EIGENVALUES OF PRODUCTS OF UNITARY MATRICES AND QUANTUM SCHUBERT CALCULUS

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ABSTRACT. We describe the inequalities on the possible eigenvalues of products of unitary matrices in terms of quantum Schubert calculus. Related problems are the existence of flat connections on the punctured two-sphere with prescribed holonomies, and the decomposition of fusion product of representations of SU(n), in the large level limit.

In the second part of the paper we investigate how various aspects of the problem (symmetry, factorization) relate to properties of the Gromov-Witten invariants.

1. INTRODUCTION

Beginning with Weyl [29], many mathematicians have been interested in the following question: given the eigenvalues of two Hermitian matrices, what are the possible eigenvalues of their sum? In a recent preprint [16], Klyachko observes that a complete solution to this problem is given by an application of Mumford's criterion in geometric invariant theory. The eigenvalue inequalities are derived from products in Schubert calculus. In particular, Weyl's inequalities correspond to Schubert calculus in projective space. The necessity of these conditions is due to Helmke and Rosenthal [12].

One of the fascinating points about the above problem are several equivalent formulations noted by Klyachko. For instance, the problem is related to the following question in representation theory: Given a collection of irreducible representations of SU(n), which irreducibles appear in the tensor product? A second equivalent problem involves toric vector bundles over the complex projective plane.

In this paper we investigate the corresponding problem for *products* of *unitary* matrices. This question also has a relationship with a representation-theoretic problem, that of the decomposition of the fusion product of representations. The solution to the multiplicative problem is also derived from geometric invariant theory, namely from the Mehta-Seshadri theory of parabolic bundles over the projective line. The main result of this paper, Theorem 3.1, shows that the eigenvalue inequalities are derived from products in *quantum* Schubert calculus. This improves a result of I. Biswas [6], who gave the first description of these inequalities. A similar result has been obtained independently by P. Belkale [2].

The proof is an application of the Mehta-Seshadri theorem. A set of unitary matrices A_1, \ldots, A_l such that each A_i lies in a conjugacy class C_i and such that their product is the identity is equivalent to a unitary representation of the fundamental group of the l

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times punctured sphere, with each generator γ_i being mapped to the conjugacy class C_i . By the Mehta-Seshadri theorem such a representation exists if and only if there exists a semi-stable parabolic bundle on \mathbb{P}^1 with l parabolic points whose parabolic weights come from the choice of conjugacy classes C_i . This last interpretation of the original eigenvalue problem can be related to the Gromov-Witten invariants of the Grassmannian and this is done in Section 5 below.

In Sections 6 and 7 we investigate how factorization and hidden symmetries of these Gromov-Witten invariants relate to the multiplicative eigenvalue problem.

2. Additive inequalities (after Klyachko and Helmke-Rosenthal)

Let $\mathfrak{su}(n)$ denote the Lie algebra of SU(n), and

$$\mathfrak{t} = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \mid \sum \lambda_i = 0\}$$

its Cartan subalgebra. Let

$$\mathfrak{t}_{+} = \{ (\lambda_1, \dots, \lambda_n) \in \mathfrak{t} \mid \lambda_i \ge \lambda_{i+1}, \quad i = 1, \dots, n-1 \}$$

be a choice of closed positive Weyl chamber. For any matrix $A \in \mathfrak{su}(n)$ let

 $\lambda(A) = (\lambda_1(A), \lambda_2(A), \cdots, \lambda_n(A)) \in \mathfrak{t}_+$

be the eigenvalues of the Hermitian matrix -iA in non-increasing order. Let $\Delta(l) \subset (\mathfrak{t}_+)^l$ denote the set

 $\Delta(l) = \{ (\lambda(A_1), \lambda(A_2), \dots, \lambda(A_l)) \mid A_1, \dots, A_l \in \mathfrak{su}(n), \ A_1 + A_2 + \dots + A_l = 0 \}.$

Define an involution

*: $\mathfrak{t}_+ \cong \mathfrak{t}_+, \ (\lambda_1, \ldots, \lambda_n) \mapsto (-\lambda_n, \ldots, -\lambda_1).$

For any $A \in \mathfrak{su}(n)$ the matrix -A has eigenvalues $\lambda(-A) = *\lambda(A)$. The set $\Delta(l)$ is invariant under the map

$$*^l: (\mathfrak{t}_+)^l \to (\mathfrak{t}_+)^l, \ (\xi_1, \dots, \xi_l) \mapsto (*\xi_1, \dots, *\xi_l)$$

and also under the action of the symmetric group S_l on $(\mathfrak{t}_+)^l$.

The set $\Delta(l)$ has interesting interpretations in symplectic geometry and representation theory. Consider the cotangent bundle $T^*SU(n)^{l-1}$ with the action of $SU(n)^l$ given by SU(n) acting diagonally on the left and $SU(n)^{l-1}$ on the right. The moment polytope of this action may be identified with $\Delta(l)$ (see Section 5.) From convexity theorems in symplectic geometry (see e.g. [27] and [17]) it follows that $\Delta(l)$ is a finitely-generated convex polyhedral cone. In particular there are a finite number of inequalities defining $\Delta(l)$ as a subset of the polyhedral cone $(\mathfrak{t}_+)^l$.

The set $\Delta(l)$ may also be described in terms of the tensor product of representations. Let

 $(,): \mathfrak{su}(n) \times \mathfrak{su}(n) \to \mathbb{R}, \quad (A,B) \mapsto -\operatorname{Tr}(AB)$

denote the basic inner product on $\mathfrak{su}(n)$, which induces an identification $\mathfrak{su}(n) \cong \mathfrak{su}(n)^*$. Let $\Lambda = \mathbb{Z}^n \cap \mathfrak{t}$ denote the integral lattice and $\Lambda^* \subset \mathfrak{t}$ its dual, the weight lattice. For each $\lambda \in \Lambda^* \cap \mathfrak{t}_+$, let V_{λ} denote the corresponding irreducible representation of SU(n). We will see in equation (9) that $\Delta(l) \cap \mathbb{Q}^l$ is the set of $(\lambda^1, \ldots, \lambda^l)$ such that for some N such that $N\lambda^1, \ldots, N\lambda^l \in \Lambda^*$, we have

$$V_{N\lambda^1} \otimes \ldots \otimes V_{N\lambda^{l-1}} \supset V^*_{N\lambda^l},$$

that is, $V_{N\lambda^1} \otimes \ldots \otimes V_{N\lambda^l}$ contains a non-zero invariant vector.

The work of Klyachko and Helmke-Rosenthal gives a complete set of inequalities describing $\Delta(l)$ in terms of Schubert calculus. Let

$$\mathbb{C}^n = F_n \supset F_{n-1} \supset \ldots \supset F_0 = \{0\}$$

be a complete flag in \mathbb{C}^n , G(r, n) the Grassmanian of r-planes in \mathbb{C}^n , and for any subset $I = \{i_1, \ldots, i_r\} \subset \{1, \ldots, n\}$ let

$$\sigma_I = \{ W \in G(r, n) \mid \dim(W \cap F_{i_j}) \ge j, \ j = 1, \dots, r \}$$

denote the corresponding Schubert variety. The Schubert cell $C_I \subset \sigma_I$ is defined as the complement of all lower-dimensional Schubert varieties contained in σ_I :

$$C_I = \sigma_I \setminus \bigcup_{\sigma_J \subset \sigma_I} \sigma_J.$$

We say that W is in *position* I with respect to the flag F_* if $W \in C_I$.

