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METABOLIC AND HYPERBOLIC FORMS

J. P. ALEXANDER, P. E. CONNER AND G. C. HAMRICK

Introduction. In our research we have made many detailed computations concerning $W_*(\boldsymbol{Z}, C)$, C a cyclic group [**1**], [**2**], [**3**]. Several ideals and subgroups of this ring have always intrigued us, particularly the ideal generated by Witt classes represented by forms on projective $\boldsymbol{Z}[C]$-modules and the image of the Wall groups. This note begins with the question of the relationship between metabolic and hyperbolic forms. This is suggested by the different ways of defining zero elements: in Witt groups metabolics are zero and in the L-groups hyperbolics are zero. Theorem 3 recovers in a very special case results of Bak [**5**], Pardon [**8**], Wall [**9**] and others, but we feel an inclusion of the straightforward argument is justified.

Our proofs are not detailed. In fact we only outline the results for the symmetric case. Most of the computations involving group cohomology, while tedious, are elementary.

Metabolic and hyperbolic forms. Let (π, V) denote a right integral representation of a finite group π on a free abelian group V. If there is a π-invariant $\boldsymbol{Z}$-nonsingular (skew) symmetric inner product $b\colon V \times V \to \boldsymbol{Z}$ then we say (π, V) is an integral orthogonal (symplectic) representation and denote this by (π, V, b).

For some time we have been concerned with relationships between the following concepts.

DEFINITION. (i) (π, V, b) is *metabolic* if and only if there is a π-invariant submodule $N \subset V$ for which $N = N^{\perp} = \{v \in V | b(v, n) = 0 \text{ for all } n \in N\}$.

(ii) (π, V, b) is hyperbolic if and only if there is an integral representation (π, N) for which (π, V) is equivalent by a π-equivariant isometry to the natural form on $(\pi, N \oplus N^*)$ given by

$$b_h((n, \varphi), (n', \varphi')) = \varphi(n') + \varphi'(n)$$

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where $N^* = \mathrm{Hom}_{\mathbf{Z}}(N, \mathbf{Z})$ with the usual right π-module action. We denote this hyperbolic form simply by $H(N)$.

Let $W_0(\mathbf{Z}, \pi)$ ($W_2(\mathbf{Z}, \pi)$) denote the Witt group of orthogonal (symplectic) representations of π where (π, V, b) is zero if it is metabolic. We say that (π, V, b) is an *even form* if for each element $g \in \pi$ such that $g^2 = e$ then $b(v, vg)$ is even for all $v \in V$.

THEOREM 1. *If $\pi = C_p$, the cyclic group of order p, p a prime and (π, V, b) is an even, metabolic form, then (π, V, b) is hyperbolic.*

To prove this theorem we introduce several new groups. Given (π, N) consider all $\mathbf{Z}[\pi]$-embeddings $0 \to (\pi, N) \to^{\rho} (\pi, V, b)$ so that $(\mathrm{im}(\rho))^{\perp} = \mathrm{im}(\rho)$ and b is even. Two embeddings $0 \to (\pi, N) \to^{\rho} (\pi, V, b)$ and $0 \to (\pi, N) \to^{\rho'} (\pi, V', b')$ are *Baer equivalent* if and only if there is a commutative diagram

$$\begin{array}{ccc} & \overset{\rho}{\nearrow} & (\pi, V, b) \\ (\pi, N) & & \downarrow J \\ & \underset{\rho'}{\searrow} & (\pi, V', b') \end{array}$$

where J is a $\mathbf{Z}[\pi]$-module isometry. The sum, $\rho + \rho'$, of two embeddings is described as follows. Consider $L \subset V \perp V'$ where $L = \{(\rho(x), -\rho'(x)) \mid x \in N\}$. ($V \perp V'$ is the orthogonal direct sum of (π, V, b) and (π', V', b'). The associated bilinear form is denoted $b \perp b'$.) Clearly L is summand and $L \subset L^{\perp}$. Let b'' denote the natural form induced on $L^{\perp}/L$ by $b \perp b'$. Then $\rho + \rho'$ is defined by the natural embedding

