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# Geometric Classification of Simplicial Structures on Topological Manifolds

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## 0. Introduction

In this paper we are interested in the triangulation problem for manifolds: when is a topological manifold $M$ homeomorphic to a polyhedron. If such a polyhedron happens to be a combinatorial manifold we say that $M$ admits a PL structure; otherwise $M$ admits a TRI structure. A precise notion of a TRI structure will be introduced shortly.

Here is an informal short review of what is known at this time about the triangulation problem. Each $m$-manifold, $m \leqq 3$, does admit a unique PL structure. The existence for 2-manifolds was proven in 1925 by T. Radó, and uniqueness in 1943 by C. D. Papakyriakopoulos. In 1952 E. E. Moise showed the existence of PL structures on all 3-manifolds, and in 1954 the uniqueness was proven independently by R. H. Bing and E. E. Moise. R. C. Kirby and L. C. Siebenmann showed in 1969 that in each dimension greater than four there exist closed topological manifolds that do not admit a PL structure. They also discovered that the PL structures are not always unique. In 1974 R. D. Edwards found noncombinatorial triangulations of $R^n$, $n \geqq 5$. D. E. Galewski and R. J. Stern investigated the question about noncombinatorial triangulations in greater detail in [2]. They obtained Theorems 0.1–0.3. T. Matumoto proved Theorem 0.1 independently in [7].

The first step in [2] is to prove a TRI product structure theorem which constructs from a given TRI manifold structure on $M \times R^q$ a related TRI manifold structure on $M$. In the second step an appropriate bundle theory technique is developed. It turns out that the work of Kirby and Siebenmann [4] provides enough material to avoid the bundle theory in [2], and the object of this paper is to provide more geometric proofs of Theorems (0.1)–(0.3).

In particular, we construct simplicial triangulations of arbitrary manifolds of sufficiently large dimensions by using the TRI product structure theorem, the handle structure of topological manifolds, and the existence of a homology 3-sphere $H^3$ as in (0.1). By using the Kirby-Siebenmann obstruction [3] to PL triangulating a topological manifold and the Cohen-Sullivan obstruction [6] to PL resolving a polyhedral homology manifold we define an obstruction cochain to

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the existence of a TRI manifold structure on a topological manifold $M$. Its cohomology class turns out to be the same as the Galewski-Stern obstruction. We also construct a difference cocycle for any pair of TRI manifold structures on $M$. The cohomology classes of all these difference cocycles form $H^4(M;\mathrm{Ker}(\alpha))$, and they are in one-to-one correspondence with TRI manifold structures on $M$. We also obtain a product formula for the Galewski-Stern obstruction.

This paper is a slightly generalized version of my dissertation written under the guidance of David E. Galewski. To him I would like to express my sincerest thanks.

We now introduce some definitions. By a *manifold* we always mean a metrizable topological manifold. A locally finite polyhedron $M$ is a TRI *manifold* if it is also a topological manifold. A TRI *manifold structure* $\Sigma$ (TRI structure, for short) on a topological manifold $M$ is a maximal covering family of piecewise-linearly related embeddings of compact polyhedra. If there is a TRI manifold structure $\Sigma$ on $M$, then there exists a locally finite complex $K$ such that $|K|$ is a TRI manifold, and a homeomorphism $h:|K|\to M$ which is piecewise-linearly related to every element of $\Sigma$. Moreover, such a homeomorphism $h$ determines a TRI manifold structure on $M$, and also provides $M$ with a simplicial triangulation. Thus, we will use "TRI manifold structure" and "simplicial triangulation" as synonyms. Note that if $\Sigma$ is a TRI manifold structure on $M$ and $U$ is an open subset of $M$, then $\Sigma$ restricts to a TRI manifold structure on $U$. Let $N$ be a codimension zero clean (see [4, p. 12] for "clean") submanifold of $\partial M$. We say that $\Sigma$ is a TRI manifold structure on $M$ *near* $N$ if $\Sigma$ is a TRI manifold structure on a neighborhood $U$ of $N$ in $M$ which restricts to a TRI manifold structure on $N$. Let $\Sigma_0$ and $\Sigma_1$ be TRI manifold structures on $M$, and $\Psi$ a TRI manifold structure on $M$ near $N$. We say that $\Sigma_0$ and $\Sigma_1$ are TRI *concordant* rel $\Psi$ provided there exists a TRI manifold structure $\Phi$ on $M\times I$ whose restriction to $M\times\{i\}$ is $\Sigma_i$, $i=0,1$, and $\Phi$ equals to $\Psi\times I$ on a neighborhood of $N\times I$ in $M\times I$. $\Phi$ is called a *TRI concordance* between $\Sigma_0$ and $\Sigma_1$. TRI concordance is obviously an equivalence relation which determines TRI *concordance* classes of TRI manifold structures on a given manifold $M$.

We denote by $\theta$ the abelian group of oriented PL $H$-cobordism classes of oriented homology 3-spheres modulo those which bound PL acyclic 4-manifolds. The group operation in $\theta$ is induced by taking connected sums. Let $\alpha:\theta\to Z_2$ be the Kervaire-Milnor-Rohlin epimorphism $\alpha(H^3)=\sigma(W^4)/(\mathrm{mod}\,2)$, where $\sigma(W^4)$ is the signature of any parallelizable PL 4-manifold $W^4$ that $H^3$ bounds.

**Theorem 0.1** [2, 7]. *Every topological $m$-manifold, $m\geqq 7$ ($m\geqq 6$ if $\partial M$ compact, or $m\geqq 5$ if $M$ is closed) can be triangulated as a simplicial complex iff there exists a homology 3-sphere $H^3$ such that*

(i) $\alpha(H^3)=1$, *and*

(ii) $H^3\# H^3$ *bounds an acyclic PL 4-manifold.*

**Theorem 0.2** [2]. (a) *Let $M$ be a topological $m$-manifold with $N$ a codimension zero clean submanifold of $\partial M$ such that a neighborhood of $N$ in $M$ is simplicially triangulated. Let $m\geqq 7$ ($m\geqq 6$ if* $\mathrm{cl}(\partial M-N)$ *is compact, or $m\geqq 5$ if $M$ is closed).*

*Then there exists a well-defined* $\nabla(M,N)\in H^5(M,N;\mathrm{Ker}(\alpha))$ *such that* $\nabla(M,N)=0$ *iff there exists a simplicial triangulation of* $M$ *compatible with the given one near* $N$.

(b) $\nabla(M)=\beta(\Delta(M))$, *where* $\Delta(M)$ *is the Kirby-Siebenmann obstruction to the existence of a PL structure on* $M$, *and* $\beta$ *is the Bockstein homomorphism associated with the short exact coefficient sequence* $0\to\mathrm{Ker}(\alpha)\to\theta\xrightarrow{\alpha}Z_2\to 0$.

**Theorem 0.3** [2]. *Let* $M$ *and* $N$ *be as in* (0.2). *Then the TRI concordance classes of simplicial triangulations of* $M$ *compatible with a given triangulation of* $N$ *are in one-to-one correspondence with the elements of* $H^4(M,N;\mathrm{Ker}(\alpha))$.

We call $\nabla(M,N)$ the Galewski-Stern obstruction.

## 1. Some Basic Theorems

In this section we recall from [2] some basic facts about the relationship between homology manifolds and topological manifolds. We will also find a simple relation between the Cohen-Sullivan obstruction to resolving a homology manifold into a PL manifold and the Kirby-Siebenmann obstruction to putting a PL manifold structure on a topological manifold (see [8] for homology manifolds, and [6] for resolutions of homology manifolds).

**Proposition 1.1** [2]. *If* $M$ *is a simplicially triangulated* $m$*-manifold, then* $M$ *is a homology* $m$*-manifold.*

**Theorem 1.2** [2]. *A homology* $m$*-manifold* $M$ *with a collared boundary,* $m\geqq 5$ ($m\geqq 6$ *if* $\partial M\neq\emptyset$) *is a topological* $m$*-manifold if and only if for each vertex* $v$ *of any triangulation* $K$ *of* $M$ *both* $\mathrm{lk}(v,K)$ *and* $\mathrm{lk}(v,\partial K)$ *are simply connected.*

**Theorem 1.3** [2]. *Let* $H^m$ *be a compact homology* $m$*-manifold without boundary having the integral homology of the* $m$*-sphere. If* $m\geqq 4$, *then* $H^m$ *bounds a contactible homology* $(m+1)$*-manifold* $V$ *such that* $V-\partial V$ ($\partial V=H^m$) *is a topological* $(m+1)$*-manifold.*

With the help of (1.2) and (1.3) we will be able to replace homological manifolds with simple homotopy equivalent topological manifolds.

We will construct triangulations using the notion of a handlebody decomposition [4]. A topological manifold of a sufficiently large dimension always has a handlebody structure. More precisely

**Theorem 1.4** [4]. *Let* $M$ *be an* $m$*-manifold and* $N$ *a (possibly empty) clean* $m$*-submanifold. Then* $M$ *is a handlebody on* $N$ *provided* $m\geqq 6$.

