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THE BORDISM RING OF MANIFOLDS
WITH INVOLUTION

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ABSTRACT. A complete set of representatives is found for the bordism ring of manifolds with involution. The multiplicative structure is expressed in terms of these representatives, and in particular, a set of ring generators can be isolated. Also the constructions involved are used to give an insight into a map of J. M. Boardman.

1. **Introduction.** In [3], P. E. Conner and E. E. Floyd defined the notion of cobordism of differentiable manifolds with involution, and they investigated the structure of the resulting bordism groups. If M is a closed differentiable manifold and $T:M \to M$ is an involution, (M, T) bounds if there exists a manifold N such that $M = \partial N$ and an involution $S:N \to N$ such that $T = S|M$. We let $\mathscr{I} = \{\mathscr{I}_n\}$ be the resulting bordism groups graded by dimension.

If $\mathscr{N} = \{\mathscr{N}_n\}$ is the unoriented bordism ring, we consider $\mathscr{N}$ a subgroup of $\mathscr{I}$ by $[M] \mapsto [M, I]$, where I is the trivial involution. Similarly, we have an augmentation $\varepsilon:\mathscr{I} \to \mathscr{N}$ defined by $[M, T] \mapsto [M]$. The ring $\mathscr{N}$ operates on $\mathscr{I}$ by $[N] \cdot [M, T] = [N \times M, I \times T]$. $\mathscr{I}$ is a free $\mathscr{N}$-module. Indeed, $\mathscr{I}$ is a commutative ring, if we let

$$[N, T_1][M_2, T_2] = [M_1 \times M_2, T_1 \times T_2].$$

Conner and Floyd completely determined the additive structure of $\mathscr{I}$ [3, Theorem 28.1] which is as follows: there is a short exact sequence of $\mathscr{N}$ modules

$$(1.1) \qquad 0 \longrightarrow \mathscr{I} \overset{\nu}{\longrightarrow} \mathscr{M} \overset{J}{\longrightarrow} \mathscr{N}_*(P^\infty) \longrightarrow 0,$$

where $\mathscr{M}_n = \sum_{j=0}^{n} \mathscr{N}_j(BO(n-j))$. The notation is standard: $\mathscr{N}_*$ is the unoriented bordism (homology) theory, $BO(r)$ is the classifying space for real r-plane bundles, and $P^\infty = BO(1)$ is infinite-dimensional real projective space. The map ν is defined as follows: given (M^n, T), let F_i be the i-dimensional fixed-point submanifold of M, and let $\nu_i:F_i \to BO(n-i)$ be

the normal bundle of F_i in M. Then $[\nu_i] \in \mathcal{N}_i(BO(n-i))$, and $\nu[M, T]$ is the sum of these classes. Note that ν is a map of degree zero. J is defined as follows: given an r-plane bundle η over a manifold M^n, let $RP(\eta)$ be the total space of the associated projective-space bundle (an $(n+r-1)$-dimensional manifold), and let $\hat{\eta}$ over $RP(\eta)$ be the canonical line bundle. $J(\eta)$ is the class of $\hat{\eta} : RP(\eta) \rightarrow P^\infty$. The map J is of degree -1.

In this note, we exhibit a complete set of representatives for $\mathcal{I}$, and we determine its ring structure. Let $P^n = \{[x_0, \cdots, x_n]\}$ be real n-dimensional projective space, and let $T_0 : P^n \rightarrow P^n$ be defined by $T_0[x_0, x_1, \cdots, x_n] = [-x_0, x_1, \cdots, x_n]$. Denote the pair (P_n, T_0) by P_0^n. We need a map $\Gamma : \mathcal{I} \rightarrow \mathcal{I}$ of degree $+1$, defined as follows: let S^1 denote the circle $|z| = 1$ in the complex plane. Given (M, T), let $M' = M \times S^1 / \sim$, where $(m, z) \sim (Tm, -z)$. Define $T' : M' \rightarrow M'$ by $T'(m, z) = (Tm, \bar{z})$. Then $\Gamma[M, T] = [M', T']$ (compare [4, §4]).

1.2. THEOREM. *The set*

$$\{1\} \cup \{\Gamma^n[P_0^{n_1} \times \cdots \times P_0^{n_r}] \mid n \geq 0, r \geq 1, n_i \geq 2\}$$

is a free $\mathcal{N}$-basis for $\mathcal{I}$.