The homology classes $[\sigma_I]$ form a basis of $H_*(G(r, n), \mathbb{Z})$. Given two Schubert cycles σ_I, σ_J , we can expand the intersection product $[\sigma_I] \cap [\sigma_J]$ in terms of this basis. We say $[\sigma_I] \cap [\sigma_J]$ contains $[\sigma_K]$ if $[\sigma_K]$ appears in this expansion with non-zero (and therefore positive) coefficient. Equivalently, let

$$*K = \{n + 1 - i_r, n + 1 - i_{r-1}, \dots, n + 1 - i_1\},\$$

so that $[\sigma_{*K}]$ is the Poincare dual of $[\sigma_K]$. Then $[\sigma_I] \cap [\sigma_J]$ contains $[\sigma_K]$ if and only if the intersection of general translates of the Schubert cycles $\sigma_I, \sigma_J, \sigma_{*K}$ is non-empty and finite.

Theorem 2.1 (Klyachko, resp. Helmke-Rosenthal). A complete (resp. necessary) set of inequalities describing $\Delta(l)$ as a subset of $(\mathfrak{t}_+)^l$ are

(1)
$$\sum_{i\in I_1}\lambda_i(A_1) + \sum_{i\in I_2}\lambda_i(A_2) + \ldots + \sum_{i\in I_l}\lambda_i(A_l) \le 0,$$

where I_1, \ldots, I_l are subsets of $\{1, \ldots, n\}$ of the same cardinality r such that $[\sigma_{I_1}] \cap \ldots \cap [\sigma_{I_{l-1}}] \supset [\sigma_{*I_l}]$, and r ranges over all values between 1 and n-1.

Note that the cases l = 1, 2 are trivial: $\Delta(1) = \{0\}$, and $\Delta(1) = \{(\mu, *\mu) \mid \mu \in \mathfrak{t}_+\}$. Klyachko also claims that these inequalities are independent. From Theorem 2.1 follows a complete set of inequalities for the possible eigenvalues of a sum of skew-Hermitian matrices. For instance, for l = 3 one obtains the inequalities

(2)
$$\sum_{i \in I} \lambda_i(A) + \sum_{j \in J} \lambda_j(B) \le \sum_{k \in K} \lambda_k(A+B),$$

where $I, J, K \subset \{1, \ldots, n\}$ range over subsets such that $[\sigma_I] \cap [\sigma_J]$ contains $[\sigma_K]$.

Example 2.2. Let r = 1 so that $G(r, n) \cong \mathbb{P}^{n-1}$ and $I = \{n - i + 1\}$, $J = \{n - j + 1\}$. Then $\sigma_I \cong \mathbb{P}^{n-i}$, $\sigma_J \cong \mathbb{P}^{n-j}$ so that $[\sigma_I] \cap [\sigma_J] \cong \mathbb{P}^{n-i-j+1} = \sigma_K$ where $K = \{n - i - j + 2\}$. One obtains

(3)
$$\lambda_{n-i+1}(A) + \lambda_{n-j+1}(B) \le \lambda_{n-i-j+2}(A+B).$$

2.1. **Duality.** Let $A_1, \ldots, A_l \in \mathfrak{su}(n)$. From (2) applied to $-A_1, \ldots, -A_l$ one obtains

(4)
$$-\sum_{i\in *I_1}\lambda_i(A_1)-\ldots-\sum_{i\in *I_l}\lambda_i(A_l)\leq 0$$

or equivalently $\sum_{i \in *I_1} \lambda_i(A_1) + \ldots + \sum_{i \in *I_l} \lambda_i(A_l) \ge 0$. By the trace condition, (4) is equivalent to

$$\sum_{i \notin *I_1} \lambda_i(A_1) + \ldots + \sum_{i \notin *I_l} \lambda_i(A_l) \le 0$$

Let $I_i^c = \{1, \ldots, n\} \setminus *I_i$. Then $[\sigma_{I_i^c}]$ is the image of $[\sigma_{I_i}]$ under the isomorphism of homology induced by $G(r, n) \cong G(n - r, n)$ (see page 197 onwards of Griffiths and Harris [9]). Thus the appearance of (4) in (1) corresponds to a product in the Schubert calculus of G(n - r, n).

Example 2.3. The dual equation to (3) is Weyl's 1912 [29] inequality

(5) $\lambda_i(A) + \lambda_j(B) \ge \lambda_{i+j-1}(A+B).$

3. Multiplicative Inequalities

Let $\mathfrak{A} \subset \mathfrak{t}_+$ be the fundamental alcove of SU(n):

 $\mathfrak{A} = \{ \lambda \in \mathfrak{t}_+ \mid \lambda_1 - \lambda_n \leq 1 \}.$

Let $A \in SU(n)$ be a unitary matrix with determinant 1. Its eigenvalues may be written $e^{2\pi i\lambda_1(A)}, e^{2\pi i\lambda_2(A)}, \dots, e^{2\pi i\lambda_n(A)}$

where $\lambda(A) = (\lambda_1(A), \dots, \lambda_n(A)) \in \mathfrak{A}$. The map $A \mapsto \lambda(A)$ induces a homeomorphism $\mathfrak{A} \cong SU(n) / \operatorname{Ad}(SU(n)).$

Let $\Delta_q(l) \subset \mathfrak{A}^l$ (q for quantum) denote the set

$$\Delta_q(l) = \{ (\lambda(A_1), \dots, \lambda(A_l)) \mid A_1, \dots, A_l \in SU(n), \ A_1 A_2 \dots A_l = I \}$$

As before, $\Delta_q(l)$ is invariant under the involution, $*^l : \mathfrak{A}^l \to \mathfrak{A}^l$, and the action of the symmetric group S_l on \mathfrak{A}^l .

The set $\Delta_q(l)$ has an interpretation as a moment polytope. Let \mathcal{M} be the space of flat SU(n)-connections on the trivial SU(n) bundle over the *l*-holed two-sphere, modulo gauge transformations which are the identity on the boundary (see [19]). The gauge group of the boundary acts on \mathcal{M} in Hamiltonian fashion and the set $\Delta_q(l)$ is the moment polytope for this action. By [19, Theorem 3.19], $\Delta_q(l)$ is a convex polytope. In fact, an analogous statement holds for arbitrary compact, simply-connected Lie groups. In particular, a finite number of inequalities describe $\Delta_q(l)$. In the case n = 2, these inequalities were given explicitly for l = 3 in Jeffrey-Weitsman [13] and for arbitrary numbers of marked points in Biswas [5]. A description of the inequalities in the arbitrary rank case was given in [6] but the description given there does not seem to be computable.

There is also an interpretation of $\Delta_q(l)$ in terms of fusion product. Let \circledast_N denote the fusion product on the Verlinde algebra $R(SU(n)_N)$ of SU(n) at level N. Then $\Delta_q(l) \cap \mathbb{Q}^l$ is the set of $(\lambda^1, \ldots, \lambda^l) \in \mathfrak{A} \cap \mathbb{Q}^l$ such that for some N such that $N\lambda^1, \ldots, N\lambda^l \in \Lambda^*$, we have

(6)
$$V_{N\lambda^1} \circledast_N \ldots \circledast_N V_{N\lambda^{l-1}} \supset V_{N*\lambda^l}$$

See Section 8.

3.1. Quantum Schubert calculus. Quantum cohomology is a deformation of the ordinary cohomology ring that was introduced by the physicists Vafa and Witten. Quantum cohomology of the Grassmannian (quantum Schubert calculus) was put on a rigorous footing by Bertram [3]. Recall that the degree of a holomorphic map $\varphi : \mathbb{P}^1 \to G(r, n)$ is the homology class $[\varphi] \in H^2(G(r, n), \mathbb{Z}) \cong \mathbb{Z}$. Let p_1, \ldots, p_l be distinct marked points in \mathbb{P}^1 . The quantum intersection product \star on $H_*(G(r, n), \mathbb{C}) \otimes \mathbb{C}[q]$ is defined by

$$[\sigma_{I_1}] \star \ldots \star [\sigma_{I_l}] = \sum_J \langle [\sigma_{I_1}], \ldots, [\sigma_{I_l}], [\sigma_J] \rangle_d \ [\sigma_{*J}] q^d,$$

where the Gromov-Witten invariant $\langle [\sigma_{I_1}], \ldots, [\sigma_{I_l}], [\sigma_J] \rangle_d$ is equal to the number of holomorphic maps $\mathbb{P}^1 \mapsto G(r, n)$ sending p_1, \ldots, p_l, p to general translates of $\sigma_{I_1}, \ldots, \sigma_{I_l}, \sigma_J$ if this number is finite, and is otherwise zero.

