive hypothesis, we suppose it true when $\dim a < k$ ($k > 0$); we must deduce it when $\dim a = k$.

We begin with the expression $a(\mathscr{E}b)$. Consider the equation of Lemma 8 (iii). The terms $(\mathscr{E}a).(\mathscr{E}b)$, and $\mathscr{E}ba$ are already polynomials in the $\mathscr{E}c$; and each term $(a_r'(\mathscr{E}b))a_r''$ can be written in that form, by the inductive hypothesis. Hence $a(\mathscr{E}b)$ can be written as a polynomial in the $\mathscr{E}c$.

If W is a polynomial in the $\mathscr{E}b$, then aW can be expanded (by linearity and the Cartan formula) in terms of expressions $c(\mathscr{E}b)$ with $\dim c \leqslant k$. Each of the expressions $c(\mathscr{E}b)$ can be written as a polynomial in the $\mathscr{E}d$, as we have just shown; hence aW can be written as a polynomial in the $\mathscr{E}d$.

This completes the inductive step, so far as aW is concerned; we turn to Wa. By Lemma 8 (ii) we have

$$Wa = (\mathscr{E}a).W - aW - \sum_{r \neq \alpha,\omega} (a_r' W)a_r''.$$

The term $(\mathscr{E}a)W$ is already a polynomial in the $\mathscr{E}b$; aW can be written in that form, as we have just shown; and so can each term $(a_r' W)a_r''$, by the inductive hypothesis. Hence Wa can be written as a polynomial in the $\mathscr{E}b$. This completes the induction, and the proof of Lemma 9.

We have yet to show that U is multiplicatively generated by the u_i ($i > 0$). Let us write $I(U) = \sum_{j>0} U^j$, and let $D(U)$ be the set of decomposable elements in U, that is, those which can be written in the form $u = \sum_r u_r' u_r''$ with $u_r' \in I(U)$, $u_r'' \in I(U)$. Our task amounts to calculating $I(U)/D(U)$.

LEMMA 10. *If $u \in D(U)$, $a \in A$, then $ua \in D(U)$.*

The proof is by induction over $\dim a$. The result is trivial when $\dim a = 0$; as an inductive hypothesis, we suppose it true when $\dim a < k$ ($k > 0$); we must deduce it when $\dim a = k$.

Consider the equation of Lemma 8 (ii). The term $(\mathscr{E}a).u$ is certainly decomposable. Since u is decomposable, au is decomposable, by the Cartan formula; and similarly, each term $a_r'u$ is decomposable. By the inductive hypothesis, each term $(a_r'u)a_r''$ is decomposable. Hence ua is decomposable. This completes the induction.

LEMMA 11. *If $\dim b > 0$, then*

$$a(\mathscr{E}b) = \mathscr{E}b\chi(a) \mod D(U).$$

The proof is again by induction over $\dim a$. The result is trivial when $\dim a = 0$; as an inductive hypothesis, we suppose it true when $\dim a < k$ ($k > 0$); we must deduce the result when $\dim a = k$.

By Lemma 8 (iii) we have

$$a(\mathscr{E}b) = (\mathscr{E}a).(\mathscr{E}b) - \mathscr{E}ba - \sum_{r \neq \alpha, \omega} (a'_r(\mathscr{E}b))a''_r.$$

Since $\dim b > 0$, the term $(\mathscr{E}a).(\mathscr{E}b)$ is decomposable. By the inductive hypothesis, the term $a'_r(\mathscr{E}b)$ yields $\mathscr{E}b\chi(a'_r)$, modulo $D(U)$. Using Lemma 10, the term $(a'_r(\mathscr{E}b))a''_r$ yields $\mathscr{E}b\chi(a'_r)a''_r$, modulo $D(U)$. Hence

$$a(\mathscr{E}b) = -\mathscr{E}ba - \sum_{r \neq \alpha, \omega} \mathscr{E}b\chi(a'_r)a''_r$$

modulo $D(U)$. Using the characteristic property of χ, we find

$$a(\mathscr{E}b) = \mathscr{E}b\chi(a)$$

modulo $D(U)$. This completes the induction.

