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Algebraic K-theory of generalized free products, Part 2

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III. Decomposition theorems for K-theory

10. *A fibration.* This section gives the main step towards relating the K-theory of a ring to that of its constituent rings, in any of the situations of Sections 1-3. The result is Proposition 10.1 below which relates the K-theory of the ring to the K-theories of the auxiliary categories of 'Mayer Vietoris splittings' and 'split modules', respectively.

We denote $R \mapsto \mathcal{P}_R$ the functor which to the ring R associates a suitable small category equivalent to the category of finitely generated projective right R-modules, cf. 6.2. In either of the three cases of Sections 1-3 (generalized free products, Laurent extensions, polynomial extensions) we let $\mathfrak{MV}$ be the category of *admissible Mayer Vietoris presentations*, that is, the full subcategory of those Mayer Vietoris presentations (cf. Sections 1-3) which satisfy the fact that any of the modules involved is in the appropriate category $\mathcal{P}_?$. For example, in the free product case, the Mayer Vietoris presentation

$$0 \longrightarrow M \longrightarrow M_A \otimes_A R \oplus M_B \otimes_B R \longrightarrow M_C \otimes_C R \longrightarrow 0$$

is in $\mathfrak{MV}$ if and only if $M \in \mathcal{P}_R$, $M_A \in \mathcal{P}_A$, etc.; $\mathfrak{MV}$ is an exact category in an evident way: a sequence is exact if and only if the induced sequences in $\mathcal{P}_R$, $\mathcal{P}_A$, etc., are exact. We have an exact forgetful map

$$f\colon \mathfrak{MV} \longrightarrow \mathcal{P}_R$$

(and more such maps $f_A\colon \mathfrak{MV} \to \mathcal{P}_A$, etc.).

Notation. $\mathcal{P}_R^*$ is the full subcategory of those modules that are in the image of the forgetful map $f\colon \mathfrak{MV} \to \mathcal{P}_R$.

$\mathcal{P}_R^*$ is a cofinal subcategory of $\mathcal{P}_R$ as, for example, it contains the free modules. In fact, it is a strongly cofinal subcategory, cf. Proposition 7.4, hence the sequence $Q\mathcal{P}_R^* \to Q\mathcal{P}_R \to \mathcal{G}$ is a fibration up to homotopy, where

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$G = \operatorname{coker}(K_0(\mathcal{P}_R^*) \to K_0(R))$, and $\mathcal{G}$ is G considered as a category.

We let $\mathcal{V}$ be the exact category of *admissible split modules*, a certain subcategory of the category of split modules (cf. Sections 1-3). We can identify $\mathcal{V}$ to the subcategory $f^{-1}(0)$ of $\mathfrak{M}\mathcal{V}$, where 0 is the distinguished zero object of $\mathcal{P}_R$.

PROPOSITION 10.1. *The sequence $Q\mathcal{V} \to Q\mathfrak{M}\mathcal{V} \to Q\mathcal{P}_R^*$ is a fibration up to homotopy.*

The proof will be given after the statement of the next lemma. Let $\mathcal{F}_R$ denote the full subcategory of free modules in $\mathcal{P}_R$, and $\mathfrak{M}\mathcal{V}'$ the subcategory $f^{-1}(\mathcal{F}_R)$ of $\mathfrak{M}\mathcal{V}$.

LEMMA 10.2. *The sequence $Q\mathcal{V} \to Q\mathfrak{M}\mathcal{V}' \to Q\mathcal{F}_R$ is a fibration up to homotopy.*

The proof will be given at the end of this section.

Proof of Proposition 10.1. If $U \in \mathfrak{M}\mathcal{V}$, and $\xrightarrow{\bar{u}} f(U)$ is a surjection from an object of $\mathcal{F}_R$, it follows from Propositions $i.1$ and $i.2$, $1 \leqq i \leqq 3$, whichever applies, that there is a surjection $\xrightarrow{u} U$ in $\mathfrak{M}\mathcal{V}$ with $f(u) = \bar{u}$. Then $U \oplus \ker(u) \in \mathfrak{M}\mathcal{V}'$. From this and the fact that $\mathcal{F}_R$ is strongly cofinal in $\mathcal{P}_R^*$, one sees that $\mathfrak{M}\mathcal{V}'$ is strongly cofinal in $\mathfrak{M}\mathcal{V}$; also that

$$\operatorname{coker}(K_0(\mathfrak{M}\mathcal{V}') \longrightarrow K_0(\mathfrak{M}\mathcal{V})) \longrightarrow \operatorname{coker}(K_0(\mathcal{F}_R) \longrightarrow K_0(\mathcal{P}_R^*))$$

is an isomorphism. Consequently by Proposition 7.4, the right hand square in the diagram

$$\begin{array}{ccccc} Q\mathcal{V} & \longrightarrow & Q\mathfrak{M}\mathcal{V}' & \longrightarrow & Q\mathcal{F}_R \\ \downarrow & & \downarrow & & \downarrow \\ Q\mathcal{V} & \longrightarrow & Q\mathfrak{M}\mathcal{V} & \longrightarrow & Q\mathcal{P}_R^* \end{array}$$

is homotopy cartesian. The upper row is a fibration up to homotopy, by Lemma 10.2, therefore so is the lower row, as asserted.

Notation. $\mathcal{S}$ is the full subcategory of those objects in $\mathfrak{M}\mathcal{V}$ that are 'standard,' i.e., isomorphic to a direct sum of Mayer Vietoris presentations of the type $\langle N, n, \Delta\rangle$ as defined in Sections 1-3. From Propositions (1-3).1 we have

Fact 10.3. If $U \in \mathcal{S}$, and $\xrightarrow{u} U \xrightarrow{v}$ are morphisms in $\mathfrak{M}\mathcal{V}$, then v is the zero map if and only if $f(v)$ is; consequently u is surjective if and only if $f(u)$ is.

Definition. $\mathcal{S}\mathcal{V}$, the category of 'semi-standard' Mayer Vietoris presentations, has $\mathrm{Ob}(\mathcal{S}\mathcal{V}) = \mathrm{Ob}(\mathcal{S}) \times \mathrm{Ob}(\mathcal{V})$, and if $W_i = (U_i, V_i) \in \mathcal{S}\mathcal{V}$, $i = 1, 2$, then a morphism from W_1 to W_2 is a morphism in $\mathfrak{M}\mathcal{V}$ from $U_1 \oplus V_1$ to $U_2 \oplus V_2$. So $W \mapsto U \oplus V$ gives an equivalence with a full subcategory of $\mathfrak{M}\mathcal{V}$. On the other hand, the component $U_1 \to V_2$ of the morphism above, is necessarily the zero map, by fact 10.3. Hence

Fact 10.4. $\mathcal{S}\mathcal{V}$ is equivalent to the category of short exact sequences in $\mathfrak{M}\mathcal{V}$,

$$U \rightarrowtail U \oplus V \twoheadrightarrow V\,, \quad U \in \mathcal{S}\,, \quad V \in \mathcal{V}\,,$$

Also, the map $U_1 \oplus V_1 \to U_2 \oplus V_2$ is surjective if and only if both $U_1 \to U_2$ and $V_1 \to V_2$ are, by fact 10.3 again.

Following the notation of Section 8, we denote $\mathcal{E}(\mathcal{S}, \mathcal{V})$ the category of epimorphisms in $\mathcal{S}$ with kernel in $\mathcal{V}$. Letting $\mathrm{Is}(\mathcal{F}_R)$ be the groupoid of isomorphisms in $\mathcal{F}_R$, we have a map $f'\colon \mathcal{E}(\mathcal{S}, \mathcal{V}) \to \mathrm{Is}(\mathcal{F}_R)$, induced from the forgetful map f.

LEMMA 10.5. *The map* $f'\colon \mathcal{E}(\mathcal{S}, \mathcal{V}) \to \mathrm{Is}(\mathcal{F}_R)$ *is a homotopy equivalence.*

Proof. It suffices to show that the left fibre f'/M is contractible for any $M \in \mathrm{Is}(\mathcal{F}_R)$. An object X of f'/M consists of an object S of $\mathcal{S}$ together with an isomorphism $x\colon f'(S) \to M$. A morphism in f'/M from X to X' is a morphism $s\colon S \to S'$ in $\mathfrak{M}\mathcal{V}$ so that $x = x' \circ f'(s)$; by 10.3, the extra condition that s be a surjection with kernel in $\mathcal{V}$ is then automatically satisfied.

In the language of Propositions (1–3).1, the object X of f'/M is equivalent to a basis $(n_1, \cdots, n_m)$ of M together with a tuple of Mayer Vietoris presentations $\langle N_j, n_j, \Delta_j\rangle$, $j = 1, \cdots, m$, and by the above, a morphism from X to X' is equivalent to a morphism in $\mathfrak{M}\mathcal{V}$,

$$\bigoplus \langle N_j, n_j, \Delta_j\rangle \longrightarrow \bigoplus \langle N'_j, n'_j, \Delta'_j\rangle$$

inducing the identity map on M. By Propositions (1–3).1, such a morphism exists if and only if, for any $j = 1, \cdots, m$, the tree Δ_j contains a certain finite tree $\Delta(n_j, X')$, and if it exists, the morphism is unique. It follows that f'/M is equivalent to the opposite category of an ordered set; consequently it is contractible, as asserted.

Again in the notation of Section 8, we have a category $\mathcal{E}(\mathfrak{M}\mathcal{V}', \mathcal{V})$ and a subcategory $\mathcal{E}(\mathcal{S}\mathcal{V}, \mathcal{V})$ (actually, it is only equivalent to a subcategory).

LEMMA 10.6. *The inclusion* $j\colon \mathcal{E}(\mathcal{S}\mathcal{V}, \mathcal{V}) \to \mathcal{E}(\mathfrak{M}\mathcal{V}', \mathcal{V})$ *is a homotopy equivalence.*

Proof. It suffices to show that, for any $U \in \mathcal{E}(\mathfrak{M}\mathcal{V}', \mathcal{V})$, the category j/U is contractible. For this in turn it is enough that any finite diagram in

j/U be contractible in j/U. Let $\mathcal{C}$ be such a finite diagram and let $v_i\colon V_i \to U$ be its objects. By Propositions (1-3).1, there exists $U_1 \in \mathcal{S}$ and a map $u_1\colon U_1 \to U$ so that $f(u_1)$ is an isomorphism, *and so that u_1 factors through any of the v_i*. Finally by Propositions (1-3).2, we can find $U_2 \in \mathcal{V}$ and $u_2\colon U_2 \to U$ so that $U_1 \oplus U_2 \to U$ is surjective. Pullback with $U_1 \oplus U_2 \to U$ now defines a functor on j/U whose restriction to $\mathcal{C}$ takes values in j/U as one easily checks. This functor from $\mathcal{C}$ to j/U admits two natural transformations, one to the embedding functor, and one to the constant functor with value $U_1 \oplus U_2 \to U$. Therefore $\mathcal{C}$ is nullhomotopic in j/U, as asserted.

From the inclusion of exact categories, $\mathcal{V} \to \mathfrak{M}\mathcal{V}'$, we have a bicategory $Q^{\mathrm{ep}}(\mathfrak{M}\mathcal{V}', \mathcal{V})$, cf. Section 7. Similarly there is a bicategory $Q^{\mathrm{ep}}(\mathcal{F}_R, 0)$, and the forgetful map f induces a map $Q^{\mathrm{ep}}(\mathfrak{M}\mathcal{V}', \mathcal{V}) \to Q^{\mathrm{ep}}(\mathcal{F}_R, 0)$.

LEMMA 10.7. *The map $Q^{\mathrm{ep}}(\mathfrak{M}\mathcal{V}', \mathcal{V}) \to Q^{\mathrm{ep}}(\mathcal{F}_R, 0)$ is a homotopy equivalence.*

Proof. This map is in a natural way the induced map of underlying objects, of a map of Γ-objects. By Proposition 6.3 and its addendum, it is sufficient to show that the 'de-loop' of the map in question, the map of simplicial bicategories

$$N_\Gamma\big(Q^{\mathrm{ep}}(\mathfrak{M}\mathcal{V}', \mathcal{V})\big) \longrightarrow N_\Gamma\big(Q^{\mathrm{ep}}(\mathcal{F}_R, 0)\big)$$

is a homotopy equivalence.

Let $N^{(\cdot)}(\mathfrak{M}\mathcal{V}', \mathcal{V})$ denote the simplicial category obtained by taking the nerve in the Q-direction of the bicategory $Q^{\mathrm{ep}}(\mathfrak{M}\mathcal{V}', \mathcal{V})$, and $N^{(\cdot)}(\mathcal{F}_R, 0)$ similarly. In order to establish the homotopy equivalence it is sufficient, by Lemma 5.1, to show that, for each m, the map of simplicial categories

$$N_\Gamma\big(N^{(m)}(\mathfrak{M}\mathcal{V}', \mathcal{V})\big) \longrightarrow N_\Gamma\big(N^{(m)}(\mathcal{F}_R, 0)\big)$$

is a homotopy equivalence.

Following the terminology of Lemma 8.4, we introduce the short hand notation $\mathcal{E}(\mathcal{B}, \mathcal{C})_{(n)}$ for the category $\mathcal{E}(\mathcal{B}_1, \cdots, \mathcal{B}_n; \mathcal{C}_1, \cdots, \mathcal{C}_n)$ in the case when $\mathcal{B}_1 = \cdots = \mathcal{B}_n = \mathcal{B}$ and $\mathcal{C}_1 = \cdots = \mathcal{C}_n = \mathcal{C}$.

An object of the category $N^{(m)}(\mathcal{F}_R, 0)$ is a sequence of morphisms of length m in $Q\mathcal{F}_R$; it is thus equivalent to a filtered object of length $2m+1$ in $\mathcal{F}_R$. Indeed the category $N^{(m)}(\mathcal{F}_R, 0)$ is equivalent to the category $\mathcal{E}(\mathcal{F}_R, 0)_{(2m+1)}$. By Lemma 8.4, the subquotient map induces a homotopy equivalence

$$N_\Gamma\big(\mathcal{E}(\mathcal{F}_R, 0)_{(2m+1)}\big) \longrightarrow \big(N_\Gamma\big(\mathcal{E}(\mathcal{F}_R, 0)\big)\big)^{2m+1} .$$

The analogous result for $N^{(m)}(\mathfrak{M}\mathcal{V}', \mathcal{V})$ requires a bit more work since exact sequences in $\mathfrak{M}\mathcal{V}'$ need not split. We define $\widetilde{\mathcal{E}}(\mathcal{B}, \mathcal{C})_{(n)}$ to be a category of epimorphisms of filtered objects, like $\mathcal{E}(\mathcal{B}, \mathcal{C})_{(n)}$, except that the filtrations involved in the objects need not split. Then $N^{(m)}(\mathcal{M}\mathcal{V}', \mathcal{V})$ is equivalent to $\tilde{\mathcal{E}}(\mathcal{M}\mathcal{V}', \mathcal{V})_{(2m+1)}$. Furthermore we define $\tilde{\mathcal{E}}'(\mathcal{B}, \mathcal{C})_{(n)}$ to be the category in which an object is an object of $\tilde{\mathcal{E}}(\mathcal{B}, \mathcal{C})_{(n)}$ together with an admissible splitting of the *last* admissible monomorphism of the filtration involved in the object. Morphisms in $\tilde{\mathcal{E}}'(\mathcal{B}, \mathcal{C})_{(n)}$ are *not* required to respect the splitting. The obvious forgetful map is therefore an equivalence with a full subcategory of $\tilde{\mathcal{E}}(\mathcal{B}, \mathcal{C})_{(n)}$.

SUBLEMMA. *The map* $\tilde{\mathcal{E}}'(\mathcal{M}\mathcal{V}', \mathcal{V})_{(n)} \to \tilde{\mathcal{E}}(\mathcal{M}\mathcal{V}', \mathcal{V})_{(n)}$ *is a homotopy equivalence.*

Proof. It suffices to show that for any $M \in \tilde{\mathcal{E}}(\mathcal{M}\mathcal{V}', \mathcal{V})_{(n)}$, the left fibre j/M is contractible, where j denotes the map in question. Let the object M be given by the filtration $M_1 \rightarrowtail \cdots \rightarrowtail M_{n-1} \rightarrowtail M_n$. By Propositions (1-3).1 and (1-3).2, there exist $N \in \mathcal{S}\mathcal{V}$ and a map $N \to M_n$ so that the composite map $N \to M_n/M_{n-1}$ is an epimorphism with kernel in $\mathcal{V}$. On replacing M_n by $M_n' = M_{n-1} \oplus N$, we obtain an object M' of $\tilde{\mathcal{E}}'(\mathcal{M}\mathcal{V}', \mathcal{V})_{(n)}$ together with an obvious map $j(M') \to M$ in $\tilde{\mathcal{E}}(\mathcal{M}\mathcal{V}', \mathcal{V})_{(n)}$. Pullback with this map gives a functor $j/M \to j/M$. There are two natural transformations of this functor, one to the identity functor, and one to the constant functor with value $(M', j(M') \to M)$. The category j/M is thus contractible, and the proof of the sublemma is complete.

In view of Lemma 8.1 and the sublemma we have that

$$N_\Gamma\big(\tilde{\mathcal{E}}(\mathcal{M}\mathcal{V}', \mathcal{V})_{(n)}\big) \longrightarrow N_\Gamma\big(\tilde{\mathcal{E}}(\mathcal{M}\mathcal{V}', \mathcal{V})_{(n-1)} \times \mathcal{E}(\mathcal{M}\mathcal{V}', \mathcal{V})\big)$$

is a homotopy equivalence and hence, by induction, that the subquotient map induces a homotopy equivalence $N_\Gamma\big(\tilde{\mathcal{E}}(\mathcal{M}\mathcal{V}', \mathcal{V})_{(n)}\big) \to \big(N_\Gamma\big(\mathcal{E}(\mathcal{M}\mathcal{V}', \mathcal{V})\big)\big)^n$.

To sum up, we have now verified that the map

$$N_\Gamma\big(N^{(m)}(\mathcal{M}\mathcal{V}', \mathcal{V})\big) \longrightarrow N_\Gamma\big(N^{(m)}(\mathcal{F}_R, 0)\big)$$

is homotopy equivalent to the map

$$\big(N_\Gamma\big(\mathcal{E}(\mathcal{M}\mathcal{V}', \mathcal{V})\big)\big)^{2m+1} \longrightarrow \big(N_\Gamma\big(\mathcal{E}(\mathcal{F}_R, 0)\big)\big)^{2m+1}$$

and consequently, that it is enough to show that $N_\Gamma\big(\mathcal{E}(\mathcal{M}\mathcal{V}', \mathcal{V})\big) \to N_\Gamma\big(\mathcal{E}(\mathcal{F}_R, 0)\big)$ is a homotopy equivalence.