Our main result is the following description of $\Delta_q(l)$:

Theorem 3.1. A complete set of inequalities for $\Delta_q(l)$ are given by

(7)
$$\sum_{i \in I_1} \lambda_i(A_1) + \sum_{i \in I_2} \lambda_i(A_2) + \ldots + \sum_{i \in I_l} \lambda_i(A_l) \le d$$

for (I_1, \ldots, I_l, d) such that $\langle [\sigma_{I_1}] \star \ldots \star [\sigma_{I_l}] \rangle_d \neq 0$, that is, $[\sigma_{I_1}] \star \ldots \star [\sigma_{I_{l-1}}] \supset q^d[\sigma_{*I_l}]$.

In the last few years several techniques have been developed for computing the coefficients of quantum Schubert calculus. See for instance Bertram, Ciocan-Fontanine, Fulton [4]. Therefore the above theorem makes the question of which inequalities occur computable in practice.

One recovers the inequalities for $\Delta(l)$ from the degree 0 Gromov-Witten invariants. This shows that $\Delta(l)$ is the cone on $\Delta_q(l)$ at the 0-vertex, i.e.

$$\Delta(l) = \mathbb{R}_+ \cdot \Delta_q(l)$$

This may be verified by several alternative methods, e.g. Remark 5.4.

The simplest example of a positive degree inequality is given by the following:

Example 3.2. Let r = 1 so that $G(r, n) = \mathbb{P}^{n-1}$, and $U, V, W \subset \mathbb{C}^n$ be subspaces in general position of dimensions i, j, n + 1 - i - j. There is a unique degree 1 map $\mathbb{P}^1 \to \mathbb{P}^{n-1}$ mapping p_1, p_2, p_3 to $\mathbb{P}(U), \mathbb{P}(V), \mathbb{P}(W)$ respectively. Together with the degree 0 inequality mentioned before, this gives

(8)
$$\lambda_{i+j-1}(AB) \le \lambda_i(A) + \lambda_j(B) \le \lambda_{i+j}(AB) + 1.$$

We will see in Section 7 that these inequalities are related by a symmetry of $\Delta_q(l)$.

Question: Are the inequalities in Theorem 3.1 independent?

4. Moduli of flags and Mumford's criterion

As a warm-up we review some of the ideas involved in Klyachko's proof. For any $\xi \in \mathfrak{t}_+$ let

$$\mathcal{O}_{\xi} = SU(n) \cdot \xi = \{ A \in \mathfrak{su}(n) \mid \lambda(A) = \xi \}$$

denote the corresponding adjoint orbit. Via the identification $\mathfrak{su}(n) \cong \mathfrak{su}(n)^*$, \mathcal{O}_{ξ} inherits a canonical symplectic structure, called the Kostant-Kirillov-Souriau two-form, and the action of SU(n) on \mathcal{O}_{ξ} is Hamiltonian with moment map given by inclusion into $\mathfrak{su}(n)$.

The diagonal action of SU(n) on $\mathcal{O}_{\xi_1} \times \ldots \times \mathcal{O}_{\xi_l}$ has moment map given by

$$(A_1,\ldots,A_l)\mapsto A_1+A_2+\ldots+A_l$$

The symplectic quotient $\mathcal{N}(\xi_1, \ldots, \xi_l) = \mathcal{O}_{\xi_1} \times \ldots \times \mathcal{O}_{\xi_l} / SU(n)$ is given by

$$\mathcal{N}(\xi_1, \dots, \xi_l) = \{ (A_1, \dots, A_l) \in \mathcal{O}_{\xi_1} \dots \mathcal{O}_{\xi_l} \mid A_1 + A_2 + \dots + A_l = 0 \} / SU(n).$$

For generic (ξ_1, \ldots, ξ_l) , that is, values where the moment map has maximal rank, the quotient $\mathcal{N}(\xi_1, \ldots, \xi_l)$ is a symplectic manifold. The *l*-tuple (ξ_1, \ldots, ξ_l) lies in $\Delta(l)$ if and only if $\mathcal{N}(\xi_1, \ldots, \xi_l)$ is non-empty.

The quotients $\mathcal{N}(\xi_1, \ldots, \xi_l)$ may be viewed as symplectic quotients of the cotangent bundle $T^*SU(n)^{l-1}$. Indeed, the symplectic quotient

$$(T^*SU(n) \times \mathcal{O}_{\xi}) / SU(n) \cong \mathcal{O}_{\xi}.$$

Therefore, the quotient of $T^*SU(n)^{l-1}$ by the right action of $SU(n)^{l-1}$ and the diagonal left action of SU(n) is

$$(T^*SU(n)^{l-1} \times \mathcal{O}_{\xi_1} \times \ldots \times \mathcal{O}_{\xi_l}) / SU(n)^l \cong \mathcal{N}(\xi_1, \ldots, \xi_l).$$

It follows that $\Delta(l)$ is the moment polytope of the action of $SU(n)^{l}$ on $T^{*}SU(n)^{l-1}$.

One can determine whether $\mathcal{N}(\xi_1, \ldots, \xi_l)$ is empty by computing its symplectic volume. This is given by a formula derived from the Duistermaat-Heckman theorem due to Guillemin-Prato (see [10] or [21, (4)]). Unfortunately the formula involves cancelations and it is not apparent what the support of the volume function is, or even that the support is a convex polytope.

The manifolds \mathcal{O}_{ξ_i} have canonical complex structures (induced by the choice of positive Weyl chamber) and are isomorphic to (possibly partial) flag varieties. Suppose that ξ_1, \ldots, ξ_l lie in the weight lattice Λ^* , so that there exist pre-quantum line bundles $L_{\xi_i} \to$

 \mathcal{O}_{ξ_i} ; i.e., equivariant line bundles with curvature equal to $2\pi i$ times the symplectic form. The sections of $L_{\xi_1} \boxtimes \ldots \boxtimes L_{\xi_l}$ define a Kähler embedding

$$\mathcal{O}_{\xi_1} \times \ldots \times \mathcal{O}_{\xi_l} \to \mathbb{P}(V_{\xi_1}^*) \times \ldots \times \mathbb{P}(V_{\xi_l}^*),$$

where $V_{\xi_1}, \ldots, V_{\xi_l}$ are the irreducible representations with highest weights ξ_1, \ldots, ξ_l . By an application of a theorem of Kirwan and Kempf-Ness (which holds for arbitrary smooth projective varieties, see [15, page 109]) the symplectic quotient is homeomorphic to the geometric invariant theory quotient

$$\mathcal{N}(\xi_1,\ldots,\xi_l)\cong \mathcal{O}_{\xi_1}\times\ldots\times\mathcal{O}_{\xi_l}/\!\!/SL(n,\mathbb{C}).$$

By definition, $\mathcal{O}_{\xi_1} \times \ldots \times \mathcal{O}_{\xi_l} /\!\!/ SL(n, \mathbb{C})$ is the quotient of the set of semi-stable points in $\mathcal{O}_{\xi_1} \times \ldots \times \mathcal{O}_{\xi_l}$ by the action of $SL(n, \mathbb{C})$, where $(F_{\xi_1}, \ldots, F_{\xi_l}) \in \mathcal{O}_{\xi_1} \times \ldots \times \mathcal{O}_{\xi_l}$ is called semi-stable if and only if for some N there is an invariant section of $(L_{\xi_1} \boxtimes \ldots \boxtimes L_{\xi_l})^{\otimes N}$ which is non-vanishing at $(F_{\xi_1}, \ldots, F_{\xi_l})$. The quotient $\mathcal{N}(\xi_1, \ldots, \xi_l)$ is therefore non-empty if and only if there exists a non-zero SU(n)-invariant vector in

(9)
$$H^{0}((L_{\xi_{1}}\boxtimes\ldots\boxtimes L_{\xi_{l}})^{\otimes N}) = V_{N\xi_{1}}\otimes\ldots\otimes V_{N\xi_{l}}.$$

This explains the representation-theoretic interpretation of $\Delta(l)$ alluded to in the introduction.