The product of $\mathcal{I}$ is thus determined when we know how it interacts with Γ.

1.3. THEOREM. *If $a, b \in \mathcal{I}$, then*

$$\Gamma(ab) = \Gamma(a) \cdot b + \varepsilon(a) \cdot \Gamma(b) = a \cdot \Gamma(b) + \Gamma(a) \cdot \varepsilon(b).$$

Let θ be transcendental over $\mathcal{N}$; we consider the ring of formal power series $\mathcal{N}[[\theta]]$. Define $\tilde{J} : \mathcal{I} \rightarrow \mathcal{N}[[\theta]]$ by $\tilde{J}[M^n, T] = \sum_{r=0}^{\infty} \varepsilon\Gamma^r[M^n, T] \cdot \theta^{n+r}$ (the coefficient of θ^s is in $\mathcal{N}_s$).

1.4. COROLLARY. *$\tilde{J}$ is a ring homomorphism.*

J. M. Boardman (unpublished; see [1], [2] for a related map) has defined a ring homomorphism

$$J' : \mathscr{F} = \sum_{i=0}^{\infty} \mathcal{N}_i(BO) \rightarrow \mathcal{N}[[\theta]],$$

where $\mathscr{F}$ has the ring structure given by the external product of bundles. Let $\iota_* : \mathcal{N}_i(BO(r)) \rightarrow \mathcal{N}_i(BO)$ be induced by inclusion. Then $\iota_* \nu$ is a ring homomorphism. We connect our results with Boardman's by the following result.

1.5. THEOREM. *$\tilde{J} = J' \iota_* \nu$.*

Although most of the results in this note are independent of Boardman's, they follow from an attempt to understand his work, and I would like to

thank him for several discussions about his work. Also, R. E. Stong has shown me some of his unpublished work wherein he obtains essentially the same result as our Theorem 1.2.

2. The map Γ. $\Gamma:\mathscr{I}_n\to\mathscr{I}_{n+1}$ is defined as follows: let S^1 denote the circle $|z|=1$ in the complex plane. Given a pair (M, T), let $M'=M\times S^1/\sim$, where $(m, z)\sim(Tm, -z)$. The involution T' is defined on M' by $T'(m, z)= (Tm, \bar{z})$. Then $\Gamma[M, T]=[M', T']$. These manifolds (M', T') (and $M^{(r)}$, $T^{(r)}$) are called $V(n, k)$ in [**4**]. Alternatively, if $I=[0, 1]$, let $M'=M\times I/\sim$, where $(m, 0)\sim(Tm, 1)$. In this notation, T' is defined by $T'(m, t)=(Tm, 1-t)$.

It is easily verified that (1) Γ is well defined, (2) Γ is additive, (3) Γ is an $\mathscr{N}$-map, (4) $\Gamma(\mathscr{N})=0$.

If F is the fixed-point set of (M, T) and ν is the normal bundle of $F\subset M$, the fixed point set of (M', T') is $F\cup M$ (here $F\equiv F\times\frac{1}{2}\subset F\times I$; $M\equiv M\times 0\subset M\times I$) with normal bundles $\nu\oplus o^1\cup o^1$, where o^r is the trivial r-plane bundle. The natural inclusion $\iota:BO(r)\to BO(r+1)$ induces $\iota_*:\mathscr{N}(BO(r))\to\mathscr{N}(BO(r+1))$, and the map $M\to$(base point) induces $\tau:\mathscr{N}_n\to\mathscr{N}_n(BO(1))$.

2.1. PROPOSITION. $\nu\Gamma=\iota_*\nu+\tau\varepsilon$.

Since ι_* is an injection, the next result is immediate.

2.2. COROLLARY. *If* $[M, T]\notin\mathscr{N}\subset\mathscr{I}$, *then* $\Gamma[M, T]\notin\mathscr{N}$. *In particular,* $\Gamma[M, T]=0$ *if and only if* $[M, T]\in\mathscr{N}$.