Let $f: B^k\times B^{m-k}\to M^m$ be the homeomorphism that determines the $k$-handle $h^k=f(B^k\times B^{m-k})$. We will use the following notation:

$S_a(h^k)=f(\partial B^k\times\{0\})$ – the attaching sphere of $h^k$,
$T_a(h^k)=f(\partial B^k\times B^{m-k})$ – the attaching tube of $h^k$,
$S_b(h^k)=f(\{0\}\times\partial B^{m-k})$ – the belt sphere of $h^k$, and
$T_b(h^k)=f(B^k\times\partial B^{m-k})$ – the belt tube of $h^k$.

We will reduce the general triangulation problem to the problem of extending TRI structures on $T_a(h^k)$ to TRI structures on the whole $k$-handle $h^k$.

**Theorem 1.5** [2]. (Product Structure Theorem.) *Let $M^m$ be a connected topological m-manifold and let $\Theta$ be a* TRI *manifold structure on $M \times R^q$. Let $N$ be a codimension zero submanifold of $\partial M$ and $\Sigma_0$ a* TRI *manifold on $M$ near $N$ such that $\Sigma_0 \times R^q$ agrees with $\Theta$ near $N \times R^q$. If $m \geqq 7$ ($m \geqq 6$ if $\mathrm{cl}(\partial M - N)$ is compact or $m \geqq 5$ if $M$ is closed), then there exists a* TRI *manifold structure $\Gamma$ on $M$ coinciding with $\Sigma_0$ near $N$, unique up to concordance* $\mathrm{rel}\,\Sigma_0$, *with $\Gamma \times (-1,1)^q$ concordant* $\mathrm{rel}\,\Sigma_0 \times (-1,1)^q$ *to $\Theta | M \times (-1,1)^q$.*

Rather than Theorem 1.5 we will use time and again the following

**Corollary 1.6.** *Let $N$ be a bicollared codimension one proper submanifold of a topological m-manifold $M$, $m \geqq 7$. Let $\Theta$ be a* TRI *manifold structure on $M$. Then there exists a* TRI *concordance $\Phi$ on $M \times I$ such that $\Phi | M \times \{0\} = \Theta$ and $\Phi | M \times \{1\}$ restricts to a* TRI *manifold structure $\Gamma$ on $N \times \{1\}$. Moreover, if $\Theta$ restricts to a TRI manifold structure $\Sigma_0$ on $\partial M$ near $\partial N = \partial N \cap \partial M$, then $\Phi$ can be chosen* $\mathrm{rel}\,\Sigma_0$. *In other words, $N$ can be made into a* TRI *submanifold of $M$ within the TRI concordance class of $\Theta$.*

*Proof.* First assume $\partial N = \emptyset$. Since $N$ is bicollared in $M$ we have that $N \times R$ is an open subset of $M$. Thus $\Theta | N \times R$ provides a TRI structure on $N \times R$. Let $\Gamma$ be the TRI structure on $N$ and $\Psi$ the TRI concordance on $N \times (-1,1) \times [0,1]$ between $\Theta | N \times (-1,1)$ and $\Gamma \times (-1,1)$ provided by Theorem 1.5 for $q=1$. Let $\Xi$ be the TRI structure on $X = M \times [-5,0] \bigcup_{N \times (-1,1) \times \{0\}} N \times (-1,1) \times [0,4]$ such that

$\Xi | M \times [-5,0] = \Theta \times [-5,0]$,

$\Xi | N \times (-1,1) \times [0,1] = \Psi$, and

$\Xi | N \times (-1,1) \times [1,4] = \Gamma \times (-1,1) \times [1,4]$.

Let $f : X \to X$ be a homeomorphism which pushes $N \times (-1,1) \times [0,4]$ into $M \times [-5,0]$. More precisely, we can and do choose $f$ such that

$f(y) = y$, $y \in M \times \{-5\}$,

$f(x,a,b) = (x,a,b-4)$, $(x,a,b) \in N \times [-\frac{1}{2},\frac{1}{2}] \times [0,4]$, and

$f(X) = M \times [-5,0] - N \times ([1,2) \cup (-2,-1]) \times \{0\}$.

Let $\Omega = f(\Xi) | M \times [-5,0)$ be a TRI structure on $M \times [-5,0)$. Note that $\Omega$ has the following properties:

$\Omega | M \times \{-5\} = \Theta$, and

$\Omega | N \times [-\frac{1}{2},\frac{1}{2}] \times [-3,-1] = \Gamma \times [-\frac{1}{2},\frac{1}{2}] \times [-3,-1]$.

Let $D = \{(x,y) | (x+2)^2 + y^2 \leqq 1\} \subset [-5,0) \times R$. It is easy to find a PL ambient isotopy $G_t$ of $[-5,0) \times R$ which keeps $[-5,0) \times R - \mathrm{int}\, D$ fixed and such that $G_1$ rotates $\frac{1}{2}D = \{(x,n) | (x+2)^2 + y^2 \leqq \frac{1}{4}\}$ around its center by 90°. Let $\mathrm{id}_M \times G_t$ be an ambient isotopy of $M \times [-5,0) \times R$, and let $\Omega_1 = (\mathrm{id}_M \times G_1)(\Omega \times R)$ be a TRI structure on $M \times [-5,0) \times R$. Properties of $\Omega$ and $G_t$ show that $\Omega_1$ restricts to a TRI structure $\Omega_2$ on $M \times [-5,-2] \times (-\frac{1}{4},\frac{1}{4})$. We have

$\Omega_2 | M \times \{-5\} \times (-\frac{1}{4},\frac{1}{4}) = \Theta \times (-\frac{1}{4},\frac{1}{4})$, and

$\Omega_2 | N \times [-\frac{1}{2},\frac{1}{2}] \times \{0\} \times (-\frac{1}{4},\frac{1}{4}) = \Gamma \times [-\frac{1}{2},\frac{1}{2}] \times (-\frac{1}{4},\frac{1}{4})$.

We now apply Theorem 1.5 to $\Omega_2$ to obtain a TRI structure $\Phi_1$ on $M \times [-5,-2]$. Let $g : [-5,-2] \to [0,1]$ be the linear map. Then $\Phi = (\mathrm{id}_M \times g)(\Phi_1)$ is the desired TRI structure on $M \times [0,1]$.

Now assume $\partial N \neq \emptyset$, and that $\Theta$ already restricts to a TRI structure $\Sigma_0$ on $\partial M$ near $\partial N$. Let $\Sigma_1$ denote the resulting TRI structure on $\partial N$. We repeat the argument

as for $\partial N=\emptyset$. However, we have to make sure that the isotopy $G_t$ preserves the structure $\Sigma_1$. To do so we only need to replace $\Theta$ by a TRI concordant structure $\Theta_1$ such that $\Theta_1|\partial N\times R=\Sigma_1\times R$. This can be done easily since $N$ is bicollared in $\partial M$. □

The following proposition can be proved using the PL Product Structure Theorem (Theorem 5.1, [4]) exactly in the same way as (1.6) was proved using (1.5).

**Proposition 1.7.** *Corollary 1.6 remains true if we use PL manifold structures and PL concordances.* □

Let $\Sigma$ be a TRI structure on $T_a(h^k)$. It follows from (1.1) that $T_a(h^k)_\Sigma$ is a homology manifold.

**Theorem 1.8.** *Let $\Sigma$ be a TRI structure on $\partial B^k\times B^{m-k}$, $m\geqq 10$. Then $\Sigma$ can be extended to a* TRI *structure on $B^k\times B^{m-k}$ iff the Cohen-Sullivan obstruction $[O(\partial B^k\times B^{m-k})_\Sigma]\in H^4(\partial B^k\times B^{m-k};\theta)$ vanishes.*

Before we start a proof of (1.8) we need to gather some supportive information.

**Proposition 1.9.** *Let $M$ be a homology $m$-manifold that admits an acyclic* PL *resolution $f:P\to M$ as constructed in* [6]. *If $m\geqq 5$ then $f$ is a simple homotopy equivalence.*

*Proof.* From the construction of $f$ in [6] and Van Kampen's theorem it follows easily that $f$ induces an isomorphism $f_*:\pi_1(P)\to\pi_1(M)$. It then follows from [3] that $f$ is a simple homotopy equivalence.

**Lemma 1.10.** *Let $M$ be a simplicially triangulated $m$-manifold, $m\geqq 5$ ($m\geqq 6$ if $\partial M\neq\emptyset$) with $\partial M$ being a* PL *manifold. Let $\alpha_*:H^4(M,\partial M;\theta)\to H^4(M,\partial M;Z_2)$ be induced by the homomorphism $\alpha:\theta\to Z_2$. Then $\alpha_*[O(M,\partial M)]=\Delta(M,\partial M)$.*

Here and in the rest of the paper $O$ will denote the Cohen-Sullivan obstruction.