In view of Lemmas 10.5 and 10.6 and the commutative diagram

$$\begin{array}{ccccc} \mathcal{E}(\mathcal{S}, \mathcal{V}) & \longrightarrow & \mathcal{E}(\mathcal{S}\mathcal{V}, \mathcal{V}) & \longrightarrow & \mathcal{E}(\mathcal{M}\mathcal{V}', \mathcal{V}) \\ & \searrow & \downarrow & \swarrow & \\ & & \operatorname{Is}(\mathcal{F}_R) = \mathcal{E}(\mathcal{F}_R, 0)\,, & & \end{array}$$

we may show instead that $N_\Gamma(\mathcal{E}(\mathcal{S}, \mathcal{V})) \to N_\Gamma(\mathcal{E}(\mathcal{S}\mathcal{V}, \mathcal{V}))$ is a homotopy equivalence. But by 10.4, $\mathcal{S}\mathcal{V}$ is equivalent to a category of filtered objects, and $\mathcal{E}(\mathcal{S}\mathcal{V}, \mathcal{V})$ is equivalent to the category $\mathcal{E}(\mathcal{S}, \mathcal{V}; \mathcal{V}, \mathcal{V})$ (cf. Section 8). So by Lemma 8.1, the subquotient map on the latter category induces a homotopy equivalence $N_\Gamma(\mathcal{E}(\mathcal{S}\mathcal{V}, \mathcal{V})) \to N_\Gamma(\mathcal{E}(\mathcal{S}, \mathcal{V})) \times N_\Gamma(\mathcal{E}(\mathcal{V}, \mathcal{V}))$. Disregarding $N_\Gamma(\mathcal{E}(\mathcal{V}, \mathcal{V}))$ which is contractible since $\mathcal{E}(\mathcal{V}, \mathcal{V})$ is, we see that the map $N_\Gamma(\mathcal{E}(\mathcal{S}, \mathcal{V})) \to N_\Gamma(\mathcal{E}(\mathcal{S}\mathcal{V}, \mathcal{V}))$ is a section of this homotopy equivalence, and so is a homotopy equivalence itself, completing the proof.

Proof of lemma 10.2. The left hand square in the diagram

$$\begin{array}{ccccc}
Q\mathcal{V} & \longrightarrow & Q^{\mathrm{ep}}(\mathcal{V}, \mathcal{V}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
Q\mathfrak{M}\mathcal{V}' & \longrightarrow & Q^{\mathrm{ep}}(\mathfrak{M}\mathcal{V}', \mathcal{V}) & \longrightarrow & Q^{\mathrm{ep}}(\mathcal{F}_R, 0)
\end{array}$$

is homotopy cartesian by Proposition 7.3, and in the right hand square both horizontal maps are homotopy equivalences, the upper one by a remark preceding Proposition 7.3, and the lower one by Lemma 10.7. Hence the large square is homotopy cartesian. This means that the lower row in the following diagram is a fibration up to homotopy

$$\begin{array}{ccccc}
Q\mathcal{V} & \longrightarrow & Q\mathfrak{M}\mathcal{V}' & \longrightarrow & Q\mathcal{F}_R \\
\downarrow \| & & \downarrow \| & & \downarrow \\
Q\mathcal{V} & \longrightarrow & Q\mathfrak{M}\mathcal{V}' & \longrightarrow & Q^{\mathrm{ep}}(\mathcal{F}_R, 0) .
\end{array}$$

The right hand vertical map in this diagram is a homotopy equivalence by another remark preceding Proposition 7.3. Hence the upper row is a fibration up to homotopy, as asserted.

11. *Decomposition theorems in the generalized free product case.* Let R be given as the free product of $\alpha: C \to A$ and $\beta: C \to B$ where α and β are both pure and their complements are free from the left, as in Section 1. These data will be assumed throughout this section. The rings and maps involved can be collected in a commutative diagram

$$\begin{array}{ccc}
C & \xrightarrow{\alpha} & A \\
{\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\bar{\alpha}} \\
B & \xrightarrow{\bar{\beta}} & R .
\end{array}$$

The categories $\mathcal{P}_?$ are as in 6.2. In particular if $M_C \in \mathcal{P}_C$, the modules $(M_C \otimes_C A) \otimes_A R$ and $M_C \otimes_C R$ are not just canonically isomorphic, they are identical.

$f: \mathfrak{M}\mathcal{V} \to \mathcal{P}_R$ is the forgetful map of the preceding section; $f_A: \mathfrak{M}\mathcal{V} \to \mathcal{P}_A$

is the forgetful map which to the Mayer Vietoris presentation

$$0 \longrightarrow M \longrightarrow M_A \otimes_A R \oplus M_B \otimes_B R \longrightarrow M_C \otimes_C R \longrightarrow 0$$

associates the A-module M_A, and f_B, f_C are similar.

Proposition 11.1. *The forgetful map* $Q\mathfrak{MV} \to Q\mathcal{P}_A \times Q\mathcal{P}_B \times Q\mathcal{P}_C$ *is a homotopy equivalence.*

Proof. We will use sections of the above forgetful maps. $s_A: \mathcal{P}_A \to \mathfrak{MV}$ associates to M_A the Mayer Vietoris presentation

$$0 \longrightarrow (M_A \otimes_A R) \xrightarrow{=} M_A \otimes_A R \longrightarrow 0 \longrightarrow 0$$

and $s_B: \mathcal{P}_B \to \mathfrak{MV}$ is similar; $s_C: \mathcal{P}_C \to \mathfrak{MV}$ associates to M_C the Mayer Vietoris presentation

$$0 \longrightarrow (M_C \otimes_C R) \xrightarrow{\Delta} (M_C \otimes_C A) \otimes_A R \oplus (M_C \otimes_C B) \otimes_B R \xrightarrow{\tilde{\Delta}} M_C \otimes_C R \longrightarrow 0$$

where Δ is the diagonal and $\tilde{\Delta}$ the skew codiagonal.

There is a natural transformation from the identity functor on $\mathfrak{MV}$ to any of the functors $s_A \circ f_A$, $s_B \circ f_B$, $s_C \circ f_C$. The latter natural transformation is given by the diagram

$$\begin{array}{ccccccc} 0 \longrightarrow & M & \xrightarrow{\iota} & M_A \otimes_A R \;\oplus\; M_B \otimes_B R & \xrightarrow{\kappa} & M_C \otimes_C R & \longrightarrow 0 \\ & \downarrow{\scriptstyle \kappa_\alpha \circ \iota} & & \downarrow{\scriptstyle \kappa_\alpha} \qquad\qquad \downarrow{\scriptstyle \kappa_\beta} & & \downarrow & \\ 0 \longrightarrow & (M_C \otimes_C R) & \xrightarrow{\Delta} & (M_C \otimes_C A) \otimes_A R \oplus (M_C \otimes_C B) \otimes_B R & \xrightarrow{\tilde{\Delta}} & M_C \otimes_C R & \longrightarrow 0 \end{array}$$

where κ_α and κ_β are the terms in the canonical decomposition $\kappa = \kappa_\alpha - \kappa_\beta$; note that the A-map inducing κ_α is unique, similarly with κ_β, so the natural transformation is well defined. The two former natural transformations are obvious.

We denote $\mathfrak{MV}''$ the full subcategory of $\mathfrak{MV}$ whose objects satisfy the fact that the natural transformation from the identity to $s_C \circ f_C$ is surjective. It is closed under extensions in $\mathfrak{MV}$, and the sections s_A, s_B, s_C can all be considered as maps to $\mathfrak{MV}''$. By definition of $\mathfrak{MV}''$ there is an exact sequence of endofunctors

$$0 \longrightarrow g \longrightarrow \mathrm{Id}_{\mathfrak{MV}''} \longrightarrow s_C \circ f_C \longrightarrow 0$$

defining g, and g is isomorphic to $s_A \circ g_A \oplus s_B \circ g_B$ for certain functors $g_A: \mathfrak{MV}'' \to \mathcal{P}_A$ and $g_B: \mathfrak{MV}'' \to \mathcal{P}_B$. By the (3×3)-lemma, g and hence also g_A and g_B are exact functors. These satisfy the fact that $g_A \circ s_A$ is the identity on $\mathcal{P}_A$, that $g_A \circ s_C$ is the zero map, and the like. So the map

$$Q(g_A, g_B, f_C): Q\mathfrak{MV}'' \longrightarrow Q\mathcal{P}_A \times Q\mathcal{P}_B \times Q\mathcal{P}_C$$

is a left inverse of

$$Q(s_A \oplus s_B \oplus s_C): Q\mathcal{P}_A \times Q\mathcal{P}_B \times Q\mathcal{P}_C \longrightarrow Q\mathfrak{M}\mathcal{V}'' .$$

But it is also a right inverse up to homotopy, for the additivity theorem applied to the above sequence of endofunctors shows that the endofunctor on $Q\mathfrak{M}\mathcal{V}''$ induced from $s_A \circ g_A \oplus s_B \circ g_B \oplus s_C \circ f_C$ is homotopic to the identity. Consequently, $Q(s_A \oplus s_B \oplus s_C)$ is a homotopy equivalence when we consider its target to be $\mathfrak{M}\mathcal{V}''$.

On the other hand, the exact subcategory $\mathfrak{M}\mathcal{V}''$ is closed in $\mathfrak{M}\mathcal{V}$ under extensions and admissible quotients; and the natural transformation in $\mathfrak{M}\mathcal{V}$ from the identity functor to the endofunctor $s_A \circ f_A \oplus s_B \circ f_B \oplus s_C \circ f_C$ is an admissible monomorphism with target and quotient in $\mathfrak{M}\mathcal{V}''$. So by the resolution theorem [20] applied to the inclusion of the dual categories (or even by the additivity theorem) it follows that the inclusion $Q\mathfrak{M}\mathcal{V}'' \to Q\mathfrak{M}\mathcal{V}$ is a homotopy equivalence. Consequently, $Q(s_A \oplus s_B \oplus s_C)$ is a homotopy equivalence when we consider its target to be $Q\mathfrak{M}\mathcal{V}$.

The composed map $(f_A, f_B, f_C) \circ (s_A \oplus s_B \oplus s_C)$ can be described by the matrix

$$\begin{pmatrix} \mathrm{Id}_{\mathcal{P}_A} & & \alpha_* \\ & \mathrm{Id}_{\mathcal{P}_B} & \beta_* \\ & & \mathrm{Id}_{\mathcal{P}_C} \end{pmatrix} .$$

Since the H-space $BQ\mathcal{P}_C$, being connected, has a homotopy inverse, the induced map on $Q\mathcal{P}_A \times Q\mathcal{P}_B \times Q\mathcal{P}_C$ is a homotopy equivalence. Consequently $Q(f_A, f_B, f_C)$ is a homotopy equivalence, as asserted.

We denote $\varphi_\alpha, \varphi_\beta: \mathcal{P}_C \to \mathcal{V}$ the two maps which take M_C to

$$(M_C \otimes_C A) \otimes_A R \xrightarrow{=} M_C \otimes_C R , \quad (M_C \otimes_C B) \otimes_B R \xrightarrow{=} M_C \otimes_C R$$

respectively. Combining them we have a map $\varphi = (\varphi_\alpha \oplus \varphi_\beta): \mathcal{P}_C \times \mathcal{P}_C \to \mathcal{V}$. Under the equivalence of $\mathcal{V}$ with a category of nilpotent objects, cf. Propositions 1.3 and 1.4, the map φ corresponds to the section i described just before the statement of Lemma 1.5. From this equivalence of categories we therefore obtain a map

$$\psi: \mathcal{V} \longrightarrow \mathcal{P}_C \times \mathcal{P}_C$$

such that the composed map $\psi \circ \varphi$ is isomorphic to the identity functor on $\mathcal{P}_C \times \mathcal{P}_C$.

THEOREM 11.2. *Suppose the ring C is regular coherent. Then the map*

$$Q\mathcal{P}_C \times Q\mathcal{P}_C \longrightarrow Q\mathcal{V}$$

is a homotopy equivalence.

Proof. Let $\mathfrak{M}_C^{\mathrm{f.p.}}$ be the category of finitely presented C-modules. By assumption about C this is an abelian category in which every object has finite projective dimension. Hence the inclusion $Q\mathcal{P}_C \to Q\mathfrak{M}_C^{\text{f.p.}}$ is a homotopy equivalence by the resolution theorem [20].

Denote $\mathcal{V}^{\text{f.p.}}$ the category of those split modules for which the C-module involved is in $\mathfrak{M}_C^{\text{f.p.}}$. Then $\mathcal{V}^{\text{f.p.}}$ is also an abelian category. (Note if $\mathcal{V}^{\text{f.p.}}$ is made into an exact category by analogy with the procedure for Mayer Vietoris presentations then the same exact structure results.) Further every object of $\mathcal{V}^{\text{f.p.}}$ may be resolved by objects of $\mathcal{V}$. In fact, Proposition 1.2 says such a resolution can be built up; and the process can terminate because objects of $\mathfrak{M}_C^{\text{f.p.}}$ have finite projective dimension. Hence by another application of the resolution theorem, the inclusion $Q\mathcal{V} \to Q\mathcal{V}^{\text{f.p.}}$ is also a homotopy equivalence.

Every object of $\mathcal{V}^{\text{f.p.}}$ has a finite filtration with subquotients in the image of $\mathfrak{M}_C^{\text{f.p.}} \times \mathfrak{M}_C^{\text{f.p.}}$. In fact, Lemma 1.5 provides such a filtration where the C-modules involved are finitely generated; but a finitely generated submodule of a finitely presented C-module is necessarily itself finitely presented as C is coherent. So

$$Q\mathfrak{M}_C^{\text{f.p.}} \times Q\mathfrak{M}_C^{\text{f.p.}} \longrightarrow Q\mathcal{V}^{\text{f.p.}}$$

is a homotopy equivalence by the devissage theorem [20]. We have estalished now that three of the maps in the diagram

$$\begin{array}{ccc} Q\mathcal{P}_C \times Q\mathcal{P}_C & \longrightarrow & Q\mathcal{V} \\ \downarrow & & \downarrow \\ Q\mathfrak{M}_C^{\text{f.p.}} \times Q\mathfrak{M}_C^{\text{f.p.}} & \longrightarrow & Q\mathcal{V}^{\text{f.p.}} \end{array}$$

are homotopy equivalences. If follows that the remaining map is also a homotopy equivalence, as asserted.

It is an interesting question if the conclusion of the theorem can be established with weaker hypotheses on the ring C. The results below show that any deviation of $Q\varphi$ from a homotopy equivalence must contribute in a very direct way to the K-theory of the ring R.

Let $\mathcal{P}_R^*$ be as defined in the preceding section (the modules which are in the image of $f\colon \mathfrak{M}\mathcal{V} \to \mathcal{P}_R$).

THEOREM 11.3. *In the diagram*

$$\begin{array}{ccc} Q\mathcal{V} & \xrightarrow{(Qf_A,\, Qf_B)} & Q\mathcal{P}_A \times Q\mathcal{P}_B \\ {\scriptstyle Qf_C}\downarrow & & \downarrow{\scriptstyle Q(\alpha_* \oplus \beta_*)} \\ Q\mathcal{P}_C & \longrightarrow & Q\mathcal{P}_R^* \end{array}$$

the two composed maps from $Q\mathcal{V}$ to $Q\mathcal{P}_R^*$ differ by the functor isomorphism *which to any object of $Q\mathcal{V}$ associates its structure map. The square is homotopy cartesian with respect to the homotopy of the composed maps given by this functor isomorphism.*

Proof. Let $\mathcal{D}$ be the exact category in which an object is a short exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_C \otimes_C R \longrightarrow 0$$

with $M_1, M_2 \in \mathrm{Ob}(\mathcal{P}_R^*)$ and $M_C \in \mathrm{Ob}(\mathcal{P}_C)$. Let $s, t\colon \mathcal{D} \to \mathcal{P}_R^*$ denote the functors 'subobject' and 'total object', and $f_C\colon \mathcal{D} \to \mathcal{P}_C$ the forgetful map. The (unnamed) map $\mathcal{P}_C \to \mathcal{D}$ associates to M_C the exact sequence

$$0 \longrightarrow 0 \longrightarrow (M_C \otimes_C R) \xrightarrow{=} M_C \otimes_C R \longrightarrow 0\ .$$

There is a natural map $\mathcal{M}\mathcal{V} \to \mathcal{D}$. The resulting square

$$\begin{array}{ccc} Q\mathcal{V} & \longrightarrow & Q\mathcal{M}\mathcal{V} \\ {\scriptstyle Qf_C}\downarrow & & \downarrow \\ Q\mathcal{P}_C & \longrightarrow & Q\mathcal{D} \end{array}$$

is not commutative. But the two composed maps from $Q\mathcal{V}$ to $Q\mathcal{D}$ differ on the total objects of objects of $Q\mathcal{D}$ only, and here the distinction is just as described in the theorem. We assert the square is homotopy cartesian with respect to this homotopy. To see this we consider the diagram

$$\begin{array}{ccccc} Q\mathcal{V} & \longrightarrow & Q\mathcal{M}\mathcal{V} & \longrightarrow & Q\mathcal{P}_R^* \\ \downarrow & & \downarrow & & \| \\ Q\mathcal{P}_C & \longrightarrow & Q\mathcal{D} & & \| \\ \downarrow & & \downarrow{\scriptstyle Q(f_C,\, s)} & & \downarrow \\ Q\mathcal{P}_C & \longrightarrow & Q\mathcal{P}_C \times Q\mathcal{P}_R^* & \longrightarrow & Q\mathcal{P}_R^*\ . \end{array}$$

Discarding the middle row, we obtain a commutative diagram in which the upper row is the fibration of Proposition 10.1, and the lower row is trivially a fibration. Hence the large left hand square is homotopy cartesian. The lower left hand square is commutative, one of its vertical maps is an identity, and the other one is a homotopy equivalence by the additivity theorem. Furthermore the functor isomorphism that measures non-commutativity of the upper left hand square, is not felt by the large left hand square. Consequently, the upper left hand square is homotopy cartesian in the way asserted.

This being established, consider the diagram

$$\begin{array}{ccccccc} Q\mathcal{V} & \longrightarrow & Q\mathcal{M}\mathcal{V} & \xrightarrow{Q(f_C,\, f_A,\, f_B)} & Q\mathcal{P}_C \times (Q\mathcal{P}_A \times Q\mathcal{P}_B) & \longrightarrow & Q\mathcal{P}_A \times Q\mathcal{P}_B \\ \downarrow & & \downarrow & & {\scriptstyle \mathrm{Id}\times}\downarrow{\scriptstyle Q(\bar\alpha_* \oplus \bar\beta_*)} & & \downarrow \\ Q\mathcal{P}_C & \longrightarrow & Q\mathcal{D} & \xrightarrow{Q(f_C,\, t)} & Q\mathcal{P}_C \times Q\mathcal{P}_R^* & \longrightarrow & Q\mathcal{P}_R^* \end{array}$$

in which the left hand square is the one just considered. The middle and right hand squares commute. In the middle square, the horizontal maps are homotopy equivalences, the upper map by Proposition 11.1, and the lower map by the additivity theorem, by the fact that $BQ\mathcal{P}_C$ has a homotopy inverse and that therefore the map

$$Q\mathcal{P}_C \times Q\mathcal{P}^*_R \longrightarrow Q\mathcal{P}_C \times Q\mathcal{P}^*_R\,, \quad (M_C, M) \longmapsto (M_C, M \oplus M_C \otimes_C R)$$

is a homotopy equivalence. In the right hand square, the horizontal maps are projections away from $Q\mathcal{P}_C$. It follows that the large square is homotopy cartesian in the way asserted by the theorem.

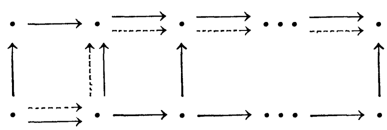
There is a technical variant of Theorem 11.3 that we will need later on. Let $\mathcal{V}.$ be the simplicial exact category which in degree n is $\mathcal{V}_n$, the category equivalent to $\mathcal{V}$ in which an object is a sequence of n composable isomorphisms in $\mathcal{V}$; similarly with $\mathcal{P}_A.$, $\mathcal{P}^*_R.$, etc.

Proposition 11.4. *The non-commutative square of simplicial categories*

$$\begin{array}{ccc} Q\mathcal{V}. & \longrightarrow & Q\mathcal{P}_A. \times Q\mathcal{P}_B. \\ \downarrow & & \downarrow \\ Q\mathcal{P}_C. & \longrightarrow & Q\mathcal{P}^*_R. \end{array}$$

is homotopy cartesian with respect to the simplicial homotopy which to any object of $\mathcal{V}$ associates its structure map.

