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On the Classification of Fiber Spaces

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I. Introduction

Let $\mathscr{C}$ be the category of CW complexes and let $[X, Y]$ be the set of free homotopy classes of maps from X into Y. If F is a finite CW complex, denote by $LF(X)$ the set of fiber homotopy equivalence classes of Hurewicz fibrations (E, p, X) with fibers the homotopy type of F. In [15] STASHEFF showed that there exists a CW complex B_H such that the functors $[\ ,B_H]$ and $LF(\ )$ are naturally equivalent, the equivalence being obtained by taking induced fibrations from a universal one over B_H.

The objective here is to prove a classification theorem similar to STASHEFF'S, assuming only that F has the homotopy type of a CW complex. Similar, in this case, means that a somewhat different functor than $LF(\ )$ is used (see Definition 2.4). This is due to the fact that we work in the category of based spaces and based maps, whereas in [15] non-based spaces and free maps are used. Restricting F to have the homotopy type of a CW complex is more or less forced by the fact that, in general, the transfer from a quasi fibration to the associated Hurewicz fibration only preserves the weak homotopy type of the fibers. Originally results of [2] were used, but [1] seems preferable.

In any case, to a space X, in the category $\mathscr{C}_0$ of CW complexes with base point, we associate a set of equivalence classes of fiber spaces $\mathscr{H}(X, F)$ and show that $\mathscr{H}(\ , F)$ is a homotopy functor from $\mathscr{C}_0$ into the category S_0 of sets with base points[1]. By BROWN's main result, this means that there exists a B_∞ in $\mathscr{C}_0$ such that $\mathscr{H}(\ , F)$ and $[\ , B_\infty]_0$ are naturally equivalent. Here $[X, B_\infty]_0$ denotes homotopy classes of maps preserving base points, and as before the equivalence is obtained by taking induced fibrations from a universal one. The $\mathscr{H}(X, F)$ we use differs from $LF(X)$ in that a fiber space over X is required to carry a homotopy equivalence $g\colon F \to p^{-1}(x_0)$, x_0 being the base point, and if $\mathscr{E}$ denotes a fiber space over X, it may be that two pairs $(\mathscr{E}_1, g_1)$ and $(\mathscr{E}_2, g_2)$ are not equivalent even if $\mathscr{E}_1 = \mathscr{E}_2$.

The paper is organized as follows: in Section II the needed definitions are given, including that of a homotopy functor, and the main result is stated, i.e., Theorem 2.1. This is then proven in Section III. Section IV considers

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[1] The morphisms in $\mathscr{C}_0$ are homotopy classes of maps preserving base points, those of S_0 maps preserving base points.

the principal fiber space associated to the universal one over B_∞, and we show that its total space is aspherical. Section V gives one application, namely a theorem of HILTON and GANEA on induced principal fibrations.

It might be mentioned that if B_∞^n denotes the classifying space for $\mathscr{H}(\ ,F^n)$ where F^n is the join of F with itself n times, there are canonical maps $B_\infty^n \to B_\infty^{n+1}$. In fact using the join operations between fiber spaces we get maps $B_\infty^n \times B_\infty^m \to B_\infty^{n+m}$. This leads to stable classes of fiber spaces and the study of $B = \bigcup B_\infty^n$. This will be considered elsewhere.

Remark. Unless stated all fiber spaces are Hurewicz fiber spaces. $\mathscr{E}_1 \underset{f}{\sim} \mathscr{E}_2$ will mean that $\mathscr{E}_1$ and $\mathscr{E}_2$ are fiber homotopy equivalent.

II. Preliminaries and Definitions

Definition 2.1. Let F be a topological space and let X be a space with base point x_0. A fiber space over X with fiber F consists of a sequence of spaces and maps

$$F \xrightarrow{g} E \xrightarrow{p} X$$

such that:

a) The triple $\mathscr{E} = (E, p, X)$ is a fiber space.

b) $g\colon F \to p^{-1}(x_0)$ is a homotopy equivalence.

Remark. If X is not assumed to be arcwise connected, we have to restrict ourselves to fiber spaces whose fibers are of the same homotopy type.

A fiber space will be written as a pair $(\mathscr{E}, g)$ or simply as $\mathscr{E}$ in cases where g has no role.

Definition 2.2. Let $(\mathscr{E}, g)$ be a fiber space over X with fiber F and let f: $Y \to X$ be a base point preserving map. Let $f^{-1}(\mathscr{E})$ be the usual fiber space over Y induced by f. Let g': $F \to f^{-1}(E)$ be given by $g'(x) = (g(x), y_0)$. Then $f^{-1}(\mathscr{E}, g)$ is defined to be $(f^{-1}(\mathscr{E}), g')$.

Definition 2.3. Let $(\mathscr{E}_1\, g_1)$ and $(\mathscr{E}_2\, g_2)$ be fiber spaces. A map $(\mathscr{E}_1, g_1) \to (\mathscr{E}_2, g_2)$ is a triple $g\colon F_1 \to F_2$, $\bar{f}\colon E_1 \to E_2$, $f\colon X_1 \to X_2$ such that:

$$\begin{tikzcd} F_1 \arrow[r, "g_1"] \arrow[d, "g"] \arrow[dr, phantom, "\mathrm{(I)}"] & E_1 \arrow[r, "p_1"] \arrow[d, "\bar{f}"] \arrow[dr, phantom, "\mathrm{(II)}"] & X_1 \arrow[d, "f"] \\ F_2 \arrow[r, "g_2"] & E_2 \arrow[r, "p_2"] & X_2 \end{tikzcd}$$

(I) is homotopy commutative and (II) is commutative.

Remark. There is no lost generality in requiring (II) to be commutative rather than homotopy commutative since the triple $(E_2\, p_2\, X_2)$ is a fiber space and using a lifting function we can replace $\bar{f}$ by a fiber map over f.