*Proof.* This lemma is only a paraphrased version of the Assertion that follows Theorem C in [9]. Let $C_*(M,\partial M)$ denote the chain complex based on dual cells of $M$. Let $\Delta_\alpha:C^4(M,\partial M)\to Z_2$ be the cochain defined by $\Delta_\alpha(D^4)=\alpha[\partial D^4]\in Z_2$, where $D^4$ is any dual 4-cell. The Assertion says that $[\Delta_\alpha]=\Delta(M,\partial M)$. Recall that the Cohen-Sullivan obstruction cochain $O(M,\partial M):C_4(M,\partial M)\to\theta$ is defined by $O(M,\partial M)(D^4)=[\partial D^4]\in\theta$. It then follows immediately that, on the cochain level, $\alpha_\#(O(M,\partial M))=\Delta_\alpha$. □

We will use (1.10) in the following situation. Consider $S^3\times S^m$, $m\geqq 5$. Since $H^3(S^3\times S^m;Z_2)\cong Z_2$ it follows from [4] that $S^3\times S^m$ admits exactly two PL structures which are not PL concordant. The structure that factors to the PL structures on $S^3$ and $S^m$ is called the standard PL structure.

**Lemma 1.11.** *The two* PL *structures on $S^3\times S^m$, $m\geqq 4$, are* TRI *concordant.*

*Proof.* We will construct a simplicial triangulation of $S^3\times S^m\times I$ that will restrict to different PL structures on $S^3\times S^m\times\{0\}$ and $S^3\times S^m\times\{1\}$.

Let $\Theta$ be a PL structure on $S^3\times S^m$. Then the Cohen-Sullivan obstruction cocycle $O(S^3\times S^m)_\Theta$ is trivial. According to [5] we can realize every element $[d]$ of $H^3(S^3\times S^m;\theta)\cong\theta$ with a PL resolution $f:P\to(S^3\times S^m)_\Theta$ as follows. Construct $f$

using the cocycle $d$. By techniques from [1] we have that the mapping cylinder $W$ of $f$ is an $H$-cobordism, and by (1.9) we have that the inclusions $P \to W$ and $S^3 \times S^m \to W$ are simple homotopy equivalences. Let $h$ denote the following composite of isomorphisms, with coefficients in $\theta$:

$$H^4(W, P \cup S^3 \times S^m) \xrightarrow{\theta^{*-1}} H^4(S^3 \times S^m \times ([0,1], \{0,1\})) \xrightarrow{D}$$

$$\xrightarrow{D} H_{m-3}(S^3 \times S^m \times [0,1]) \xrightarrow{p_1} H_{m-3}(S^3 \times S^m) \xrightarrow{D^{-1}} H^3(S^3 \times S^m),$$

where $g$ is the acyclic resolution induced by $f$, $D$ is the Poincaré duality isomorphism, and $p_1$ is the projection to the first factor. We then have $h([O(W, P \cup S^3 \times S^m)]) = [d]$.

Next we change $W$ into a simple homotopy equivalent topological manifold, keeping $\partial W$ fixed ($\partial W$ is already a topological manifold). According to (1.2) we need to make sure that each top dimensional link is simply connected. We may assume that $\partial W$ is collared in $W$ (if not we add an exterior collar). Thus, let $v$ be a vertex from $\operatorname{int} W$ such that $\operatorname{lk}(v, W')$ is not simply connected. Here, $W'$ denotes the first barycentric subdivision of $W$. $\operatorname{lk}(v, W')$ is a homology sphere, which bounds, by (1.3), a homology manifold, say $V_v$, such that $\operatorname{int} V_v$ is a topological manifold. Remove $\operatorname{st}(v, W')$ and glue $V_v$ in its place. Denote the resulting homology manifold by $W_v$. There is a contractible resolution $k : W_v \to W$ that identifies $V_v$ to $v \in W$, and is an identity on the complement of $\operatorname{st}(v, W')$. $k$ obviously induces an isomorphism of fundamental groups and is therefore a simple homotopy equivalence [3]. Let $\tilde{W}$ denote the resulting topological manifold after alteration has been done for all non-simply connected top dimensional links. Then $\tilde{W}$ is an $s$-cobordism between $P$ and $S^3 \times S^m$, and by the $s$-cobordism theorem [9] it is homeomorphic to $S^3 \times S^m \times I$. Therefore $P \approx S^3 \times S^m$. It then follows from [6] that $[O(\tilde{W}, P \cup S^3 \times S^m)]$ corresponds to $[O(W, P \cup S^3 \times S^m)]$ under the simple homotopy equivalence $\tilde{W} \to W$.

Now assume that $[d] \notin \operatorname{Ker}(\alpha)$. Then (1.10) shows that $\tilde{W}$ has two different PL structures on its ends. □

We say that the TRI structure on $S^3 \times S^m \times I \approx \tilde{W}$ from (1.11) has been *constructed from* $[d] \in \theta$, and conclude that

(1.12) PL structures $\Sigma_0$ and $\Sigma_1$ on $S^3 \times S^m$ are PL concordant iff the TRI structure on $S^3 \times S^m \times I$ between them is constructed with an element from $\operatorname{Ker}(\alpha)$. □

*Proof of* (1.8). Let $\Sigma$ be a TRI structure on $\partial B^k \times B^{m-k}$ such that its Cohen-Sullivan obstruction vanishes. Then we can construct an acyclic resolution $f : P \to \partial B^k \times B^{m-k}_{\Sigma}$ where $P$ is a PL $m$-manifold. As in the proof of (1.11) we change the mapping cylinder of $f$ into a trivial $s$-cobordism $W$ between $P$ and $\partial B^k \times B^{m-k}$. The trivialization of $W$ provides us with a TRI structure $\Psi$ on $\partial B^k \times B^{m-k} \times I$ which extends $\Sigma$ on $\partial B^k \times B^{m-k} \times \{0\}$ and restricts to a PL structure $\Phi$ on $\partial B^k \times B^{m-k} \times \{1\}$.

Assume $k \neq 4$. Then $H^3(\partial B^k \times B^{m-k}; Z_2) = 0$, and it follows from [4] that $\partial B^k \times B^{m-k}$ can have only one PL structure which therefore extends to a PL structure on $B^k \times B^{m-k} \times \{1\}$. The resulting triangulation of $\partial B^k \times B^{m-k} \times I \cup \partial B^k \times B^{m-k} \times \{1\}$ is then a desired extension of $\Sigma$.

Now let $k=4$. If the PL structure $\Phi$ is the standard one we can extend it trivially. Otherwise $\Phi$ restricts to the non-standard PL structure $\Xi$ on $\partial B^4 \times \partial B^{m-k} \times \{1\}$. Let $\Gamma$ be a TRI structure on $\partial B^4 \times \partial B^{m-4} \times [1,2]$ between $\Xi$ and the standard PL structure on $\partial B^4 \times \partial B^{m-4} \times \{2\}$ as provided by (1.11). Recall from (1.12) that $\Gamma$ must be constructed from an element from $\theta$ which is not in $\mathrm{Ker}(\alpha)$. Let $\Theta$ denote the PL structure on $B^4 \times \partial B^{m-k} \times \{2\}$ extending the one on $\partial B^4 \times \partial B^{m-4} \times \{2\}$. Then $\Psi \cup \mathrm{cone}(\Phi \cup \Gamma \cup \Theta)$ is a desired extension of $\Sigma$.

For the converse let us assume that a TRI structure $\Sigma$ on $\partial B^k \times B^{m-k}$ has an extension to a TRI structure $\Psi$ on $B^k \times B^{m-k}$. Since $H^4(B^k \times B^{m-k}; \theta)=0$ we have that $[O(B^k \times B^{m-k})_\Psi]=0$. Thus, there exists an acyclic resolution of $B^k \times B^{m-k}$; in particular, there is an acyclic resolution of $\partial(B^k \times B^{m-k})$, so that $[O(\partial(B^k \times B^{m-k})_\Psi)]=0$. Let $j: \partial B^k \times B^{m-k} \to \partial(B^k \times B^{m-k})$ be the inclusion and recall that $[O(\partial B^k \times B^{m-k})]=j^*[O(\partial(B^k \times B^{m-k}))]$ (cf. Proposition 1.2 of [5]). Thus $[O(\partial B^k \times B^{m-k})]=0$. □

## 2. Simplicial Triangulations of Topological Manifolds

In this section we show how to triangulate topological manifolds of sufficiently large dimension $m$ provided there is an element $g \in \theta$ with $\alpha(g)=1$, and $2g=0$. Inductively, a triangulation of manifold's $(k-1)$-handle skeleton will, with the help of (1.6) or (1.7), provide a triangulation of the attaching tubes of the $k$-handles (which are homeomorphic to $\partial B^k \times B^{m-k}$). This triangulation will then be extended, using (1.8), to a triangulation of the $k$-handles. Details that follow are essentially a computation of the Cohen-Sullivan obstruction needed for applications of (1.8).