Proof. An object of $\mathcal{V}_n$ determines a commutative diagram in $\mathcal{P}^*_R$



where each row has n arrows, and the vertical arrows are the structure maps of the objects of $\mathcal{V}$ involved. One can obtain from this diagram a sequence of $n+1$ objects in $\mathcal{P}^*_{R(n+1)}$ as indicated by the broken arrows in one case. By definition, this sequence gives the value of the simplicial homotopy on the object of $\mathcal{V}_n$ in question. It is clear that the homotopy described is indeed a simplicial homotopy between the two composed maps from $Q\mathcal{V}.$ to $Q\mathcal{P}^*_R.$; we have to verify that the square is homotopy cartesian with respect to this homotopy.

Considering $Q\mathcal{V}$ as a simplicial category in a trivial way, we can consider it as a simplicial subcategory of $Q\mathcal{V}.$, and the inclusion is a homotopy equivalence. It is thus sufficient to compare the homotopy of Theorem 11.3 to the restriction to $Q\mathcal{V}$ of the present homotopy. The two homotopies are not the same exactly, as one is simplicial and the other one is in the category direction. But the two homotopies will indeed become identical when we pass to nerves and take the diagonal simplicial sets of these. This establishes the proposition.

Combining Theorems 11.2 and 11.3 we have

COROLLARY 11.5. *If C is regular coherent, there is a commutative homotopy cartesian square*

$$\begin{array}{ccc} Q\mathcal{P}_C \times Q\mathcal{P}_C & \xrightarrow{Q\alpha_* \times Q\beta_*} & Q\mathcal{P}_A \times Q\mathcal{P}_B \\ \big\downarrow {\scriptstyle Q(\mathrm{Id} \oplus \mathrm{Id})} & & \big\downarrow {\scriptstyle Q(\bar{\alpha}_* \oplus \bar{\beta}_*)} \\ Q\mathcal{P}_C & \longrightarrow & Q\mathcal{P}_R^* . \end{array}$$

Remark. The appearance of $Q\mathcal{P}_R^*$ rather than $Q\mathcal{P}_R$ in this corollary is a bit unsatisfying. Its significance is the following. In view of the connectedness of the bottom terms in the square, the square is homotopy cartesian if and only if the map of homotopy theoretic fibres of the vertical maps is a homotopy equivalence. Hence the homotopy fibration associated to the right vertical map gives the following long exact sequence of homotopy groups

$$\cdots \longrightarrow K_1(R) \longrightarrow K_0(C) \longrightarrow K_0(A) \oplus K_0(B) \longrightarrow K_0(R)$$

in view of Proposition 7.4. Replacing $Q\mathcal{P}_R^*$ by $Q\mathcal{P}_R$ in the corollary would amount to the additional assertion that $K_0(A) \oplus K_0(B) \to K_0(R)$ is surjective. One can achieve this at the expense of a stronger hypothesis in the corollary. Namely assume that not just C but also the (ordinary) polynomial and Laurent polynomial ring on C are regular coherent. The *contracted functor* device of Bass [3] gives a continuation of the above sequence to the right, keeping it exact,

$$\longrightarrow K_0(A) \oplus K_0(B) \longrightarrow K_0(R) \longrightarrow K_{-1}(C) \longrightarrow .$$

Furthermore one has a vanishing theorem for $K_{-1}(C)$ under suitable hypotheses, or stronger still, a theorem that

$$K_0(C) \longrightarrow K_0(C[t, t^{-1}])$$

is surjective. Two proofs of such a theorem are given in [3] under the assumption that C be regular noetherian. One checks that the first of these proofs can be modified to apply in our present situation when $C[t]$ and $C[t, t^{-1}]$ are assumed to be regular coherent: one just works with finitely presented modules instead of finitely generated ones. So under this stronger assumption, $Q\mathcal{P}_R^*$ can indeed be replaced by $Q\mathcal{P}_R$ in the corollary.

On the level of geometric realizations, Theorem 11.3 can be given a more explicit form.

Notation. $\tilde{B}Q\mathcal{V}$ is the homotopy theoretic fibre of the retraction

$$BQ\psi\colon BQ\mathcal{V} \longrightarrow BQ(\mathcal{P}_C \times \mathcal{P}_C) ;$$

$(-\mathrm{Id})\colon BQ\mathcal{P}_C \to BQ\mathcal{P}_C$ is a homotopy inverse on the H-space $BQ\mathcal{P}_C$. Similarly, $-BQ\beta_*\colon BQ\mathcal{P}_C \to BQ\mathcal{P}_B$ is the composition of $BQ\beta_*\colon BQ\mathcal{P}_C \to BQ\mathcal{P}_B$ with a homotopy inverse (on either $BQ\mathcal{P}_C$ or $BQ\mathcal{P}_B$; this does not matter).

THEOREM 11.6. *There is a natural splitting, up to homotopy,*

$$BQ\mathcal{V} \xrightarrow{\simeq} \tilde{B}Q\mathcal{V} \times BQ(\mathcal{P}_C \times \mathcal{P}_C) .$$

The sequence

$$\tilde{B}Q\mathcal{V} \times BQ\mathcal{P}_C \xrightarrow{\left(\substack{\text{pt.} \\ (BQ\alpha_*,\ -BQ\beta_*)}\right)} BQ\mathcal{P}_A \times BQ\mathcal{P}_B \longrightarrow BQ\mathcal{P}^*_R$$

has the homotopy type of a fibration. The map $\Omega BQ\mathcal{P}^*_R \to \tilde{B}Q\mathcal{V}$ *has a canonical section, up to homotopy.*

Proof. By Corollary 7.2 the diagram of simplicial categories

$$\begin{array}{ccc} Q\mathcal{V} & \longrightarrow & QF.(\varphi) \\ {\scriptstyle Q\psi}\downarrow & & \downarrow \\ Q\mathcal{P}_C \times Q\mathcal{P}_C & \longrightarrow & QF.(\psi \circ \varphi) \end{array}$$

is homotopy cartesian, and $QF.(\psi \circ \varphi)$ is contractible since $\psi \circ \varphi$ is isomorphic to the identity functor. Hence $BQ\mathcal{V} \xrightarrow{\simeq} BQF.(\varphi) \times BQ\mathcal{P}_C \times BQ\mathcal{P}_C$ and $BQF.(\varphi) \xrightarrow{\simeq} \tilde{B}Q\mathcal{V}$, which is the first assertion.

It was seen in the proof of Proposition 1.3 that for any object V of $\mathcal{V}$ there are *canonical* isomorphisms $M_A \to M_1 \otimes_C A$, $M_B \to M_2 \otimes_C B$ where $(M_1, M_2) = \psi(V) \in \mathcal{P}_C \times \mathcal{P}_C$. Let $\mathcal{V}'$ be the full subcategory of those objects in $\mathcal{V}$ for which these canonical isomorphisms are identities. Then $\mathcal{V}'$ is equivalent to $\mathcal{V}$ by the inclusion. By definition of $\mathcal{V}'$ the diagram

$$\begin{array}{ccc} BQ\mathcal{V}' & \longrightarrow & BQ\mathcal{P}_A \times BQ\mathcal{P}_B \\ \downarrow & & \downarrow \\ BQ\mathcal{P}_C & \longrightarrow & BQ\mathcal{P}^*_R \end{array}$$

induced by the diagram of Theorem 11.3 is commutative. By 11.3 this diagram is homotopy cartesian, though not as a commutative diagram but with respect to the homotopy described in 11.3.

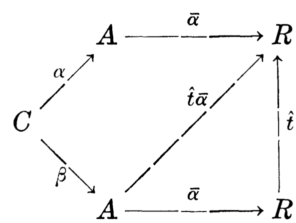
Homotopy cartesianness of the diagram gives a map between the homotopy fibrations associated to the vertical maps, inducing a homotopy equivalence between the homotopy theoretic fibres. The homotopy fibration associated to the left vertical map is, by the first assertion, homotopy equivalent to the sequence

$$\tilde{B}Q\mathcal{V} \times BQ\mathcal{P}_C \xrightarrow{\mathrm{Id}\times(\mathrm{Id},\,-\mathrm{Id})} \tilde{B}Q\mathcal{V} \times BQ\mathcal{P}_C \times BQ\mathcal{P}_C \xrightarrow{\begin{pmatrix}\mathrm{pt.}\\ \mathrm{Id}\\ \mathrm{Id}\end{pmatrix}} BQ\mathcal{P}_C \,.$$

So one obtains the asserted homotopy fibration.

But there is another map between these homotopy fibrations, because of the commutativity of the diagram. The two maps agree on the base, on the total space, and on the part $BQ\mathcal{P}_C$ of the fibre. Using that the map $\mathrm{Id} + (-\mathrm{Id})$ on a homotopy everything H-space is nullhomotopic by a homotopy that itself is unique up to homotopy, one can therefore take the 'difference' of the two maps, and one obtains this way the required map $\tilde{B}Q\mathcal{V} \to \Omega BQ\mathcal{P}_R^*$.

12. *Decomposition theorems in the Laurent extension case.* Let the ring R be given as the Laurent extension of A with respect to $\alpha, \beta: C \to A$ where α and β are both pure and their complements are free from the left, as in Section 2. By definition of the Laurent extension, there exists an embedding $\bar{\alpha}: A \to R$ and a unit t of R so that



commutes, where $\hat{t}$ denotes conjugation by t, $\hat{t}: R \to R$, $\hat{t}(r) = trt^{-1}$.

If we let $\mathcal{P}_R$ be as in 6.2 (so $? \mapsto \mathcal{P}_?$ is a functor), $\hat{t}$ induces an automorphism of $\mathcal{P}_R$. This automorphism is inner; in tensor product notation, an isomorphism j_t from the identity functor to $\hat{t}_*$ is given by

$$j_t: M \longrightarrow M \otimes_{R\,\hat{t}} R,\ m \cdot r \longmapsto m \cdot r \otimes t = m \otimes (trt^{-1})t = m \otimes tr \,.$$

In agreement with the convention in Section 2, we consider R as a left A-module via $\bar{\alpha}: A \to R$ and as a left C-module via $\bar{\alpha} \circ \alpha: C \to R$. Thus the natural transformation

$$(M_C \otimes_{C\,\alpha} A) \otimes_A R \longrightarrow M_C \otimes_C R$$

is the identity, and a natural transformation

$$(M_C \otimes_{C\,\beta} A) \otimes_A R \longrightarrow M_C \otimes_C R$$

is given by the isomorphism j_t described before. Note that j_t is nowhere the identity, except on zero.

An object of $\mathfrak{M}\mathcal{V}$, the category of admissible Mayer Vietoris presentations, is an exact sequence

$$0 \longrightarrow M \longrightarrow M_A \otimes_A R \xrightarrow{\kappa} M_C \otimes_C R \longrightarrow 0$$

with $M_C \in \mathcal{P}_C$, etc. We have the obvious forgetful maps $f_A: \mathcal{M}\mathcal{V} \to \mathcal{P}_A$, $f_C: \mathcal{M}\mathcal{V} \to \mathcal{P}_C$, the obvious section s_A of f_A, and the obvious natural transformation from the identity on $\mathcal{M}\mathcal{V}$ to $s_A \circ f_A$. The section s_C of f_C associates to M_C the Mayer Vietoris presentation

$$0 \longrightarrow (M_C \otimes_C R) \xrightarrow{(\mathrm{Id},\, j_t^{-1})} (M_C \otimes_{C\,\alpha} A \oplus M_C \otimes_{C\,\beta} A) \otimes_A R \xrightarrow{\binom{\mathrm{Id}}{-j_t}} M_C \otimes_C R \longrightarrow 0$$

and the natural transformation from the identity on $\mathcal{M}\mathcal{V}$ to $s_C \circ f_C$ is given by the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{\iota} & M_A \otimes_A R & \xrightarrow{\kappa} & M_C \otimes_C R & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \kappa_\alpha \circ \iota} & & {\scriptstyle \kappa_\alpha} \swarrow \quad \searrow {\scriptstyle j_t^{-1} \circ \kappa_\beta} & & \downarrow & & \\
0 & \longrightarrow & (M_C \otimes_C R) & \longrightarrow & (M_C \otimes_{C\,\alpha} A \oplus M_C \otimes_{C\,\beta} A) \otimes_A R & \longrightarrow & M_C \otimes_C R & \longrightarrow & 0
\end{array}$$

where $\kappa = \kappa_\alpha - \kappa_\beta$ is the canonical decomposition, cf. Section 2, the point being that $j_t^{-1} \circ \kappa_\beta$ is indeed induced from an A-map.

There are maps $\varphi_\alpha, \varphi_\beta: \mathcal{P}_C \to \mathcal{V}$ which send $M_C \in \mathrm{Ob}(\mathcal{P}_C)$ to

$$0 \longrightarrow (M_C \otimes_{C\,\alpha} A) \otimes_A R \xrightarrow{=} M_C \otimes_C R \longrightarrow 0\ ,$$
$$0 \longrightarrow (M_C \otimes_{C\,\beta} A) \otimes R \xrightarrow{j_t} M_C \otimes_C R \longrightarrow 0$$

respectively. These combine to $\varphi = (\varphi_\alpha \oplus \varphi_\beta): \mathcal{P}_C \times \mathcal{P}_C \to \mathcal{V}$. From Section 2 we have a map $\psi: \mathcal{V} \to \mathcal{P}_C \times \mathcal{P}_C$, and the composed map $\psi \circ \varphi$ is isomorphic to the identity functor on $\mathcal{P}_C \times \mathcal{P}_C$.

The arguments of the preceding section now carry over to the present situation, with trivial alteration. We obtain corresponding results.

PROPOSITION 12.1. *The map $Q(f_A, f_C): Q\mathcal{M}\mathcal{V} \to Q\mathcal{P}_A \times Q\mathcal{P}_C$ is a homotopy equivalence.*

THEOREM 12.2. *Suppose the ring C is regular coherent. Then $Q\mathcal{P}_C \times Q\mathcal{P}_C \to Q\mathcal{V}$ is a homotopy equivalence.*

THEOREM 12.3. *In the square*

$$\begin{array}{ccc}
Q\mathcal{V} & \xrightarrow{Qf_A} & Q\mathcal{P}_A \\
{\scriptstyle Qf_C}\downarrow & & \downarrow \\
Q\mathcal{P}_C & \longrightarrow & Q\mathcal{P}_R^*
\end{array}$$

the two composed maps from $Q\mathcal{V}$ to $Q\mathcal{P}_R^*$ differ by the functor isomorphism which to any object of $Q\mathcal{V}$ associates its structure map. The square is homotopy cartesian with respect to the homotopy of the composed maps given by this functor isomorphism.

PROPOSITION 12.4. *The non-commutative square of simplicial categories*

$$\begin{array}{ccc} Q\mathcal{V}_{\cdot} & \longrightarrow & Q\mathcal{P}_{A\cdot} \\ \downarrow & & \downarrow \\ Q\mathcal{P}_{C\cdot} & \longrightarrow & Q\mathcal{P}^*_{R\cdot} \end{array}$$

is homotopy cartesian with respect to the simplicial homotopy which to any object of $\mathcal{V}$ associates its structure map.

COROLLARY 12.5. *In the square*

$$\begin{array}{ccc} Q\mathcal{P}_C \times Q\mathcal{P}_C & \xrightarrow{Q(\alpha_* \oplus \beta_*)} & Q\mathcal{P}_A \\ \Big\downarrow {\scriptstyle Q(\mathrm{Id} \oplus \mathrm{Id})} & & \Big\downarrow \\ Q\mathcal{P}_C & \longrightarrow & Q\mathcal{P}^*_R \end{array}$$

the two composed maps from $Q\mathcal{P}_C \times Q\mathcal{P}_C$ to $Q\mathcal{P}^*_R$ differ by the functor isomorphism which is the identity on $Q\mathcal{P}_C \times 0$, and is given by the isomorphism j_t on $0 \times Q\mathcal{P}_C$. If C is regular coherent then the square is homotopy cartesian with respect to the homotopy of the composed maps given by this functor isomorphism.

Notation. $\tilde{B}Q\mathcal{V}$ is the homotopy fibre of the retraction

$$BQ\psi\colon BQ\mathcal{V} \longrightarrow BQ(\mathcal{P}_C \times \mathcal{P}_C)\ .$$

The map $BQ\alpha_* + (-BQ\beta_*)\colon BQ\mathcal{P}_C \to BQ\mathcal{P}_A$ is the sum (with respect to the H-space structure on $BQ\mathcal{P}_A$) of the maps $BQ\alpha_*$ and $-BQ\beta_*$, the composition of $BQ\beta_*$ with a homotopy inverse.

THEOREM 12.6. *There is a natural splitting, up to homotopy,*

$$BQ\mathcal{V} \xrightarrow{\simeq} \tilde{B}Q\mathcal{V} \times BQ\mathcal{P}_C \times BQ\mathcal{P}_C\ .$$

The sequence

$$\tilde{B}Q\mathcal{V} \times BQ\mathcal{P}_C \xrightarrow{\begin{pmatrix}\text{pt.}\\ BQ\alpha_* + (-BQ\beta_*)\end{pmatrix}} BQ\mathcal{P}_A \longrightarrow BQ\mathcal{P}^*_R$$

has the homotopy type of a fibration. The map $\Omega BQ\mathcal{P}^*_R \to \tilde{B}Q\mathcal{V}$ has a canonical section, up to homotopy.

13. *Decomposition theorems in the polynomial extension case.* Let the ring R be given as the tensor algebra of the C-bimodule S where S is free as left C-module and finitely generated projective as right C-module, as in Section 3.

An object of $\mathfrak{M}\mathcal{V}$ is a short exact sequence

$$0 \longrightarrow M \longrightarrow M_2 \otimes_C R \longrightarrow M_1 \otimes_C R \longrightarrow 0$$

with M_1, $M_2 \in \mathcal{P}_C$, etc. The two forgetful maps $f_1, f_2\colon \mathfrak{M}\mathcal{V} \to \mathcal{P}_C$ and the section s_2 of f_2 are the obvious ones. The value of the section s_1 of f_1 on the module M_C is, by definition, the short exact sequence associated to the commutative diagram

$$\begin{array}{ccc} ((M_C \otimes_C S) \otimes_C R) & \longrightarrow & M_C \otimes_C R \\ \downarrow & & \downarrow \\ ((M_C \otimes_C S) \otimes_C R) & \longrightarrow & M_C \otimes_C R \end{array}$$

in which the horizontal map is induced from the inclusion $S \subset R$; we are using here the condition that S be finitely generated projective from the right (actually, the generality could be pushed a bit). There are natural transformations from the identity functor on $\mathcal{M}\mathcal{V}$ to $s_2 \circ f_2$ and to $s_1 \circ f_1$. The latter natural transformation is given by the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{\iota} & M_2 \otimes_C R & \xrightarrow{\kappa} & M_1 \otimes_C R & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \kappa_0 \circ \iota} & & {\scriptstyle \kappa_0} \swarrow \quad \searrow {\scriptstyle \kappa_1} & & \downarrow & & \\ 0 & \longrightarrow & ((M_1 \otimes_C S) \otimes_C R) & \longrightarrow & (M_1 \oplus M_1 \otimes_C S) \otimes_C R & \longrightarrow & M_1 \otimes_C R & \longrightarrow & 0\,. \end{array}$$

As in the two preceding sections, one now deduces

PROPOSITION 13.1. *The map $Q(f_2, f_1)\colon Q\mathcal{M}\mathcal{V} \to Q\mathcal{P}_C \times Q\mathcal{P}_C$ is a homotopy equivalence.*

THEOREM 13.2. *Suppose the ring C is regular coherent. Then $Q\mathcal{P}_C \to Q\mathcal{V}$ is a homotopy equivalence.*

THEOREM 13.3. *In the square*

$$\begin{array}{ccc} Q\mathcal{V} & \xrightarrow{Qf_2} & Q\mathcal{P}_C \\ {\scriptstyle Qf_1}\downarrow & & \downarrow \\ Q\mathcal{P}_C & \longrightarrow & Q\mathcal{P}^*_R \end{array}$$

the two composed maps from $Q\mathcal{V}$ to $Q\mathcal{P}^*_R$ differ by the functor isomorphism which to any object of $Q\mathcal{V}$ associates its structure map. The square is homotopy cartesian with respect to the homotopy of the composed maps given by this functor isomorphism.