In order to obtain the inequalities in Theorem 2.1, one applies the criterion of Mumford, which says that a point is semi-stable if and only if it is semi-stable for all oneparameter subgroups [22, Chapter 2], see also [15, Lemma 8.8]. Let us assume that the ξ_i are generic. An application of the criterion gives that an *l*-tuple of complete flags $(F_1, \ldots, F_l) \in \mathcal{O}_{\xi_1} \times \ldots \times \mathcal{O}_{\xi_l}$ is semi-stable if and only if for all subspaces $W \subset \mathbb{C}^n$, one has

$$\sum_{i\in I_1}\xi_{1,i}+\ldots+\sum_{i\in I_l}\xi_{l,i}\leq 0,$$

where I_j is the position of W with respect to the flag F_j . The proof similar to that for Grassmannians given in Section 4.4 of [22].

The set of semi-stable points is dense if non-empty. It follows that $\mathcal{N}(\xi_1, \ldots, \xi_l)$ is non-empty if and only if the above inequality holds for every intersection $\sigma_{I_1} \cap \ldots \cap \sigma_{I_l}$ of Schubert cycles in general position. Any inequality corresponding to a positive dimensional intersection must be redundant. Indeed, since the intersection is a projective variety, it cannot be contained in any of the Schubert cells. The boundary of σ_{I_l} consists of Schubert varieties σ_J with J such that $j_k \leq i_k$ for $k = 1, \ldots, r$, where i_1, \ldots, i_r and j_1, \ldots, j_r are the elements of I_l and J in increasing order. The inequality obtained from an intersection $\sigma_{I_1} \cap \ldots \cap \sigma_{I_{l-1}} \cap \sigma_J \neq \emptyset$ therefore implies the inequality obtained from $\sigma_{I_1} \cap \ldots \cap \sigma_{I_l} \neq \emptyset$.

5. Application of the Mehta-Seshadri theorem

For any $\xi \in \mathfrak{A}$, let

$$\mathcal{C}_{\xi} = \{ A \in SU(n) \mid \lambda(A) = \xi \}$$

denote the corresponding conjugacy class. The mapping $A \mapsto \lambda(A)$ induces a homeomorphism $SU(n)/\operatorname{Ad}(SU(n)) \cong \mathfrak{A}$.

Let $p_1, \ldots, p_l \in \mathbb{P}^1$ be distinct marked points and $\mathcal{M}(\xi_1, \ldots, \xi_l)$ the moduli space of flat SU(n)-connections on $\mathbb{P}^1 \setminus \{p_1, \ldots, p_l\}$ with holonomy around p_i lying in \mathcal{C}_{ξ_i} . Since the fundamental group of $\mathbb{P}^1 \setminus \{p_1, \ldots, p_l\}$ has generators the loops $\gamma_1, \ldots, \gamma_l$ around the punctures, with the single relation $\gamma_1 \cdot \ldots \cdot \gamma_l = 1$,

$$\mathcal{M}(\xi_1,\ldots,\xi_l)\cong\{(A_1,\ldots,A_l)\in\mathcal{C}_{\xi_1}\times\ldots\times\mathcal{C}_{\xi_l}\mid A_1A_2\cdots A_l=I\}/SU(n).$$

In particular $\mathcal{M}(\xi_1, \ldots, \xi_l)$ is non-empty if and only if $(\xi_1, \ldots, \xi_l) \in \Delta_q(l)$. In theory one can determine if $\mathcal{M}(\xi_1, \ldots, \xi_l)$ is non-empty by computing its symplectic volume by the formulae stated in Witten [30, (4.11)], Szenes [28], and [20, Theorem 5.2].

For rational ξ_1, \ldots, ξ_l the space $\mathcal{M}(\xi_1, \ldots, \xi_l)$ has an algebro-geometric description due to Mehta-Seshadri [18]. Let C be a Riemann surface with marked points $p_1, \ldots, p_l \in C$ and let $\mathcal{E} \to C$ be a holomorphic bundle. A parabolic structure without multiplicity on \mathcal{E} consists of the following data at each marked point p_i : a complete ascending flag

$$0 = \mathcal{E}_{p_i,0} \subset \mathcal{E}_{p_i,1} \subset \mathcal{E}_{p_i,2} \ldots \subset \mathcal{E}_{p_i,n} = \mathcal{E}_{p_i}$$

in the fiber \mathcal{E}_{p_i} and a set of *parabolic weights*

$$\lambda_{i,1} > \lambda_{i,2} > \ldots > \lambda_{i,n}$$

satisfying $\lambda_{i,1} - \lambda_{i,n} \leq 1$. In [18] the weights are required to lie in the interval [0, 1), but the definitions work without this assumption. A parabolic bundle is a holomorphic bundle with a parabolic structure. Recall that the degree deg(\mathcal{E}) of \mathcal{E} is the first Chern class $c_1(\mathcal{E}) \in H^2(C, \mathbb{Z}) \cong \mathbb{Z}$. The parabolic degree pardeg(\mathcal{E}) is defined by

$$\operatorname{pardeg}(\mathcal{E}) = \operatorname{deg}(\mathcal{E}) + \sum_{i=1,j=1}^{l,n} \lambda_{i,j}.$$

The parabolic slope $\mu(\mathcal{E})$ is

$$\mu(\mathcal{E}) = \frac{\operatorname{pardeg}(\mathcal{E})}{\operatorname{rk}(\mathcal{E})}$$

Given a holomorphic sub-bundle $\mathcal{F} \subset \mathcal{E}$ of rank r one obtains a parabolic structure on \mathcal{F} as follows. An ascending flag in the fiber \mathcal{F}_{p_i} at each marked point p_i is obtained by removing from

$$\mathcal{F}_{p_i} \cap \mathcal{E}_{p_i,1} \subseteq \mathcal{F}_{p_i} \cap \mathcal{E}_{p_i,2} \subseteq \ldots \subseteq \mathcal{F}_{p_i} \cap \mathcal{E}_{p_i,n} = \mathcal{F}_{p_i}$$

those terms for which the inclusion is not strict. The parabolic weights for \mathcal{F} are

$$\mu_{i,j} = \lambda_{i,k_j},$$

where k_j is the minimal index such that $\mathcal{F}_{p_i,j} \subseteq \mathcal{E}_{p_i,k_j}$. Let $K_i = \{k_1, \ldots, k_r\}$. The fiber \mathcal{F}_{p_i} may be viewed as a element of the Grassmannian of *r*-planes in \mathcal{E}_{p_i} , and *K* is the position of \mathcal{F}_{p_i} with respect to the flag $\mathcal{E}_{p_i,*}$. The parabolic degree of \mathcal{F} is

$$\operatorname{pardeg}(\mathcal{F}) = \operatorname{deg}(\mathcal{F}) + \sum_{i, k \in K_i} \lambda_{i,k}.$$

A parabolic sub-bundle of \mathcal{E} is a holomorphic sub-bundle $\mathcal{F} \subset E$ whose parabolic structure is the one induced from the inclusion. A parabolic bundle $\mathcal{E} \to C$ is called parabolic semi-stable if $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$ for all parabolic sub-bundles $\mathcal{F} \subset \mathcal{E}$. There is a natural equivalence relation on parabolic bundles: Two bundles are said to be grade equivalent if the associated graded bundles are isomorphic as parabolic bundles. See [18] for more details.