Let $M^m$ be a topological $m$-manifold and $N$ a clean $m$-submanifold. By (1.4) $M$ is a handlebody on $N$. It follows from [4] that we can arrange handles so that they are attached in order of increasing index. Also, handles of the same index can be made disjoint. $M^{(k)}=N \bigcup_{i \leq k} \{\text{all } i\text{-handles from } M\}$ is called the $k$-handle skeleton of $M$. The homology groups $C_k=H_k(M^{(k)}, M^{(k-1)})$ are free with one generator for each $k$-handle. We will use the same symbol for the handle and corresponding preferred generator. Let $C(M,N)=\{C_k\}$ be the chain complex with $\partial: C_k \to C_{k-1}$ being the composite homomorphism $H_k(M^{(k)}, M^{(k-1)}) \to H_{k-1}(M^{(k-1)}) \to H_{k-1}(M^{(k-1)}, M^{(k-2)})$. Then $H_*(C(M,N))=H_*(M,N)$ (cf. [11]). Let $h$ and $\mu$ be the homomorphisms entering in the universal coefficient theorems for cohomology as in [10, (5.5.3), (5.5.10)]. Then $h: H^k(M^{(k)}, M^{(k-1)}) \to \mathrm{Hom}(H_k(M^{(k)}, M^{(k-1)})), Z)$ is an isomorphism showing $H^k(M^{(k)}, M^{(k-1)})$ to be free with one generator for each $k$-handle. Similarly, $\mu: H^k(M^{(k)}, M^{(k-1)}) \otimes \theta \to H^k(M^{(k)}, M^{(k-1)}; \theta)$ is an isomorphism. In the rest of the paper we will use $h$ and $\mu$ in the sense just described. We also reserve the letter $v$ for the isomorphisms in the following situations: Let $X$ be a space with $H^i(X) \cong Z$ for some $i$. Then $v: H^i(X) \to Z$ will denote that isomorphism that maps the preferred generator of $H^i(X)$ to $1 \in Z$.

The expression "a TRI structure $\Psi$ on a submanifold $N$ of $M$ can be extended to a TRI structure on $M$" will mean that $\Phi$ is TRI concordant to a TRI structure that restricts to $\Psi$.

**Theorem 2.1.** *Let $M^m$ be a topological m-manifold, $m \geqq 10$, and $N$ a clean m-submanifold with a given* TRI *structure $\Sigma$. If there exists an element $g \in \theta$ such that $\alpha(g)=1$, and $2g=0$, then $\Sigma$ can be extended to a* TRI *structure on $M$.*

*Proof.* Assume that $M^{(k-1)}$, $k \geqq 1$, already has a TRI structure $\Sigma_{k-1}$ that extends $\Sigma$ on $N=M^{(0)}$. Let $h^k$ be any $k$-handle (since $k$-handles are disjoint we can carry out our construction for all handles simultaneously). First we replace $\Sigma_{k-1}$ by a TRI concordant structure $\tilde{\Sigma}_{k-1}$ in such a way that $T_a(h^k)$ becomes a TRI submanifold of $\tilde{\Sigma}_{k-1}$. Recall that $\partial T_a(h^k)$ is a codimension one bicollared submanifold of $\partial M^{(k-1)}$. Using (1.6) we can find a TRI structure $\Phi$ on $\partial M^{(k-1)} \times I$ such that $\Phi | \partial M^{(k-1)} \times \{0\} = \Sigma_{k-1} | \partial M^{(k-1)}$ and $\Phi | \partial M^{(k-1)} \times \{1\}$ has $\partial T_a(h^k)$ and therefore also $T_a(h^k) \times \{1\}$ as a TRI submanifold. Then $\tilde{\Sigma}_{k-1} = \Sigma_{k-1} \cup \Phi$ is a desired TRI structure on $M^{(k-1)}$.

If $k \neq 5$, $H^4(T_a(h^k); \theta)=0$. Thus $[O(T_a(h^k))]=0$, and by (1.8) we can extend the TRI structure on $T_a(h^k)$ to a TRI structure on $h^k$.

When $k=5$ we will use $g \in \theta$ to construct a TRI structure on $M^{(4)}$ in such a way that the Cohen-Sullivan obstruction will vanish for the attaching tube of every 5-handle. As before, (1.8) will then provide triangulations of 5-handles, and the proof will be completed. In analyzing the attaching tubes of 5-handles we will use the following

**Proposition 2.2.** *Every* TRI *structure on $M^{(4)}$ is* TRI *concordant to a* TRI *structure, say $\Sigma_4$, which restricts to a* TRI *structure on $M^{(3)}$. Moreover, in $\Sigma_4$ each $T_a(h_i^4)$ has a* PL *structure, and each $T_a(h^5)$ is a* TRI *submanifold in $\partial M^{(4)}$ with* $\mathrm{cl}(T_a(h^5) - \bigcup_j T_b(h_j^4))$ *being a* PL *submanifold.*

*Proof.* By the argument above we may assume that the TRI structure on $M^{(4)}$ is already such that the attaching tubes of 5-handles are TRI submanifolds. Fix a 5-handle $h^5$. Let $h_i^4$ be a 4-handle such that $h_i^4 \cap h^5 \neq \emptyset$. We can isotope (cf. [4]) $S_a(h^5)$ by an arbitrary small topological ambient isotopy so that it meets $S_b(h_i^4)$ transversally with respect to the normal microbundle $\xi^4$ of $S_b(h_i^4)$ in $\partial M^{(4)}$. This way, $S_a(h^5) \cap S_b(h_i^4)$ consists of finitely many points $p_{ij}$, $j=1, \ldots, n_i$. Since $\xi^4$ is trivial with the fiber $R^4$ we may assume that the total space $E(\xi^4)=T_b(h_i^4)$, and write

$$T_a(h^5) \cap (T_b(h_i^4), \partial T_b(h_i^4)) = \bigcup_{j=1}^{n_i} (A_{ij}, \bar{\partial} A_{ij}) \approx (B^4, \partial B^4) \times (B_1^{m-5} \cup \ldots \cup B_{n_i}^{m-5}),$$

where $\bar{\partial} A_{ij} = A_{ij} \cap \partial T_b(h_i^4)$. Since the isotopy of $S_a(h^5)$ can be made arbitrary small with the support near $S_t(h_i^4)$ we can do this process simultaneously for all 4-handles that meet $h^5$, and consequently for all 5-handles, since they are disjoint.

We now change $\bigcup_i T_a(h_i^5)$ into a TRI submanifold of $\partial M^{(4)}$ as follows. Using (1.6) we first make $(\bigcup_i T_a(h_i^5)) \cap (\bigcup_j \partial T_b) h_j^4))$ into a TRI submanifold of $\bigcup_j \partial T_b(h_j^4)$, and then $\mathrm{cl}(\bigcup_i T_a(h_i^5) - \bigcup_j T_b(h_j^4))$ into a TRI submanifold of $\mathrm{cl}(\partial M^{(4)} - \bigcup_j T_b(h_j^4))$. Let $C_j$ denote a collar of $\partial T_b(h_j^4)$ in $T_b(h_j^4)$. Then $X = [\bigcup_i T_a(h_i^5) - \bigcup_j T_b(h_j^4)] \cup \bigcup_j C_j$ is a TRI submanifold of $\partial M^{(4)}$ having trivial 4th cohomology group with any coefficients, and therefore it can be resolved into a PL manifold. We change, as in the proof of (1.11), the mapping cylinder of this resolution into a trivial topological $s$-cobordism. This enables us to consider $X$ as a PL manifold. We now use (1.7) to make $(\bigcup_i T_a(h_i^5)) \cap (\bigcup_j \partial T_b(h_j^4))$ into a PL

submanifold of $X$. Using (1.6) we now make $(\bigcup_i T_a(h_i^5))\cap(\bigcup_j T_b(h_j^4))$ into a TRI submanifold of $\bigcup_j T_b(h_i^4)$, keeping the TRI structure on $\bigcup_j \partial T_b(h_j^4)$ (which is PL at this stage!) fixed. Since $H^4(T_a(h_i^4), \partial T_a(h_i^4); \theta)=0$ we can resolve the TRI structure on $T_a(h_i^4)$ into a PL structure, keeping the PL structure on $\partial T_a=\partial T_b$ fixed. Alteration of the mapping cylinder of this resolution into a trivial $s$-cobordism then gives the desired TRI structure $\Sigma_4$ on $M^{(4)}$. □

Let $\Theta$ be the TRI structure on $M^{(3)}$ to which $\Sigma_4$ from (2.2) restricts. We can extend $\Theta$ back to different TRI structures on $M^{(4)}$ by constructing TRI structures on $T_b(h_j^4)$ using different elements of $\theta$ as shown in (1.11). Let $q_j\in\theta$ denote the element used in triangulating $T_b(h_j^4)$. Recall from (1.12) that

(2.3) $q_j\in\mathrm{Ker}(\alpha)$ iff $\partial T_b(h_j^4)$ has the standard PL structure in $\Theta$.