The composition of $\mathcal{P}_C \to \mathcal{V}$ with either f_1 or f_2 is the identity. Thus instead of just a commutative homotopy cartesian square, one obtains by combining Theorems 13.2 and 13.3,

COROLLARY 13.4. *Suppose that C is regular coherent. Then $Q\mathcal{P}_C \to Q\mathcal{P}^*_R$ is a homotopy equivalence.*

As in the remark after Corollary 11.5, one can sharpen this to a homotopy equivalence $Q\mathcal{P}_C \to Q\mathcal{P}_R$ when one assumes $C[t]$ and $C[t, t^{-1}]$ are regular coherent.

The analogue of Theorems 11.6 and 12.6 also takes a sharpened form here.

THEOREM 13.5. *The loop space* $\Omega BQ\mathcal{P}_R^*$ *is naturally homotopy equivalent to the product of* $\tilde{B}Q\mathcal{V}$ (the homotopy theoretic fibre of $BQ\mathcal{V} \to BQ\mathcal{P}_C$) *and the homotopy theoretic fibre of*

$$BQ\mathcal{P}_C \xrightarrow{(\mathrm{Id},\, -BQS_*)} BQ\mathcal{P}_C \times BQ\mathcal{P}_C$$

where $S_*(M_C) = M_C \otimes_C S$. *Thus it is homotopy equivalent to* $\tilde{B}Q\mathcal{V} \times \Omega BQ\mathcal{P}_C$.

IV. K-theory and homology

14. *The homology theory associated to a* Γ*-category.* In Section 6, a *small* Γ*-category* was defined as a covariant functor $F: \mathcal{S}_* \to$ (small categories) satisfying

(i) $F\{*\}$ is the category with one object and one morphism,

(ii) For any two pointed sets $(X, *)$, $(Y, *)$, the natural map

$$F\big((X, *) \vee (Y, *)\big) \longrightarrow F(X, *) \times F(Y, *)$$

is an equivalence of categories.

Using direct limit one can extend F to a functor on the category of pointed sets, not necessarily finite. Then (i) and (ii) continue to hold. We keep the notation F for the extended functor.

One can further extend to a functor

$$(\text{pointed sets})^{\Delta^{\mathrm{op}}} \longrightarrow (\text{categories})^{\Delta^{\mathrm{op}}},$$

that is, to a functor from pointed simplicial sets to simplicial categories (which actually are also pointed). We continue to denote this functor F. Then F satisfies

(ii′) For any two pointed simplicial sets $(X, *)$, $(Y, *)$, the natural map

$$F\big((X, *) \vee (Y, *)\big) \longrightarrow F(X, *) \times F(Y, *)$$

is a weak equivalence of simplicial categories, that is, it is an equivalence of categories in each degree.

The following is essentially a translation of a result by Anderson [2]. One should note that Segal's de-looping of $F\{1 \cup *\}$, cf. Proposition 6.3, corresponds to the special case of the cofibration

$$(S^0, *) \longrightarrow (\Delta^1, *) \longrightarrow \operatorname{coker}(S^0 \longrightarrow \Delta^1).$$

LEMMA 14.1. *Let* F *be a small* Γ*-category. Suppose the underlying category* $F\{1 \cup *\}$ *is connected. Then* F *sends cofibrations to fibrations up to homotopy.*

Proof. The assertion means, if $(A, *)$ is a simplicial subset of $(X, *)$, and $(X/A, *)$ the quotient simplicial set, then $F(A, *) \to F(X, *) \to F(X/A, *)$ is a fibration up to homotopy. To see this, one notes that in each degree n, the sequence

$$F(A_n, *) \longrightarrow F(X_n, *) \longrightarrow F\big((X/A)_n, *\big)$$

is trivially a fibration up to homotopy, by (ii) above. By hypothesis, and again by (ii), $F\big((X/A)_n, *\big)$ is connected. So the assertion follows from Lemma 5.2.

LEMMA 14.2. *Let F be a small Γ-category, with connected underlying category.*

(1) *Let $f, g: (X, *) \to (Y, *)$ be maps of pointed simplicial sets. Suppose the geometric realizations of f and g are homotopic. Then the geometric realizations of $F(f)$ and $F(g)$ are homotopic.*

(2) *Let $h: (V, *) \to (W, *)$ be a map of pointed simplicial sets. Suppose h is a weak homotopy equivalence. Then $F(h)$ is a weak homotopy equivalence.*

Proof. If in (1), f and g are *simplicially* homotopic, we know the assertion (1) to be true because the functor F, being extended from the category of basepointed sets, preserves simplicial homotopies between maps of simplicial objects. Consequently, we know (2) to be true if both V and W satisfy the Kan condition. To prove the lemma, it suffices thus to show that $F(j)$ is a weak homotopy equivalence when j is the natural transformation $j: (X, *) \to (Ex^\infty X, *)$ of Kan. Since F commutes with direct limit, up to homotopy, the latter follows if we show $F(e)$ is a weak homotopy equivalence when e is an elementary *anodyne extension*, that is, an inclusion

$$e: (X, *) \longrightarrow (X \cup_{\Lambda^i} \Delta^n, *)$$

where Δ^n is the simplicial set *n-simplex*, and Λ^i the i^{th} *horn* of Δ^n, the union of all the $(n-1)$-faces of Δ^n except the i^{th} one. By the preceding lemma, the sequence

$$F(X, *) \xrightarrow{F(e)} F(X \cup_{\Lambda^i} \Delta^n, *) \longrightarrow F(\Delta^n/\Lambda^i, *)$$

is a fibration up to homotopy. But $(\Delta^n/\Lambda^i, *)$ is contractible by simplicial homotopy. So $F(\Delta^n/\Lambda^i, *)$ is contractible, and $F(e)$ is a weak homotopy equivalence, as asserted.

In view of Lemmas 14.1 and 14.2, the functor $(X, *) \mapsto F(X, *)$ is a *homology theory* on the category of pointed simplicial sets; that is, its homotopy groups satisfy the Eilenberg-Steenrod axioms except for the dimension axiom. We will also need the corresponding unreduced homology theory.

Notation. If F is a Γ-category, we denote F^+ the functor on *unbased* simplicial sets which is the composition of F with the functor that adds a basepoint.

There is the following formulation of *excision* for F^+.

LEMMA 14.3. *Let*

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

be a cocartesian square of simplicial sets so that at least one of the maps originating at A is a monomorphism. Then the square

$$\begin{array}{ccc} F^+(A) & \longrightarrow & F^+(B) \\ \downarrow & & \downarrow \\ F^+(C) & \longrightarrow & F^+(D) \end{array}$$

is homotopy cartesian.

Proof. For the sake of argument we assume the horizontal maps are monomorphisms. Then in the diagram

$$\begin{array}{ccccc} F^+(A) & \longrightarrow & F^+(B) & \longrightarrow & F(B/A, *) \\ \downarrow & & \downarrow & & \downarrow \\ F^+(C) & \longrightarrow & F^+(D) & \longrightarrow & F(D/C, *) \end{array}$$

the rows are fibrations up to homotopy, by Lemma 14.1, and the right vertical map is an isomorphism. Hence the left hand square is homotopy cartesian, as asserted.

Amalgamated free products of groups. Let $\alpha: G_0 \to G_1$ and $\beta: G_0 \to G_2$ be monomorphisms of groups and $G = G_1 *_{G_0} G_2$ the resulting amalgamated free product (the pushout of α and β). Let N denote the functor *nerve.*

LEMMA 14.4. *Let F be a small Γ-category with connected underlying category. Then the commutative diagram of simplicial categories*

$$\begin{array}{ccc} F^+(NG_0 \cup NG_0) & \longrightarrow & F^+(NG_1 \cup NG_2) \\ \downarrow & & \downarrow \\ F^+(NG_0) & \longrightarrow & F^+(NG) \end{array}$$

is homotopy cartesian.

Proof. In view of Lemmas 14.2 and 14.3, it suffices to verify that the map $NG_1 \cup_{NG_0} NG_2 \to NG$ induces a homotopy equivalence of geometric realizations. To see this one notes first that the functor *fundamental groupoid* commutes with colimits. So the map induces an isomorphism of fundamental groups in view of the very definition of the amalgamated free product. Since NG is an Eilenberg-MacLane space, it is thus sufficient to show that $NG_1 \cup_{NG_0} NG_2$ is an Eilenberg-MacLane space, too. But this is easily seen from the Mayer Vietoris sequence of homology in the universal covering.

HNN *extensions of groups.* Let $\alpha, \beta\colon G_0 \to G_1$ be monomorphisms of groups. The HNN extension of G_1 with respect to (α, β) can be defined as the pushout in the category of groupoids in the diagram

$$\begin{array}{ccc} G_0 \cup G_0 & \xrightarrow{(\alpha,\beta)} & G_1 \\ \downarrow & & \downarrow \\ G_0 \times I & \longrightarrow & G \end{array}$$

where $\cup$ is the coproduct in the category of groupoids (disjoint union) and I is the connected groupoid with two vertices and trivial vertex groups.

LEMMA 14.5. *Let F be a small Γ-category with connected underlying category. Then the commutative square of simplicial categories*

$$\begin{array}{ccc} F^+(NG_0 \cup NG_0) & \longrightarrow & F^+(NG_1) \\ \downarrow & & \downarrow \\ F^+(NG_0 \times NI) & \longrightarrow & F^+(NG) \end{array}$$

is homotopy cartesian.

Proof. As in the preceding lemma one verifies that the map

$$(NG_0 \times NI) \cup_{(NG_0 \cup NG_0)} NG_1 \longrightarrow NG$$

induces a homotopy equivalence of geometric realizations.

15. *Whitehead groups.* Let R denote a ring, and G a group. A functor from pairs (R, G) to spaces will be constructed,

$$(R, G) \longmapsto \mathrm{Wh}^R(G)$$

whose homotopy groups give the Whitehead groups of G, taken relative to the ring R.

Notation. The Γ-category $\Gamma_{\mathcal{P}_R}$ of Section 6 will be denoted Γ_R henceforth.

By direct limit, we can assume that Γ_R is defined on pointed sets which are not necessarily finite. Γ_R can be considered in a natural way as a functor with values in the category of exact categories, so we can compose with the Q-construction.

LEMMA 15.1. *$Q\Gamma_R$ is a Γ-category.*

Proof. The Q-construction commutes with products and filtering direct limits, up to equivalence, and it preserves equivalences. So the assertion is immediate from the definition of a Γ-category.

Notation. Γ_R^+ is the composition of Γ_R with the functor that adds a basepoint.

If X is a set without basepoint, an object of Γ_R^+ thus consists of (an equivalence class of) the following data:

(i) An object $P = P_{X'}$ of $\mathcal{P}_R$ where X' is some non-empty finite subset of X (resp., $P = P_{X'} = 0$, the distinguished zero, if $X' = \emptyset$),

(ii) The data required to express P as a direct sum

$$P = \bigoplus_{x \in X'} P_{\{x\}} \ ,$$

(iii) A choice, for any $Y \subset X'$ not mentioned so far, of an object P_Y in the isomorphism class of $\bigoplus_{x \in Y} P_{\{x\}}$.

The equivalence relation on these data is generated by allowing X' to be replaced by a larger finite subset X'' of X, but insisting that if $Y'' \subset X''$ one have $P_{Y''} = P_{(Y'' \cap X')}$ (equality, not just isomorphism). Note in particular that equivalent data involve the same P.

We will now consider Γ_R^+ as a functor from simplicial sets (without basepoint) to simplicial categories. As before, we let $\mathcal{P}_{R\cdot}$ be the simplicial category which in degree n is $\mathcal{P}_{R(n)}$, the category equivalent to $\mathcal{P}_R$ in which an object is a sequence of n composable isomorphisms in $\mathcal{P}_R$. Let G be a group, NG its nerve, and RG the group algebra over R.

PROPOSITION 15.2. *There is a natural transformation of functors from pairs (R, G) to simplicial exact categories*

$$\big((R, G) \longmapsto \Gamma_R^+(NG)\big) \xrightarrow{W(R,\, G).} \big((R, G) \longmapsto \mathcal{P}_{RG\cdot}\big) \ .$$

If G is the trivial group, $W(R, G).$ is a weak equivalence, that is, it is in each degree an equivalence of categories.

Proof. In view of the description of $\Gamma_R^+(X)$ above, we see that in particular for $X = (NG)_n = G^n$, an object of $\Gamma_R^+\big((NG)_n\big)$ consists of

(i) an object P of $\mathcal{P}_R$,

(ii) a direct sum decomposition indexed by the n-tuples in G,

$$P = \bigoplus_{(g_1, \cdots, g_n) \in G^n} P_{(g_1, \cdots, g_n)} \ ,$$

(iii) certain other data.

The proposition requires us to associate to this object a sequence of isomorphisms in $\mathcal{P}_{RG}$. But if $P \in \mathcal{P}_R$ and $i_*\colon \mathcal{P}_R \to \mathcal{P}_{RG}$, $i_*(P) = P \otimes_R RG$, the left action of G on RG extends to an action

$$m: G \longrightarrow \mathrm{Is}\big(i_*(P)\big) .$$

A canonical choice of the natural transformation $W(R, G)_n$ is then made by associating to the object in question, the sequence of isomorphisms on $i_*(P)$ which on the summand indexed by $(g_1, \cdots, g_n)$ is given by

$$m(g_1), \cdots, m(g_n) .$$

It is clear that $W(R, G)_n$ has the asserted properties.

Applying Corollary 7.2 in the case when $f = \mathrm{Id}_n$ is the identity map on $\Gamma_R^+((NG)_n)$, and $g = W(R, G)_n$, we obtain the homotopy cartesian square of simplicial categories

$$\begin{array}{ccc} Q\Gamma_R^+((NG)_n) & \longrightarrow & QF.(\mathrm{Id}_n) \\ \downarrow & & \downarrow \\ Q\mathcal{P}_{RG(n)} & \longrightarrow & QF.\big(W(R, G)_n\big) \end{array}$$

and $QF.(\mathrm{Id}_n)$ is contractible. Assembling these squares for varying n, we obtain a commutative square of bisimplicial categories.

PROPOSITION 15.3. *The square of bisimplicial categories*

$$\begin{array}{ccc} Q\Gamma_R^+((NG).) & \longrightarrow & QF.(\mathrm{Id}.) \\ \downarrow & & \downarrow \\ Q\mathcal{P}_{RG\cdot} & \longrightarrow & QF.\big(W(R, G).\big) \end{array}$$

is homotopy cartesian. The bisimplicial category $QF.(\mathrm{Id}.)$ *is contractible. The square is natural in* (R, G).

Proof. Naturality is clear. $QF.(\mathrm{Id}.)$ is a simplicial object of contractible things (the $QF.(\mathrm{Id}_n)$) hence is itself contractible. The square is homotopy cartesian by Lemma 5.2. To see this, let X_n (resp. Y_n) denote the homotopy theoretic fibre at $*$ of the left (resp. right) vertical map in the square preceding the proposition. Then $X_n \to Y_n$ is a homotopy equivalence since this square is homotopy cartesian. Similarly let X and Y denote the homotopy theoretic fibres of the vertical maps in the square of the proposition. Then the natural map $B(X.) \to X$ is a homotopy equivalence by Lemma 5.2 since $Q\mathcal{P}_{RG(n)}$ is connected, for every n. Similarly $B(Y.) \to Y$ is a homotopy equivalence. $B(X.) \to B(Y.)$ is a homotopy equivalence by Lemma 5.1. Hence $X \to Y$ is a homotopy equivalence, and the square is homotopy cartesian, as asserted.

Definition 5.4. The *Whitehead space of* G *relative to* R, denoted $\mathrm{Wh}^R(G)$, is given by $\Omega BQF.\big(W(R, G).\big)$, the loop space of the geometric realization of the bisimplicial category $QF.(W(R, G).)$. Its homotopy groups are the *Whitehead groups of G relative to R.*

Notice that the Whitehead groups of the trivial group are always trivial (essentially by definition). Letting, for short, $K(R) = \Omega BQ\mathcal{P}_R$ and $K(X; R) = \Omega BQ\Gamma_R^+(X)$ for a simplicial set X, we can restate Proposition 15.3 to say

COROLLARY 15.5. *There is a canonical homotopy equivalence of* $K(NG; R)$ *with the homotopy theoretic fibre of the map* $K(RG) \to \mathrm{Wh}^R(G)$.

Or put otherwise, *the functor* $\mathrm{Wh}^R(G)$ *is so defined that it measures to what extent the functor* $G \mapsto K(RG)$ *deviates from a homology theory evaluated on* NG.

Remark. It can be shown that the bisimplicial *set* $\mathrm{Ob}(QF.(W(R, G).))$ or what is the same by definition, the bisimplicial set $\mathrm{Ob}(F.(W(R, G).))$, is naturally homotopy equivalent to $\mathrm{Wh}^R(G)$. The fundamental group of this bisimplicial set can easily be computed by hand. In the case when R is the ring of integers, it is amusing to see how not just the usual Whitehead group appears, but almost its definition in terms of elementary expansions.

Definition 15.6. The *Whitehead groups* of a group G are given by

$$\mathrm{Wh}_i(G) = \pi_i \mathrm{Wh}^Z(G)$$

where Z is the ring of integers.

Here are a few comments on this definition. Below we verify that for $i = 0, 1, 2$, this definition is the correct one in the sense that it recovers the Whitehead groups hitherto considered which have been used in relating algebraic K-theory to problems of geometric topology (finiteness obstructions, h-cobordisms, concordances). In higher dimensions, the geometric theory branches into two cases corresponding to whether one considers smooth manifolds or piecewise linear manifolds (resp. topological manifolds —for the matter at hand this amounts to the same thing as considering piecewise linear manifolds). The theory of [27] shows that for application to higher concordance groups in the piecewise linear case, the present definition of $\mathrm{Wh}_*(G)$ appears to be the correct one. It should be noted though that the natural transformation from concordance groups to Whitehead groups is very far from being an isomorphism in general (as opposed to the situation for $i = 0$ or 1). That so much information is thrown away is due to two facts: Firstly, the Whitehead groups here defined depend only on groups (that is, the fundamental groups of any spaces considered, not the higher homotopy groups). Secondly, K-theory (from which after all the Whitehead groups are derived) is really concerned with *linear* phenomena, but the geometry involves non-linear phenomena as well.

The first defect can be rectified by extending the definition of $\mathrm{Wh}_*(G)$ to simplicial groups. It would be tempting here to just use the degreewise extension (that is, evaluate the functor $G \mapsto \mathrm{Wh}^Z(G)$ in each degree of the simplicial group and then pass to the geometric realization of the resulting simplicial space). This does *not* give the correct result however (for instance the functor so constructed does not take a weak homotopy equivalence of simplicial groups to an isomorphism of the Whitehead groups—as it should do if it were the correct one). The correct procedure is to use a suitable definition of K-theory for simplicial rings (not the degreewise extension) [27]. This K-theory coincides with Quillen's for a ring considered as a simplicial ring in a trivial way (as of course does the degreewise extension), and it preserves weak homotopy equivalences. The second defect, that of ignoring non-linear phenomena, can also be rectified [27].