$$(\pi, N) \xrightarrow{\rho + \rho'} (\pi, L^{\perp}/L, b'').$$

The hyperbolic form $H(N)$ plays the role of the identity and the inverse of $(\pi, N) \to^{\rho} (\pi, V, b)$ is given by $(\pi, N) \to^{-\rho} (\pi, V, -b)$. The group of embeddings $(\pi, N) \to (\pi, V, b)$ is denoted $\mathrm{Met}_{\pi}(N)$. This is a covariant functor with respect to $\mathbf{Z}[\pi]$-module homomorphisms. A natural transformation $\mathrm{Met}_{\pi}(N) \to \mathrm{Ext}^1_{\mathbf{Z}\pi}(N^*, N)$ is given by associating to each embedding $\rho: (\pi, N) \to (\pi, V, b)$ the resulting short exact sequence $0 \to N \to V \to N^* \to 0$. If N is projective as a $\mathbf{Z}[\pi]$-module then this map is a *monomorphism*! Furthermore if π has odd order then this is a monomorphism for any integral representation N.

There are two important constructions involving $\mathrm{Met}_{\pi}(N)$.

CONSTRUCTION 1. Given (π, N_1) and (π, N_2) there is a homomorphism

$$\mathrm{Ext}^1_{\pi}(N_2^*, N_1) \to \mathrm{Met}_{\pi}(N_1 \oplus N_2)$$

given as follows. If $0 \to N_1 \to^{\alpha} K \to^{\beta} N_2^* \to 0$ is an extension, then its dual is $0 \to N_2 \to^{\beta^*} K^* \to^{\alpha^*} N_1^* \to 0$. Adding these sequences together we get

$$0 \to N_1 \oplus N_2 \xrightarrow{\alpha \oplus \beta^*} K \oplus K^* \to N_2^* \oplus N_1^* \to 0.$$

$K \oplus K^*$ supports the hyperbolic form $H(K)$ and $\mathrm{im}(\alpha \oplus \beta^*)^{\perp} = \mathrm{im}(\alpha \oplus \beta^*)$. This gives us an element of $\mathrm{Met}_{\pi}(N_1 \oplus N_2)$.

CONSTRUCTION 2. If $(\pi, N) \to^{f} (\pi, N')$ is an epimorphism then the induced homomorphism $\mathrm{Met}_{\pi}(N) \to \mathrm{Met}_{\pi}(N')$ can be described as follows. Given $[(\pi, N) \to$

$(\pi, V, b)] \in \mathrm{Met}_\pi(N)$, $\ker f$ is a π-invariant subspace of V and $\ker f \subset (\ker f)^\perp$. Clearly there is an embedding

$$(\pi, N') \to (\pi, (\ker f)^\perp/\ker f, b').$$

This gives us an element of $\mathrm{Met}_\pi(N')$.

LEMMA 2. *The following is a split exact sequence*

$$0 \to \mathrm{Ext}^1_\pi(N_2^*, N_1) \to \mathrm{Met}_\pi(N_1 \oplus N_2) \to \mathrm{Met}_\pi(N_1) \oplus \mathrm{Met}_\pi(N_2) \to 0. \quad \square$$

This reduces the problem of analyzing $\mathrm{Met}_\pi(N)$ to N an indecomposable π-representation. Except when $\pi = C_p$, p a prime, our ignorance here is almost complete. Theorem 1 is proved by first using the Reiner-Dietrichsen decomposition for integral C_p-representations to show that for an indecomposable representation that $\mathrm{Met}_{C_p}(N) = 0$. This is followed by an application of Lemma 2 and the fact that the elements in the image of Construction 1 are hyperbolic.

By restricting (π, V) we can extend this result in the following manner.

THEOREM 3. *If $\pi = C_{p^n}$, p a prime, C_{p^n} the cyclic group of order p^n, V is a projective $\mathbf{Z}[\pi]$-module and b is a metabolic even form, then (π, V, b) is hyperbolic.* $\square$

This theorem is proved by induction on n and uses the following result about group cohomology.

THEOREM 4 [**4**, PP. 112–113]. *If π is a p-group, and A is a π-module without p-torsion then the following are equivalent*:

(i) *A is cohomologically trivial,*
(ii) *$\hat{H}^q(\pi, A) = \hat{H}^{q-1}(\pi, A) = 0$ for some $q \in \mathbf{Z}$,*
(iii) *A/pA is a free $\mathbf{Z}_p[\pi]$-module,*
(iv) *A is a projective π-module.* $\square$

Theorem 3 is well known for $n = 0$.