1.3′. THEOREM. *For* $[M_i, T_i]\in\mathscr{I}$ $(i=1, 2)$,

$$\Gamma([M_1, T_1]\cdot[M_2, T_2]) = [M_1, T_1]\cdot\Gamma[M_2, T_2] + \Gamma[M_1, T_1]\cdot\varepsilon[M_2, T_2].$$

PROOF. By (1.1), we need to compare bordism classes of fixed-point bundles. Let (M_i, T_i) have fixed-point sets F_i with normal bundles ν_i. The fixed-point bundle of $((M_1, T_1)\times(M_2, T_2))'$ is

$$((\nu_1\times\nu_2)\oplus o^1)\cup o^1 \quad\text{over}\quad (F_1\times F_2)\cup(M_1\times M_2).$$

The fixed-point bundle of $(M_1, T_1)\times(M_2', T_2')$ is

$$((\nu_1\times\nu_2)\oplus o^1)\cup(\nu_1\times o^1) \quad\text{over}\quad (F_1\times F_2)\cup(F_1\times M_2).$$

The fixed-point bundle of $(M_1', T_1')\times(M_2, \text{trivial})$ is

$$(\nu_1\oplus o^1)\cup o^1 \quad\text{over}\quad (F_1\times M_2)\cup(M_1\times M_2).$$

The theorem now follows if we take bordism classes.

1.4′. COROLLARY. $\tilde{J}:\mathscr{I}\to\mathscr{N}[[\theta]]$ *is a ring homomorphism.*

This is an easy consequence of a fomal manipulation with coefficients of power series.

3. $Z_2 \otimes_{\mathscr{N}} \mathscr{I}$. Let Z_2 be the trivial $\mathscr{N}$-module concentrated in grading 0. For an $\mathscr{N}$-module $\mathscr{R}$, let $\hat{\mathscr{R}} = Z_2 \otimes_{\mathscr{N}} \mathscr{R}$ which is a Z_2-module. If $f: \mathscr{R}_1 \to \mathscr{R}_2$ is a map of $\mathscr{N}$ modules, let $\hat{f} \hat{\mathscr{R}}_1 \to \hat{\mathscr{R}}_2$ be the induced Z_2 map.

Since all terms in (1.1) are free $\mathscr{N}$-modules, we have an exact sequence

$$(3.1) \qquad 0 \longrightarrow \hat{\mathscr{I}} \xrightarrow{\hat{v}} \hat{\mathscr{M}} \xrightarrow{\hat{J}} (\mathscr{N}_*(P^\infty))^\wedge \longrightarrow 0.$$

Since $(\mathscr{N}_*(X))^\wedge = H_*(X)$ $(=H_*(X;Z_2))$ for each space X, we get the exact sequence

$$(3.2) \qquad 0 \longrightarrow \hat{\mathscr{I}}_n \xrightarrow{\hat{v}} \sum_{j=0}^{n} H_j(BO(n-j)) \xrightarrow{\hat{J}} H_{n-1}(P^\infty) \longrightarrow 0.$$

Let $\iota_*: H_*(BO(r)) \to H_*(BO(r+1))$ be induced by the natural inclusion. The next two results are immediate consequences of Proposition (2.1).

3.3. PROPOSITION. $\hat{v}\hat{\Gamma} = \iota_* \hat{v}$ in positive gradings.

3.4. COROLLARY. $\hat{\Gamma}$ is injective in positive gradings.

Let $[M, T]$ denote the class of (M, T) in $\hat{\mathscr{I}}$.

3.5. COROLLARY. $[M, T] \in$ image $\hat{\Gamma}$ implies $\hat{v}[M, T] \in$ image ι_*.

$\hat{v}: \hat{\mathscr{I}} \to \hat{\mathscr{M}}$ is the following map: Let F_i be the i-dimensional fixed-point set of (M^n, T). Let $\{F_i\} \in H_i(F_i)$ be the fundamental class, and let $v_i: F_i \to BO(n-i)$ be the normal bundle of $F_i \subset M^n$. Then $\hat{v}[M, T] = \sum (v_i)_* \{F_i\}$.