PROPOSITION 15.7. (0) $\mathrm{Wh}_0(G) = \widetilde{K}_0(ZG)$, *the reduced projective class group.*

(1) $\mathrm{Wh}_1(G)$ *is the usual Whitehead group.*

(2) $\mathrm{Wh}_2(G)$ *coincides with the quotient of* $K_2(ZG)$ *considered in* [13]; *hence it coincides with the second Whitehead group of* [9].

Proof. From Corollary 15.5, one has the long exact sequence of homotopy groups

$$\cdots \to K_1(RG) \to \pi_1 \mathrm{Wh}^R(G) \to \pi_0 K(NG; R) \to K_0(RG) \to \pi_0 \mathrm{Wh}^R(G) \to 0 .$$

Let H_* denote ordinary homology. Since $\pi_* K(?; R)$ is a generalized homology theory one has

$$\pi_0 K(NG; R) \xrightarrow{=} H_0(NG, K_0(R)) \xrightarrow{=} H_0(\mathrm{pt.}, K_0(R)) \xrightarrow{=} K_0(R)$$

from which part (0) of the proposition is immediate. Furthermore one sees easily from the spectral sequence of a generalized homology theory that

$$\pi_1 K(NG; R) \xrightarrow{=} H_0(NG, K_1(R)) \oplus H_1(NG, K_0(R)) \xrightarrow{=} K_1(R) \oplus H_1(NG, K_0(R)).$$

Assertion 15.8. $\pi_1 K(NG; Z) \to K_1(ZG)$ *is the usual map.*

It is immediate from this assertion that

$$\mathrm{Wh}_1(G) = \mathrm{coker}\big(K_1(Z) \oplus H_1(NG, Z) \longrightarrow K_1(ZG)\big)$$

is the usual Whitehead group.

The usual map $K_1(Z) \oplus H_1(NG, Z) \to K_1(ZG)$ is injective. For abelian G, this is clear from the existence of the determinant homomorphism and the ensuing diagram

$$\begin{array}{ccc} Z_2 \oplus G & \longrightarrow & GL_1(ZG) \\ \downarrow & & \downarrow \\ \text{units}(ZG) & \longleftarrow & K_1(ZG) \end{array} .$$

For non-abelian G one reduces to this case by abelianization and the diagram

$$\begin{array}{ccc} Z_2 \oplus H_1(NG,\, Z) & \longrightarrow & K_1(ZG) \\ \| \downarrow & & \downarrow \\ Z_2 \oplus H_1(NG^{\mathrm{ab}},\, Z) & \longrightarrow & K_1(ZG^{\mathrm{ab}}) \end{array} .$$

Thus by the preceding assertion, $K_2(ZG) \to \mathrm{Wh}_2(G)$ is surjective, and

$$\mathrm{Wh}_2(G) = \operatorname{coker}\big(\pi_2 K(NG;\, Z) \longrightarrow K_2(ZG)\big) .$$

Assertion 15.9. *$\pi_* K(NG;\, Z) \to K_*(ZG)$ coincides with the corresponding map in* [13].

In view of this assertion, our $\mathrm{Wh}_2(G)$ is the same quotient of $K_2(ZG)$ as the one in [13] which Loday has shown to coincide with the second Whitehead group of Hatcher and Wagoner [10]. Modulo the two assertions above, the proof of the proposition is thus complete. The assertions will be dealt with in the next section.

Remark 15.10. The natural transformation of 15.2 can be put into a more general framework, as follows. Suppose $\mathcal{A} \times \mathcal{B} \to \mathcal{C}$ is a bi-exact pairing in the sense of 9.2. Let G be a group. To the pair $(G, \mathcal{B})$ one can associate an exact category $\mathrm{Rep}_{\mathcal{B}}(G)$, the category of G-representations in $\mathcal{B}$; it may be defined as the category of functors $\mathcal{G} \to \mathcal{B}$ where $\mathcal{G}$ is G considered as a category.

The map of 15.2 then has an analogue which is a bi-exact pairing of simplicial exact categories, 'evaluation',

$$\Gamma_{\mathcal{A}}(NG \cup *) \times \mathrm{Rep}_{\mathcal{B}}(G) \longrightarrow \mathcal{C} .$$

from which other pairings may be deduced as in 9.2. The map of 15.2 itself can be recovered as the induced map

$$\Gamma_R(NG \cup *) \times \sigma \longrightarrow \mathcal{P}_{RG} .$$

where $\sigma \in \mathrm{Rep}_{\mathcal{P}_{ZG}}$ is the standard representation of G on ZG.

16. *Comparison of homology theories.* The purpose of this section is to prove assertions 15.8 and 15.9. The proof involves showing that certain *a priori* different ways of manufacturing a homology theory from K-theory lead in fact to the same result. Also, one must chase analogues of the natural transformation $W(R, G)$. of 15.2, through comparison theorems for K-theory.

If $X.$ and $Y.$ are simplicial sets we denote $X. \otimes Y.$ the bisimplicial set which in bidegree (m, n) is $X_m \times Y_n$. If $X.$ and $Y.$ are pointed we can form an analogue of the usual smash product from $X. \otimes Y.$ by collapsing $X. \otimes * \cup * \otimes Y.$; we denote this $X. \wedge Y.$; a similar notation will be used for multi-simplicial sets.

LEMMA 16.1. *Let X and Y be pointed simplicial sets. Let $\mathcal{A}$ be a small exact category, pointed by a zero object* 0. *There is a natural transformation of trisimplicial sets*

$$NQ\Gamma_{\mathcal{A}}(X) \wedge Y \longrightarrow NQ\Gamma_{\mathcal{A}}(X \wedge Y) .$$

If $Y = S^0$ this is a homotopy equivalence. If $Y = S^1$ (any pointed simplicial circle) *the associated map*

$$BQ\Gamma_{\mathcal{A}}(X) \longrightarrow \Omega BQ\Gamma_{\mathcal{A}}(X \wedge S^1)$$

is a homotopy equivalence.

Proof. This has been described by Anderson [2] in a more general context. Suffice it to point out here the following. An element $\neq 0$ in bidegree (m, n) of the bisimplicial set $\mathrm{Ob}\big(Q\Gamma_{\mathcal{A}}(X)\big) \wedge Y$ is represented by

(i) $(U \subset X_m - *, y \in Y_n - *)$ where U is finite,

(ii) certain data indexed by the category of subsets of U.

Similarly, an element $\neq 0$ in bidegree (m, n) of $\mathrm{Ob}\big(Q\Gamma_{\mathcal{A}}(X \wedge Y)\big)$ is represented by

(i) $V \subset (X_m - *) \times (Y_n - *)$ where V is finite and non-empty,

(ii) certain data indexed by the category of subsets of V.

The asserted natural transformation is simply given by $(U, y) \mapsto U \times \{y\}$.

It is clear that $NQ\Gamma_{\mathcal{A}}(X) \wedge S^0 \to NQ\Gamma_{\mathcal{A}}(X \wedge S^0)$ is an isomorphism. The third assertion need be proved only for a particular simplicial circle, in view of Lemma 14.2. Applying the natural transformation to the cofibration sequence $S^0 \to \Delta^1 \to S^1$, we obtain a diagram

$$\begin{array}{ccccc} NQ\Gamma_{\mathcal{A}}(X) \wedge S^0 & \longrightarrow & NQ\Gamma_{\mathcal{A}}(X) \wedge \Delta^1 & \longrightarrow & NQ\Gamma_{\mathcal{A}}(X) \wedge S^1 \\ \downarrow & & \downarrow & & \downarrow \\ NQ\Gamma_{\mathcal{A}}(X \wedge S^0) & \longrightarrow & NQ\Gamma_{\mathcal{A}}(X \wedge \Delta^1) & \longrightarrow & NQ\Gamma_{\mathcal{A}}(X \wedge S^1) \end{array}$$

in which the bottom row is a fibration up to homotopy by Lemma 14.1. The simplicial nullhomotopy of $X \wedge \Delta^1$ induces a nullhomotopy of $NQ\Gamma_A(X \wedge \Delta^1)$, hence a map $NQ\Gamma_A(X) \wedge S^1 \to NQ\Gamma_A(X \wedge S^1)$. On the one hand, this map is the same as the right vertical map in the diagram; on the other hand, its adjoint is the homotopy equivalence from $BQ\Gamma_A(X)$ to $\Omega BQ\Gamma_A(X \wedge S^1)$ given by the bottom row.

As pointed out by Anderson [2] the lemma signifies in particular that

$$NQ\Gamma_{\mathcal{A}}(S^n) \wedge S^1 \longrightarrow NQ\Gamma_{\mathcal{A}}(S^n \wedge S^1)$$

gives an Ω-spectrum, and that the two reduced homology theories

$$\pi_* \varinjlim_n \Omega^n B\big(NQ\Gamma_{\mathcal{A}}(S^n) \wedge X\big) \quad \text{and} \quad \pi_* BQ\Gamma_{\mathcal{A}}(X)$$

are the same in view of the natural transformation

$$\Omega^n B\big(NQ\Gamma_{\mathcal{A}}(S^n) \wedge X\big) \longrightarrow \Omega^n BQ\Gamma_{\mathcal{A}}(S^n \wedge X) \xleftarrow{\simeq} BQ\Gamma_{\mathcal{A}}(X)$$

and the fact that this natural transformation is an isomorphism on the 'coefficients', the case $X = S^0$.

As in 9.2, let $\mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a bi-exact functor of small exact categories, each pointed by 0. So there is a map of bicategories

$$Q\mathcal{A} \otimes \mathrm{Is}\,(\mathcal{B}) \longrightarrow Q\mathcal{C}^{\mathrm{Is}}$$

inducing an embedding of bisimplicial sets

$$NQ\mathcal{A} \wedge N\mathrm{Is}\,(\mathcal{B}) \longrightarrow NQ\mathcal{C}.$$

where $\mathcal{C}_n$ is the category equivalent to $\mathcal{C}$ in which an object is a sequence of n composable isomorphisms in $\mathcal{C}$.

LEMMA 16.2. *In this situation there is a natural transformation of simplicial exact categories*

$$\Gamma_{\mathcal{A}}\big(N\mathrm{Is}\,(\mathcal{B})\big) \longrightarrow \mathcal{C}.$$

satisfying that the following diagram of bisimplicial sets, involving the natural transformation 16.1, *commutes* (the source trisimplicial set is diagonalized along the S^0 and $N\mathrm{Is}\,(\mathcal{B})$ directions, to get a bisimplicial set):

$$\begin{array}{ccc} \mathrm{diag}_{(2,3)}\big(NQ\Gamma_{\mathcal{A}}(S^0) \wedge N\mathrm{Is}\,(\mathcal{B})\big) & \longrightarrow & NQ\Gamma_{\mathcal{A}}\big(N\mathrm{Is}\,(\mathcal{B})\big) \\ \| \downarrow & & \downarrow \\ NQ\mathcal{A} \wedge N\mathrm{Is}\,(\mathcal{B}) & \longrightarrow & NQ\mathcal{C}. \end{array}$$

If in particular $\mathcal{P}_R \times \mathcal{P}_{ZG} \to \mathcal{P}_{RG}$ *is induced from* $RG = R \otimes_Z ZG$, *and* $G \subset \mathrm{Is}\,(ZG) \subset \mathrm{Is}\,(\mathcal{P}_{ZG})$, *the following diagram also commutes*

$$\begin{array}{ccc} \Gamma_R(NG \cup *) & \xrightarrow{W(R,\,G)} & \mathcal{P}_{RG}. \\ \downarrow & & \downarrow \| \\ \Gamma_R\big(N\mathrm{Is}\,(\mathcal{P}_{ZG})\big) & \longrightarrow & \mathcal{P}_{RG}. \end{array}$$

Proof. The construction of the natural transformation is entirely analogous to that of the natural transformation $W(R, G)$ in 15.2. The asserted properties are immediate from the definitions.

The bi-exact pairing $\mathcal{A} \times \mathcal{B} \to \mathcal{C}$ induces, for every pointed simplicial set X, a bi-exact pairing

$$\Gamma_{\mathcal{A}}(X) \times \mathcal{B} \longrightarrow \Gamma_{\mathcal{C}}(X)$$

as well as an embedding of trisimplicial sets (using that $\Gamma_{\mathcal{C}}(X). \xrightarrow{=} \Gamma_{\mathcal{C}.}(X)$):

$$NQ\Gamma_{\mathcal{A}}(X) \wedge N\mathrm{Is}(\mathcal{B}) \longrightarrow NQ\Gamma_{\mathcal{C}.}(X)\,.$$

LEMMA 16.3. *There is a natural transformation of bisimplicial exact categories*

$$\Gamma_{\mathcal{A}}\big(X \wedge N\mathrm{Is}(\mathcal{B})\big) \longrightarrow \Gamma_{\mathcal{C}.}(X)$$

satisfying that

$$\begin{array}{ccc} & NQ\Gamma_{\mathcal{A}}\big(X \wedge N\mathrm{Is}(\mathcal{B})\big) & \\ \nearrow & & \searrow \\ NQ\Gamma_{\mathcal{A}}(X) \wedge N\mathrm{Is}(\mathcal{B}) & \longrightarrow & NQ\Gamma_{\mathcal{C}.}(X) \end{array}$$

commutes.

Proof. For pointed sets U, V, there is an equivalence of categories

$$\Gamma_{\mathcal{A}}(U \wedge V) \longrightarrow \Gamma_{(\Gamma_{\mathcal{A}}(U))}(V)$$

which forgets some of the choices. Taking this map in each degree gives the vertical arrow in the following diagram;

$$\begin{array}{ccccc} & & NQ\Gamma_{\mathcal{A}}\big(X \wedge N\mathrm{Is}(\mathcal{B})\big) & & \\ & \nearrow & \big\downarrow & \searrow & \\ NQ\Gamma_{\mathcal{A}}(X) \wedge N\mathrm{Is}(\mathcal{B}) & \dashrightarrow & \big| & \dashrightarrow & NQ\Gamma_{\mathcal{C}.}(X)\,. \\ & \searrow & \downarrow & \nearrow & \\ & & NQ\Gamma_{(\Gamma_{\mathcal{A}}(X))}\big(N\mathrm{Is}(\mathcal{B})\big) & & \end{array}$$

The solid arrow from bottom to right is the natural transformation of 16.2, taken in each degree in X, and the other solid arrow comes from 16.1. The broken arrows are filled in to make the diagram commutative.

The asserted natural transformation is defined as the broken arrow from top to right. The upper triangle in the diagram looks like the one whose commutativity has been asserted, but we are left to identify the horizontal arrow. That this arrow is as asserted is seen by applying the preceding lemma, in each degree in X, to the lower triangle.

Proof of assertion 15.8, *that* $\pi_1 K(NG; Z) \to K_1(ZG)$ *is the usual map.* The summand $K_1(Z)$ of $\pi_1 K(NG; Z) \approx K_1(Z) \oplus H_1\big(NG, K_0(Z)\big)$ is mapped correctly because of functoriality. To see how the other summand is mapped, we consider the diagram

$$\begin{array}{ccccc}
B((N\mathrm{Id}_R \cup *) \wedge (NG \cup *)) & \xrightarrow{=} & BG \cup * & & \\
\downarrow & & \downarrow & & \\
B(N\mathrm{Is}(\mathcal{P}_R) \wedge (NG \cup *)) & \longrightarrow & B\mathrm{Is}(\mathcal{P}_{RG\cdot}) & \searrow & \\
\downarrow & & | & & K_0(RG) \times BGL^+(RG) \, . \\
\Omega B(NQ\mathcal{P}_R \wedge (NG \cup *)) & \searrow & \downarrow & \swarrow & \\
\downarrow & & & & \\
\Omega BQ\Gamma_R(NG \cup *) & \longrightarrow & \Omega BQ\mathcal{P}_{RG\cdot} & &
\end{array}$$

The subdiagram of the solid arrows is commutative in view of Lemmas 16.2 and 9.2.4 and the inclusion $NG \cup * \subset N\mathrm{Is}(ZG) \cup * \subset N\mathrm{Is}(\mathcal{P}_{ZG})$. The broken arrow is a homotopy equivalence and the triangle containing it is homotopy commutative, by Lemma 9.3.3. Assume now that $R = Z$. The composed vertical map on the left is a kind of Hurewicz map. It sends $\pi_1 BG$ into the summand $H_1(NG, K_0(Z))$ of $\pi_1 K(NG; Z)$ because this summand can be identified with the kernel of $\pi_1 K(NG; Z) \to \pi_1 K(\mathrm{pt.}; Z)$. We claim that the map

$$G = \pi_1 BG \longrightarrow H_1(NG, K_0(Z)) \xrightarrow{=} G^{\mathrm{ab}}$$

is surjective; in fact it ought to be abelianization. Granting this, the assertion is now immediate from the commutativity of the diagram and the fact pointed out in the preceding section, that the induced map on the right, $G^{\mathrm{ab}} \to K_1(ZG)$, is injective.

To justify the claim one could check more details about the Hurewicz map. A quicker way is this. By the vanishing theorem for Whitehead groups, in Section 19, we know if F is a free group, $\pi_1 K(NF; Z) \to K_1(ZF)$ is an isomorphism. Hence by the commutative diagram above, $F \to H_1(NF, K_0(Z))$ is surjective. Let $F \to G$ be a surjection from a free group. Then $H_1(NF, K_0(Z)) \to H_1(NG, K_0(Z))$ is the induced surjection of the abelianized groups, and the diagram

$$\begin{array}{ccc}
F^{\mathrm{ab}} & \longrightarrow & H_1(NF, K_0(Z)) \\
\downarrow & & \downarrow \\
G^{\mathrm{ab}} & \longrightarrow & H_1(NG, K_0(Z))
\end{array}$$

establishes the claim.

Proof of assertion 15.9, *that* $\pi_* K(NG; Z) \to K_*(ZG)$ *coincides with the map in* [13]. In the diagram

$$\begin{array}{ccc}
N\mathrm{Is}(\mathcal{P}_R) \wedge N\mathrm{Is}(\mathcal{P}_{ZG}) & \longrightarrow & N\mathrm{Is}(\mathcal{P}_{RG}) \\
\downarrow & & \downarrow \\
(K_0(R) \times BGL^+(R)) \wedge (K_0(ZG) \times BGL^+(ZG)) & \dashrightarrow & K_0(RG) \times BGL^+(RG) \\
\downarrow & & | \\
\Omega BQ\mathcal{P}_R \wedge \Omega BQ\mathcal{P}_{ZG} & & |\simeq \\
\downarrow & & \downarrow \\
\Omega\Omega(BQ\mathcal{P}_R \wedge BQ\mathcal{P}_{ZG}) & \longrightarrow & \Omega\Omega BQQ\mathcal{P}_{RG} ,
\end{array}$$

the subdiagram of the solid arrows is commutative up to basepoint preserving homotopy, in view of 9.2.6 and 9.3.3, and there is a unique way, up to homotopy, to fill in the broken arrow so that the diagram stays homotopy commutative. Because of the homotopy commutativity of the upper square, the pairing given by the broken arrow coincides with the pairing in [13] in view of the very definition of the latter (actually, the pairing in [13] is well defined up to 'weak homotopy' only).

Let Σ: (rings) $\rightarrow$ (rings) denote the functor *suspension* of Karoubi and Σ^n its n-fold iteration. The only things we have to know about Σ are that it has been used by Loday in a way described below, and that there is a homotopy equivalence, due to Wagoner [24]

$$K_0(R) \times BGL^+(R) \longrightarrow \Omega BGL^+(\Sigma R)$$

which is functorial in R, further, that the resulting Ω-spectrum is connected if R is regular noetherian, especially if $R = Z$ [24] (due to the vanishing of the K_{-1} of Bass [3]).