Consider (C_{p^n}, V, B). First some notation. C_p will be the subgroup of order p in C_{p^n} and $\tilde{C}_{p^{n-1}} = C_{p^n}/C_p$. C_p is generated by $T^{p^{n-1}}$. Let $\Delta = 1 - T^{p^{n-1}}$ and $\Sigma = 1 + T^{p^{n-1}} + T^{2p^{n-1}} + \cdots + T^{(p-1)p^{n-1}}$. Now $U = \{x \in V \mid \Sigma x = 0\}$ is a projective $\mathbf{Z}(\lambda_{p^n})$-module ($\mathbf{Z}(\lambda_k)$ are the cyclotomic integers, $\lambda_k = \exp(2\pi i/k)$). Since V is metabolic, so is $U \otimes \mathbf{Z}(1/p)$ and hence there is a C_{p^n}-invariant summand $W \subset U$ such that $W = W^\perp \cap U$. W is clearly a $\mathbf{Z}(\lambda_{p^n})$ projective module.

LEMMA 5. *$W^\perp/W$ is a projective $\mathbf{Z}[\tilde{C}_{p^{n-1}}]$-module.*

PROOF. Recall that for a C_{p^n}-module N that $H^1(C_p; N) = \ker \Sigma/\mathrm{im}\, \Delta$ and that $H^2(C_p; N) = \ker \Delta/\mathrm{im}\, \Sigma$. Since V is a projective $\mathbf{Z}[C_{p^n}]$-module and W is a projective $\mathbf{Z}(\lambda_{p^n})$-module, we can use the exact sequence $0 \to W^\perp \to V \to W^* \to 0$ to compute $H^*(C_p; W^\perp)$. Notice that since $\mathrm{im}\, \Delta \subset U$ and $W = W^\perp \cap U$ that C_p acts trivially on $W^\perp/W$. The six-term exact sequence for $0 \to W \to W^\perp \to W^\perp/W \to 0$ reduces to

$$0 \to H^2(C_p; W^\perp) \to H^2(C_p; W^\perp/W) \to H^1(C_p; W) \to 0.$$

Now, $H^2(C_p; W^\perp/W) \approx W^\perp/W \otimes \mathbf{Z}_p$ and the first and last terms are free $\mathbf{Z}_p[\tilde{C}_{p^{n-1}}]$-modules.

By Theorem 4 this shows $W^{\perp}/W$ is a projective $\tilde{C}_{p^{n-1}}$-module. □

By induction $W^{\perp}/W$ is hyperbolic. If $A \subset W^{\perp}/W$ let $Y = \{x \in W^{\perp} \mid x + W \in A\}$ (i. e., $0 \to W \to Y \to A \to 0$ is exact).

LEMMA 6. *There exists* $A \subset W^{\perp}/W$ *so that*
(i) $W^{\perp}/W = H(A)$,
(ii) $\delta\colon H^2(C_{p^n}; A) \to H^1(C_{p^n}; W)$ *is an isomorphism,*
(iii) Y *is a* $\boldsymbol{Z}[C_{p^n}]$*-projective module and* $Y = Y^{\perp}$,
(iv) $V = H(Y)$.

PROOF. First consider the exact cohomology sequence $0 \to H^2(C_{p^n}; W^{\perp}) \to^{\rho} H^2(C_{p^n}; W^{\perp}/W) \to H^1(C_{p^n}; W) \to 0$. There is a nonsingular finite form on $X = H^2(C_{p^n}; W^{\perp}/W)$ induced by cup products in cohomology and the bilinear form on $W^{\perp}/W$. Because $H^2(C_{p^n}; \boldsymbol{Z}_p[\tilde{C}_{p^{n-1}}]) = \boldsymbol{Z}_p$ this form has values in $\boldsymbol{Z}_p$. $W^{\perp}/W$ is hyperbolic; therefore X is also hyperbolic. There are two ways of describing X as a hyperbolic module. The first comes from choosing $B \subset W^{\perp}/W$ so that $W^{\perp}/W = H(B)$. If $\sigma\colon H^2(C_{p^n}; B) \to H^2(C_{p^n}; W^{\perp}/W)$ then $X = H(\text{im } \sigma)$. Also $X = H(\text{im } \rho)$. Notice that if $\text{im } \sigma \cap \text{im } \rho = \{0\}$ then the composition $H^2(C_{p^n}; B) \to H^2(C_{p^n}; W^{\perp}/W) \to H^1(C_{p^n}; W)$ is an isomorphism. Take $A = B$ and the lemma would be proven. In general find B', a projective submodule of B, so that $\sigma(H^2(C_{p^n}; B'))\colon m\sigma \cap \text{im } \rho$. Then $B = B' \oplus B''$ and

$$W^{\perp}/W = B' \oplus B'' \oplus (B')^* \oplus (B'')^* = H(B'' \oplus (B')^*).$$