Since $\hat{\mathscr{I}}$ is a two-sided $\mathscr{N}$ module, $\hat{\mathscr{I}}$ inherits a ring structure. Further, the sum of bundles induces maps

$$H_j(BO(n-j)) \otimes H_k(BO(m-k)) \to H_{j+k}(BO(m+n-j-k));$$

i.e., $\hat{\mathscr{M}}_n \otimes \hat{\mathscr{M}}_m \to \hat{\mathscr{M}}_{m+n}$. Thus $\hat{\mathscr{M}}$ is a graded ring. The map $\hat{v}$ is a ring homomorphism. Let y_j be the nonzero element in $H_j(P^\infty)$, let

$$\lambda: H_j(P^\infty) = H_j(BO(1)) \subset \hat{\mathscr{M}}_{j+1}$$

be the inclusion, and let $x_j = \lambda y_{j-1}$. Since the y_j $(j \geq 1)$ are algebraically independent in $H_*(BO)$, the x_j $(j \geq 1)$ are algebraically independent in $\hat{\mathscr{M}}$.

3.6. PROPOSITION. $\hat{\mathscr{M}} = Z_2[x_1, x_2, \cdots]$.

PROOF. The remarks above establish that $Z_2[x_1, x_2, \cdots]$ is a subring of $\hat{\mathscr{M}}$. The result is established once we count the dimensions (over Z_2) of

the two sides. Let $\pi(r)$ be the number of partitions of the integer r. We claim

(3.7) $$\dim \mathcal{M}_r = \pi(r);$$

this will finish the proof.

Let $\pi(r; k)$ denote the number of partitions of r with largest element precisely k. Formula (3.7) follows immediately from the following three facts (the verifications of which we leave to the reader):

(1) $\pi(r) = \sum_{k=1}^{r} \pi(r; k)$,

(2) $\pi(r; k) = \pi_k(r-k)$, where $\pi_k(n)$ is the number of partitions of n with elements at most k,

(3) $\dim_{Z_2} H_j(BO(n-j)) = \pi_{n-j}(j)$.

1.2′. THEOREM. $\{1\} \cup \{\hat{\Gamma}^n(P_0^{n_1} \times \cdots \times P_0^{n_r}) | n \geq 0, r \geq 1, n_i \geq 2\}$ form a Z_2-basis of $\mathscr{I}$.

PROOF. First we establish that these elements are linearly independent. It is sufficient, by Corollary (3.4), to establish that $\{P_0^2, P_0^3, \cdots\}$ are algebraically independent. The fixed point set of P_0^n is the set $P^{n-1} \cup P^0$ with normal bundle $\xi_{n-1} \cup o^n$ over $P^{n-1} \cup P^0$. Here ξ_{n-1} is the canonical bundle. Thus $\tilde{v}P_0^n = x_n + x_1^n$. By Proposition (3.6), $\{P_0^2, P_0^3, \cdots\}$ are algebraically independent.

Now we count the number of elements and compare it with the number required by (3.2) and (3.7). We use generating functions in $Z[[x]]$. Let $d_n(m)$ denote the number of elements

$$\{\hat{\Gamma}^m[P_0^{n_1} \times \cdots \times P_0^{n_r}] \mid r > 0, n_i \geq 2\}$$

in $\mathscr{I}$, and let $d_n = \sum_{m=0}^{\infty} d_n(m)$.

Clearly,

$$1 + \sum_{n=1}^{\infty} d_n(0) x^n = \prod_{r \geq 2} \frac{1}{1 - x^r}.$$

Thus

$$\sum_{n=0}^{\infty} d_n(m) x^n = x^m \prod_{r \geq 2} \frac{1}{1 - x^r} - x^m,$$

and

$$1 + \sum_{n=1}^{\infty} d_n x^n = (1 + x + x^2 + \cdots) \prod_{r \geq 2} \frac{1}{1 - x^r} - (x + x^2 + x^3 + \cdots)$$

$$= \prod_{r \geq 1} \frac{1}{1 - x^r} - \frac{x}{1 - x} = 1 + \sum_{n=1}^{\infty} (\pi(n) - 1) x^n.$$

The theorem is proved.

Using this result, we can improve some earlier results.

Note that since $\hat{\Gamma}(a \cdot b) = a \cdot \hat{\Gamma}(b)$ (from (1.3)), $\hat{\Gamma}\mathscr{I}$ is an ideal in $\mathscr{I}$. The most elemental collection of elements seems to be $\mathscr{I}/\hat{\Gamma}\mathscr{I}$.