In view of 9.3.3 and 9.2.5, and the inclusion $BG \cup * \subset B\mathrm{Is}(ZG) \cup * \subset B\mathrm{Is}(\mathcal{P}_{ZG})$, we have a homotopy commutative diagram

$$\begin{array}{l}
\varinjlim_m \Omega^m((K_0(\Sigma^m R) \times BGL^+(\Sigma^m R)) \wedge (BG \cup *)) \\
\qquad\qquad\downarrow \\
\varinjlim_m \Omega^m((K_0(\Sigma^m R) \times BGL^+(\Sigma^m R)) \wedge (K_0(ZG) \times BGL^+(ZG)))
\end{array}$$

$$\begin{array}{ccc}
\xrightarrow{\simeq} \varinjlim_m \Omega^m(\Omega BQ\mathcal{P}_{\Sigma^m R} \wedge (BG \cup *)) & \longrightarrow & \varinjlim_m \Omega^{m+1}(BQ\mathcal{P}_{\Sigma^m R} \wedge (BG \cup *)) \\
\downarrow & & \downarrow \\
\xrightarrow{\simeq} \lim_m \Omega^m(\Omega BQ\mathcal{P}_{\Sigma^m R} \wedge \Omega BQ\mathcal{P}_{ZG}) & \longrightarrow & \varinjlim_m \Omega^{m+1}(BQ\mathcal{P}_{\Sigma^m R} \wedge \Omega BQ\mathcal{P}_{ZG}) \\
& & \longrightarrow \varinjlim_m \Omega^{m+1} BQ\mathcal{P}_{\Sigma^m RG}. \\
& & \qquad\downarrow\simeq \\
& & \longrightarrow \varinjlim_m \Omega^{m+2} BQQ\mathcal{P}_{\Sigma^m RG}. \ .
\end{array}$$

The homotopy equivalences are those of Quillen's comparison theorem, cf. 9.3.3, and the de-looping of 9.1.2, respectively. As pointed out above, the pairing involved coincides with the one in [13]; therefore the composed map in the diagram, from the upper left to the right, gives the transformation of [13] in question, by definition of the latter (actually, only the induced map of homotopy groups is defined in [13]).

The map

$$\varinjlim_m \Omega^m(\Omega BQ\mathcal{P}_{\Sigma^m R} \wedge BX) \longrightarrow \varinjlim_m \Omega^{m+1}(BQ\mathcal{P}_{\Sigma^m R} \wedge BX)$$

is a weak homotopy equivalence for any pointed simplicial set X because it is a transformation of homology theories inducing an isomorphism of the coefficients. Therefore the transformation of [13] can also be given by the map

$$\varinjlim_m \Omega^{m+1} B\big(NQ\mathcal{P}_{\Sigma^m R} \wedge (NG \cup *)\big) \longrightarrow \varinjlim_m \Omega^{m+1} BQ\mathcal{P}_{\Sigma^m RG}.$$

Because of 16.1 and 16.3 we have the following commutative diagram whose bottom row is the de-loop of the map just given:

$$\begin{array}{ccccc}
 & & BQ\Gamma_R(NG \cup *) & \longrightarrow & BQ\mathcal{P}_{RG} \\
 & & \downarrow \simeq & & \downarrow \simeq \\
\varinjlim_n \Omega^n B\big(NQ\Gamma_R(S^n) \wedge (NG \cup *)\big) & \xrightarrow{\simeq} & \varinjlim_n \Omega^n BQ\Gamma_R\big(S^n \wedge (NG \cup *)\big) & \longrightarrow & \varinjlim_n \Omega^n BQ\Gamma_{\mathcal{P}_{RG}}(S^n) \\
\downarrow & & & & \downarrow \simeq \\
\varinjlim_{m,n} \Omega^{m+n} B\big(NQ\Gamma_{\Sigma^m R}(S^n) \wedge (NG \cup *)\big) & & \longrightarrow & & \varinjlim_{m,n} \Omega^{m+n} BQ\Gamma_{\Sigma^m RG}(S^n) \\
\uparrow & & & & \uparrow \simeq \\
\varinjlim_m \Omega^m B\big(NQ\mathcal{P}_{\Sigma^m R} \wedge (NG \cup *)\big) & & \longrightarrow & & \varinjlim_m \Omega^m BQ\mathcal{P}_{\Sigma^m RG}.
\end{array}$$

The claim is that the two vertical maps on the left are homotopy equivalences if $R = Z$; assertion 15.9 follows immediately from this. These maps are transformations of homology theories, evaluated on $(NG \cup *)$. If $R = Z$ both homology theories are connected, so it suffices to show the transformations are homotopy equivalences when evaluated at S^0; that is, the maps

$$\varinjlim_n \Omega^n BQ\Gamma_R(S^n) \longrightarrow \varinjlim_{m,n} \Omega^{m+n} BQ\Gamma_{\Sigma^m R}(S^n) \longleftarrow \varinjlim_n \Omega^m BQ\mathcal{P}_{\Sigma^m R}$$

are homotopy equivalences. By induction on Lemma 16.1, and direct limit, this diagram is homotopy equivalent to

$$BQ\Gamma_R(S^0) \longrightarrow \varinjlim_m \Omega^m BQ\Gamma_{\Sigma^m R}(S^0) \longleftarrow \varinjlim_m \Omega^m BQ\mathcal{P}_{\Sigma^m R}\ .$$

By induction on Wagoner's homotopy equivalence $BQ\mathcal{P}_R \to \Omega BQ\mathcal{P}_{\Sigma R}$, and direct limit, it follows that the first one of the latter maps is a homotopy equivalence. The second map is a homeomorphism. This completes the proof.

Remark. The argument shows in particular that generally the homology theory $X \mapsto \pi_* BQ\Gamma_R(X)$ coincides with the homology theory obtainable from

$$X \longmapsto \pi_* \lim_{\underset{m}{\longrightarrow}} \Omega^m B(NQ\mathcal{P}_{\Sigma^m R} \wedge X)$$

by making the latter connected.

17. *Decomposition theorems for Whitehead groups.*

17.1. *The case of free products with amalgamation.* Let $\alpha: G_0 \to G_1$, $\beta: G_0 \to G_2$ be monomorphisms of groups and $G = G_1 *_{G_0} G_2$ the associated free product with amalgamation. Let R be a fixed ring. We follow the notation of Section 15, $\Gamma_R^+(NG) = \Gamma_R(NG \cup *) = \Gamma_{\mathcal{P}_R}(NG \cup *)$.

By definition of a Γ-category there is a canonical map

$$Q\Gamma_R^+(NG_1 \cup NG_2) \longrightarrow Q\Gamma_R^+(NG_1) \times Q\Gamma_R^+(NG_2)$$

and this map is a homotopy equivalence since it is a weak equivalence of simplicial categories. The direct sum map on the category $\mathcal{P}_R$ induces a section of this map which is also a homotopy equivalence. Hence Lemma 14.4 may be reformulated thus;

LEMMA 17.1.1. *The commutative square of simplicial categories*

$$\begin{array}{ccc} Q\Gamma_R^+(NG_0) \times Q\Gamma_R^+(NG_0) & \longrightarrow & Q\Gamma_R^+(NG_1) \times Q\Gamma_R^+(NG_2) \\ \downarrow & & \downarrow \\ Q\Gamma_R^+(NG_0) & \longrightarrow & Q\Gamma_R^+(NG) \end{array}$$

is homotopy cartesian.

With the group algebras over R of the groups G_0, G_1, G_2, G, we are in a position to apply Proposition 11.4. Proposition 15.2 gives a natural transformation from the square 17.1.1 to the square 11.4 where by definition the transformation on the upper left term is the composed map

$$Q\Gamma_R^+(NG_0) \times Q\Gamma_R^+(NG_0) \longrightarrow Q\mathcal{P}_{RG_0^\cdot} \times Q\mathcal{P}_{RG_0^\cdot} \longrightarrow Q\mathcal{V}. .$$

The only non-commutativity in the resulting cubical diagram is in the square 11.4. Furthermore the homotopy between the two composed maps in this square restricts to the trivial homotopy on $Q\Gamma_R^+(NG_0) \times Q\Gamma_R^+(NG_0)$. We can thus formulate

THEOREM 17.1.2. *The non-commutative square of bisimplicial categories*

$$\begin{array}{ccc} QF.\big(\Gamma_R^+(NG_0) \times \Gamma_R^+(NG_0) \to \mathcal{V}.\big) & \longrightarrow & QF.\big(\Gamma_R^+(NG_1) \times \Gamma_R^+(NG_2) \to \mathcal{P}_{RG_1^\bullet} \times \mathcal{P}_{RG_2^\bullet}\big) \\ \downarrow & & \downarrow \\ QF.\big(\Gamma_R^+(NG_0) \to \mathcal{P}_{RG_0^\bullet}\big) & \longrightarrow & QF.\big(\Gamma_R^+(NG) \to \mathcal{P}^*_{RG^\bullet}\big) \end{array}$$

is homotopy cartesian with respect to a specific homotopy of the composed maps (the simplicial homotopy which to any object of $\mathcal{V}$ *associates its structure map).*

Proof. This is a formal consequence of other results. Technically, the proof is a special case of a generality on homotopy cartesian squares. The latter is most easily understood in a more general framework. So we give a little theory of 'homotopy cartesian cubes', and point out in the end how this implies the theorem.

Let I be the ordered set $(0 < 1)$ considered as a category, and $I^n = I \times \cdots \times I$ (n times). An n-ad *of spaces* is a functor

$$X: I^n \longrightarrow (\text{topological spaces}) .$$

Consider the n-cube as an n-ad C_n with

$$C_n(i_1, \cdots, i_n) = [0, i_1] \times \cdots \times [0, i_n]$$

where $[0, i]$ is the closed interval from 0 to i for $i = 0$ or 1. The *base* bC_n is the sub-n-ad of C_n given by those points in the cube which have at least one coordinate equal to 1. Let

$$f: bC_n \longrightarrow X$$

be a fixed map of n-ads. The *homotopy fibre* of X at f is the space X_f of maps

$$C_n \longrightarrow X$$

which restrict to f at bC_n. We say X is a *homotopy cartesian n-cube* if X_f is contractible for every f.

Let S be a proper subset of $\{1, \cdots, n\}$, of cardinality m say, and S' its complement. Denote by $X(S', 0)$ the $(n-m)$-ad given by restriction of X to

$$I^{S'} \times (0, \cdots, 0)$$

and let

$$f_0: bC_{n-m} \longrightarrow X(S', 0)$$

be a fixed map. Define the *derived* m-ad $X' = X(S/f_0)$ by letting $X'(i, \cdots, i_m)$ be the homotopy fibre of the $(n-m)$-ad

$$X \,|\, \big(I^{S'} \times (i_1, \cdots, i_m)\big)$$

at f_0 (resp. the image of f_0 in the $(n-m)$-ad under consideration).

Let $f_1: bC_m \to X(S/f_0)$. Then f_0 and f_1 combine to give a map (which is stationary on the coordinates indexed by S'),

$$f_0 \cup f_1: bC_n \longrightarrow X$$

and

$$X_{(f_0 \cup f_1)} \approx (X(S/f_0))_{f_1}$$

by the exponential law for mappings.

By the homotopy extension theorem, any f is homotopic to a map of the type $f_0 \cup f_1$, and again if f and f' are homotopic then $X_f \simeq X_{f'}$. Hence

LEMMA. *Let X be an n-ad, and S a proper subset of $\{1, \cdots, n\}$, of cardinality m. Then the following are equivalent*:

(i) *X is a homotopy cartesian n-cube;*

(ii) *For any $f_0: bC_{n-m} \to X(S', 0)$, the derived m-ad $X(S/f_0)$ is a homotopy cartesian m-cube.*

Consequence 1. Let the $(m + p)$-ad X be given as an m-ad of homotopy cartesian p-cubes. Then X is homotopy cartesian.

Consequence 2. Let Y be a p-ad satisfying that there is only one homotopy class of maps $bC_p \to Y$. Suppose there is an m-ad of p-ads one of which is Y, and where the other ones are homotopy cartesian p-cubes. Suppose further the $(m + p)$-ad is homotopy cartesian. Then Y is homotopy cartesian.

These observations extend to diagrams which are not necessarily commutative, but which are equipped with commuting homotopies in a suitable sense.

As to the theorem, each of the terms involved embeds in a homotopy cartesian square obtainable from Corollary 7.2; for example,

$$\begin{array}{ccc} Q\Gamma_R^+(NG) & \longrightarrow & QF.(\mathrm{Id}_{(\ldots)}) \\ \downarrow & & \downarrow \\ Q\mathcal{P}_{RG}^*. & \longrightarrow & QF.(\Gamma_R^+(NG) \to \mathcal{P}_{RG}^*.) \end{array}$$

is one of these. Putting these squares together, we obtain a $(2 + 2)$-ad which is homotopy cartesian by consequence 1 above. The theorem results now by application of consequence 2 above to this $(2 + 2)$-ad considered as a 2-ad in the other way: One of the squares is that of the theorem, and the other three squares are homotopy cartesian. In fact, these are just the two squares of Proposition 11.4 and Lemma 17.1.1, respectively, and another square in which all terms are contractible.

Under an extra assumption we can replace Theorem 11.3 by Corollary 11.5 in deriving the preceding theorem, and obtain a stronger result. We use the notation

$$W(R, G_i).: \Gamma_R^+(NG_i) \longrightarrow \mathcal{P}_{RG_i^\cdot}$$

of 15.2; also

$$W(R, G)^*_\cdot : \Gamma_R^+(NG) \longrightarrow \mathcal{P}^*_{RG^\cdot}\,.$$

Observe that

$$QF.(W(R, G_1).\times W(R, G_2).) \xrightarrow{\approx} QF.(W(R, G_1).) \times QF.(W(R, G_2).)\,.$$

COROLLARY 17.1.3. *Suppose the group algebra RG_0 is regular coherent. Then the commutative square of bisimplicial categories*

$$\begin{array}{ccc} QF.(W(R, G_0).) \times QF.(W(R, G_0).) & \longrightarrow & QF.(W(R, G_1).) \times QF.(W(R, G_2).) \\ \downarrow & & \downarrow \\ QF.(W(R, G_0).) & \longrightarrow & QF.(W(R, G)^*_\cdot) \end{array}$$

is homotopy cartesian.

Remarks 1. There is a fibration up to homotopy

$$QF.(W(R, G)^*_\cdot) \longrightarrow QF.(W(R, G).) \longrightarrow \mathcal{T}$$

where $\mathcal{T}$ is the group $\operatorname{coker}\big(K_0(RG_1) \oplus K_0(RG_2) \to K_0(RG)\big)$ considered as a bisimplicial category in a trivial way.

2. Taking geometric realization and passing to loop spaces, the corollary says that

$$\begin{array}{ccc} \mathrm{Wh}^R(G_0) \times \mathrm{Wh}^R(G_0) & \longrightarrow & \mathrm{Wh}^R(G_1) \times \mathrm{Wh}^R(G_2) \\ \downarrow & & \downarrow \\ \mathrm{Wh}^R(G_0) & \longrightarrow & \mathrm{Wh}^R(G)^* \end{array}$$

is homotopy cartesian, where $\mathrm{Wh}^R(G)^*$ is a certain union of components of $\mathrm{Wh}^R(G)$.

3. The argument of Theorem 11.6 carries over to show that in general one has a sequence of the homotopy type of a fibration

$$\Omega\widetilde{B}Q\mathcal{V} \times \mathrm{Wh}^R(G_0) \xrightarrow{\left(\substack{\text{pt.} \\ \mathrm{Wh}^R(\alpha_*),\ -\mathrm{Wh}^R(\beta_*)}\right)} \mathrm{Wh}^R(G_1) \times \mathrm{Wh}^R(G_2) \longrightarrow \mathrm{Wh}^R(G)^*$$

and that $\widetilde{B}Q\mathcal{V}$ is canonically a direct factor of $\mathrm{Wh}^R(G)^*$, up to homotopy.

17.2. *The case of* HNN *extensions.* Let $\alpha, \beta\colon G_0 \to G_1$ be monomorphisms of groups, and G the associated HNN extension, the pushout in the diagram of groupoids

$$\begin{array}{ccc} G_0 \cup G_0 & \xrightarrow{(\alpha,\beta)} & G_1 \\ \downarrow & & \downarrow \\ G_0 \times I & \longrightarrow & G\,. \end{array}$$

We identify G_1 with a subgroup of G by means of the right vertical map. The morphism in I from 0 to 1 maps to an element t of G. Let $\hat{t}$ denote conjugation by t, $\hat{t}(g) = tgt^{-1}$. Then $\alpha = \hat{t} \circ \beta$. Let R be a ring.

LEMMA 17.2.1. *The non-commutative square of simplicial categories*

$$\begin{array}{ccc} Q\Gamma_R^+(NG_0) \times Q\Gamma_R^+(NG_0) & \longrightarrow & Q\Gamma_R^+(NG_1) \\ \downarrow & & \downarrow \\ Q\Gamma_R^+(NG_0) & \longrightarrow & Q\Gamma_R^+(NG) \end{array}$$

is homotopy cartesian with respect to the simplicial homotopy of the composed maps from the upper left to the lower right, which is trivial on $Q\Gamma_R^+(NG_0) \times 0$, *and on* $0 \times Q\Gamma_R^+(NG_0)$ *is given by* $\hat{t}$.

Proof. In the diagram

$$\begin{array}{ccccc} Q\Gamma_R^+(NG_0) \times Q\Gamma_R^+(NG_0) & \longrightarrow & Q\Gamma_R^+(NG_0 \cup NG_0) & \longrightarrow & Q\Gamma_R^+(NG_1) \\ \downarrow & & \downarrow & & \downarrow \\ Q\Gamma_R^+(NG_0) & \longrightarrow & Q\Gamma_R^+(N(G_0 \times I)) & \longrightarrow & Q\Gamma_R^+(NG) \end{array}$$

the right hand square is the commutative homotopy cartesian square of Lemma 14.5. The upper left horizontal map is the section induced from the direct sum map on $\mathcal{P}_R$, of the natural homotopy equivalence which goes the other way. The lower left horizontal map is induced from the inclusion $\{0\} \subset \mathrm{Ob}(I)$ and is a homotopy equivalence by Lemma 14.2. The failure of commutativity of the left hand square is measured by the simplicial homotopy of maps

$$Q\Gamma_R^+(NG_0) \times Q\Gamma_R^+(NG_0) \longrightarrow Q\Gamma_R^+(N(G_0 \times I))$$

induced from the simplicial homotopy of maps $\{0 \cup 1\} \to NI$ which is trivial on 0, and moves 1 along NI. This gives precisely the asserted effect.