LEMMA 12. *$I(U)/D(U)$ is spanned by the elements u_i, $i > 0$.*

We will give the proof for the case $p > 2$. We easily see (using Lemma 9) that $I(U)/D(U)$ is spanned by the elements $\mathscr{E}a$, where $\dim a > 0$. In fact, it is spanned by the elements $\mathscr{E}a$ as a runs over a set of elements which span $I(A) = \sum_{i>0} A^i$.

Consider the set S of elements

$$P^{k_1}P^{k_2}\dots P^{k_l}$$

such that $k_1 \geqslant pk_2$, $k_2 \geqslant pk_3, \dots, k_{l-1} \geqslant pk_l > 0$. It is easy to show from the Adem relations that these elements do span $I(A)$. It is well known (3) that they form a base for it; but we do not need this fact here.

The set S contains the elements P^k $(k > 0)$. Every other element in S can be written in the form P^lc with $0 < \dim c < 2l$. As a runs over S, χa runs over a set χS which also spans $I(A)$. The set χS contains the elements χP^k $(k > 0)$; every other element in χS can be written in the form $d\chi(P^l)$ with $0 < \dim d < 2l$. By Lemma 11, we have

$$\mathscr{E}d\chi(P^l) = P^l(\mathscr{E}d) \mod D(U).$$

But $P^l(\mathscr{E}d)$ is zero, because $\dim(\mathscr{E}d) < 2l$. We have shown that $\mathscr{E}d\chi(P^l)$ is decomposable. Hence $I(U)/D(U)$ is spanned by the elements $\mathscr{E}\chi(P^k)$ $(k > 0)$. This completes the proof in case $p > 2$.

The proof in case $p = 2$ is closely similar. We begin with the set S of elements

$$Sq^{i_1}Sq^{i_2}\dots Sq^{i_l}$$

such that $i_1 \geqslant 2i_2$, $i_2 \geqslant 2i_3, \dots, i_{l-1} \geqslant 2i_l > 0$. The set χS contains the elements χSq^i $(i > 0)$; every other element in χS can be written in the form $d\chi(Sq^j)$ with $0 < \dim d < j$; $Sq^j(\mathscr{E}d)$ is zero, and therefore $\mathscr{E}d\chi(Sq^j)$ is decomposable. This completes the proof of Lemma 12.

Lemma 12 shows that U is multiplicatively generated by the elements u_i $(i > 0)$. This completes the proof of Theorem 1.

3. In this section we shall complete the description of U, by describing its operations from A (on the left and on the right). This description is given in Theorem 13 below. After proving this theorem, the section concludes by remarking that U can be given the structure of a Hopf algebra.

THEOREM 13. *Let $P(u_1, u_2,...)$ be a polynomial in the u_i. Then in U we have*

$$a(P(u_1, u_2,...)) = Q(u_1, u_2,...),$$
$$(P(u_1, u_2,...))a = R(u_1, u_2,...),$$

where the polynomials Q and R are constructed according to the method given below.

We now give the method for constructing Q and R. If $p = 2$, take $N \geqslant \dim a + \dim P$, and take a cartesian product of N copies of RP^∞, with fundamental classes $x_1, x_2,..., x_N$. Let σ_i be the ith elementary symmetric function in $x_1, x_2,..., x_N$; set $X = x_1 x_2 ... x_N$. Solve the equations

$$a(P(\sigma_1, \sigma_2,...)) = Q(\sigma_1, \sigma_2,...),$$
$$(\chi a)(XP(\sigma_1, \sigma_2,...)) = XR(\sigma_1, \sigma_2,...)$$

for Q and R.

If $p > 2$, take $2N \geqslant \dim a + \dim P$, and take a cartesian product of N copies of CP^∞, with fundamental classes $y_1, y_2,..., y_N$. Let σ_i be the ith elementary symmetric function in $(y_1)^{p-1}, (y_2)^{p-1},..., (y_N)^{p-1}$; set $Y = y_1 y_2 ... y_N$. Solve the equations

$$a(P(\sigma_1, \sigma_2,...)) = Q(\sigma_1, \sigma_2,...),$$
$$(\chi a)(YP(\sigma_1, \sigma_2,...)) = YR(\sigma_1, \sigma_2,...)$$

for Q and R.