3.8. COROLLARY. $\mathscr{I}/\hat{\Gamma}\mathscr{I} = Z_2[P_0^2, P_0^3, \cdots]$.

3.9. COROLLARY. $[M, T] \in \hat{\Gamma}\mathscr{I}$ *if and only if* $\nu[M, T] \in$ *image* ι_*.

4. **Connection with Boardman's results.**   First we give a definition of Boardman's map J'. The diagram

$$
\begin{array}{ccccc}
\mathscr{N}_i(BO(n+1)) & \xrightarrow{\ \ J\ \ } & \mathscr{N}_{i+n}(BO(1)) & = & \mathscr{N}_{i+n}(P^{i+n}) \\
\uparrow{\iota_*} & & \downarrow{\Delta} & & \downarrow{\iota_!} \\
\mathscr{N}_i(BO(n)) & \longrightarrow & \mathscr{N}_{i+n-1}(BO(1)) & = & \mathscr{N}_{i+n-1}(P^{i+n-1})
\end{array}
$$

(4.1)

is commutative [3, Theorem 26.4]. Here Δ is defined as in [3], and $\iota_!$ is the bordism dual to the map ι^* induced in cobordism by inclusion. If for $N \geq r$, $\kappa_{r,N} : P^r \to P^N$ is the natural inclusion, then

(4.2)
$$
\iota_![\kappa_{r,N}] = 0 \qquad\qquad (r = 0),
$$
$$
= [\kappa_{r-1,N-1}] \qquad (r > 0).
$$

The $\{[\kappa_{r,N}] \,|\, 0 \leq r \leq N\}$ constitute a free $\mathscr{N}$-basis for $\mathscr{N}(P^N)$. Also, if $\theta_N = [\kappa_{N-1,N}]$, the intersection product is given by

(4.3)
$$
[\kappa_{r,N}] = \theta_N^{N-r} \qquad (0 \leq r \leq N).
$$

For $[\eta] \in \mathscr{N}_i(BO(n))$, the formula $J\iota_*^m[\eta] = \sum_{r=0}^N \alpha_{N-r,m}[\kappa_{r,N}]$ holds for a unique collection of $\alpha_{N-r,m}$ in $\mathscr{N}_{N-r}$ ($N = i+n+m-1$). By (4.1) and (4.2), $\alpha_{N-r,m} = \alpha_{N-r-1,m-1}$, hence we can let $\alpha_r = \alpha_{N-r,m} \in \mathscr{N}_r$. Using (4.3), we obtain the formula

$$
J\iota_*^m[\eta] = \sum_{r=0}^N \alpha_r \theta_N^r.
$$

Let $m \to \infty$. Let $[\eta]_\infty = \text{inj lim } \iota_*^m[\eta]$ and $\theta = \text{proj lim } \theta_N$. Then

$$
J'[\eta]_\infty = \sum_{r=0}^\infty \alpha_r \theta^r
$$

defines Boardman's J'.

1.5'. THEOREM. $\tilde{J} = J'\iota_* \nu$.

PROOF.   We must establish that for each (M^n, T) and $m \geq 1$,

(4.4)
$$
J\iota_*^m \nu[M, T] = \sum_{r=0}^{m-1} (\varepsilon\Gamma^r[M, T])[\kappa_{m-r-1,n+m-1}].
$$

For $m=1$, (4.4) is true by [3, Theorem 24.2]. Note that the pair (M^{n+1}, T_3) constructed in the proof of [3, Theorem 24.2] is simply the (M', T') of §2. Thus, iterating $(m - 1)$ times the proof of [3, Theorem 24.2], we establish (4.4) for m. Hence Theorem 1.5' is established.

There is another, more geometric connection between $\tilde{J}[M, T]$ and $\nu[M, T]$. Let ν be the normal bundle over the fixed-point set F of (M^n, T). Let $RP_r(\nu)$ be the total space of the associated projective bundle of $\nu \oplus o^r$. The element $[RP_r(\nu)] \in \mathcal{N}_{r+n-1}$ is well determined by the element $[M, T] \in \mathscr{I}_n$. Let

$$K[M, T] = \sum_{r=0}^{\infty} [RP_r(\nu)]\theta^{r+n-1} \in \mathcal{N}[[\theta]].$$

Let

$$P = \sum_{r=0}^{\infty} [RP^{2r}]\theta^{2r} \in \mathcal{N}[[\theta]].$$

PROPOSITION. $\tilde{J} = P^{-1} \cdot K$.

PROOF. It is an immediate consequence of [4, Theorem 4.1].

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