Definition 2.4. Let $(\mathscr{E}_1, g_1)$ and $(\mathscr{E}_2, g_2)$ be fiber spaces over X with fiber F. We say that they are equivalent if there exists a map $(\mathscr{E}_1, g_1) \to (\mathscr{E}_2, g_2)$ of the form $(1, \alpha, 1)$, and we write $(\mathscr{E}_1, g_1) \sim (\mathscr{E}_2, g_2)$.

Lemma 2.1. $\sim$ *is an equivalence relation.*

Proof. By Theorem 6.3 of [3] α must be a fiber homotopy equivalence and hence has an inverse β: $E_2 \to E_1$. This shows that $\sim$ is symmetric. That it is reflexive and transitive is clear.

We will denote the set of equivalence classes of fiber spaces by $\mathscr{H}(X, F)$. In $\mathscr{H}(X, F)$ there is a natural base point, namely the class of the trivial fibration

$$F \xrightarrow{i} X \times F \xrightarrow{\pi} X$$

which is also the class of any $(\mathscr{E}, g)$ for which $\mathscr{E}$ is fiber homotopy equivalent to a product.

Given $[f] \in [X, Y]_0$ define:

$$\mathscr{H}[f]: \mathscr{H}(Y, F) \to \mathscr{H}(X, F)$$

by

$$\mathscr{H}[f]\{(\mathscr{E}, g)\} = \{f^{-1}(\mathscr{E}, g)\}.$$

$\{\}$ denoting equivalence class. It is easy to see that $\mathscr{H}[f]$ is well defined and that $\mathscr{H}$ is a contravariant functor from the category $\mathscr{C}_0$ to the category S_0.

Example 2.1. Let X be a space such that for some $n \geqq 2$, $\pi_n(X, x_0)$ contains an element which is not of order 2. Let $P(X, x_0)$ be the space of paths starting at x_0, and let $\Omega(X, x_0)$ be the corresponding loop space. Consider the map μ: $\Omega(X, x_0) \to P(X, x_0)$ given by $\mu(w) = w^{-1}$, where $w^{-1}(t) = w(1-t)$, $0 \leqq t \leqq 1$. If i: $\Omega(X, x_0) \to P(X, x_0)$ is the inclusion we obtain two fiber spaces

$$\Omega(X, x_0) \xrightarrow{i} P(X, x_0) \xrightarrow{p} X$$

$$\Omega(X, x_0) \xrightarrow{\mu} P(X, x_0) \xrightarrow{p} X,$$

where as usual p is given by $p\,w = w(1)$ $w \in P(X, x_0)$. Now $(\mathscr{E}, i)$ and $(\mathscr{E}, g)$ cannot be equivalent. To see this, suppose there did exist a fiber map α: $P(X, x_0) \to P(X, x_0)$ such that $\alpha(w)(1) = w(1)$ and $\alpha\,\mu \sim i$. Let $w_s(t) = w(s\,t)$, $0 \leqq s \leqq 1$, $0 \leqq t \leqq 1$, and consider the map

$$H: \Omega(X, x_0) \times I \to \Omega(X, x_0)$$

given by:

$$H(w, s) = w_s * [\alpha(w_s)]^{-1}$$

($*$ denotes the usual path multiplication). Then: $H(w, 0) = w_0 * [\alpha(w_0)]^{-1} = x_0^* =$ constant loop. $H(w, 1) = w_1 * [\alpha(w_1)]^{-1} = w * [\alpha(w)]^{-1}$ and $H(w, 1)$ is homotopic to the square map $w \to w^2$. But if the square map is homotopically trivial, every element of $\pi_n(X) \approx \pi_{n-1}(\Omega(X))$ will be of order 2 for all n's. This contradicts the choice of X.

We now turn to the notion of homotopy functors.

Definition 2.5. According to BROWN [1], also [2], a contravariant functor $\mathscr{H}$: $\mathscr{C}_0 \to S_0$ is called a homotopy functor if the following conditions are satisfied:

Wedge Condition. If $\vee X_\alpha$ is a wedge of spaces and $h_\alpha\colon X_\alpha \to \vee X_\alpha$ are the injections then:

$$\Pi\,\mathscr{H}(h_\alpha)\colon\quad \mathscr{H}(\vee X_\alpha) \approx \Pi\,\mathscr{H}(X_\alpha).$$

Equalizer Condition. Suppose given spaces A, Z, X_i, $i=1,2$, and maps $f_i\colon A\to X_i$, $g_i\colon X_i\to Z$ such that $g_1 f_1 \sim g_2 f_2$. Suppose further that if Z' and $g_i'\colon X_i \to Z'$ are such that $g_1' f_1 \sim g_1' f_2$ then there exists an $h\colon Z\to Z'$ for which $g_i' \sim h\, g_i$. In $\mathscr{C}_0$ such a Z always exists given f_i, A, X_i, $i=1,2$. Under these conditions if $u_i \in \mathscr{H}(X_i)$ satisfy $\mathscr{H}(f_1)\,u_1 = \mathscr{H}(f_2)\,u_2$ it is required that there be a $v\in\mathscr{H}(Z)$ such that $\mathscr{H}(g_i)\,v = u_i$.

In the next section it will be shown that $\mathscr{H}(X, F)$ is a homotopy functor if F has the homotopy type of a CW complex. Using the main result of Brown on the realizibility of homotopy functors, Theorem 10, [1], we will have:

Theorem 2.1. *There exists $B_\infty \in \mathscr{C}_0$ and a fiber space over B_∞ with fiber F, $(\mathscr{E}_\infty, g_\infty)$, such that the natural transformation T: $[X, B_\infty]_0 \to \mathscr{H}(X, F)$ given by*

$$T[f] = f^{-1}(\mathscr{E}_\infty, g_\infty)$$

is an isomorphism for all $X\in\mathscr{C}_0$.