Now $\text{im}(H^2(C_{p^n}; B'' \oplus (B')^*) \cap \text{im } \rho = \{0\}$. Set $A = B'' \oplus (B')^*$ and $Y = \{w \in W^{\perp} \mid w + W \in A\}$. □

Witt groups and L-groups. If $L^h_*(\pi)$ is the Wall group for surgery to a homotopy equivalence then there is an obvious homomorphism

$$L^h_0(\pi) \to W_0(\boldsymbol{Z}, \pi).$$

One application of Theorem 3 is the following result, well known in much more generality by Bak [**5**], Pardon [**8**], Wall [**9**], and Karoubi [**6**].

COROLLARY 7. *For p a prime the following sequence is exact*:

$$0 \to H^1(C_2, \tilde{K}_0(C_p)) \to L^h_0(C_{p^n}) \to W_0(\boldsymbol{Z}, C_{p^n}).$$

PROOF. The involution of $\tilde{K}_0(C_{p^n})$ is defined by sending a projective module V to its dual V^*. An element of $H^1(C_2; \tilde{K}_0(C_{p^n}))$ is represented by a projective module V such that $V \oplus V^*$ is free. The homomorphism $H^1(C_2; \tilde{K}_0(C_{p^n})) \to L^h_0(C_{p^n})$ sends $[V] \mapsto [H(V)]$. Theorem 3 says that if $[U] \in L^h_0(C_{p^n})$ goes to zero in $W_0(\boldsymbol{Z}, C_{p_n})$ then it is hyperbolic, that is $U = H(V)$ where V is projective and $V \oplus V^* = U$ is free. A careful analysis of the units in $\boldsymbol{Z}[C_{p^n}]$ prove the left-hand homomorphism is a monomorphism. □

In fact since $W_*(\boldsymbol{Z}, C_{p^n})$ is torsion free, p odd [**1**], we see that $L_0(C_{p^n})$ is torsion free up to projective kernels. Even though $W_0(\boldsymbol{Z}, C_{2^n})$ is not torsion free, the invariant $H^2(C_{2^n}; V)$ detects the torsion elements. For V projective, this invariant must vanish so in all cases the image of $L_0(C_{p^n}) \to W_0(\boldsymbol{Z}, C_{p^n})$ is torsion free.

It is obvious that the image of the Wall groups in the Witt groups is contained in the ideal generated by elements with representatives (π, V, b) where V is a projective $\boldsymbol{Z}[\pi]$-module. Let $P_*(\boldsymbol{Z}, \pi)$ denote this ideal.

Given an arbitrary $(C_{p^n}, V, b) \in W_0(\boldsymbol{Z}, C_{p^n})$ there is a symmetric finite form defined on $H^2(C_{p^k}; V)$, $1 \leqq k \leqq n$, by the composition

$$H^2(C_{p^k}; V) \times H^2(C_{p^k}; V) \to H^4(C_{p^k}; V \otimes V) \to H^4(C_{p^k}; \boldsymbol{Z}) \to \boldsymbol{Z}/p^k\boldsymbol{Z}$$

where the first map is induced by cup products in cohomology, the second by the coefficient pairing $v \otimes v' \mapsto b(v, v')$, and the last by the natural identification of $H^4(C_{p^k}; \boldsymbol{Z})$ with $\boldsymbol{Z}/p^k\boldsymbol{Z}$. Standard arguments involving finite forms [**3**, 1.7] show that this form can be considered as an element of $W(\boldsymbol{Z}_p)$. Consider the group homomorphisms φ_k: $W_0(\boldsymbol{Z}, C_{p^n}) \to W(\boldsymbol{Z}_p)$, p odd, $1 \leqq k \leqq n$, given by

$$\varphi_k((C_{p^n}, V, b)) = H^2(C_{p^k}; V) + H^2(C_{p^{k-1}}, V) \in W(\boldsymbol{Z}_p).$$

PROPOSITION 8. *φ_k is a ring homomorphism, $P_0(\boldsymbol{Z}, \pi) = \bigcap_k \ker \varphi_k$, and*

$$W_0(\boldsymbol{Z}, C_{p^n})/P_0(\boldsymbol{Z}, C_{p^n}) \approx \bigoplus_1^n W(\boldsymbol{Z}_p). \quad \square$$

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UNIVERSITY OF TEXAS AT AUSTIN