Let $\lambda_i$ be the coefficients such that for $\partial: C_5\to C_4$ we have $\partial(h^5)=\Sigma_i\lambda_i h_i^4$. If $\varepsilon_{ij}$ denotes the intersection number of $S_a(h^5)$ and $S_b(h_i^4)$ at $p_{ij}$, we have $\lambda_i=\sum_{j=1}^{n_i}\varepsilon_{ij}$.

**Lemma 2.4.** $(v\times \mathrm{id}_\theta)\circ\mu^{-1}([O(T_a(h^5))])=\Sigma_i\lambda_i q_i$.

*Proof.* It follows from (2.2) that we can obtain from the TRI structure on $\partial M^{(4)}$ a dual cell complex in such a way that $T_a(h^5)$, $A_{ij}$, $T_b(h_i^4)$, $t_b(h_i^4)$, and $C_i$ ($t_b$ and $C_i$ to be introduced shortly) are all cell subcomplexes. Recall that dual cells are used to define the Cohen-Sullivan obstruction cocycles. All the cells of $\mathrm{cl}(T_a(h^5)-\bigcup_{i,j}A_{ij})$ are PL cells [cf. (2.2)] so that we can write the Cohen-Sullivan obstruction cocycle $O(T_a(h^5))$ as

$$O(T_a(h^5))=\Sigma_{i,j}\varepsilon_{ij}O(A_{ij},\bar\partial A_{ij}). \tag{2.5}$$

To compute $O(A_{ij},\bar\partial A_{ij})$, let $C_i$ be a collar of $\partial T_b(h_i^4)$ in $T_b(h_i^4)$. Write $T_b(h_i^4)=C_i\cup t_b(h_i^4)$. Then $\partial C_i=\partial_0 C_i\cup\partial_1 C_i$ where $\partial_0 C_i=\partial T_b(h_i^4)$, and $\partial_1 C_i=C_i\cap t_b(h_i^4)=\partial t_b(h_i^4)$. Recalling how a TRI structure on $T_b(h_i^4)$ was constructed with $q_i\in\theta$, we have that $t_b(h_i^4)$ has a PL structure, and

$$(v\otimes \mathrm{id}_\theta)\circ\mu^{-1}([O(C_i,\bar\partial C_i)])=q_i. \tag{2.6}$$

Consider the inclusions

$$i_1:(C_i,\bar\partial C_i)\to(T_b(h_i^4),t_b(h_i^4)\cup\partial T_b(h_i^4)),$$

$$i_2:(T_b(h_i^4),\partial T_b(h_i^4))\to(T_b(h_i^4),\partial T_b(h_i^4)\cup t_b(h_i^4)),$$

and

$$i_j:(A_{ij},\bar\partial A_{ij})\to(T_b(h_i^4),\partial T_b(h_i^4)).$$

It is easy to see that they induce isomorphisms of 4th cohomology groups with any coefficients ($i_1$ and $i_3$ are excisions, and for $i_2$ use that $m\geqq 10$). On the cochain level we now have the following:

$$\begin{aligned}O(C_i,\bar\partial C_i)&=i_1^{\#}(O(T_b(h_i^4),t_b(h_i^4)\cup\partial T_b(h_i^4)),O(T_b(h_i^4),\partial T_b(h_i^4)))\\&=i_2^{\#}(O(T_b(h_i^4)\,t_b(h_i^4)\cup\partial T_b(h_i^4)),\end{aligned}$$

and

$$O(A_{ij}\,\bar\partial A_{ij})=i_3^{\#}(O(T_b(h_i^4),\partial T_b(h_i^4))).$$

In the second equation we use the fact that $t_b(h_i^4)$ has a PL structure so that the Cohen-Sullivan obstruction vanishes on its dual cells. Thus $[O(A_{ij}, \bar{\partial}A_{ij})] = i_3^* \circ i_2^* i_1^{*-1}([O(C_i, \partial C_i)])$, and (2.5) and (2.6) give $(v \otimes \mathrm{id}_0) \circ \mu^{-1}([O(T_a(h^5)]) = \Sigma_i \lambda_i q_i$. □

Now recall that $T_a(h^5) \approx S^4 \times B^{m-5}$. Since $S^4 \times B^{m-5}$ does have a PL structure, it follows that the Kirby-Siebenmann obstruction $\Delta(T_a(h^5)) = 0$, and (1.10) shows that $[O(T_a(h^5))] \in \mathrm{Ker}(\alpha)$. From (2.4) we thus conclude

$$\Sigma_i \lambda_i q_i \in \mathrm{Ker}(\alpha). \tag{2.7}$$

We reorder subscripts so that for some $s$, $q_i \notin \mathrm{Ker}(\alpha)$ if $1 \leqq i \leqq s$, and $q_i \in \mathrm{Ker}(\alpha)$ for $i \geqq s+1$. Now alter the existing TRI structure on $M^{(4)}$ as follows: with $0 \in \theta$ we triangulate all those 4-handles that were previously triangulated with elements from $\mathrm{Ker}(\alpha)$ [i.e., they receive the standard PL structure – see (1.12)], and with $g \in \theta$ all remaining 4-handles. For this new triangulation (2.7) becomes $\sum_{i=1}^{s} \lambda_i g \in \mathrm{Ker}(\alpha)$. Since $\alpha(g) = 1$ it follows that $\sum_{i=1}^{s} \lambda_i$ is an even integer. But $2g = 0$ implies $\left(\sum_{i=1}^{s} \lambda_i\right) g = 0$ so that $[O(T_a(h^5))] = 0$ for every 5-handle $h^5$. Now (1.8) provides triangulation of all 5-handles, and the proof of Theorem 2.1 is completed. □

For the sufficiency part of Theorem 0.1 we need to lower the dimension $m$ of $M$. Let $k$ be such that $m + k \geqq 10$. Then (2.1) provides a TRI structure on $M \times R^k$, and (1.5) a TRI structure on $M$. □

## 3. Obstruction to Existence of TRI Manifold Structures

Let $\Sigma_0$ be a TRI structure on a clean codimension zero submanifold $N$ of a topological manifold $M$. We will investigate when $\Sigma_0$ can be extended to a TRI structure on $M$. In what follows, we restrict our attention only to those TRI structures that extend $\Sigma_0$. Let $m = \dim M \geqq 10$.

**Lemma 3.1.** *Let $\Phi$ and $\Psi$ be* TRI *structures on $M^{(3)}$. Then they are* TRI *concordant.*

*Proof.* Consider the TRI manifold

$$(M^{(3)} \times [-1, 0])_{\Phi \times [-1,1]} \cup (N \times [0, 1])^0_{\Sigma \times [0,1]} \cup M^{(3)} \times [1, 2]_{\Psi \times [1,2]}.$$

Assume inductively that this triangulation has been extended to one on

$$M^{(3)} \times [-1, 0] \cup M^{(k)} \times [0, 1] \cup M^{(3)} \times [1, 2], k < 3.$$

Let $h^k$ be a $k$-handle of $M$. We can consider $h^k \times [0, 1]$ as a $(k+1)$-handle. Using (1.6), keeping the TRI structure on $M^{(3)} \times \{-1, 2\}$ fixed, we may assume that for each $h^k$, $h^k \times \{0\} \cup T_a(h^k) \times [0, 1] \cup h^k \times \{1\} = T_a(h^k \times [0, 1])$ is a TRI submanifold. It then follows from (1.8) and (1.11) that we can extend the triangulation of $T_a(h^k \times [0, 1])$ to one on $h^k \times [0, 1]$. $k = 3$ gives the desired TRI concordance. □

**Corollary 3.2.** *Any* TRI *structure $\Phi$ on $M^{(4)}$ is* TRI *concordant to a* TRI *structure $\Xi$ that restricts to any given* TRI *structure $\Psi$ on $M^{(3)}$.* □

We will call $\Xi$ from (3.2) a $\Psi$-*standardized* structure on $M^{(4)}$. Let $\{\lambda_{ij}\}$ be integers such that for $\partial: C_5 \to C_4$ we have $\partial(h_i^5) = \Sigma_j \lambda_{ij} h_j^4$. Thus, the $\lambda_{ij}$'s are the same coefficients as $\lambda_i$'s in (2.4). Now that we have a hold on $\lambda_{ij}$'s we will suppress isomorphisms $(v \otimes \mathrm{id}_\theta) \circ \mu^{-1}$ to simplify notation. Using the last paragraph of the proof of (2.4) we can rewrite (2.4) as

$$[O(T_a(h_i^5))] = \Sigma_j \lambda_{ij} [O(T_b(h_j^4), \partial T_b(h_j^4))]. \tag{3.3}$$

Let $\Sigma$ be a TRI structure on $M^{(4)}$. We may assume that $\Sigma$ is as in (2.2) and $\Psi$-standardized for some $\Psi$. For extending $\Sigma$ to a TRI structure on $M^{(5)}$ (1.8) suggests to define a cochain

$$\nabla_\Sigma = \nabla_\Sigma(M, N): C_5 \to \mathrm{Ker}(\alpha) \quad \text{by} \quad \nabla_\Sigma(M, N)(h_i^5) = [O(T_a(h_i^5))]. \tag{3.4}$$

Note that $[O(T_a(h_i^5))] \in \mathrm{Ker}(\alpha)$ (cf. 2.13).