With the group algebras over R of the groups G_0, G_1, G, we are in a position to apply Proposition 12.4. Proposition 15.2 gives a natural transformation from the square 17.2.1 to the square 12.4 where by definition the transformation on the upper left term is the composed map

$$Q\Gamma_R^+(NG_0) \times Q\Gamma_R^+(NG_0) \longrightarrow Q\mathcal{P}_{RG_0^\bullet} \times Q\mathcal{P}_{RQ_0^\bullet} \longrightarrow \mathcal{V}_\bullet\,.$$

The only non-commutativity in the resulting cubical diagram is in the squares 17.2.1 and 12.4 themselves. We can thus formulate

THEOREM 17.2.2. *The non-commutative square of bisimplicial categories*

$$\begin{array}{ccc} QF_\bullet(\Gamma_R^+(NG_0) \times \Gamma_R^+(NG_0) \to \mathcal{V}_\bullet) & \longrightarrow & QF_\bullet(\Gamma_R^+(NG_1) \to \mathcal{P}_{RG_1^\bullet}) \\ \downarrow & & \downarrow \\ QF_\bullet(\Gamma_R^+(NG_0) \to \mathcal{P}_{RG_0^\bullet}) & \longrightarrow & QF_\bullet(\Gamma_R^+(NG) \to \mathcal{P}^*_{RG\bullet}) \end{array}$$

is homotopy cartesian with respect to a specific homotopy of the composed maps (the simplicial homotopy which to any object of $\mathcal{V}$ *associates its structure map).*

Proof. This is exactly the same as the proof of Theorem 17.1.2 except for one more point; namely the homotopies involved in the squares 17.1.1 and 12.4 must be compatible under the transformation. It suffices to check this on $Q\Gamma_R^+(NG_0) \times 0$ (where it is trivial) and on $0 \times Q\Gamma_R^+(NG_0)$. Let $(P, \cdots)$ be an object in degree n of $0 \times Q\Gamma_R^+(NG_0)$. The homotopy of the square 17.2.1 is given on $(P, \cdots)$ by the isomorphism

$$j_t\colon P \otimes_R RG \longrightarrow (P \otimes_R RG) \otimes_{RG\,\hat{t}} RG\ .$$

The compatibility condition is that under the transformation, j_t should go to the structure map of the object $\varphi_\beta(P)$ of $\mathcal{V}$ which is

$$0 \longrightarrow \big((P \otimes_R RG_0) \otimes_{RG_0\,\beta} RG_1\big) \otimes_{RG_1} RG \longrightarrow (P \otimes_R RG_0) \otimes_{RG_0} RG \longrightarrow 0\ .$$

But the structure map of this object is j_t, by definition.

COROLLARY 17.2.3. *Suppose the group algebra* RG_0 *is regular coherent. Then the non-commutative square of bisimplicial categories*

$$\begin{array}{ccc} QF.\big(W(R, G_0).\big) \times QF.\big(W(R, G_0).\big) & \longrightarrow & QF.\big(W(R, G_1).\big) \\ \downarrow & & \downarrow \\ QF.\big(W(R, G_0).\big) & \longrightarrow & QF.\big(W(R, G)^*_.\big) \end{array}$$

is homotopy cartesian with respect to the homotopy of the composed maps which is trivial on $QF.\big(W(R, G_0).\big) \times 0$, *and on* $0 \times QF.\big(W(R, G_0).\big)$ *is given by the isomorphism* j_t, *that is, the inner automorphism induced by conjugation by* t.

Remarks 1. The isomorphism j_t equals the identity only on zero objects.

2. There is a fibration up to homotopy

$$QF.\big(W(R, G)^*_.\big) \longrightarrow QF.\big(W(R, G).\big) \longrightarrow \mathcal{T}$$

where $\mathcal{T}$ is the group $\operatorname{coker}\big(K_0(RG_1) \to K_0(RG)\big)$ considered as a bisimplicial category in a trivial way.

3. In analogy to Theorem 12.6 one has a sequence of the homotopy type of a fibration

$$\Omega\tilde{B}Q\mathcal{V} \times \mathrm{Wh}^R(G_0) \xrightarrow{\binom{\text{pt.}}{\mathrm{Wh}(\alpha_*)+(-\mathrm{Wh}(\beta_*))}} \mathrm{Wh}^R(G_1) \longrightarrow \mathrm{Wh}^R(G)^*$$

and $\tilde{B}Q\mathcal{V}$ is canonically a direct factor of $\mathrm{Wh}^R(G)^*$, up to homotopy.

18. *The fundamental theorem.* This relates the K-theory of the ordinary Laurent extension of a ring to the K-theory of the polynomial extension. The interesting feature is that mention of exotic terms can be avoided altogether. The most direct formulation of the result is Theorem 18.1 below. On passage to geometric realization, or homotopy groups, the result can be given a more explicit formulation, especially if the product in K-theory is used.

Let R be a ring. We use two polynomial rings on R whose indeterminates we denote t and t^{-1}, respectively. Using the suggestive notation $R[t, t^{-1}]$ for the Laurent polynomial ring, we have natural embeddings

$$R[t] \longrightarrow R[t, t^{-1}] \longleftarrow R[t^{-1}] .$$

Let NZ be the nerve of the infinite cyclic group (the standard simplicial circle). Identifying $R[t, t^{-1}]$ to the group algebra RZ we have from 15.2 a map of simplicial exact categories

$$\Gamma_R(NZ \cup *) \longrightarrow \mathcal{P}^*_{R[t,t^{-1}]}.$$

where the star signifies that we are considering a certain cofinal subcategory. On checking the definition, cf. Section 10, one sees that an object of $\mathcal{P}_{R[t,t^{-1}]}$ is in $\mathcal{P}^*_{R[t,t^{-1}]}$ if and only if it is stably isomorphic to a projective that comes from $\mathcal{P}_R$.

From the natural embedding $\mathcal{P}_R \to \Gamma_R(NZ \cup *)$ (one can think of it as the composition of the isomorphism $\mathcal{P}_R \overset{\approx}{\leftarrow} \Gamma_R(\text{pt.} \cup *)$ and the map induced from pt. $\cup * \to NZ \cup *$), used twice, we obtain the left vertical map in the following diagram in which the terms in the upper row are regarded as simplicial exact categories in a trivial way:

$$\begin{array}{ccc} \mathcal{P}_R \times \mathcal{P}_R & \longrightarrow & \mathcal{P}_{R[t]} \times \mathcal{P}^*_{R[t^{-1}]} \\ \downarrow & & \downarrow \\ \Gamma_R(NZ \cup *) & \longrightarrow & \mathcal{P}^*_{R[t,t^{-1}]}. \end{array}$$

The upper horizontal map goes component by component. The vertical maps involve the direct sum map. The diagram is commutative.

THEOREM 18.1. *The commutative diagram of simplicial categories*

$$\begin{array}{ccc} Q\mathcal{P}_R \times Q\mathcal{P}_R & \longrightarrow & Q\mathcal{P}^*_{R[t]} \times Q\mathcal{P}^*_{R[t^{-1}]} \\ \downarrow & & \downarrow \\ Q\Gamma_R(NZ \cup *) & \longrightarrow & Q\mathcal{P}^*_{R[t,t^{-1}]}. \end{array}$$

is homotopy cartesian.

Proof. This results from formally putting together previous results. By Corollary 7.2 applied to the rows of the diagram, we obtain two homotopy cartesian squares

$$\begin{array}{ccc} Q\mathcal{P}_R \times Q\mathcal{P}_R & \longrightarrow & Q\mathcal{P}^*_{R[t]} \times Q\mathcal{P}^*_{R[t^{-1}]} \\ \downarrow & & \downarrow \\ QF.(\mathrm{Id}_{(\ldots)}) & \longrightarrow & QF.(\mathcal{P}_R \times \mathcal{P}_R \to \mathcal{P}^*_{R[t]} \times \mathcal{P}^*_{R[t^{-1}]}) \end{array}$$

and

$$\begin{array}{ccc} Q\Gamma_R(NZ \cup *) & \longrightarrow & Q\mathcal{P}^*_{R[t,t^{-1}]}. \\ \downarrow & & \downarrow \\ QF.(\mathrm{Id}_{(\ldots)}) & \longrightarrow & QF.\big(\Gamma_R(NZ \cup *) \to \mathcal{P}^*_{R[t,t^{-1}]}.\big) \end{array}$$

together with a map from the former square to the latter. Hence (cf. the generalities in the proof of Theorem 17.1.2) the square formed by the upper rows will be homotopy cartesian if and only if the square formed by the lower rows is. The latter square involves two contractible terms, so it will be homotopy cartesian if and only if the map

$$QF.(\mathcal{P}_R \times \mathcal{P}_R \to \mathcal{P}^*_{R[t]} \times \mathcal{P}^*_{R[t^{-1}]}) \longrightarrow QF.\big(\Gamma_R(NZ \cup *) \to \mathcal{P}^*_{R[t,t^{-1}]}.\big)$$

is a homotopy equivalence. We will prove this.

Applying Theorem 17.2.2 in the special case when $G_0 = G_1$ is the trivial group (and hence G the infinite cyclic group), and using that in this case $\Gamma_R(NG_0 \cup *)$ is just $\mathcal{P}_R$ considered as a simplicial category in a trivial way, we obtain a square of bisimplicial categories

$$\text{(T)} \qquad \begin{array}{ccc} QF.(\mathcal{P}_R \times \mathcal{P}_R \to \mathcal{V}.) & \longrightarrow & QF.(\mathcal{P}_R \to \mathcal{P}_R.) \\ \downarrow & & \downarrow \\ QF.(\mathcal{P}_R \to \mathcal{P}_R.) & \longrightarrow & QF.\big(\Gamma_R(NZ \cup *) \to \mathcal{P}^*_{R[t,t^{-1}]}.\big) \end{array}$$

and this square is homotopy cartesian with respect to a specific homotopy of the composed maps from the upper left to the lower right (that is to any object of $\mathcal{V}$ is associated its structure map).

The square (T) has an analogue (t) for the polynomial extension $R[t]$ (which is the special case of Sections 3 and 13 in which the bimodule is the ground ring itself). That is there is a natural transformation from the square made up of $\mathcal{P}_R$ and identity maps, to the square of Theorem 13.3. By the argument of the proof of Theorem 17.2.2, this gives a square

$$\text{(t)} \qquad \begin{array}{ccc} QF.(\mathcal{P}_R \to \mathcal{V}^t_.) & \longrightarrow & QF.(\mathcal{P}_R \to \mathcal{P}_R.) \\ \downarrow & & \downarrow \\ QF.(\mathcal{P}_R \to \mathcal{P}_R.) & \longrightarrow & QF.(\mathcal{P}_R \to \mathcal{P}^*_{R[t]}.) \end{array}$$

which is homotopy cartesian with respect to a specific homotopy of the composed maps. The notation $\mathcal{V}^t$ is used to avoid confusion with the $\mathcal{V}$ above.

The obvious natural transformation from the square (t) to the square (T) is compatible with the homotopies of the composed maps involved.

Interchanging the roles of t an t^{-1} we obtain another homotopy cartesian square (t^{-1}) and a natural transformation from (t^{-1}) to (T).

It was pointed out at the end of Section 2 that the natural map

$$\mathcal{V}^t \times \mathcal{V}^{t^{-1}} \longrightarrow \mathcal{V}$$

is an equivalence of categories. Hence the map $\mathcal{V}^t_\bullet \times \mathcal{V}^{t^{-1}}_\bullet \to \mathcal{V}_\bullet$ is a weak equivalence of simplicial categories. Applying Proposition 7.1 to the rows of the diagram

$$\begin{array}{ccc} \mathcal{P}_R \times \mathcal{P}_R & \longrightarrow & \mathcal{V}^t_\bullet \times \mathcal{V}^{t^{-1}}_\bullet \\ {\scriptstyle =}\downarrow & & \downarrow \\ \mathcal{P}_R \times \mathcal{P}_R & \longrightarrow & \mathcal{V}_\bullet \end{array}$$

gives the fact that

$$QF_\bullet(\mathcal{P}_R \to \mathcal{V}^t_\bullet) \times QF_\bullet(\mathcal{P}_R \to \mathcal{V}^{t^{-1}}_\bullet) \longrightarrow QF_\bullet(\mathcal{P}_R \times \mathcal{P}_R \to \mathcal{V}_\bullet)$$

is a homotopy equivalence.

Thus if we combine the squares (t) and (t^{-1}) by taking the cartesian product of corresponding terms, the natural transformation from (t) $\times$ (t^{-1}) to (T) is a homotopy equivalence on the upper left term. But it is a homotopy equivalence on two more terms, for in any of the squares (t), (t^{-1}), (T), the lower left and upper right terms are contractible, by Proposition 7.1. Hence the natural transformation is a homotopy equivalence on all four terms, and in particular we have proved that

$$QF_\bullet(\mathcal{P}_R \to \mathcal{P}^*_{R[t]\bullet}) \times QF_\bullet(\mathcal{P}_R \to \mathcal{P}^*_{R[t^{-1}]\bullet}) \longrightarrow QF_\bullet\big(\Gamma_R(NZ \cup *) \to \mathcal{P}^*_{R[t,t^{-1}]\bullet}\big)$$

is a homotopy equivalence. In view of the natural homotopy equivalence

$$QF_\bullet(\mathcal{P}_R \times \mathcal{P}_R \to \mathcal{P}^*_{R[t]} \times \mathcal{P}^*_{R[t^{-1}]}) \longrightarrow QF_\bullet(\mathcal{P}_R \to \mathcal{P}^*_{R[t]\bullet}) \times QF_\bullet(\mathcal{P}_R \to \mathcal{P}^*_{R[t^{-1}]\bullet})$$

this completes the proof of the theorem.

COROLLARY 18.2. *For $i \geqq 1$, there is a natural exact sequence*

$$0 \longrightarrow K_i(R) \xrightarrow{\Delta'} K_i(R[t]) \oplus K_i(R[t^{-1}]) \xrightarrow{\tilde{\Delta}} K_i(R[t, t^{-1}]) \longrightarrow K_{i-1}(R) \longrightarrow 0$$

where Δ' and $\tilde{\Delta}$ *denote the skew diagonal and the codiagonal of the natural maps, respectively, together with a natural splitting of the map*

$$K_i(R[t, t^{-1}]) \longrightarrow K_{i-1}(R)\ .$$

Proof. The cofibration sequence pt. $\cup * \to NZ \cup * \to (NZ, *)$ gives a fibration up to homotopy upon application of $Q\Gamma_R$, by Lemma 14.1. Because of the retraction $NZ \cup * \to$ pt. $\cup *$, we can compare this fibration to a split fibration and hence obtain the homotopy equivalence

$$Q\Gamma_R(NZ \cup *) \xrightarrow{\simeq} Q\mathcal{P}_R \times Q\Gamma_R(NZ, *)$$

using the fact that $Q\Gamma_R(\text{pt.} \cup *) \xrightarrow{\approx} Q\mathcal{P}_R$.

In view of this homotopy equivalence, the long exact sequence of homotopy groups of the homotopy theoretic fibration associated to the map

$$Q\mathcal{P}_R \times Q\mathcal{P}_R \longrightarrow Q\Gamma_R(NZ \cup *)$$

decomposes into exact sequences

$$0 \longrightarrow \pi_i BQ\mathcal{P}_R \xrightarrow{\Delta'} \pi_i(BQ\mathcal{P}_R \times BQ\mathcal{P}_R) \xrightarrow{\tilde{\Delta}} \pi_i BQ\Gamma_R(NZ \cup *) \longrightarrow \pi_i BQ\Gamma_R(NZ, *) \longrightarrow 0 .$$

Furthermore one has a canonical splitting of the map

$$\pi_i BQ\Gamma_R(NZ \cup *) \longrightarrow \pi_i BQ\Gamma_R(NZ, *) .$$

Also $BQ\mathcal{P}_R \xrightarrow{\simeq} \Omega BQ\Gamma_R(NZ, *)$ canonically, by 6.3 or 14.1.

On the other hand, the map of homotopy fibrations associated to the transformation between the vertical maps of the square of Theorem 18.1 induces a homotopy equivalence between the homotopy theoretic fibres since the square is homotopy cartesian. Also if $i \geqq 2$ then

$$\pi_i BQ\mathcal{P}^*_{R[t]} \xrightarrow{\approx} \pi_i BQ\mathcal{P}_{R[t]}$$

etc., by Proposition 7.4. The assertion of the corollary therefore follows by comparison of the long exact sequences of homotopy groups of the two homotopy fibrations.

Addendum 18.3. *The splitting $K_{i-1}(R) \to K_i(R[t, t^{-1}])$ of the map $K_i(R[t, t^{-1}]) \to K_{i-1}(R)$ of* 18.2, *can be induced by the product with the element of $K_1(Z[t, t^{-1}])$ represented by $t \in (Z[t, t^{-1}])$.*

Proof. In view of Remark 15.10 and associativity of the smash product one has a commutative square, for any group G,

$$\begin{array}{ccc} BQ\Gamma_Z(NG \cup *) \wedge BQ\mathcal{P}_R & \longrightarrow & BQQ\Gamma_R(NG \cup *) \\ \downarrow & & \downarrow \\ BQ\mathcal{P}_{ZG^\cdot} \wedge BQ\mathcal{P}_R & \longrightarrow & BQQ\mathcal{P}_{RG^\cdot} . \end{array}$$

Because of the homotopy equivalence similar to one of the preceding lemma

$$BQQ\Gamma_R(NG \cup *) \xrightarrow{\simeq} BQQ\mathcal{P}_R \times BQQ\Gamma_R(NG, *)$$

and the homotopy equivalence obtainable from Lemma 14.1,

$$BQQ\mathcal{P}_R \xrightarrow{\simeq} \Omega BQQ\Gamma_R(S^1, *) ,$$

this gives in the case when G is the infinite cyclic group, a homotopy commutative diagram

$$\begin{array}{ccc} BQ\mathcal{P}_Z \wedge BQ\mathcal{P}_R & \longrightarrow & BQQ\mathcal{P}_R \\ \simeq\downarrow & & \Big| \\ \Omega BQ\Gamma_Z(S^1, *) \wedge BQ\mathcal{P}_R & & \Big| \\ \downarrow & & \downarrow \\ \Omega BQ\mathcal{P}_{Z[t,t^{-1}]} \wedge BQ\mathcal{P}_R & \longrightarrow & \Omega BQQ\mathcal{P}_{R[t,t^{-1}]} \end{array}$$

in which the right vertical map is (a de-loop of) the section used in the preceding lemma, by definition of this section.

By 9.2.4 we have a commutative diagram

$$\begin{array}{ccc} B(\mathrm{Id}_Z \cup 0) \wedge BQ\mathcal{P}_R & \xrightarrow{=} & BQ\mathcal{P}_R \\ \downarrow & & \downarrow\simeq \\ \Omega BQ\mathcal{P}_Z \wedge BQ\mathcal{P}_R & \longrightarrow & \Omega BQQ\mathcal{P}_R \end{array}$$

in which the left hand vertical map is given by associating to $B(\mathrm{Id}_Z)$ the loop in $BQ\mathcal{P}_Z$ which is given by the pair of arrows $(0 \twoheadleftarrow Z; 0 \rightarrowtail Z)$ in $Q\mathcal{P}_Z$.

The assertion of the addendum is now immediate by combining this diagram with the one obtained from the preceding diagram by taking loop spaces.

19. *A vanishing theorem.* Recall a ring R is *regular coherent* if any finitely presented (right) R-module has a finite resolution by finitely generated projective R-modules, and *regular noetherian* if, in addition, any finitely generated R-module is finitely presented. For example, a theorem of Hilbert says if R is regular noetherian then so are the polynomial ring and Laurent polynomial ring on R, cf. [3].

Definition. A group G is *regular coherent* if, *for any regular noetherian ring* R, the group algebra RG is regular coherent. Similarly, G is *regular noetherian* if RG is, for any regular noetherian R.

For example, a finitely generated free abelian group is regular noetherian, by the theorem of Hilbert above. More generally, poly-Z-groups are regular noetherian (G is a poly-Z-group if there is a sequence of subgroups $1 = G_0 \subset G_1 \subset \cdots \subset G_n = G$, each normal in the next, so that each of the groups G_j/G_{j-1} is isomorphic to Z).

THEOREM 19.1. *Any of the following conditions* (i) *to* (iv) *is sufficient for the group G to be regular coherent.*

(i) $G = G_1 *_{G_0} G_2$ *where G_1 and G_2 are regular coherent and G_0 is regular noetherian.*

(ii) *G is the* HNN *extension constructed from embeddings* $G_0 \rightrightarrows G_1$ *where*

G_1 is regular coherent and G_0 is regular noetherian.