III. Proof of Theorem 2.1.

1. The Equalizer Condition

We begin with a construction in the category $\mathscr{C}$, i.e. base points will be ignored, and consider the following situation:

Let $f_1\colon A\to X_1$, $f_2\colon A\to X_2$ be given maps. Let $\mathscr{E}_1 = (E_1, p_1, X_1)$ and $\mathscr{E}_2 = (E_2, p_2, X_2)$ be fiber spaces and suppose that $f_1^{-1}(\mathscr{E}_1) \underset{f}{\sim} f_2^{-1}(\mathscr{E}_2)$. Consider the space $Z_{f_1 f_2} = X_1 \cup_{f_1} A\times I \cup_{f_2} X_2$, i.e., $Z_{f_1 f_2}$ is the quotient space obtained from $X_1 \cup A\times I \cup X_2$ by the identification

$$(1)\qquad \begin{cases} (a,0) = f_1(a) & a\in A \\ (a,1) = f_2(a). \end{cases}$$

Let $k\colon X_1\cup A\times I\cup X_2 \to Z_{f_1 f_2}$ be the quotient map and let $i_1\colon X_1 \to Z_{f_1 f_2}$, $i_2\colon X_2\to Z_{f_1 f_2}$ be the canonical inclusions. Also let $m_i\colon f_i^{-1}(E_i)\to E_i$, $i=1,2$ be the canonical maps and finally let α and β be fiber maps

$$(2)\qquad \begin{tikzcd} f_1^{-1}(E_1) \arrow[rr, shift left, "\alpha"] \arrow[dr] & & f_2^{-1}(E_2) \arrow[ll, shift left, "\beta"] \arrow[dl] \\ & A & \end{tikzcd}$$

such that $\alpha\,\beta \underset{f}{\sim} 1$ $\beta\,\alpha \underset{f}{\sim} 1$. Define maps $n_1\colon f_1^{-1}(E_1)\to E_2$, $n_2\colon f_2^{-1}(E_2)\to E_1$ by $n_1 = m_2\,\alpha$, $n_2 = m_1\,\beta$.

Now form the space $Z_{m_1 n_1} = E_1 \cup_{m_1} f_1^{-1}(E_1) \times I \cup_{n_1} E_2$ letting $\bar{k}$ be the quotient map, and define $q\colon Z_{m_1 n_1} \to Z_{f_1 f_2}$ by:

(3.1) $$q(e_1) = p_1(e_1) \qquad e_1 \in E_1$$

(3.2) $$q(e_1, a, t) = (a, t) \qquad (e_1, a) \in f_1^{-1}(E_1)$$

(3.3) $$q(e_2) = p_2(e_2) \qquad e_2 \in E_2 .$$

Let $\mathscr{E}$ be the triple $(Z_{m_1 n_1}, q, Z_{f_1 f_2})$.

Lemma 3.1. *$\mathscr{E}$ is a quasifibration.*

Proof. We use arguments similar to those given in [*4*], Proposition 2.3, or [*11*], Theorem 1.2. By [*5*], Satz 2.2, $\mathscr{E}$ will be a quasifibration if $Z_{f_1 f_2}$ can be written as the union of two open sets U and V such that the triples $(q^{-1}(U), q, U)$, $(q^{-1}(V), q, V)$, $(q^{-1} U \cap V, q, U \cap V)$ are quasifibrations, i.e. provided U, V and $U \cap V$ are "distinguished". Let $U = X_1 \cup_{f_1} A \times [0, \frac{3}{4})$, $V = X_2 \cup_{f_2} A \times (\frac{1}{4}, 1]$. Considered as subsets of $Z_{f_1 f_2}$ U and V are open and $Z_{f_1 f_2} = U \cup V$. Furthermore, U and V can be deformed onto $i_1(X_1)$ and $i_2(X_2)$, while $g^{-1}(U)$ and $g^{-1}(V)$ can be deformed in a fiber-wise manner onto $g^{-1}(i_1(X_1))$ and $g^{-1}(i_2(X_2))$ respectively, and since α and β were homotopy equivalences on each fiber, the same is true of any stages of the deformations. Hilfssatz 2.10 of [*5*] then implies that U and V are "distinguished" if $i_1(X_1)$ and $i_2(X_2)$ are. But this is certainly the case since $\mathscr{E}_1$ and $\mathscr{E}_2$ are fiber spaces. Since $U \cap V$ is clearly "distinguished" the proof of the lemma is complete.

Remark. The fiber homotopy type of $Z_{m_1 n_1}$ will depend on the maps α and β that are used. There may be many possible choices.

We need to replace $\mathscr{E}$ by a true Hurewicz fibration whose fibers will have the same homotopy type as those of $\mathscr{E}_1$ and $\mathscr{E}_2$. This is done in the usual manner, i.e., let

(4) $$E = \{(z, w) \mid z \in Z_{m_1 n_1}, \, w \in (Z_{f_1 f_2})^I, \, q(z) = w(0)\}$$

and define $p\colon E \to Z_{f_1 f_2}$ by

(5) $$p(z, w) = w(1) .$$

The triple $\tilde{\mathscr{E}} = (E, p, Z_{f_1 f_2})$ is a fiber space and there is a fiber-wise inclusion:

(6)
$$\begin{tikzcd} Z_{m_1 n_1} \arrow[rr, "\mu"] \arrow[dr, "q"'] & & E \arrow[dl, "p"] \\ & Z_{f_1 f_2} & \end{tikzcd}$$

$$\mu(z) = (z, q(z))$$

and μ is a homotopy equivalence.

The only difficulty involves the homotopy type of the fibers of $\tilde{\mathscr{E}}$ and this is why some restriction must be imposed on the fibers of $\mathscr{E}_1$ and $\mathscr{E}_2$, which of course are of the same homotopy type. Let us assume, then, that all fibers have the homotopy type of a CW complex. By Proposition 0, [*15*] the same is true of E_1 and E_2 and hence of $Z_{m_1 n_1}$, and E. By Corollary 13, [*15*] $p^{-1}(z)$ has the homotopy type of a CW complex for any $z \in Z_{f_1 f_2}$. Since $\mathscr{E}$ is a quasifibration μ is a weak homotopy equivalence on each fiber and hence a homotopy equivalence, Theorem 3, [*16*]. This means that the fibers of $\tilde{\mathscr{E}}$ have the required homotopy type.