From (1.8) and the proof of (2.1) it follows immediately that

(3.5) $\Sigma$ can be extended to a TRI structure on $M$ iff $\nabla_\Sigma = 0$.

**Proposition 3.6.** *$\nabla_\Sigma$ is a cocycle.*

*Proof.* We define a cohain $O_\Sigma = O_\Sigma(M, N): C_4 \to \theta$ by $O_\Sigma(h_i^4) = [O(T_b(h_i^4), \partial T_b(h_i^4))] \in \theta$. Then (3.3) shows $\nabla_\Sigma = \delta O_\Sigma$ so that $\delta \mu_\Sigma = \delta \delta O_\Sigma = 0$. Note that $\nabla_\Sigma$ is not necessarily a coboundary since $O_\Sigma$ has coefficients in $\theta$ not in $\mathrm{Ker}(\alpha)$.

**Lemma 3.7.** *Let $\Sigma$ and $\Xi$ be* TRI *structures on $M^{(4)}$. Then $[\nabla_\Sigma] = [\nabla_\Sigma] \in H^5(M, N;$ $\mathrm{Ker}(\alpha))$.*

*Proof.* We may assume that $\Sigma$ and $\Xi$ are $\Psi$-standardized for some $\Psi$. Let $d_i = [O(T_b(h_i^4), \partial T_b(h_i^4))_\Xi] - [O(T_b(h_i^4), \partial T_b(h_i^4))_\Xi]$. Then $d_i \in \mathrm{Ker}(\alpha)$ since both its summands belong simultaneously to $\mathrm{Ker}(\alpha)$ or to its complement [see (1.12)]. We define a cochain $d_{\Sigma\Xi}: C_4 \to \mathrm{Ker}(\alpha)$ by $d_{\Sigma\Xi}(h_i^4) = d_i$. Then (3.3) shows $\nabla_\Sigma - \nabla_\Xi = \delta d_{\Sigma\Xi}$. □

Since the cohomology class $[\nabla_\Sigma]$ does not depend on a particular TRI structure on $M^{(4)}$, we denote it by $\nabla(M, N)$.

**Theorem 3.8.** *Let $M$ be a topological $m$-manifold, $m \geqq 10$, and $N$ a clean submanifold with a given* TRI *structure $\Sigma_0$. Then $\Sigma_0$ extends to a* TRI *structure on $M$ iff $\nabla(M, N) = 0$.*

*Proof.* First assume that $\Sigma_0$ extends to a TRI structure on $M$. Using (1.6), we can find a concordant TRI structure on $M$ that restricts to a TRI structure $\Theta$ on $M^{(5)}$. Applying (1.6) to $M^{(5)}$ we exchange $\Theta$ for a concordant TRI structure which restricts to a TRI structure $\Xi$ on $M^{(4)}$. $\Xi$ provides us with a TRI structure on each $h^5$. Using (3.2) we replace $\Xi$ by a concordant $\Psi$-standardized TRI structure $\Sigma$. Let $\Gamma$ be a TRI structure on $M^{(4)} \times I$ between $\Xi$ and $\Sigma$. Using (1.6), we may assume that $T_a(h_i^5) \times I$ is a TRI submanifold of $(\partial M^{(4)} \times I)_\Gamma$. Therefore $[O(T_a(h_i^4))_\Xi]$ $= [O(T_a(h_i^5))_\Sigma] = 0$, and we have $\nabla_\Sigma = 0$. Thus $\nabla(M, N) = 0$.

Now assume $\nabla(M, N) = 0$. Let $\Sigma$ be a $\Psi$-standardized TRI structure on $M^{(4)}$ such that $[\nabla_\Sigma(M, N)] = \nabla(M, N)$. Since $\nabla(M, N) = 0$, we have $\nabla_\Sigma = \delta C_\Sigma$ for some cochain $C_\Sigma: C_4 \to \mathrm{Ker}(\alpha)$. We also have $\nabla_\Sigma = \delta O_\Sigma$ where $O_\Sigma$ is as in the proof of (3.6).

This shows that

$$O_\Sigma|\mathrm{Im}\,\partial = C_\Sigma|\mathrm{Im}\,\partial, \tag{3.9}$$

where $\partial: C_5 \to C_4$ is the boundary operator. We replace $\Sigma$ on $M^{(4)}$ by another TRI structure $\Xi$ which is constructed by triangulating each 4-handle $h_j^4$ using element $O_\Sigma(h_j^4) - C_\Sigma(h_j^4) \in \theta$. Then we have

$$\begin{aligned} \nabla_\Xi(h_i^5) = [O(T_a(h_i^5))_\Xi] &= \Sigma_j \lambda_{ij}[O(T_b(h_j^4), \partial T_b(h_j^4))_\Xi] \\ &= \Sigma_j \lambda_{ij}(O_\Sigma(h_j^4) - C_\Sigma(h_j^4) = 0, \end{aligned}$$

using (3.3), (2.5), and (3.9) for the second, third, and fourth equality, respectively. It follows now from (3.2) that $\Xi$ has an extension to $M^{(5)}$ and thus to $M$. □

**Theorem 3.10.** *Let $M$ be a topological $m$-manifold and $N$ a clean submanifold. Let $m \geqq 7$ ($m \geqq 6$ if $\mathrm{cl}(\partial M - N)$ is compact, $m \geqq 5$ if $M$ is closed and $N = \emptyset$) and let $\Sigma_0$ be a* TRI *structure on $N$. Then there is an element $\nabla(M, N) \in H^5(M, N; \mathrm{Ker}(\alpha))$ such that $\Sigma_0$ extends to a* TRI *structure on $M$ iff $\nabla(M, N) = 0$.*

*Proof.* We only need to consider the case $m < 10$. Let $k$ be an integer such that $m + k \geqq 10$. Let $p: M \times R^k \to M$ be the projection. Then $p$ induces an isomorphism of cohomology groups with any coefficients. Define $\nabla(M, N) = p^{*-1}(\nabla(M \times R^k, N \times R^k))$. Then (3.8) and (1.5) complete the proof. □

Now we would like to examine the relation between $\nabla(M)$ and $\Delta(M)$. Let $\Sigma$ be a TRI structure on $M^{(4)}$. Since $H^4(M^{(3)}; Z_2) = 0$, we have $\Delta(M^{(3)}) = 0$ so that $M^{(3)}$ does admit a PL structure, say $\Psi$. We may assume that $\Sigma$ is $\Psi$-standardized, so that $T_a(h_i^4)$ is a PL submanifold of $\partial M^{(3)}$ for each $h_i^4$. Define a cochain

(3.11) $\Delta_\Sigma: C_4 \to Z_2$ by

$$\Delta_\Sigma(h_i^4) = \begin{cases} 0 & \text{if } T_a(h_i^4) \text{ has the standard PL structure} \\ 1 & \text{otherwise.} \end{cases}$$

Then (2.3) shows that

$$\nabla_\Sigma = \alpha_\#(O_\Sigma). \tag{3.12}$$

It follows from Theorem 5.5.3 in [10] that $[O_\Sigma] = [O(M^{(4)})_\Sigma]$. Combining (1.1) with (3.12) we obtain

$$[\Delta_\Sigma] = \Delta(M^{(4)}) \in H^4(M^{(4)}; Z_2). \tag{3.13}$$

The cochain $\Delta_\Sigma$ also represents an element $[\Delta_\Sigma]_M \in H^4(M; Z_2)$. Let $i: M^{(4)} \to M$ be the inclusion. Then $i^*: H^4(M; Z_2) \to H^4(M^{(4)}; Z_2)$ is a monomorphism with $i^*([\Delta_\Sigma]_M) = [\Delta_\Sigma] = \Delta(M^{(4)})$. Using the naturality of the Kirby-Siebenmann obstruction we conclude

$$[\Delta_\Sigma]_M = \Delta(M). \tag{3.14}$$

Let $G$ be an abelian group and denote $C^i(G) = \mathrm{Hom}(C_i, G)$. With this notation, the short exact sequence $0 \to \mathrm{Ker}(\alpha) \xrightarrow{i} \theta \xrightarrow{\alpha} Z_2 \to 0$ gives a short exact sequence $0 \to C^i(\mathrm{Ker}(\alpha)) \xrightarrow{i_\#} C^i(\theta) \xrightarrow{\alpha_\#} C^i(Z_2) \to 0$, all $i$. Let $\beta: H^k(M; Z_2) \to H^{k+1}(M; \mathrm{Ker}(\alpha))$ denote the corresponding Bockstein homomorphism. Then we have the following

