On the Chains of a Complex and Their Duals

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Proceedings of the National Academy of Sciences of the United States of America, Volume 21, Issue 8 (Aug. 15, 1935), 509-511.

ON THE CHAINS OF A COMPLEX AND THEIR DUALS

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Communicated July 8, 1935

Let C be a finite or infinite topological complex. With each i-cell of C we shall introduce a pair of symbols $E^{(i)}_s$ and $F^s_{(i)}$. An *i-chain* will be any finite linear combination of symbols of the first kind,

$$K^{(i)} = \sum_s \alpha^s E^{(i)}_s, \tag{1}$$

where the coefficients α^s are elements of any preassigned additive group A. An *i-function* will be any linear combination, finite or infinite, of symbols of the second kind,

$$L_{(i)} = \sum_s \beta_s F^s_{(i)}, \tag{2}$$

where the coefficients β_s are elements of the character group B of A (the group of all representations of A on the unit circle in the complex plane). If we designate the points of the unit circle by a real parameter λ, modulo 1, we can construct a bilinear form

$$(\beta_s, \alpha^t) = \lambda^t_s, \quad (\text{mod } 1), \tag{3}$$

such that λ^t_s is the point on which the character β_s maps the group element α^t. We can then form the expression

$$(L_{(i)}, K^{(i)}) = \sum_s (\beta_s, \alpha^s), \quad (\text{mod } 1), \tag{4}$$

which will be called the *integral* of $L_{(i)}$ over $K^{(i)}$.

The classical theory of connectivity may be summarized as follows. We orient the cells of C and denote by

$$\{E^{(i)}_s, F^t_{(i-1)}\}, \quad (= 0, 1, -1), \tag{5}$$

the incidence number between the oriented i-cell associated with $E^{(i)}_s$ and the oriented $(i-1)$-cell associated with $F^t_{(i-1)}$. We then define a linear operator β transforming an i-chain $K^{(i)}$ into an $(i-1)$-chain

$$\beta K^{(i)} = \sum_{s,t} \alpha^s \{E^{(i)}_s, F^t_{(i-1)}\} E^{(i-1)}_t, \quad \begin{pmatrix} \alpha^s.1 = \alpha^s \\ \alpha^s.0 = 0 \end{pmatrix}, \tag{6}$$

called the *boundary* of $K^{(i)}$. The chain $K^{(i)}$ is said to be *closed* if its boundary vanishes; it is said to be *bounding* if there exists any $K^{(i+1)}$ such that $\beta K^{(i+1)} = K^{(i)}$. The incidence numbers (5) satisfy the well-known relations

$$\sum_t \{E_s^{(i)}, F_{(i-1)}^t\} \{E_t^{(i-1)}, F_{(i-2)}^u\} = 0, \quad (7)$$

from which we obtain, at once,

$$\beta^2 K^{(i)} = 0. \quad (8)$$

By (8), the group Ψ^i of all bounding i-cycles is a subgroup of the group Φ^i of all closed i-cycles. The group Φ^i mod Ψ^i is the *i-th connectivity group* of C.

We now propose to introduce a new linear operator δ dual to β. The operator δ will transform an i-function $L_{(i)}$ into an $(i + 1)$-function

$$\delta L_{(i)} = \sum_{s,t} \beta_s \{E_t^{(i+1)}, F_{(i)}^s\} F_{(i+1)}^t, \quad (9)$$

which will be called the *derived* of $L_{(i)}$. We shall say that $L_{(i)}$ is *exact* if its derived vanishes and that it is *derived* if there exists any $L_{(i-1)}$ such that $\delta L_{(i-1)} = L_{(i)}$. By (7), the operator δ satisfies the relation

$$\delta^2 L_{(i)} = 0, \quad (10)$$

dual to (8). Thus, we see that the group Ψ_i of all derived i-functions is a subgroup of the group Φ_i of all exact i-functions. We call the group Φ_i mod Ψ_i the *dual* of the i-th connectivity group of C. *This group is easily seen to be the character group of the i-th connectivity group.*