Theorem 5.1 (Mehta-Seshadri). Suppose the parabolic weights $\lambda_{i,j}$ are rational and lie in the interval [0, 1). Then the moduli space $\mathcal{M}(\lambda_1, \ldots, \lambda_l)$ of grade equivalence classes of semi-stable parabolic bundles with parabolic weights $\lambda_{i,j}$ and parabolic degree 0 is a normal, projective variety, homeomorphic to the moduli space of flat unitary connections over $C \setminus \{p_1, \ldots, p_r\}$ such that the holonomy of a small loop around p_i lies in \mathcal{C}_{λ_i} .

In fact, the Mehta-Seshadri theorem also holds without the assumption that the parabolic weights lie in [0, 1). One can see this either through the theory of elementary transformations, or through the extension of the Mehta-Seshadri theorem to non-zero parabolic degree given in Boden [7].

The explanation using elementary transformations goes as follows. Let \mathcal{Q} denote the skyscraper sheaf with fiber $\mathcal{E}_{p_i}/\mathcal{E}_{p_i,n-1}$ at p_i . One has an exact sequence of sheaves

$$0 \to \mathcal{E}' \to \mathcal{E} \to \mathcal{Q} \to 0.$$

The kernel \mathcal{E}' is a sub-sheaf of a locally free sheaf and therefore locally free. Since degree is additive in short exact sequences $\deg(\mathcal{E}') = \deg(\mathcal{E}) - 1$. One calls the \mathcal{E}' an elementary transformation of \mathcal{E} at p_i . There is a canonical line $\mathcal{E}'_{p_i,1}$ in the fiber \mathcal{E}'_{p_i} which is the kernel of the fiber map $\pi : \mathcal{E}'_{p_i} \to \mathcal{E}_{p_i}$. One extends the canonical line to a complete flag by taking $\mathcal{E}'_{p_{i,j}} = \pi^{-1}(\mathcal{E}'_{p_{i,j-1}})$ for j > 1. Finally one takes as parabolic weights at p_i the set $\lambda_{i,n} + 1, \lambda_{i,1}, \ldots, \lambda_{i,n-1}$. With this parabolic structure the bundle \mathcal{E}' is parabolic semi-stable of the same parabolic degree as \mathcal{E} . Details, in a slightly different form, can be found in Boden and Yokogawa [8].

The following is the key lemma in the derivation of Theorem 5.3 from Mehta-Seshadri. Let $d = \deg(\mathcal{E}) = -\sum \lambda_{i,j}$ denote the degree of any element $\mathcal{E} \in \mathcal{M}(\lambda_1, \ldots, \lambda_l)$.

Lemma 5.2. Suppose that there is some ordinary semi-stable bundle on C of degree d. Then the set of equivalence classes of parabolic semi-stable bundles of parabolic degree 0 whose underlying holomorphic bundle is ordinary semi-stable is Zariski dense in $\mathcal{M}(\lambda_1, \ldots, \lambda_l)$.

Proof. Recall from the construction of $\mathcal{M}(\lambda_1, \ldots, \lambda_l)$ in [18] that for some integer N there exists an SL(N)-equivariant bundle $\tilde{R} \xrightarrow{\pi} R$ whose fibers are products of l complete flag varieties, such that the geometric invariant theory quotients of \tilde{R}, R are

$$\dot{R}/\!\!/SL(N) = \mathcal{M}(\lambda_1, \dots, \lambda_l), \quad R/\!\!/SL(N) = \mathcal{M},$$

where \mathcal{M} denotes the moduli space of ordinary semi-stable bundles on C of degree d. By definition,

$$\tilde{R}/\!\!/SL(N) = \tilde{R}^{\rm ss}/SL(N), \qquad R/\!\!/SL(N) = R^{\rm ss}/SL(N)$$

where \tilde{R}^{ss} , R^{ss} denote the Zariski dense set of semi-stable points in \tilde{R} , R respectively. The inverse image $\pi^{-1}(R^{ss}) \cap \tilde{R}^{ss}/SL(N)$ is therefore dense in $\tilde{R}^{ss}/SL(N) = \mathcal{M}(\lambda_1, \ldots, \lambda_l)$.

Now we specialize to the case $C = \mathbb{P}^1$ with l marked points p_1, p_2, \ldots, p_l . Let $\xi_1, \ldots, \xi_l \in \mathfrak{A}^l \cap \mathbb{Q}^l$. By Lemma 5.2, $\mathcal{M}(\xi_1, \ldots, \xi_l)$ is non-empty if and only there exists a parabolic semi-stable \mathcal{E} with parabolic degree 0 and weights ξ_1, \ldots, ξ_l whose underlying holomorphic bundle is semi-stable. Since the sum of the parabolic weights is zero, the degree of \mathcal{E} is also zero. By Grothendieck's theorem, \mathcal{E} is holomorphically trivial. A sub-bundle $\mathcal{F} \subset \mathcal{E}$ of rank r is given by a holomorphic map

$$\varphi_{\mathcal{F}}: \mathbb{P}^1 \to G(r, n).$$

Since $\varphi_{\mathcal{F}}$ is the classifying map of the quotient \mathcal{E}/\mathcal{F} , the degree of \mathcal{F} is minus the degree of $\varphi_{\mathcal{F}}$. The parabolic slope of \mathcal{F} is given by

$$\mu(\mathcal{F}) = -\deg(\varphi_{\mathcal{F}}) + \sum_{i \in I_1(\varphi)} \xi_{1,i} + \ldots + \sum_{i \in I_1(\varphi)} \xi_{l,i},$$

where $I_i(\varphi)$ is the position of the subspace $\varphi(p_i) \subset \mathcal{E}_{p_i}$ with respect to the flag $\mathcal{E}_{p_i,*}$ above. The parabolic bundle \mathcal{E} is called parabolic semi-stable if and only if for all such $F, \mu(F) \leq 0$, that is,

$$\sum_{i \in I_1(\varphi)} \xi_{1,i} + \ldots + \sum_{i \in I_1(\varphi)} \xi_{l,i} \le \deg(\varphi)$$

for all maps $\varphi : \mathbb{P}^1 \to G(r, n)$.

The following result was obtained independently by P. Belkale [2].

Theorem 5.3. A complete set of inequalities for $\Delta_q(l)$ as a subset of \mathfrak{A}^l is given by

(10)
$$\sum_{i \in I_1} \lambda_i(A_1) + \sum_{i \in I_2} \lambda_i(A_2) + \ldots + \sum_{i \in I_l} \lambda_i(A_l) \le d$$

for subsets $I_1, \ldots, I_l \subset \{1, \ldots, n\}$ of the same cardinality r and non-negative integers d such that there exists a rational map $\mathbb{P}^1 \to G(r, n)$ of degree d mapping p_1, \ldots, p_l to the Schubert cells C_{I_1}, \ldots, C_{I_l} in general position.

Proof. If $\mathcal{M}(\xi_1, \ldots, \xi_l)$ is non-empty, then a trivial bundle with a general choice of flags will be parabolic semi-stable. Indeed, by the above discussion the fiber Flag^l of $\pi: \tilde{R} \to R$ over a trivial bundle intersects \tilde{R}^{ss} , so $\tilde{R}^{ss} \cap \operatorname{Flag}^l$ is open in Flag^l . Therefore, $\mathcal{M}(\xi_1, \ldots, \xi_l)$ is non-empty if and only if

$$\sum_{i\in I_1}\xi_{1,i}+\ldots+\sum_{i\in I_l}\xi_{l,i}\leq d$$

for all subsets I_1, \ldots, I_l and integers d such that there exists a degree d map sending p_1, \ldots, p_l to general translates of the Schubert cells C_{I_1}, \ldots, C_{I_l} .

Remark 5.4. For sufficiently small parabolic weights $\lambda_{i,j}$ any parabolic semi-stable bundle on \mathbb{P}^1 is necessarily ordinary semi-stable of degree 0, and therefore trivial. It follows that the moduli spaces $\mathcal{M}(\lambda_1, \ldots, \lambda_l)$ and $\mathcal{N}(\lambda_1, \ldots, \lambda_l)$ are isomorphic. This shows that Klyachko's result is implied by Theorem 5.3.