Finally if we consider the fiber spaces $i_1^{-1}(\tilde{\mathscr{E}})$ and $i_2^{-1}(\tilde{\mathscr{E}})$ we see that μ induces fiber maps

$$(7) \qquad \begin{tikzcd} E_1 \arrow[rr, "\mu_1"] \arrow[dr, "p_1"'] & & i_1^{-1}(E) \arrow[dl, "\tilde{p}_1"] \\ & X_1 & \end{tikzcd} \qquad \begin{tikzcd} E_2 \arrow[rr, "\mu_1"] \arrow[dr, "p_2"'] & & i_2^{-1}(E) \arrow[dl, "\tilde{p}_2"] \\ & X_2 & \end{tikzcd}$$

which are homotopy equivalences on the fibers. Hence by Theorem 6.3 [*3*] $\mathscr{E}_1 \underset{f}{\simeq} i_1^{-1}(\tilde{\mathscr{E}})$, $\mathscr{E}_2 \underset{f}{\simeq} i_2^{-1}(\tilde{\mathscr{E}})$.

We have proved:

Theorem 3.1. *Let $\mathscr{E}_i = (E_i, p_i, X_i)$ $i=1,2$ be fiber spaces with fibers of the homotopy type of a CW complex. Let f_i: $A \to X_i$ be maps such that $f_1^{-1}(\mathscr{E}_1) \underset{f}{\simeq} f_2^{-1}(\mathscr{E}_2)$. Then there exists a fiber space $\tilde{\mathscr{E}} = (E, p, B)$ together with inclusion maps i_1: $X_1 \to B$, i_2: $X_2 \to B$ such that $i_1^{-1}(\tilde{\mathscr{E}}) \underset{f}{\simeq} \mathscr{E}_1$, $i_2^{-1}(\tilde{\mathscr{E}}) \underset{f}{\simeq} \mathscr{E}_2$.*

Corollary 3.1. *Let A, X_i, Z, f_i, g_i, $i=1,2$, be as in Definition 2.5 and let $\mathscr{E}_i$, $i=1,2$, be as in Theorem 3.1. Under these conditions there is a fiber space $\mathscr{E}' = (E', p', Z)$ such that $g_1^{-1}(\mathscr{E}') \underset{f}{\simeq} \mathscr{E}_1$, $g_2^{-1}(\mathscr{E}') \underset{f}{\simeq} \mathscr{E}_2$.*

Proof. Construct $Z_{f_1 f_2}$ and let i_s: $X_i \to Z_{f_1 f_2}$, $s=1,2$ be the inclusions. Since $i_1 f_1 \sim i_2 f_2$ we can consider $Z_{f_1 f_2}$ as Z' and conclude that there exists a map h: $Z \to Z_{f_1 f_2}$ such that $i_s \sim h\, g_s$, $s=1,2$. Let $\mathscr{E}' = h^{-1}(\tilde{\mathscr{E}})$. By Theorem 3.1 $i_1^{-1}(\tilde{\mathscr{E}}) \underset{f}{\simeq} \mathscr{E}_1$, $i_2^{-1}(\tilde{\mathscr{E}}) \underset{f}{\simeq} \mathscr{E}_2$. But $i_s^{-1}(\tilde{\mathscr{E}}) \underset{f}{\simeq} (h\, g_s)^{-1}(\tilde{\mathscr{E}})$ and $(h\, g_s)^{-1}(\tilde{\mathscr{E}}) \underset{f}{\simeq} g_s^{-1}(h^{-1}(\tilde{\mathscr{E}})) \underset{f}{\simeq} g_s^{-1}(\mathscr{E}')$.

Remark. Let $LF(X)$ be the functor of [*15*], p. 239, where F has the homotopy type of a CW complex. Then on $\mathscr{C}$ LF is a homotopy functor. For in $\mathscr{C}$ the analog of the wedge is the disjoint union and the wedge condition is automatically satisfied, while the equalizer condition holds by Corollary 3.1.

2. *The Base Point Case*

Let f_i: $A \to X_i$ be as before and let a_0, x_1, x_2 be corresponding base points. Then $f_i(a_0) = x_i$. We suppose given fiber spaces $(\mathscr{E}_1, g_1)$ and $(\mathscr{E}_2, g_2)$ over X_1 and X_2 respectively such that $f_1^{-1}(\mathscr{E}_1, g_1) \sim f_2^{-1}(\mathscr{E}_2, g_2)$.

Let $\bar{Z}_{f_1 f_2}$ be obtained from $Z_{f_1 f_2}$ by "collapsing" $a_0 \times I$ to a point and let ε: $Z_{f_1 f_2} \to \bar{Z}_{f_1 f_2}$ be the quotient map. For a base point in $\bar{Z}_{f_1 f_2}$ choose $\bar{z} = \varepsilon(a_0 \times I)$. Since we are dealing with CW complexes ε is a homotopy equivalence and letting ν be a homotopy inverse for ε such that $\nu(\bar{z})=i_1(x_1)$ we consider the fiber space $\bar{\mathscr{E}}=\nu^{-1}(\tilde{\mathscr{E}})=(\bar{E}, \bar{p}, \bar{Z}_{f_1 f_2})$.