Example. Take $p = 2$, $P = \mathscr{E}$, $a = Sq^3$; we will calculate R. We have

$$\begin{aligned}(\chi a)X &= (Sq^2 Sq^1)(x_1 x_2 ... x_N) \\ &= Sq^2(\textstyle\sum x_1^2 x_2 ... x_N) \\ &= \textstyle\sum x_1^4 x_2 ... x_N + \sum x_1^2 x_2^2 x_3^2 x_4 ... x_N \\ &= X(\textstyle\sum x_1^3 + \sum x_1 x_2 x_3).\end{aligned}$$

But $\sum x_1^3 + \sum x_1 x_2 x_3 = \sigma_1^3 + \sigma_1 \sigma_2$; therefore $\mathscr{E} Sq^3 = u_1^3 + u_1 u_2$.

Caution. This representation of U (for $p = 2$) does not throw w_k onto the elementary symmetric function σ_k.

We give the proof of Theorem 13 for the case $p > 2$; the case $p = 2$ is closely similar. Lemma 7 shows that U is faithfully represented (up to any given dimension) in $H^*(D; Z_p)$, where D is a certain cartesian product of projective spaces CP^m with $m = p^s - 2$. Moreover, in this representation, u_i becomes the ith elementary symmetric function σ_i in $(y_1)^{p-1}$, $(y_2)^{p-1},..., (y_N)^{p-1}$. If we find formulae $aP = Q$, $Pa = R$ which hold in

$H^*(D; Z_p)$, then these formulae must hold in U. This establishes the construction for Q; it remains to establish the construction for Pa in $H^*(D; Z_p)$.

Let us write the iterated diagonal in A in the form

$$\Delta^{N-1}a = \sum a^{(1)} \otimes a^{(2)} \otimes \dots \otimes a^{(N)},$$

without indicating the parameter of summation. Then we have

$$\begin{aligned}\langle (y_1^{j_1} y_2^{j_2} \dots y_N^{j_N})a \,.\, Y \,.\, y_1^{k_1} y_2^{k_2} \dots y_N^{k_N}, \mu\rangle & \\ &= \langle y_1^{j_1} \dots y_N^{j_N} \,.\, a(y_1^{k_1+1} \dots y_N^{k_N+1}), \mu\rangle \\ &= \sum \langle y_1^{j_1} \dots y_N^{j_N} \,.\, a^{(1)} y_1^{k_1+1} \dots a^{(N)} y_N^{k_N+1}, \mu\rangle \\ &= \sum \langle y_1^{j_1} \,.\, a^{(1)} y_1^{k_1+1}, \mu_1\rangle \dots \langle y_N^{j_N} \,.\, a^{(N)} y_N^{k_N+1}, \mu_N\rangle.\end{aligned}$$

Using Lemma 4 and the fact that $m = p^s - 2$, this yields

$$\begin{aligned}\sum \langle (\chi a^{(1)}) y_1^{j_1+1} \,.\, y_1^{k_1}, \mu_1\rangle \dots \langle (\chi a^{(N)}) y_N^{j_N+1} \,.\, y_N^{k_N}, \mu_N\rangle & \\ &= \sum \langle (\chi a^{(1)}) y_1^{j_1+1} \dots (\chi a^{(N)}) y_N^{j_N+1} \,.\, y_1^{k_1} \dots y_N^{k_N}, \mu\rangle \\ &= \langle (\chi a)(y_1^{j_1+1} \dots y_N^{j_N+1}) \,.\, y_1^{k_1} \dots y_N^{k_N}, \mu\rangle \\ &= \langle (\chi a)(Y y_1^{j_1} y_2^{j_2} \dots y_N^{j_N}) \,.\, y_1^{k_1} y_2^{k_2} \dots y_N^{k_N}, \mu\rangle.\end{aligned}$$

Therefore $(Pa).Y = (\chi a)(YP)$. This establishes the construction of R.

Instead of using this representation of U, it would be possible (and almost equivalent) to use the theory of Hopf algebras (**8**). One would first have to make U into a Hopf algebra; this is done below. In order to give the structure maps

$$A \otimes U \to U, \qquad U \otimes A \to U,$$

one would then consider the dual maps

$$A_* \otimes U_* \leftarrow U_*, \qquad U_* \otimes A_* \leftarrow U_*$$

defined on the dual U_* of U. One would remark that these maps are multiplicative, and one would finish by giving their values on the generators of U_*. We will now provide the foundations for this alternative approach.