We now show that the existence of the maps described in Theorem 5.3 may be detected by Gromov-Witten invariants. Let $\sigma_{I_1}, \ldots, \sigma_{I_l}$ be some collection of Schubert varieties, and consider the expansion

$$[\sigma_{I_1}] \star [\sigma_{I_2}] \dots \star [\sigma_{I_l}] = \sum_i q^i \alpha_i$$

where $\alpha_i \in H_*(G(r, n))$. (Question: is this product always non-zero?) We say that q^d divides $[\sigma_{I_1}] \star [\sigma_{I_2}] \ldots \star [\sigma_{I_l}]$ if $\alpha_i = 0$ for all i < d. The following lemma is stated in Ravi [25].

Lemma 5.5. Let d be the lowest degree of a map $\mathbb{P}^1 \to G(r, n)$ sending p_1, \ldots, p_l to general translates of $\sigma_{I_1}, \ldots, \sigma_{I_l}$ respectively. Then q^d is the maximal power of q dividing $[\sigma_{I_1}] \star \ldots \star [\sigma_{I_l}]$.

Proof. Let \mathcal{M}_d denote the space of maps $\mathbb{P}^1 \to G(r, n)$ of degree d, $\mathrm{ev}^l : \mathcal{M}_d \to G(r, n)^l$ the evaluation map, and $\sigma_{I_*}(p_*) = (\mathrm{ev}^l)^{-1}(\sigma_{I_*})$ the subset of maps sending p_j to σ_{I_j} for $j = 1, \ldots, l$. By [3, Moving Lemma 2.2A], $\sigma_{I_*}(p_*)$ is a quasi-projective variety, of the expected codimension in \mathcal{M}_d . By choosing enough additional marked points p'_1, \ldots, p'_m , we can insure that the corresponding evaluation map $\mathrm{ev}^m : \mathcal{M}_d \to G(r, n)^m$ is injective when restricted to $\sigma_{I_*}(p_*)$. Let $Y \subset G(r, n)^l \times G(r, n)^m$ be the closure of $(\mathrm{ev}^l \times \mathrm{ev}^m)(\mathcal{M}_d)$, and let $\phi : G(r, n)^l \times G(r, n)^m \to G(r, n)^m$ be the projection.

Since the homology class $[\phi(Y \cap \sigma_{I_*})]$ is non-trivial [9, page 64], $\phi(Y \cap \sigma_{I_*})$ must intersect some Schubert variety

$$\sigma_{J_*} = \sigma_{J_1} \times \sigma_{J_2} \times \ldots \times \sigma_{J_m} \subset G(r, n)^m$$

of complementary dimension. By Kleiman's lemma, [11, Theorem 10.8 page 273], the singular locus of $\phi(Y \cap \sigma_{I_*})$ does not intersect a general translate of σ_{J_*} , and similarly the singular locus of σ_{J_*} does not intersect $\phi(Y \cap \sigma_{I_*})$. Therefore the intersection occurs in the smooth loci of $\phi(Y \cap \sigma_{I_*})$ and σ_{J_*} , and another application of the lemma implies that the intersection is finite.

For generic translates of σ_{J_*} , the intersection is contained in $\operatorname{ev}^m(\sigma_{I_*}(p_*))$. Indeed, let $\overline{\sigma}_{I_*}(p_*)$ be the compactification of $\sigma_{I_*}(p_*)$ given in [3], and $\Gamma \subset \overline{\sigma}_{I_*}(p_*) \times G(r,n)^m$ the closure of the graph of ev^m . Let $Z \subset \Gamma$ be the complement of the graph of ev^m . The projection $\pi(Z)$ of Z in $G(r,n)^m$ is a closed sub-variety of $\phi(Y \cap \sigma_{I_*})$. By Kleiman's lemma, for generic translates of σ_{J_*} the intersection of $\pi(Z)$ and σ_{J_*} is empty, so the intersection is contained in $\operatorname{ev}^m(\sigma_{I_*}(p_*))$.

Because $\operatorname{ev}^m | \sigma_{I_*}(p_*)$ is injective, the intersection $\sigma_{I_*}(p_*) \cap \sigma_{J_*}(p'_*)$ is finite and nonempty. Since the homology class $[\phi(Y \cap \sigma_{I_*})]$ is independent of the choice of general translate of σ_{I_*} , the above intersection is finite and non-empty for general translates of the σ_{I_i} and σ_{J_i} . This implies that Gromov-Witten invariant

$$\langle [\sigma_{I_1}], \ldots, [\sigma_{I_l}], [\sigma_{J_1}], \ldots, [\sigma_{J_m}] \rangle_d \neq 0.$$

In terms of the quantum product

$$[\sigma_{I_1}] \star \ldots \star [\sigma_{I_l}] \star [\sigma_{J_1}] \star \ldots \star [\sigma_{J_{m-1}}] \supset q^d \sigma_{*J_m}$$

which implies that $[\sigma_{I_1}] \star \ldots \star [\sigma_{I_l}]$ contains a term with coefficient q^i with $i \leq d$. That is, for some Schubert variety σ ,

$$\langle [\sigma_{I_1}], [\sigma_{I_2}], \ldots, [\sigma_{I_l}], [\sigma] \rangle_i \neq 0.$$

To prove the lemma it suffices to show that i = d. By [3, Moving Lemma 2.2], for general translates of the Schubert varieties the degree i moduli space $\sigma_{I_1}(p_1) \cap \ldots \cap \sigma_{I_l}(p_l) \cap \sigma(p)$ is finite and consists of maps sending p_1, \ldots, p_l, p to the corresponding Schubert cells. Since d is minimal, i = d.

6. FACTORIZATION

In this section we show that a relationship between the polytopes for different numbers of marked points is related to factorization of Gromov-Witten invariants (i.e. associativity of quantum multiplication). A similar, easier, discussion holds for the additive polytopes $\Delta(l)$. A consideration of a "trivial" factorization completes the proof of Theorem 3.1.

Suppose that l can be written l = j + k - 2 for positive integers $j, k \ge 2$. It is easy to see that $\Delta_q(l)$ are projections of a section of $\Delta_q(j) \times \Delta_q(k)^1$.

$$\Delta_q(l) = \{ (\mu_1, \dots, \mu_{j-1}, \nu_1, \dots, \nu_{k-1}) \mid (\mu, \nu) \in \Delta_q(j) \times \Delta_q(k), \quad \mu_j = *\nu_k \}$$

To show the forward inclusion, note that if $A_1A_2 \dots A_l = I$ then letting $B = A_jA_{j+1} \dots A_l$ we have

$$(\lambda(A_1), \dots, \lambda(A_{j-1}), \lambda(B)) \in \Delta_q(j), (\lambda(B^{-1}), \lambda(A_j), \dots, \lambda(A_l)) \in \Delta(k).$$

In particular this means that any face of $\Delta_q(l)$ is a projection of a face (usually not of codimension 1) of $\Delta_q(j) \times \Delta_q(k)$. Any face is the intersection of codimension 1 faces. This shows that any defining inequality of $\Delta_q(l)$ is implied by a finite set of defining inequalities for $\Delta_q(j)$ and $\Delta_q(k)$.

¹In fact, the volume functions satisfy the factorization properties

$$\operatorname{Vol}(\mathcal{N}(\mu_1, \dots, \mu_{j-1}, \nu_1, \dots, \nu_{k-1})) = \int_{\mathfrak{t}_+} \operatorname{Vol}(\mathcal{N}(\mu_1, \dots, \mu_{j-1}, *\lambda)) \operatorname{Vol}(\mathcal{N}(*\lambda, \nu_1, \dots, \nu_{k-1})) d\lambda$$
$$\operatorname{Vol}(\mathcal{M}(\mu_1, \dots, \mu_{j-1}, \nu_1, \dots, \nu_{k-1})) = \int_{\mathfrak{A}} \operatorname{Vol}(\mathcal{M}(\mu_1, \dots, \mu_{j-1}, *\lambda)) \operatorname{Vol}(\mathcal{M}(*\lambda, \nu_1, \dots, \nu_{k-1})) d\lambda$$

The second formula is implicit in Witten [30, p.51], proved in [14], and generalized in [19].