**Theorem 3.15.** $\beta(\Delta(M))=\nabla(M)$.

*Proof.* Consider the commutative diagram

$$\begin{array}{ccccccc} 0\to C^4(\mathrm{Ker}(\alpha)) & \xrightarrow{i_\#} & C^4(\theta) & \xrightarrow{\alpha_\#} & C^4(Z_2)\to 0 \\ \downarrow\delta & & \downarrow\delta & & \downarrow\delta \\ 0\to C^5(\mathrm{Ker}(\alpha)) & \xrightarrow{i_\#} & C^5(\theta) & \xrightarrow{\alpha_\#} & C^5(Z_2)\to 0 \end{array}$$

which is used to define $\beta: H^4(\ ;Z_2)\to H^5(\ ;\mathrm{Ker}(\alpha))$. Let $\Sigma$ be a standardized TRI structure on $M^{(4)}$ so that $\Delta(M)=[\Delta_\Sigma]_M$. From (3.12) we have $\Delta_\Sigma=\alpha_\#(O_\Sigma)$. Recall also that $i_\#(\nabla_\Sigma)=\delta(O_\Sigma)$. It now follows from the definition of $\beta$ that $\beta([\Delta_\Sigma]_M)=[\nabla_\Sigma]=\nabla(M)$. □

We have from (0.2)(b) and (3.15) that $\nabla(M)$ that we developed here is really the Galewski-Stern obstruction.

We conclude this section with a product formula. Let $M$ and $N$ be topological manifolds of appropriate dimensions so that $\nabla(M)$ and $\nabla(N)$ are defined. Let $p_1: M\times N\to M$ and $p_2: M\times N\to N$ be the projections.

**Theorem 3.16.** $\nabla(M\times N)=p_1^*(\nabla(M))+p_2^*(\nabla(N))$.

*Proof.* Assume that the handles of $M$ and $N$ are already ordered in increasing index. Let $h^i$, $h^i_M$, and $h^i_N$ denote the $i$-handles from $M\times N$, $M$, and $N$, respectively. We use the handlebody decomposition of $M\times N$ where each $k$-handle has the form $h^k=h^i_M\times h^{k-i}_N$. It is easy to see that handles $h^k$ are also ordered in increasing index.

Let $\Sigma$ and $\Xi$ be TRI structures on $M^{(4)}$ and $N^{(4)}$, respectively, such that the attaching tubes of 5-handles are already TRI submanifolds. Observe that the TRI structure $\Sigma\times\Xi$ on $M^{(4)}\times N^{(4)}$ provides triangulation of all those 5-handles of $M\times N$ that can be written as $h^i_M\times h^{5-i}_N$, $i\neq 0,5$. By repeated application of (1.6), using (2.3), and the product formula for the Cohen-Sullivan obstruction [6] we can obtain a TRI structure $\Phi$ on $(M\times N)^{(4)}\times[-2,2]$ such that $T_a(h^0_M\times h^5_N\times[1,2])$ $(=h^0_M\times T_a(h^5_N)_\Xi\times[1,2])$, $T_a(h^5_M\times h^0_N\times[-2,-1])$ $(=T_a(h^5_M)_\Sigma\times h^0_N\times[-2,-1])$, and $T_a(h^i_M\times h^{5-i}_N\times[-1,1])$, $i\neq 0,5$, are TRI submanifolds with

$$\begin{aligned} &[O(T_a(h^0_M\times h^5_N\times[1,2]))_\Phi]=[O(T_a(h^5_N))_\Xi], \\ &[O(T_a(h^5_M\times h^0_N\times[-2,-1]))_\Phi]=[O(T_a(h^5_M))_\Sigma], \\ &[O(T_a(h^i_M\times h^{5-i}_N\times[-1,1]))_\Phi]=0, \qquad i\neq 0,5. \end{aligned} \tag{3.17}$$

In order to determine

$$\nabla_\Phi: H_5((M\times N\times[-2,2])^{(5)},(M\times N\times[-2,2])^{(4)})\to\mathrm{Ker}(\alpha)$$

observe that

$$\begin{aligned} (M\times N\times[-2,2])^{(5)}=(M\times N)^{(5)}\times[-2,2]&\approx(M\times N)^{(4)}\times[-2,2] \\ &\cup(\bigcup h^0_M\times h^5_N\times[1,2])\cup(\bigcup h^5_M\times h^0_N\times[-2,-1]) \\ &\cup(\bigcup_{i\neq 0,5} h^i_M\times h^{5-i}_N\times[-1,1]). \end{aligned}$$

Then (3.17) shows $\nabla_\Phi(M\times N\times[-2,2])=p_1^\#(\nabla_\Sigma)+p_2^\#(\nabla_\Xi)$, which gives $\nabla(M\times N)$ $=p_1^*(\nabla(M))+p_2^*(\nabla(N))$. □

## 4. Classification of TRI Manifold Structures

Let $N$ be a clean submanifold of a topological manifold $M$ with a given TRI structure $\bar{\Sigma}$. In this section we again agree that every TRI structure to be considered is one extending $\bar{\Sigma}$.

**Proposition 4.1.** *Let $\Sigma'_0$ and $\Sigma'_1$ be* TRI *structures on $M$, $\dim M \geqq 10$. Then $\Sigma'_0$ and $\Sigma'_1$ are* TRI *concordant iff they are* TRI *concordant to* TRI *structures $\Sigma_0$ and $\Sigma_1$ on $M$ which restrict to* TRI *concordant* TRI *structures $\Xi_0$ and $\Xi_1$ on $M^{(4)}$.*

*Proof.* First assume that $\Sigma'_0$ and $\Sigma'_1$ are TRI concordant so that there is a TRI structure $\Theta$ on $M \times [0,1]$ restricting to $\Sigma'_1$ on $M \times \{i\}$, $i=0,1$. Using (1.6) we find a TRI structure $\Theta_0$ (resp. $\Theta_1$) on $M \times [-1,0]$ (resp. $M \times [1,2]$) that extends $\Sigma'_0$ (resp. $\Sigma'_1$) on $M \times \{-1\}$ (resp. $M \times \{2\}$) where $\Sigma_i$ restricts to a TRI structure $\Xi_i$ on $M^{(4)}$, $i=0,1$. Using (1.6) again we exchange the TRI structure $\Theta_0 \cup \Theta \cup \Theta_1$ on $M \times [-1,2]$, keeping $\Sigma_0$ and $\Sigma_1$ fixed, for a TRI structure on $M \times [-1,2]$ that restricts to a TRI structure on $M^{(4)} \times [-1,2]$ thus providing a TRI concordance between $\Xi_0$ and $\Xi_1$.

For the converse, let $\Phi_0$ be a TRI structure on $M \times [-1,0]$ between $\Sigma'_0$ on $M \times \{-1\}$ and $\Sigma_0$ on $M \times \{0\}$. Similarly, let $\Phi_1$ be a TRI structure on $M \times [1,2]$ between $\Sigma_1$ on $M \times \{1\}$ and $\Sigma'_1$ on $M \times \{2\}$. Let $\Phi$ be a TRI structure on $M^{(4)} \times [0,1]$ between $\Xi_0$ on $M^{(4)} \times \{0\}$ and $\Xi_1$ on $M^{(4)} \times \{1\}$. Let $\Gamma = \Phi_0 \cup \Phi \cup \Phi_1$ be a TRI structure on the manifold $X = M \times [-1,0] \cup M^{(4)} \times [0,1] \cup M \times [1,2]$. Let $h_i^5$ be any 5-handle of $M^{(5)}$. Consider $T_i = h_i^5 \times \{0\} \cup T_a(h_i^5) \times [0,1] \cup h_i^5 \times \{1\}$. Note that $T_i \approx S^5 \times B^{m-5}$. $T_i$'s are disjoint since $h_i^5$'s are disjoint. They are submanifolds in $\partial X$, and using (1.6) we can exchange $\Gamma$, keeping $\Sigma'_0$ and $\Sigma'_1$ fixed, for another TRI structure $\Gamma'$ on $X$ such that they ($T_i$'s) become TRI submanifolds. We can regard $h_i^5 \times [0,1]$ as a 6-handle which is attached to $X$ via its attaching tube $T_i$. Then the proof of (2.1) shows that we can extend TRI structure $\Gamma'$ on $X$ to a TRI structure on $X \times (\bigcup_i h_i^5 \times [0,1]) = M \times [-1,0] \cup M^{(5)} \times [0,1] \cup M \times [1,2]$. We continue inductively with $M^{(k)}$, $k \geqq 5$. Then $k=m$ gives us a TRI concordance between $\Sigma'_0$ and $\Sigma'_1$ as desired. □

Let $\Psi$ be any TRI structure on $M^{(3)}$. We may and do assume that $\Psi$ is as in (2.2). Let $[\Sigma]$ be a TRI concordance class on $M^{(4)}$ represented by a TRI structure $\Sigma$. We will compare any other class $[\Xi]$ on $M^{(4)}$ to $[\Sigma]$. Using (3.2) we may assume that the representatives $\Sigma$ and $\Xi$ are $\Psi$-standardized, so that they coincide on $M^{(3)}$, and in particular on $T_a(h_i^4)$. We also assume that both $\Sigma$ and $\Xi$ have extensions to TRI structures $\tilde{\Sigma}$ and $\tilde{\Xi}$ on $M$. We will again use the chain complex $C(M,N) = \{C_i\}$.