We first note that if H' and H'' are algebras satisfying the axioms given above, then $H' \otimes H''$ is another such. This leads to the following lemma.

LEMMA 14. *For each u in U there is one and only one element*

$$\Delta u = \sum_s u_s' \otimes u_s''$$

in $U \otimes U$ such that

$$u(H' \otimes H'') = \sum_s u_s'(H') \otimes u_s''(H'')$$

for all H' and H''. The map $\Delta\colon U \to U \otimes U$ makes U into a Hopf algebra. We have

$$\Delta(au) = \sum_{r,s} a'_r u'_s \otimes a''_r u''_s,$$

$$\Delta(ua) = \sum_{r,s} u'_s a'_r \otimes u''_s a''_r.$$

The proof is obvious.

It follows, in particular, that Δ is given on the generators of U by

$$\Delta u_k = \sum_{i+j=k} u_i \otimes u_j,$$

where $u_0 = 1$. From this it follows that the dual U_* of U is again a polynomial algebra. We may describe generators g_j in U_* as follows: we set $\langle (u_1)^j, g_j \rangle = 1$, $\langle \mathrm{M}, g_j \rangle = 0$ for any other monomial M in the u_i.

4. In the preceding sections, we have been primarily concerned with 'characteristic classes' in abstract cohomology rings. One may ask how this theory applies to the cohomology of differentiable manifolds; the answer is that in this case those classes, which were defined in the abstract case in § 2, can be calculated in terms of the classical characteristic classes.

Let us write O for the 'infinite' orthogonal group, and BO for its classifying space. If M is a differentiable manifold, then its tangent bundle τ induces a map

$$\tau^*\colon H^*(BO;\, Z_p) \to H^*(M;\, Z_p).$$

Let the algebra U be as in §§ 2, 3. Then one can define a (unique) monomorphism

$$\nu\colon U \to H^*(BO;\, Z_p)$$

of Hopf algebras, with the following property: if $u \in U$, and M is any differentiable manifold, then the value of u in $H^*(M;\, Z_p)$ is given by the characteristic class $\tau^* \nu u$. In other words, the (universal) characteristic class νu gives a universal formula for calculating the value of u in differentiable manifolds.

By using classical representations of $H^*(BO;\, Z_p)$, it is possible to present ν in a form convenient for calculation. In this direction we present the following formulae as a sample; the u_i which occur are the generators of U considered in §§ 2, 3.

If $p = 2$, $\sum_0^\infty \nu(u_i)$ is given by the multiplicative sequence of polynomials (**4**, **5**) in the Stiefel–Whitney classes corresponding to the power-series $1/(1+x)$.

If $p > 2$, $\sum_0^\infty \nu(u_i)$ is given by the multiplicative sequence of polynomials in the mod p Pontryagin classes corresponding to the power-series $1/(1+y^{p-1})$.

In order to state our next remark, we recall that in § 2 we made

$H^*(M; Z_p)$ into a right A-module (for each M). There is a unique way of making $H^*(BO; Z_p)$ into a right A-module so that

$$\tau^*: H^*(BO; Z_p) \to H^*(M; Z_p)$$

is always a map of right A-modules. In elementary terms, this means the following. Let $c \in H^*(M; Z_p)$ be a (classical) characteristic class of M, and take $a \in A$. Then there is a universal formula for finding a second characteristic class d in M such that

$$\langle c.ah, \mu\rangle = \langle d.h, \mu\rangle$$

for all h.

It is possible to present the right A-module structure of $H^*(BO; Z_p)$ in a form convenient for calculation.

In the first draft of this paper I developed these remarks at some length, with proofs; I now forbear to do so, for three reasons. The first is space; and the second is that they belong to a cadre of ideas which is much more generally known and understood than that studied in §§ 2, 3 of this paper. The third is that they overlap to some extent with work of F. Hirzebruch (4); a paper on the subject is being prepared by Atiyah and Hirzebruch. I am very grateful to Hirzebruch for sending me a copy of an unpublished manuscript of great elegance.

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