There are diagrams:

$$\begin{array}{ccccccc} E_1 & \xrightarrow{\bar{i}_1} & Z_{m_1 n_1} & \xrightarrow{\mu} & E & \xrightarrow{\bar{\varepsilon}} & \bar{E} \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \bar{p}} \\ X_1 & \xrightarrow{i_1} & Z_{f_1 f_2} & \xrightarrow{1} & Z_{f_1 f_2} & \xrightarrow{\varepsilon} & \bar{Z}_{f_1 f_2} \end{array} \tag{8.1}$$

$$\begin{array}{ccccccc} E_2 & \xrightarrow{\bar{i}_2} & Z_{m_1 n_1} & \xrightarrow{\mu} & E & \xrightarrow{\bar{\varepsilon}} & \bar{E} \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \bar{p}} \\ X_2 & \xrightarrow{i_2} & Z_{f_1 f_2} & \xrightarrow{1} & Z_{f_1 f_2} & \xrightarrow{\varepsilon} & \bar{Z}_{f_1 f_2} \end{array} \tag{8.2}$$

where $\bar{i}_1$ and $\bar{i}_2$ are fiber maps over i_1 and i_2, and $\bar{\varepsilon}$ is a fiber map over ε constructed by using the fact that ε and ν are homotopy inverses of each other. $\bar{\varepsilon}$ is a homotopy equivalence in each fiber, and hence so are $\bar{\varepsilon}\,\mu\,\bar{i}_2$ and $\bar{\varepsilon}\,\mu\,\bar{i}_1$. This means that $(\varepsilon\, i_2)^{-1}(\bar{\mathscr{E}}) \underset{f}{\sim} \mathscr{E}_2$ and $(\varepsilon\, i_1)^{-1}(\bar{\mathscr{E}}) \underset{f}{\sim} \mathscr{E}_1$.

However, we are interested in equivalence in the sense of Definition 2.4 and this means that we have to assign a suitable map $g\colon F \to \bar{E}$ to $\bar{\mathscr{E}}$. Since $f_1^{-1}(\mathscr{E}_1, g_1) \sim f_2^{-1}(\mathscr{E}_2, g_2)$ we can assume that the maps α and β used in constructing $Z_{m_1 n_1}$ are such that:

$$\alpha\, g_1' \sim g_2', \quad \beta\, g_2' \sim g_1' \tag{9}$$

where for $i=1, 2$, $g_i'\colon F \to f_i^{-1}(E_i)$ is given by

$$g_i'(x)=\left(g_i(x), a_0\right). \tag{10}$$

Consider the chain:

$$F \xrightarrow{g_1} E \xrightarrow{\bar{i}_1} Z_{m_1 n_1} \xrightarrow{\mu} E \xrightarrow{\bar{\varepsilon}} \bar{E} \tag{11}$$

and let $g=\bar{\varepsilon}\,\mu\,\bar{i}_1\, g_1$.

Lemma 3.2. $(\varepsilon\, i_1)^{-1}(\bar{\mathscr{E}}, g) \sim \mathscr{E}_1$ *and* $(\varepsilon\, i_2)^{-1}(\bar{\mathscr{E}}, g) \sim \mathscr{E}_2$.

Proof. By definition $(\varepsilon\, i_1)^{-1}(\bar{\mathscr{E}}, g)$ is assigned the map $\bar{g}_1\colon F \to (\varepsilon\, i_1)^{-1}(\bar{E})$ given by

$$\bar{g}_1(x)=\left(\bar{\varepsilon}\,\mu\,\bar{i}_1\, g_1(x), x_1\right) \tag{12}$$

and this is compatible with the fiber map $\alpha_1\colon E_1 \to (\varepsilon\, i_1)^{-1}(\bar{E})$ given by:

$$\alpha_1(e_1)=\left(\bar{\varepsilon}\,\mu\,\bar{i}_1(e_1), p_1(e_1)\right). \tag{13}$$

This proves the first half of the lemma.

Now let $\alpha_2\colon E_2 \to (\varepsilon\, i_2)^{-1}(\bar{E})$ be given by:

$$\alpha_2(e_2)=\left(\bar{\varepsilon}\,\mu\,\bar{i}_2(e_2), p_2(e_2)\right). \tag{14}$$

Composing with g_2 we get

$$\alpha_2\, g_2(x) = (\bar{\varepsilon}\,\mu\,\bar{i}_2\, g_2(x), x_2). \tag{15}$$

Since $(\varepsilon\, i_2)^{-1}(\bar{\mathscr{E}}, g)$ is assigned the map $\bar{g}_2$ corresponding to (12) we have only to prove that

$$\bar{\varepsilon}\,\mu\,\bar{i}_1\, g_1 \sim \bar{\varepsilon}\,\mu\,\bar{i}_2\, g_2 \tag{16}$$

as maps from F into $\bar{p}^{-1}(\bar{z})$.

It is sufficient to prove that $\bar{i}_1\, g_1$ and $\bar{i}_2\, g_2$ are homotopic as maps into $q^{-1}(a_0 \times I)$. By (9) there is a homotopy γ_t, $0 \leqq t \leqq 1$, such that $\gamma_0 = \alpha\, g_1'$, $\gamma_1 = g_2'$. Define H: $F \times I \to Z_{m_1 n_1}$ by :

$$\begin{cases} H(x, t) = \bar{k}((g_1(x), a_0), 2t) & 0 \leqq t \leqq \frac{1}{2} \\ H(x, t) = \bar{k}\, m_2\, \gamma_{2t-1}(x) & \frac{1}{2} \leqq t \leqq 1. \end{cases} \tag{17}$$

Then $H(x, 0) = \bar{i}_1\, g_1$ and $H(x, 1) = \bar{i}_2\, g_2$.