Let $K^{(i)}$, cf. (1), be an arbitrary i-chain and $L_{(i-1)}$ an arbitrary $(i-1)$-function,

$$L_{(i-1)} = \sum_s \gamma_s F_{(i-1)}^s. \quad (11)$$

Then, by a simple computation, we obtain the relation

$$(\delta L_{(i-1)}, K^{(i)}) = (L_{(i-1)}, \beta K^{(i)}) = \sum_{s,t} \{E_t^{(i)}, F_{(i-1)}^s\} (\gamma_s . \alpha^t), \quad (12)$$

which says that *the integral of the derived of $L_{(i-1)}$ over $K^{(i)}$ is equal to the integral of $L_{(i-1)}$ itself over the boundary of $K^{(i)}$.* By (8), (10) and (12), we also have the corollary: *the integral of a derived function $\delta L_{(i-1)}$ over a closed chain $K^{(i)}$ is always zero; likewise, the integral of an exact function $L_{(i-1)}$ over a bounding chain $\beta K^{(i)}$.* The integral of an exact $L_{(i)}$ over a closed $K^{(i)}$ will be called the *period* of $L_{(i)}$ over $K^{(i)}$. It will, in general, be different from zero.

Formula (12) is analogous to the well-known theorem of Stokes on multiple integrals. The coefficients α^s and γ_s in (1) and (11) need not be chosen in exactly the manner indicated above. If we take the coefficients α^s to be arbitrary integers and the coefficients γ_s to be arbitrary real numbers, and if we write, in place of (3), the relation

$$(\gamma_s, \alpha^t) = \gamma_s \alpha^t, \text{ (not reduced mod 1)}, \quad (13)$$

then (12) becomes the exact analogue of the theorem on multiple integrals and reduces to the latter by a passage to the limit.

ON THE RING OF A COMPACT METRIC SPACE

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Communicated July 8, 1935

Let C be a compact metric space. We pick an arbitrary additive group A and define an *i-function* of C to be any single-valued, skew symmetric function,

$$F(p_0p_1 \ldots p_i), \tag{1}$$

of $i + 1$ variable points of C, where the values of F are elements of A. (For the case $i = 0$ the function F is an ordinary point function on C.) The function F will be said to be *locally zero*,

$$F = 0 \text{ (locally)}, \tag{2}$$

if there exists an $\epsilon > 0$ such that F vanishes whenever the distances between the points p_s ($s = 0, 1, \ldots, i$) are all less than ϵ. Given any i-function F we construct an $(i + 1)$-function δF which we call the *derived* of F and which we define by the formula

$$\delta F(p_0p_1 \ldots p_{i+1}) = \frac{1}{(i+1)^i} \sum_s (-1)^s F(p_0p_1 \ldots p_{s-1}p_{s+1} \ldots p_{i+1}). \tag{3}$$

We say that F is *exact* if δF is locally zero. For the case $i > 0$, we say that F is *derived* if there exists any $(i-1)$-function G such that $\delta G = F$. For the case $i = 0$, we say that F is *derived* if it is constant over C. By the skew symmetry of F we have, at once,

$$\delta^2 F = 0, \tag{4}$$

from which we conclude that *every derived function is exact.* Now, the exact i-functions on C form a group Φ_i with respect to addition and, by (4), the derived i-functions form a subgroup Ψ_i of Φ_i. We form the group Φ_i mod Ψ_i. *This group is a topological invariant analogous to the i-th connectivity group of C,* as defined by Vietoris. In fact, if the coefficients of the Vietoris cycles are taken to be elements of the character group B of A, then *the i-th connectivity group of C may be identified with the character group of the group* Φ_i *mod* Ψ_i.

A more interesting invariant can be obtained when A is the group of a ring, so that the elements of A can be both added and multiplied. We