Using associativity of quantum cohomology one can be more specific about which inequalities for $\Delta_q(j)$, $\Delta_q(k)$ are needed to imply an inequality for $\Delta_q(l)$. Suppose that a Gromov-Witten invariant $\langle \sigma_{I_1}, \ldots, \sigma_{I_l}, \sigma_J \rangle_d \neq 0$ so that one has an inequality for $\Delta(l)$ given by

(11)
$$\sum_{i \in I_1} \lambda_{1,i} + \ldots + \sum_{i \in I_l} \lambda_{l,i} \le d.$$

Associativity of quantum multiplication says that

$$\langle \sigma_{I_1}, \ldots, \sigma_{I_l}, \sigma_J \rangle_d = \sum_{d_1+d_2=d, |K|=r} \langle \sigma_{I_1}, \ldots, \sigma_{I_{j-1}}, \sigma_K \rangle_{d_1} \langle \sigma_{*K}, \sigma_{I_j}, \ldots, \sigma_{I_l}, \sigma_J \rangle_{d_2}.$$

In particular there exist some d_1, d_2 with $d_1 + d_2 = d$ and some Schubert variety σ_K such that

$$\langle \sigma_{I_1}, \ldots, \sigma_{I_{j-1}}, \sigma_K \rangle_{d_1} \neq 0, \quad \langle \sigma_{*K}, \sigma_{I_j}, \ldots, \sigma_{I_l}, \sigma_J \rangle_{d_2} \neq 0$$

From the non-vanishing of these Gromov-Witten invariants one deduces the inequalities for $\Delta_q(j), \Delta_q(k)$:

(12)
$$\sum_{i \in I_1} \mu_{1,i} + \ldots + \sum_{i \in I_{j-1}} \mu_{j-1,i} + \sum_{k \in K} \mu_{j,k} \le d_1;$$

(13)
$$\sum_{k \in *K} \nu_{1,k} + \sum_{i \in I_j} \nu_{2,i} + \ldots + \sum_{i \in I_l} \nu_{k,i} \le d_2.$$

Restricting to the section $\mu_j = *\nu_1$ one has that

$$\sum_{k \in *K} \nu_{1,k} = -\sum_{k \in K} (*\nu_1)_k = -\sum_{k \in K} \mu_{j,k},$$

so by adding the two inequalities one obtains (11).

Using the trivial factorization l = (l + 2) - 2 we complete the proof of Theorem 3.1.

Lemma 6.1. Any inequality for $\Delta_q(l)$ corresponding to a Gromov-Witten invariant

$$\langle [\sigma_{I_1}], ..., [\sigma_{I_l}], [\sigma_K] \rangle_d \neq 0$$

is a consequence of an inequality corresponding to a Gromov-Witten invariant of the form $% \mathcal{G} = \mathcal{G} = \mathcal{G} = \mathcal{G}$

$$\langle [\sigma_{I_1}], ..., [\sigma_{I_{l-1}}], [\sigma_J] \rangle_{d_1} \neq 0$$

for some $J \subset \{1, \ldots, n\}$ and $d_1 \leq d$.

Proof. Suppose that

$$\langle [\sigma_{I_1}], ..., [\sigma_{I_l}], [\sigma_K] \rangle_d \neq 0.$$

Taking k = 2 we obtain that for some J and $d_1 \leq d$

$$\langle [\sigma_{I_1}], \dots, [\sigma_{I_{l-1}}], [\sigma_J] \rangle_{d_1} \langle [\sigma_{*J}], [\sigma_{I_l}], [\sigma_K] \rangle_{d_2} \neq 0.$$

Thus the inequality

(14)
$$\sum_{i \in I_1} \lambda_{1,i} + \ldots + \sum_{i \in I_l} \lambda_{l,i} \le d.$$

follows from the inequalities

(15)
$$\sum_{i \in I_1} \lambda_{1,i} + \ldots + \sum_{i \in I_{l-1}} \lambda_{l-1,i} + \sum_{j \in J} \lambda_{l,j} \le d_1$$

for $\lambda \in \Delta_q(l)$ and

(16)
$$\sum_{j \in *J} (*\lambda_l)_j + \sum_{i \in I_l} (\lambda_l)_i \le d_2.$$

The last equation is a tautology for $\lambda_l \in \mathfrak{A}$ by the l = 2 case of Theorem 5.3. In other words, (16) is implied by the equations $\lambda_{l,i} \geq \lambda_{l,i+1}$, $\lambda_{l,1} - \lambda_{l,n} \leq 1$. Thus (14) follows from (15) and the inequalities defining \mathfrak{A}^l .

7. HIDDEN SYMMETRY

An interesting aspect of the multiplicative problem is that it possesses a symmetry not present in the additive case, related to the symmetry of the fundamental alcove \mathfrak{A} of SU(n). Let $Z \cong \mathbb{Z}/n\mathbb{Z}$ denote the center of SU(n), with generator $c \in SU(n)$ the unique element of SU(n) with

$$\lambda(c) = (1/n, 1/n, \dots, 1/n, (1-n)/n).$$

The action of Z on SU(n) induces an action on $\mathfrak{A} \cong SU(n) / \operatorname{Ad}(SU(n))$, given by

$$c \cdot (\lambda_1, \ldots, \lambda_n) = (\lambda_2 + 1/n, \lambda_3 + 1/n, \ldots, \lambda_n + 1/n, \lambda_1 - (n-1)/n).$$

Let $C(l) \subset SU(n)^l$ denote the subgroup

$$C(l) = \{(z_1, \ldots, z_l) \subset Z^l \mid z_1 z_2 \ldots z_l = 1\} \cong Z^{l-1}.$$

The action of C(l) on \mathfrak{A}^l leaves the polytope $\Delta_q(l)$ invariant.

This symmetry of the polytope $\Delta_q(l)$ implies a symmetry on the facets of $\Delta_q(l)$. Let c act on subsets of $\{1, 2, \ldots, n\}$ via the action of $(12 \ldots n)^{-1} \in S_n$:

$$c^{m}\{i_{1},\ldots,i_{r}\} = \{i_{s+1}-m,\ldots,i_{r}-m,i_{1}-m+n,\ldots,i_{s}-m+n\}$$

where s is the largest index for which $i_s - m \ge 1$

Suppose an l + 1-tuple (I_1, \ldots, I_l, d) defines a facet of $\Delta_q(l)$ via the inequality (10). Under the action of $(c^{m_1}, \ldots, c^{m_l}) \in C(l)$, (10) becomes the inequality corresponding to $(c^{m_1}I_1, \ldots, c^{m_l}I_l, d')$ where d' is defined by

(17)
$$\sum_{i=1}^{l} |c^{m_i} I_i| + nd' = \sum_{i=1}^{l} |I_i| + nd.$$

Example 7.1. From the degree 0 inequality $\lambda_n(A) + \lambda_n(B) \leq \lambda_n(AB)$ we obtain by the action of $(c^{-i}, c^{-j}, c^{i+j})$, $i+j \leq n$ the degree 1 inequality (8).

Equation (17) defines a C(l) action on the set of l + 1-tuples (I_1, \ldots, I_l, d) defining facets of $\Delta_q(l)$. It is an interesting fact that the Gromov-Witten invariants $\langle \sigma_{I_1}, \ldots, \sigma_{I_l} \rangle_d$ are invariant under this action:

Proposition 7.2. Let $(c^{m_1}, \ldots, c^{m_l}) \in C(l)$. Then $\langle \sigma_{I_1}, \ldots, \sigma_{I_l} \rangle_d = \langle \sigma_{c^{m_1}I_1}, \ldots, \sigma_{c^{m_l}I_l} \rangle_{d'}$.