We have, then, the analog of Theorem 3.1 with fiber homotopy equivalence being replaced by equivalence of pairs $(\mathscr{E}, g)$. Using maps and homotopies preserving base points, Corollary 3.1 can be suitably modified and we have:

Theorem 3.2. *The functor $\mathscr{H}(\ , F)$ satisfies the equalizer condition on $\mathscr{C}_0$, if F is of the homotopy type of a CW complex.*

3. The Wedge Condition

Let (X_ν, x_ν) be a collection of spaces with base points and let $X = \vee X_\nu$ be the corresponding wedge. In X we choose a base point x_0 such that if h_ν: $X_\nu \to X$ is the canonical inclusion $h_\nu(x_\nu) = x_0$. We have to prove that:

$$\Pi\, \mathscr{H}(h_\nu):\quad \mathscr{H}(X, F) \approx \Pi\, \mathscr{H}(X_\nu, F). \tag{18}$$

Lemma 3.3. *$\Pi\, \mathscr{H}(h_\nu)$ is onto.*

Proof. For each index ν let y_ν be a point and let $A = \bigcup y_\nu$. Let $X_1 = \bigcup X_\nu$, i.e., the disjoint union, and let X_2 be a single point which we denote by $*$. Also let f_1: $A \to X_1$ be given by $f_1(y_\nu) = x_\nu$ and let f_2: $A \to *$ be the constant map. Finally let i_ν: $X_\nu \to X_1$ be the inclusion map. By the usual property of the wedge there is a map h: $X \to Z_{f_1 f_2}$ such that

$$h(\bigcup h_\nu) \sim i_1(\bigcup i_\nu), \quad h(x_0) = * = i_2(*). \tag{19}$$

Further h is a homotopy equivalence and has an inverse j: $Z_{f_1 f_2} \to X$ such that

$$j\, k(A \times I) = x_0. \tag{20}$$

For each ν select a representative $(\mathscr{E}_\nu, g_\nu)$ from $\mathscr{H}(X_\nu, F)$. If $\mathscr{E}_\nu = (E_\nu, p_\nu, X_\nu)$ let $\mathscr{E}_1 = (\bigcup E_\nu, \bigcup p_\nu, X_1)$ and let $\mathscr{E}_2 = (F, p_2, *)$. The fiber spaces $f_1^{-1}(\mathscr{E}_1)$ and $f_2^{-1}(\mathscr{E}_2)$ are certainly fiber homotopy equivalent since $f_1^{-1}(E_1) = \bigcup (p_\nu^{-1}(x_\nu), y_\nu)$ and $f_2^{-1}(E_2) = \bigcup (F, y_\nu)$. Recalling that we can select any fiber homotopy equivalence α let:

(21) $$\alpha(e_\nu, y_\nu) = \big(\bar{g}_\nu(e_\nu), y_\nu\big),$$

where $\bar{g}_\nu$ is some inverse for g_ν and $e_\nu \in p_\nu^{-1}(x_\nu)$. Now as before construct $\tilde{\mathscr{E}} = (E, p, Z_{f_1 f_2})$ and let $\bar{\mathscr{E}} = h^{-1}(\tilde{\mathscr{E}})$. For each ν we have similarly to (8.1) and (8.2)

(22.1) $$\begin{array}{ccccccc} E_\nu & \xrightarrow{\bar{i}_\nu} & Z_{m_1 n_1} & \xrightarrow{\mu} & E & \xrightarrow{\bar{j}} & \bar{E} \\ \downarrow{\scriptstyle p_\nu} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \bar{p}} \\ X_\nu & \xrightarrow{i_\nu} & Z_{f_1 f_2} & \xrightarrow{1} & Z_{f_1 f_2} & \xrightarrow{j} & X \end{array}$$

(22.2) $$\begin{array}{ccccccc} F & \xrightarrow{\bar{i}_2} & Z_{m_1 n_1} & \xrightarrow{\mu} & E & \xrightarrow{\bar{j}} & \bar{E} \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \bar{p}} \\ * & \xrightarrow{i_2} & Z_{f_1 f_2} & \xrightarrow{1} & Z_{f_1 f_2} & \xrightarrow{j} & X \end{array}$$

and as in the proof of Lemma 3.2 we can show that

(23) $$\bar{j}\,\mu\,\bar{i}_2 \sim \bar{j}\,\mu\,\bar{i}_\nu\,g_\nu.$$

Hence if we assign the map $\bar{g} = \bar{j}\,\mu\,\bar{i}_2$ to $\bar{\mathscr{E}}$ we see that $h_\nu^{-1}(\bar{\mathscr{E}}, g) \sim (\mathscr{E}_\nu, g_\nu)$. This proves the lemma.

To show that $\Pi\,\mathscr{H}(h_\nu)$ is $1-1$ we need

Lemma 3.4. *Let $(\mathscr{E}, g)$ be a fiber space over $X = \vee X_\nu$. Let $(\mathscr{E}_\nu, g_\nu) = h_\nu^{-1}(\mathscr{E}, g)$. Let $(\bar{\mathscr{E}}, \bar{g})$ be the fiber space constructed in the proof of Lemma* 3.3. *Then $(\mathscr{E}, g) \sim (\bar{\mathscr{E}}, \bar{g})$.*

Proof. Using part of diagram 22.1 we see that we can define a map $\tilde{h}$: $\bar{E} \to Z_{m_1 n_1}$ such that the diagram

(24) $$\begin{array}{ccc} \bar{E} & \xrightarrow{\tilde{h}} & Z_{m_1 n_1} \\ \downarrow{\scriptstyle \bar{p}} & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{h} & Z_{f_1 f_2} \end{array}$$

is homotopy commutative. $\tilde{h}$ is *not* a fiber map because the homotopy inverse of μ is not. Using the inclusion $E_\nu \to E$ and the map g: $F \to E$ one can construct a fiber map h': $Z_{m_1 n_1} \to E$ such that the diagram:

(25) $$\begin{array}{ccc} Z_{m_1 n_1} & \xrightarrow{h'} & E \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ Z_{f_1 f_2} & \xrightarrow{j} & X \end{array}$$

is commutative. Combining (25) and (24) we see that $p\,h'\,\tilde{h} \sim j\,h\,\bar{p} \sim \bar{p}$. Since we are dealing with Hurewicz fiber spaces we can replace $h'\,\tilde{h}$ by a fiber map f

over the identity. Since all maps are homotopy equivalences on the fibers we have $\bar{\mathscr{E}} \underset{f}{\sim} \mathscr{E}$. Recalling the definition of $\bar{g}$ from Lemma 3.3 we see that $\tilde{h}\,\bar{g} \sim i_2$ and hence $h'\,\tilde{h}\,\bar{g} \sim g$ as maps from F into E.