Let $s_i = [O(h_i^4, T_a(h_i^4))_\Sigma] \in \theta$, and $t_i = [O(h_i^4, T_a(h_i^4))_\Xi] \in \theta$. Let $Y$ and a TRI structure $\Gamma$ on $Y$ be defined by

$$Y_\Gamma = M^{(4)}_\Sigma \times [-1,0] \cup M^{(3)}_\Psi \times [0,1] \cup M^{(4)}_\Xi \times [1,2].$$

Let

$$T_i = h^4 \times \{0\} \cup T_a(h^4) \times [0,1] \cup h^4 \times \{1\} \approx \partial B^5 \times B^{m-4}.$$

Then $T_i$'s are already TRI submanifolds of $Y$.

**Proposition 4.2.** $[O(T_i)_\Gamma] = s_i - t_i \in \mathrm{Ker}(\alpha)$.

*Proof.* $[O(T_i)_\Gamma]=s_i-t_i$ follows immediately from the fact that $T_a(h_i^4)\times[0,1]$ is a PL submanifold in $\partial M^{(3)}\times[0,1]$ [since $\Psi$ is as in (2.2)], and that $h_i^4\times\{0\}$ and $h_i^4\times\{1\}$ have opposite orientation in the boundary of $h_i^4\times[0,1]$. $[O(T_i)_\Gamma]\in\mathrm{Ker}(\alpha)$ by (1.10). □

We now define a cochain

$$d_{\tilde\Xi}: C_4\to\mathrm{Ker}(\alpha) \quad \text{by} \quad d_{\tilde\Xi}(h_i^4)=[O(T_i)_\Gamma]. \tag{4.3}$$

Obviously, $d_{\tilde\Xi}=O_\Sigma-O_\Xi$. From (3.3) we have $\nabla_\Sigma=\delta O_\Sigma$ and $\nabla_\Xi=\delta O_\Xi$. Since $\Sigma$ and $\Xi$ have extensions $\tilde\Sigma$ and $\tilde\Xi$, (2.3) shows $\nabla_\Sigma=\nabla_\Xi=0$. Thus, $\delta d_{\tilde\Xi}=0$, i.e., $d_{\tilde\Xi}$ is a cocycle and it represents an element $[d_{\tilde\Xi}]\in H^4(M,N;\mathrm{Ker}(\alpha))$.

**Proposition 4.4.** *$[d_{\tilde\Xi}]$ does not depend on choice of representative $\tilde\Xi$ of the* TRI *concordance class $[\tilde\Xi]$.*

*Proof.* We need to show that if $\tilde\Sigma$ and $\tilde\Xi$ are TRI concordant, then $[d_{\tilde\Xi}]=0$. We consider an equivalent extension problem for the pair $(\bar M,\bar N)$, where $M=M^{(4)}\times[-1,2]$, and $N_\Phi=M_\Sigma^{(4)}\times[-1,0]\cup M^{(4)}\times[1,2]$. First we give $(\bar M,\bar N)$ the handle structure $\bar M^k=\bar M^{(k-1)}\cup(\bigcup_i h_i^{k-1}\times[0,1])$, where the union is taken over all $(k-1)$-handles $h_i^{k-1}$ of $M^{(4)}$. Let $\Gamma$ be the TRI structure from (4.2). Consider $\nabla_\Gamma: H_5(\bar M^{(5)},\bar M^{(4)})\to\mathrm{Ker}(\alpha)$. Let $\bar h_i^5=h_i^4\times[0,1]$ be a 5-handle of $\bar M^{(5)}$. It follows from (4.2) and (4.3) that $\nabla_\Phi(\bar h_i^5)=d_{\tilde\Xi}(h_i^4)$. The handle structure of $(\bar M,\bar N)$ shows that $H_k(\bar M^{(k)},\bar M^{(k-1)}\cong H_{k-1}(M^{(k-1)},M^{(k-2)})$ under the correspondence $\bar h_i^5\to\bar h_i^4$, and that the diagram

$$\begin{array}{ccc} H_k(\bar M^{(k)},\bar M^{(k-1)}) & \xrightarrow{\partial} & H_{k-1}(\bar M^{(k-1)},\bar M^{(k-2)}) \\ \cong\downarrow & & \cong\downarrow \\ H_{k-1}(M^{(k-1)},M^{(k-2)}) & \xrightarrow{\partial} & H_{k-2}(M^{(k-2)},M^{(k-3)}) \end{array}$$

commutes for each $k$. Thus, $d_{\tilde\Xi}$ is a coboundary iff $\nabla_\Phi$ is a coboundary. But since we are assuming that $\tilde\Xi$ and $\tilde\Sigma$ (and thus $\Xi$ and $\Sigma$) are TRI concordant, we have from (3.10) that $[\nabla_\Phi]=0$. □

We can now state the following classification

**Theorem 4.5.** *Let $N$ be a clean* TRI *submanifold of a topological $m$-manifold $M$. Let $m\geqq 7$ ($m\geqq 6$ if $\mathrm{cl}(\partial M-N)$ is compact, $m\geqq 5$ if $M$ is closed and $N=\emptyset$). Then there is a one-to-one correspondence between* TRI *structures on $M$ extending the given* TRI *structure on $N$ and the elements of $H^4(M,N;\mathrm{Ker}(\alpha))$.*

*Proof.* Let $\tilde\Sigma$, $\tilde\Xi$, $\Sigma$, $\Xi$, $\Phi$, and $\Gamma$ be as in the proof of (4.4). Assume $[d_{\tilde\Xi}]=0$. We will show that $\Sigma$ and $\Xi$ are TRI concordant. Then (4.1) shows that $\tilde\Sigma$ and $\tilde\Xi$ are TRI concordant. From the proof of (4.4) we have that $d_{\tilde\Xi}$ is a coboundary iff $\nabla_\Gamma$ is a coboundary, so that $\nabla(\bar M,\bar N)=0$. Thus, the TRI structure $\Phi$ on $\bar N$ can be extended to a TRI structure on $\bar M$ which is a TRI concordance between $\Sigma$ and $\Xi$.

It remains to show that each element in $H^4(M,N;\mathrm{Ker}(\alpha))$ can be realized by some TRI structure $\tilde\Xi$ on $M$. Let $s_i=[O(h_i^4,T_a(h_i^4)_\Sigma]$, and let $[d]\in H^4(M,N;\mathrm{Ker}(\alpha))$ be any element represented, say, by a cocycle $d: H_4(M^{(4)},M^{(3)})\to\mathrm{Ker}(\alpha)$. Now triangulate each 4-handle $h_i^4$ with the element $s_i+d(h_i^4)$, and denote by $\Xi$ the resulting TRI structure on $M^{(4)}$. We need to show that $\Xi$ extends to a TRI structure on $M$. We have $\nabla_\Xi=\delta(O_\Xi)=\delta(O_\Sigma+d)=\nabla_\Sigma+\delta d=\nabla_\Sigma$. But $\nabla_\Sigma=0$ since $\Sigma$

has extension $\tilde{\Sigma}$ on $M$. Thus $\nabla_{\tilde{g}}=0$. Let $\tilde{\Xi}$ denote an extension of $\Xi$. Then obviously $[d_{\tilde{g}}]=[d]$. This completes the proof for $m\geqq 10$.

Now let $k$ be such that $\dim(M\times R^k)\geqq 10$. Then (1.5) shows that two TRI structures on $M\times R^k$ extending the TRI structure $\bar{\Sigma}\times R^k$ on $N\times R^k$ are TRI concordant rel $\bar{\Sigma}\times R^k$ iff they are TRI concordant to TRI structures whose restrictions to $M\times\{0\}$ are TRI concordant. In other words, TRI concordance classes on $(M,N)$ are in one-to-one correspondence with those on $(M,N)\times R^k$. The proof is now complete since

$$H^4(M,N;\mathrm{Ker}(\alpha))\cong H^4((M,N)\times R^k;\mathrm{Ker}(\alpha)).\quad\square$$

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