Proof. Let $\sigma_c = \sigma_{r,r+1,\ldots,n-1}$ denote the Schubert variety isomorphic to the Grassmannian G(r, n-1) of r-planes contained in n-1-space. We claim that quantum multiplication by σ_c is given by the following formula:

(18)
$$[\sigma_c] \star [\sigma_I] = q^{(|cI|+r-|I|)/n} [\sigma_{cI}].$$

The exponent (|cI| + r - |I|)/n equals 1 if $1 \in I$, and equals 0 otherwise. In particular $[\sigma_c]^{\star n} = q^r$.

The lemma then follows by associativity of the quantum product. Without loss of generality it suffices to show that the Gromov-Witten invariants are invariant under an element of the form $(c, c^{-1}, 1, ..., 1) \in C(l)$. Given that

$$[\sigma_{I_1}] \star \ldots \star [\sigma_{I_{l-1}}] \supset \langle \sigma_{I_1}, \ldots, \sigma_{I_l} \rangle_d [\sigma_{*I_l}] q^d$$

multiplying by $[\sigma_c]$ on both sides yields

$$[\sigma_{cI_1}] \star \ldots \star [\sigma_{I_{l-1}}] \supset \langle \sigma_{I_1}, \ldots, \sigma_{I_l} \rangle_d [\sigma_{c(*I_l)}] q^{d'} = \langle \sigma_{I_1}, \ldots, \sigma_{I_l} \rangle_d [\sigma_{*c^{-1}I_l}] q^{d'}.$$

The formula (18) may be proved using either the canonical isomorphism of quantum Schubert calculus with the Verlinde algebra of U(r),

$$QH^*(G(r,n))/(q=1) \cong R(U(r)_{n-r,n}).$$

given a mathematical proof in Agnihotri [1], or using the combinatorial formula of Bertram, Ciocan-Fontanine and Fulton [4]. $R(U(r)_{n-r,n})$ denotes the Verlinde algebra of U(r) at SU(r) level n - r and U(1) level n, and is the quotient of the tensor algebra R(U(r)) by the relations

$$V_{\lambda} \sim (-1)^{l}(w) V_{w(\lambda+\rho)-\rho}, w \in W_{\text{aff}}$$

and if $\lambda_1 - \lambda_r \leq n - r$ then

$$V_{(\lambda_1,\dots,\lambda_r)} \sim V_{(\lambda_2-1,\lambda_3-1,\dots,\lambda_r-1,\lambda_1-(n-r+1))}$$

Here W_{aff} acts on Λ^* at level n, and ρ is the half-sum of positive roots. The Verlinde algebra $R(U(r)_{n-r,n})$ has as a basis the (equivalence classes of the) representations V_{λ} , where $\lambda = (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r$, $0 \leq \lambda_i \leq n-r$ are dominant weights of U(r) at level n-r.

The canonical isomorphism is given by $\sigma_I \mapsto V_{\lambda}$, where λ is defined by

$$\lambda_j = n - r + j - i_j.$$

The key point is that the sub-algebra $R(U(1)) \subset R(U(r))$ descends to a sub-algebra $R(U(1)_n) \subset R(U(r)_{n-r,r})$ generated by the representation $V_c := V_{(1,1,\dots,1)}$, which maps

under the isomorphism to the Schubert variety σ_c . From the description of the algebra given above one sees that $V_c \circledast V_{\lambda} = V_{\lambda'}$ where

$$\lambda' = \begin{array}{c} (\lambda_1 + 1, \lambda_2 + 1, \dots, \lambda_r + 1) & \text{if } \lambda_1 < n - r \\ (\lambda_2, \dots, \lambda_r, \lambda_1 - n + r) & \text{if } \lambda_1 = n - r \end{array}$$

Since $V_{\lambda'}$ maps to σ_{cI} under the canonical isomorphism, this proves (18).

Alternatively, (18) can be derived from the combinatorial rim-hook formula of [4, p. 8]. Let λ^t denote the transpose of λ , so that σ_{λ^t} is the image of σ_{λ} under the isomorphism $G(r, n) \cong G(n - r, n)$. The ordinary (resp. quantum) Littlewood-Richardson numbers are invariant under transpose

$$N_{\lambda^t \mu^t}^{\rho^t} = N_{\lambda\mu}^{\rho}, \quad N_{\lambda^t \mu^t}^{\rho^t}(n-r,r) = N_{\lambda\mu}^{\rho}(r,n-r).$$

It follows from [4, Corollary] that

$$N^{\rho}_{\lambda\mu}(r,n-r) = \sum \epsilon(\rho^t/\nu^t) N^{\rho}_{\lambda\mu}$$

where ρ ranges over all diagrams of height $\leq r$ that can be obtained by adding m rim-hooks.

If $\mu = (1, 1, ..., 1)$ then

$$V_{\mu} \otimes V_{\lambda} = V_{\rho}, \quad \rho = (\lambda_1 + 1, \dots, \lambda_r + 1).$$

If $\lambda_1 < n - r$, then since the height of ρ is $\leq r$, there are no rim *n*-hooks in ρ . On the other hand, if $\lambda_1 = n - r$, then it is easy to see that there is a unique rim *n*-hook in ρ , whose complement is λ' above.

We have learned from A. Postnikov that formula similar to (18) holds for the full flag variety [24]. A deeper reason for the appearance of symmetry is given by Seidel [26]. \Box

This symmetry simplifies the computation of many Gromov-Witten invariants. For sufficiently small n and l all Gromov-Witten invariants are equivalent to degree 0 ones. An example of a Gromov-Witten invariant not equivalent via symmetry to a degree 0 invariant is the degree 1 invariant for G(5, 10)

$$\langle \sigma_{\{2,4,6,8,10\}}, \sigma_{\{2,4,6,8,10\}}, \sigma_{\{1,3,5,7,9\}} \rangle_1$$

which may be computed using the formula of [4]. That is, for sufficiently large r, n, not all of the inequalities are related to "classical" inequalities via symmetry.

8. Verlinde Algebras

Finally we want to explain the representation-theoretic interpretation of $\Delta_q(l)$ in terms of the Verlinde algebra of SU(n). Denote by Λ_N^* the set of dominant weights of SU(n)at level N:

$$\Lambda_N^* = \{ (\lambda_1, \dots, \lambda_n) \in (\mathbb{Z}/n)^n \mid \lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0}, \ \lambda_1 - \lambda_n \leq N \}.$$

The Verlinde algebra $R(SU(n)_N)$ is the free group $\mathbb{Z}[\Lambda_N^*]$ on the generators $V_{\xi}, \xi \in \Lambda_N^*$. The algebra structure is given by "fusion product"

$$V_{\xi_1} \circledast_N \ldots \circledast_N \mathcal{V}_{\xi_l} = \sum_{\nu \in \Lambda_N^*} m^N(\xi_1, \ldots, \xi_l, \nu) V_{*\nu}$$

where the coefficients $m^N(\xi_1, \ldots, \xi_l)$ are defined as follows. There is a positive line bundle $L^N(\xi_1, \ldots, \xi_l) \to \mathcal{M}(\xi_1/N, \ldots, \xi_l/N)$ which descends from the polarizing line bundle on \tilde{R} (see Pauly [23, Section 3]). The coefficient $m^N(\xi_1, \ldots, \xi_l)$ is defined by

$$m^N(\xi_1,\ldots,\xi_l) = \dim(H^0(L^N(\xi_1,\ldots,\xi_l)).$$

Since L^N is positive, $\mathcal{M}(\xi_1/N, \ldots, \xi_l/N)$ is non-empty if and only if for some k

$$\dim(H^0(L^k(\xi_1,\ldots,\xi_l)^{\otimes N})) = \dim(H^0(L^{kN}(k\xi_1,\ldots,k\xi_l)) \neq 0,$$

that is, $m^{kN}(k\xi_1, ..., k\xi_l) \neq 0$.

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