Now f is obtained as follows:

$$f(\bar{e}) = \Lambda(h'\,\tilde{h}(\bar{e}), w(\bar{e}))(1) \tag{26}$$

where λ is a lifting function for $\mathscr{E}$ and $w(\bar{e})$ is the path in X connecting $p\,h'\,\tilde{h}(\bar{e})$ to $\bar{p}(\bar{e})$. Using (26) it is easy to see that $f\,\bar{g}$ and g are homotopic as maps from F into $p^{-1}(x_0)$, and therefore $(\mathscr{E}, g) \sim (\bar{\mathscr{E}}, \bar{g})$.

Theorem 3.3. *$\Pi\,\mathscr{H}(h_\nu)$ is an isomorphism.*

Proof. Only the one to oneness has to be proved. Let u_1 and u_2 be elements of $\mathscr{H}(X, F)$ such that $\Pi\,\mathscr{H}(h_\nu)\,u_1 = \Pi\,\mathscr{H}(h_\nu)\,u_2$. This means that for every index ν $h_\nu^{-1}(\mathscr{E}^1, g^1) \sim h_\nu^{-1}(\mathscr{E}^2, g^2)$ where $(\mathscr{E}^1, g^1)$ and $(\mathscr{E}^2, g^2)$ are representatives of u_1 and u_2. If $h_\nu^{-1}(\mathscr{E}^1) = (E_\nu^1, p_\nu^1, X_\nu)$ and $h_\nu^{-1}(\mathscr{E}^2) = (E_\nu^2, p_\nu^2, X_\nu)$ there are fiber maps f_ν: $E_\nu^1 \to E_\nu^2$ such that:

$$f_\nu\, g_\nu^1 \sim g_\nu^2 \tag{27}$$

where g_ν^i $i=1, 2$ is given by $g_\nu^i(x) = (g_\nu(x), x_\nu)$. If $Z_{m_1 n_1}^1$ and $Z_{m_1 n_1}^2$ correspond to $\mathscr{E}^1$ and $\mathscr{E}^2$ respectively the homotopies of (27) will allow us to construct a fiber map

$$\begin{tikzcd} Z_{m_1 n_1}^1 \arrow[rr, "f"] \arrow[dr, "q^1"'] & & Z_{m_1 n_1}^2 \arrow[dl, "q^2"] \\ & Z_{f_1 f_2} & \end{tikzcd} \tag{28}$$

which is a homotopy equivalence on each fiber. By Proposition 5, [*15*] $\tilde{\mathscr{E}}^1$ and $\tilde{\mathscr{E}}^2$ and hence $\bar{\mathscr{E}}^1$ and $\bar{\mathscr{E}}^2$ will be fiber homotopy equivalent. In fact it is easy to see that $(\bar{\mathscr{E}}^1, \bar{g}^1) \sim (\bar{\mathscr{E}}^2, \bar{g}^2)$. The theorem follows from Lemma 3.4.

Definition 3.1. Let $\bar{\mathscr{C}}_0$ be the subcategory whose objects are objects X of $\mathscr{C}_0$ such that for any $Y \in \mathscr{C}_1$ = finite CW complexes, f: $X \to X'$ is an equivalence if and only if f^*: $[Y, X]_0 \approx [Y, X']_0$.

We have shown that $\mathscr{H}(\ , F)$ is a homotopy functor on the category $\mathscr{C}_0$. By Theorem 10 of [*1*] there is a $B_\infty \in \mathscr{C}_0$ and a $u \in \mathscr{H}(B_\infty, F)$ such that T_u: $[\ , B_\infty]_0 \to \mathscr{H}(\ , F)$ given by:

$$T_u[f] = \mathscr{H}(f)\,u \tag{29}$$

is an equivalence if $B_\infty \in \bar{\mathscr{C}}_0$. ($T_u$ is always an isomorphism on $\mathscr{C}_1$).

As pointed out by Brown, $\bar{\mathscr{C}}_0$ = category of connected CW complexes, and we need only to show that B_∞ is connected. T_u is an isomorphism in the category $\mathscr{C}_1$ of finite CW complexes. On the other hand, if X is a space consisting of two points with base point x_0 $\mathscr{H}(X, F)$ consists of one element. This means that B_∞ is pathwise connected. This completes the proof of Theorem 2.1.

IV. The Associated Principal Fiber Space

Definition 4.1. If $\mathscr{E}=(E, p, X)$ is a fiber space with fiber F let E^F consist of all maps from F into E which are a homotopy equivalence into *some* $p^{-1}(x)$ [2]. Let $\tilde{p}f=f(x)$, $f\in E^F$, and set $\mathscr{E}^F=(E^F, \tilde{p}, X)$.

Proposition 4.1. *If F is locally compact $\mathscr{E}^F$ is a fiber space, and if $H(F)$ is the space of homotopy equivalences of F into itself, there is a fiber-wise operation μ: $E^F\times H(F)\to E^F$ given by $\mu(f, h)=fh$. If F is first axiom, then $\mathscr{E}^F$ is a Serre fiber space.*

Proof. If F is locally compact, the proposition is Lemma 7 of [*15*]. See also [*4*] p. 302. Suppose that F is first axiom. Let Z be a first axiom space and suppose given a homotopy G: $Z\times I\to X$ and a map f: $Z\times\{0\}\to E^F$ such that $\tilde{p}f=G_0$. The map $\bar{f}$: $Z\times F\to E$ associated to f is continuous, [*7*], or [*10*], p. 160. Since $\mathscr{E}$ is a fiber space we can apply the covering homotopy theorem to $\bar{f}$ and $\bar{G}$ [3] and extend $\bar{f}$ and hence f to $Z\times I$ as required.