(iii) *G is the union of an increasing sequence of regular coherent subgroups.*

(iv) *G can be embedded in a regular coherent group.*

For example, iterated HNN extension from $G_0 \rightrightarrows G_1$ with G_0 trivial, and starting with G_1 trivial, gives precisely the finitely generated free groups. Thus free groups are regular coherent. As another example, let M be a connected 2-dimensional manifold. If M is not closed (i.e., compact without boundary), $\pi_1 M$ is free. If M is closed, and different from the 2-sphere and projective plane, $\pi_1 M$ is an HNN extension from $G_0 \rightrightarrows G_1$ where G_1 is finitely generated free, and G_0 is free cyclic. Thus with the exception only of the cyclic group of order 2, any such $\pi_1 M$ is regular coherent.

Proof of Theorem 19.1. A general fact to be noted is, if G_0 is a subgroup of G_1, there is a canonical splitting of RG_0-bimodules,

$$RG_1 \xrightarrow{\approx} RG_0 \oplus \widehat{RG}_1$$

where $\widehat{RG}_1$ is generated either as a left or as a right RG_0-module, by the elements of G_1 not in G_0. The bimodule $\widehat{RG}_1$ is both left free and right free. In particular, tensor product with RG_1 over RG_0 is an exact functor. In view of this remark, case (iii) of the theorem is obvious, and cases (i) and (ii) are special cases of Corollary 4.2.

To prove (iv), let R be a regular noetherian ring and suppose G is a subgroup of the regular coherent group G'. Let M be a finitely presented RG-module. Then M has a projective resolution over RG,

$$\longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

which is *i-good*, for some $i \geqq 1$, in the sense that P_j is finitely generated for $j \leqq i$. Then

$$\ker(P_i \to P_{i-1}) \otimes_{RG} RG' \xrightarrow{\approx} \ker(P_i \otimes_{RG} RG' \to P_{i-1} \otimes_{RG} RG')$$

is finitely generated since RG' is coherent and $P_i \otimes_{RG} RG'$ is finitely generated. Hence $\ker(P_i \to P_{i-1})$ is finitely generated. By induction it follows that M has a resolution which is i-good for any i. There is some n so that $\ker(P_n \to P_{n-1}) \otimes_{RG} RG'$ is projective since RG' is regular coherent. In view of the splitting $RG' = RG \oplus \widehat{RG}'$, this implies $\ker(P_n \to P_{n-1})$ is projective. It follows that M has the required kind of resolution, and the proof is complete.

Definition 19.2. Cl *is the smallest class of groups satisfying*:

(1) *The trivial group is in* Cl.

(2) *An* HNN *extension from* $G_0 \rightrightarrows G_1$ *is in* Cl *if* $G_1 \in \mathrm{Cl}$ *and* G_0 *is regular coherent.*

(3) *An amalgamated free product* $G_1 *_{G_0} G_2$ *is in* Cl *if* $G_1, G_2 \in \mathrm{Cl}$ *and* G_0 *is regular coherent.*

(4) Cl *is closed under filtering direct limit.*

PROPOSITION 19.3. (1) *The class* Cl *is closed under taking subgroups.*

(2) *The same class* Cl *is obtained if in cases* (2) *and* (3) *of Definition* 19.2, *it is required in addition that* $G_0 \in \mathrm{Cl}$.

Proof. Let a *splitting* of a group G consist of a pair of CW complexes (X, Y) satisfying

(i) Y is bicollared in X; that is, there is an open embedding $Y \times R^1 \to X$ where R^1 is the Euclidean line, so that $Y \times [-1, +1]$ is closed in X and $Y \times 0 = Y$.

(ii) X is connected, $\pi_1 X \approx G$, and $\pi_j X = 0$ if $j > 1$.

(iii) $\pi_j Y = 0$ at any basepoint if $j > 1$.

(iv) For any component Y_0 of Y, $\pi_1 Y_0 \to \pi_1 X$ is injective.

If Y is assumed connected, and a basepoint in Y and a normal direction to Y in X are chosen, then a homotopy equivalence class of splittings (preserving basepoint and normal direction) is the same as an expression of $\pi_1 X$ as an HNN extension from $\pi_1 Y \rightrightarrows \pi_1(X - Y)$ if $X - Y$ is connected; respectively, the same as an expression of $\pi_1 X$ as the free product with amalgamation from $\pi_1 X_1 \leftarrow \pi_1 Y \to \pi_1 X_2$ if $X - Y$ has components X_1 and X_2, and where the maps are the obvious ones.

In general then, a splitting can be thought of as a number of HNN extensions and/or amalgamated free products constructed one after the other and enumerated by the components of Y; cf. [26] for a fuller discussion of this. Thus if we generalize Definition 19.2 by allowing splittings in general, instead of just HNN extensions and amalgamated free products, we still obtain the same class Cl.

On the other hand, splittings are more flexible than HNN extensions and amalgamated free products. Specifically, if (X, Y) is a splitting of G, and G' any subgroup of G, we can form the pair (X', Y') where X' is the covering space of X with $\pi_1 X' = G'$, and Y' is the induced covering space of Y. Thus splittings are inherited by subgroups. In view of part (iv) of Theorem 19.1, this immediately implies the first assertion of the proposition. By tracking the construction of any particular group in Cl, it also implies the second assertion.

THEOREM 19.4. *If* R *is regular noetherian, and* $G \in \mathrm{Cl}$, *then* $\mathrm{Wh}^R(G)$ *is* *contractible.*

Proof. The definition of Cl can alternatively be interpreted as a construction of any of its members by transfinite induction, and in view of the preceding proposition, only such groups are used in the process which have been constructed before. The proof of the theorem can therefore be given by checking the assertion in any of the particular constructions to be performed. Of these, (1), that Wh^R(trivial group) is contractible, is true essentially by definition of $\mathrm{Wh}^R(G)$, and (4) is true since $\mathrm{Wh}^R(G)$ commutes with filtering direct limit.

(3) The assumption is that $G = G_1 *_{G_0} G_2$ where G_0 is regular coherent and $\mathrm{Wh}^R(G_i)$ is contractible for $i = 0, 1, 2$, and any regular noetherian ring R. By Corollary 17.1.3 there is a homotopy cartesian square in which three of the terms are contractible in view of the assumption just stated. The fourth term is $\mathrm{Wh}^R(G)^*$ which therefore is also contractible. Let R' be the group algebra over R of the infinite cyclic group. Then R' is regular noetherian since R is. So $\mathrm{Wh}^{R'}(G)^*$ is contractible by what has been established. In particular $\pi_1\mathrm{Wh}^{R'}(G)^* = 0$. But $\pi_1\mathrm{Wh}^{R'}(G)^*$ contains $\pi_0\mathrm{Wh}^R(G)$ as a direct summand, by Theorem 17.2.2 (cf. Remark 3 after Corollary 17.2.3). So $\pi_0\mathrm{Wh}^R(G) = 0$. Putting this together with the contractibility of $\mathrm{Wh}^R(G)^*$ shows that $\mathrm{Wh}^R(G)$ is contractible (cf. Remark 1 after Corollary 17.1.3).

(2) The argument of the preceding case carries over to this case. One just replaces Corollary 17.1.3 by Corollary 17.2.3 in the argument.

THEOREM 17.5. *The class* Cl *contains*

(1) *free groups,*

(2) *free abelian groups,*

(3) *poly-Z-groups,*

(4) *torsion free one-relator groups,*

(5) *fundamental groups $\pi_1 M$ if M is a 2-manifold different from the projective plane,*

(6) *fundamental groups $\pi_1 M$ if M is a compact orientable 3-manifold any irreducible summand of which either has non-empty boundary, or is simply connected, or is 'sufficiently large' in the sense of* [25],

(7) *fundamental groups $\pi_1 M$ if M is any submanifold of the 3-sphere,*

(8) *subgroups of the groups listed before.*

More curious examples are groups concocted à la [16] to illustrate unsolvability of certain decision problems in group theory. These have a marked tendency to be members of Cl. Of course this need not mean that

Cl is a very large class, it can mean as well that the groups in Cl are particularly tractable.

Proof of Theorem 19.5. (8) follows from the other assertions in view of Proposition 19.3. The cases (1), (2), (3) are clear.

Case (4). *One-relator groups.* It is implicit in the analysis of one-relator groups in [14], and will be made explicit in the lemma below, that if G is a torsion free one-relator group then there exists a sequence of groups

$$G = G_0, G'_0, G_1, G'_1, \cdots, G_n, G'_n, G_{n+1} = 1$$

so that G_j is a subgroup of G'_j and G'_j is an HNN extension obtained from $H_{j+1} \rightrightarrows G_{j+1}$ where H_{j+1} is a free group. By case (2) of Definition 19.2, $G_{j+1} \in \mathrm{Cl}$ implies $G'_j \in \mathrm{Cl}$ since a free group is regular coherent, and by Proposition 19.3, $G'_j \in \mathrm{Cl}$ implies $G_j \in \mathrm{Cl}$. So $G \in \mathrm{Cl}$, as asserted.

LEMMA. *Let G be presented as a one-relator group, with relator R. Suppose R is cyclically reduced and involves more than one generator. Then there exist one-relator groups G' and G_1, with relators R', R_1 so that*

(i) *G is a subgroup of G',*

(ii) *G' is an* HNN *extension obtained from $H \rightrightarrows G_1$ where H is free and finitely generated,*

(iii) *the relator R_1 of G_1 has smaller length than the relator R of G,*

(iv) *R is a k^{th} power if and only if R' is, if and only if R_1 is.*

Proof [14]. Let F be a free group on generators $\{t; a, b, c, \cdots\}$. To every element f of F one associates an integer $e_t(f)$, the *t-exponent sum*; it is the exponent of t in f after abelianizing and collecting.

Case 1. The assumption is that R involves t, and $e_t(R) = 0$. By definition, $G' = G$ in this case. There is a unique way of writing R as a reduced word of conjugates $t^k x t^{-k}$ of generators x different from t. We let F_1 be the free group on free generators

$$\{a_k, b_k, c_k, \cdots; n_1 \leqq k \leqq n_2\}$$

corresponding to the generators $\{a, b, c, \cdots\}$ of F different from t, where n_1 and n_2 are given, respectively, by the minimum and maximum numbers k occurring in the expression of R by the $t^k x t^{-k}$ above. There is a unique cyclically reduced word R_1 in the generators of F_1 so that on substituting

$$x_k \longmapsto t^k x t^{-k}$$

we recover R. By definition, G_1 is the one-relator group with generators F_1 and relator R_1. We can recover G from G_1 as the group with presentation

$$\langle F_1, t; R_1 = 1, tHt^{-1} = H' \rangle$$

where H and H' are the subgroups of F_1 with generators

$$\{a_k, b_k, \cdots; n_1 \leqq k \leqq n_2 - 1\}, \quad \{a_k, b_k, \cdots; n_1 + 1 \leqq k \leqq n_2\}$$

respectively. n_2 has been chosen so that there is at least one generator of F_1 which occurs in R_1 but is not contained in H. Therefore $H \to G_1$ is an embedding by the 'Freiheitssatz' [14, Theorem 4.10]. Similarly $H' \to G_1$ is an embedding. Hence G is an HNN extension in the way asserted.

Case 2. The assumption is that $e_x(R) \neq 0$ for any generator x of F involved in R. Suppose that $e_t(R) = \tau$, $e_a(R) = \alpha$, both $\neq 0$. We let F' be the free group on the same generators as F, except that t is replaced by u, and we embed F in F' by sending t to u^α. We let R' be the relator obtained by replacing t in R by u^α, and by definition, G' in the one-relator group with generators F' and relator R'. The induced map $G \to G'$ is an embedding by [14, Corollary 4.10.2]. Next we replace the generator a of F' by $v = au^\tau$. By substituting accordingly $a = vu^{-\tau}$ in R', the relator is changed to R'', and $e_u(R'') = 0$. We can apply case 1 now. To finish the argument we have to check that the relator R_1 finally obtained has smaller length than R. Indeed, its length is equal to the length of R with all occurrences of t discarded.

Case (5) *of theorem*: *2-manifold groups*. Ignoring that the present case is a special case of (4), we will give a more geometric argument. The reason is that the same argument applies in the case of 3-manifolds where however the simplicity of the argument is obscured somewhat by the tools that must be used to justify its working.

Let M be a compact connected 2-manifold and C a properly embedded circle in M which is 2-sided (that is, separates a neighborhood). Then we can *cut M at C*, to produce a manifold M' which (hopefully) is simpler. The cutting process can be made very concrete when one draws a picture and uses scissors. Mathematically it is easier to describe the reverse process of reconstructing M from M'. Namely, M' comes equipped with two embeddings of S^1 onto boundary curves C_1 and C_2, and

$$M = \operatorname{colim}(S^1 \rightrightarrows M')$$

and the common image of C_1 and C_2 in M is just C.

The fundamental groupoid commutes with colimits, so

$$\pi M = \operatorname{colim}(\pi S^1 \rightrightarrows \pi M') .$$

Choosing basepoints and connecting paths as required, we may also talk of fundamental groups. The preceding formula implies if both $\pi_1 C_1 \to \pi_1 M'$

and $\pi_1 C_2 \to \pi_1 M'$ are injective then $\pi_1 C \to \pi_1 M$ is injective, and conversely.

Hence provided that $\pi_1 C \to \pi_1 M$ is injective, the cutting of M at C gives rise to the reverse process of building up $\pi_1 M$ from simpler constituents by either amalgamated free product or HNN extension (depending on whether C does or does not separate M).

If the boundary of M is not empty, there is a variant of the cutting process where we take C to be a properly embedded arc. The condition that $\pi_1 C \to \pi_1 M$ be injective is trivially satisfied in this case. So the same conclusion about $\pi_1 M$ is valid.

The assertion of case (5) is that, in general, $\pi_1 M$ can be entirely built up from the trivial group by iterating the process of taking either an HNN extension or an amalgamated free product (with the extra condition that the amalgamation groups must be regular coherent). The assertion follows at once if we can show that, in general, the cutting process described can be performed on M, and can be iterated to reduce M to a disk (or a number of disks).

A hypothesis is required to get the cutting process started, namely that M be sufficiently large to contain the C required, with $\pi_1 C \to \pi_1 M$ injective (there are two exceptional cases: the 2-sphere and the projective plane). But once the process has been started it can be continued. For after the first step we are left with a 2-manifold with non-empty boundary, and if this is not a disk already, we can find an appropriate embedded arc since a compact 2-manifold with boundary may be represented as a disk with bands.

The amalgamation groups $\pi_1 C$ that occur in the process are either trivial or free cyclic. So in addition to what case (5) asserts we can also conclude that $\pi_1 M$ is regular coherent.

Case (6) *of theorem*: *3-manifold groups*. Let M be a compact connected 3-manifold which for simplicity we assume is orientable. Suppose we can find a properly embedded connected 2-manifold F in M which is 2-sided and satisfies that $\pi_1 F \to \pi_1 M$ is injective. Then as in case (5) above we can draw the conclusion that $\pi_1 M$ is either an amalgamated free product or an HNN extension, depending on whether F does or does not separate M.

Further, if by iterated cutting of this kind we can reduce the given M to simply connected pieces then it follows that $\pi_1 M$ can be built up from the trivial group by iterated HNN extension and/or amalgamated free product. This gives the conclusion of case (6). For the amalgamation groups are 2-manifold groups $\pi_1 F'$ where any F' is a 2-sided submanifold in an orientable manifold and hence itself orientable. In particular F' cannot be the projective plane, so $\pi_1 F'$ is regular coherent.

The fact is that a surprisingly large number of 3-manifolds can be cut to pieces in the way required. This depends on a number of theorems in 3-manifold theory (which, e.g., can be found in [10]). Here are a few relevant notions and a review of the argument. Recall we are assuming M is compact (and connected) and orientable.

M is called a connected sum of M_1 and M_2 (notation $M \approx M_1 \# M_2$) if it can be obtained by removing the interior of a 3-ball from each of M_1 and M_2, and gluing the resulting 2-sphere boundary components. This is equivalent to the possibility of cutting M at a properly embedded 2-sphere into two parts, obtainable by removing an open ball from each of M_1 and M_2, respectively. In particular,

$$\pi_1(M_1 \# M_2) \approx \pi_1 M_1 * \pi_1 M_2 .$$

A classical result of H. Kneser says that M has a (maximal) connected sum decomposition

$$M \approx M_1 \# \cdots \# M_m$$

where each of the summands is non-trivial (i.e., different from the 3-sphere) and prime for connected sum, that is, it cannot itself be non-trivially obtained by connected sum. Note that if the Poincaré conjecture were known to be true (that a closed M is necessarily S^3 if it is simply connected) the existence of a maximal decomposition would follow from Grushko's theorem. Other facts in this context are that connected sum is well defined if orientations are taken care of, and that the maximal decomposition is essentially unique; but we do not need these.

One says M is *irreducible* if every properly embedded 2-sphere in M bounds a 3-ball in M. If M is irreducible then it is prime for connected sum, and the converse is almost true: $S^1 \times S^2$ is the only M which is prime but not irreducible. Hence we may (and will) restrict attention to irreducible manifolds.

If M is irreducible and has non-empty boundary then $\pi_1 M \in \mathrm{Cl}$.

For if M has a 2-sphere boundary component it must be isomorphic to the 3-ball (why ?). If M has a boundary component different from the 2-sphere, a beautiful application of Poincaré duality (noted already in the classical book by Seifert and Threlfall) shows $H^1 M \neq 0$. Hence there is a non-trivial map $M \to S^1$ and consequently, by transversality, there is a properly embedded 2-manifold F in M which is dual to a non-trivial element of H^1M. Using the theorems of Papakyriakopoulos, one can modify F, if necessary, to satisfy that $\pi_1 F \to \pi_1 M$ is injective. So M can indeed be cut in the way required. Denoting M' the manifold obtained by cutting, it may be shown that M' is irreducible again, and of course M' has non-empty boundary. So we are in the same position as before, but with M replaced by M', and we may proceed inductively to cut M', and so on. On the other hand, a theorem of Haken [8] says that the cutting process cannot be continued indefinitely. But as mentioned before, the only way for the cutting process to stop is that only a ball (or a number of balls) is left over. This completes the argument.

Let M be a closed irreducible manifold and suppose it contains a properly embedded 2-sided 2-manifold F (connected and $\neq S^2$) so that $\pi_1 F \to \pi_1 M$ is injective. Then we may cut M at F to obtain M', say. The latter is irreducible again and has non-empty boundary. And so the cutting process can be performed all the way, by the preceding argument applied to M'. But this means that everything depends just on the existence of that very first F. The manifold M is called *sufficiently large* if such an F exists.

For example, F cannot exist if $\pi_1 M$ is finite. If on the other hand $\pi_1 M$ is not finite (and M irreducible) it appears that 'in general' M may be expected to be sufficiently large: Certainly it is very difficult to construct M which are not, and only a very few such are known to date; besides, each of the known examples has a finite covering space which *is* sufficiently large.

Not very surprisingly, M is sufficiently large if and only if $\pi_1 M$ is either non-trivially an amalgamated free product or an HNN extension (or both). For example if $\pi_1 M$ is an HNN extension then $H^1M \neq 0$ and this implies M is sufficiently large, as indicated before.

Case (7) *of theorem*: *Submanifolds of the* 3-*sphere*. By a direct limit argument it suffices to consider compact submanifolds. Let M be one. If M is not irreducible it must be non-prime since $S^1 \times S^2$ does not embed in S^3. So suppose $M \approx M_1 \# M_2$. By the Schoenfliess theorem, any proper embedding of S^2 in S^3 is equivalent to the standard embedding (or what amounts to the same thing, the S^2 decomposes S^3 into two 3-balls). This implies that M_1 and M_2 also embed in S^3. As $\pi_1 M \approx \pi_1 M_1 * \pi_1 M_2$ we can thus inductively reduce to the irreducible case. So assume M is irreducible, and embeds in S^3. Then either $M = S^3$ and there is nothing to prove, or M has non-empty boundary and the preceding case applies.

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