This means that the CHP holds for $\mathscr{E}^F$ with respect to first axiom spaces, and these include finite polyhedra.

Remark. I do not know whether or not $\mathscr{E}^F$ is a fiber space if no restriction is made on F. Perhaps an argument using quasi topologies as in [*6*], would do the trick.

Now let $(\mathscr{E}_\infty, g_\infty)$ be the universal fiber space representing the functor $\mathscr{H}(\ , F)$. The only restriction we have made on F is that of being of the homotopy type of a CW complex. We can assume therefore that F is a metric space [*12*], Theorem 2, and hence first axiom; and we can consider $\mathscr{E}^F_\infty$ as being at least a Serre fibration.

Theorem 4.1. *If $X\in\mathscr{C}_0$ and E_∞ is the total space of $\mathscr{E}_\infty$ then $[X, E^F_\infty]=1$* [4].

Letting $F_\infty=p_\infty^{-1}(b_\infty)$ we see that $\tilde{p}_\infty^{-1}=F^F_\infty$ the space of homotopy equivalences from F into F_∞. To prove Theorem 4.1 we first prove

Lemma 4.1. *Let i: $F_\infty\to E_\infty$ be the inclusion, then i^*: $[X, F^F_\infty]\to[X, E^F_\infty]$ is onto.*

Proof. Since only the homotopy type of X is involved, we can assume that X is first axiom. Let α: $X\to E^F_\infty$ be given. If x_0 is the base point of X we can assume that $\tilde{p}_\infty\,\alpha(x_0)=b_\infty$. α gives rise to the diagram:

$$\begin{array}{ccc} X\times F & \xrightarrow{\bar{\alpha}} & E_\infty \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle p_\infty} \\ X & \xrightarrow{\tilde{p}_\infty\alpha} & B_\infty \end{array} \tag{1}$$

where π: $X\times F\to X$ is the projection. Since $\bar{\alpha}$ is a homotopy equivalence on each fiber, it follows that $(\tilde{p}_\infty\,\alpha)^{-1}(\mathscr{E}_\infty)$ is fiber homotopy equivalent to a

[2] All function spaces have the CO-topology and subsets have the relative CO-topology.

[3] $\bar{G}(z, x, t)=G(z, t)$ $x\in F$, $z\in Z$, $0\leqq t\leqq 1$.

[4] This means that $[X, E^F_\infty]$ is the set consisting of one element.

product, and since $\mathscr{H}(X, F)$ and $[X, B_\infty]_0$ are equivalent it follows that $\tilde{p}_\infty \alpha$ is homotopic to the constant map $X \to b_\infty$ (keeping base point fixed). Let H be a homotopy such that

$$(2) \qquad \begin{cases} H(x,0) = \tilde{p}_\infty \alpha(x) & x \in X,\ 0 \leqq t \leqq 1 \\ H(x,1) = b_\infty \\ H(x_0, t) = b_\infty . \end{cases}$$

Let λ_∞ be a lifting function for $\mathscr{E}_\infty$ and define $\bar{H}$: $X \times F \times I \to E_\infty$ by

$$(3) \qquad \begin{cases} \bar{H}(x,y,t) = \lambda_\infty(\bar{\alpha}(x,y), H_s(x))(t) & x \in X,\ y \in F,\ 0 \leqq t \leqq 1 \\ \quad H_s(x) = \text{path}\, H(x,s) & 0 \leqq s \leqq 1. \end{cases}$$

Then $\bar{H}(x,y,0) = \bar{\alpha}(x,y)$ and $\bar{H}(x,y,1) \in F_\infty$. The usual properties of a lifting function show that $\bar{H}$ induces a map H': $X \times I \to E_\infty^F$ such that $H_0' = \alpha$, $H_1'(X) \subset F_\infty^F$.

Remark. If we select g_∞: $F \to E_\infty$ as a base point for E_∞^F then using a regular lifting function we get that i^*: $[X, F_\infty^F]_0 \to [X, E_\infty^F]_0$ is onto.

Theorem 4.1 will then follow from the next lemma.

Lemma 4.2. $i^*[X, F_\infty^F] = 1$.

Proof. Let φ: $X \to F_\infty^F$ be given and let $\bar{\varphi}$: $X \times F \to F_\infty$ be the associated map. Let π: $X \times F \to F$ be the projection and let x_1, x_2 be two points disjoint from X. Consider:

$$\begin{array}{ccccc} F \cup_\pi & X \times F \times I & \cup_{\bar{\varphi}} F_\infty & = & Z_{\pi,\bar{\varphi}} \\ \downarrow & \downarrow & \downarrow & & \downarrow{\scriptstyle q} \\ x_1 \cup_{f_1} & X \times I & \cup_{f_2} x_2 & = & S(X) \end{array}$$

where f_i, $i = 1, 2$ are the constant maps, and $S(X) = Z_{f_1 f_2}$ is the unreduced suspension. We take $i_2(x_2) = x_2$ as the base point of $S(X)$. As in part 3 we replace the quasifibration $(Z_{\pi,\bar{\varphi}}, q, S(X))$ by a fibration: $\mathscr{E} = (E, p, S(X))$ and we have the usual map

$$(4) \qquad \begin{tikzcd} Z_{\pi,\bar{\varphi}} \arrow[rr, "\mu"] \arrow[dr, "q"'] & & E \arrow[dl, "p"] \\ & S(X) & \end{tikzcd}$$

Let g: $F \to E$ be given by

$$(5) \qquad g = \mu\, \bar{i}_2\, g_\infty$$

where $\bar{i}_2$: $F_\infty \to Z_{\pi,\bar{\varphi}}$ is the inclusion, and consider the pair $(\mathscr{E}, g)$. $\mathscr{E}_\infty$ being a classifying fiber space there exist maps f and $\bar{f}$

$$(6) \qquad \begin{tikzcd} E \arrow[r, "\bar{f}"] \arrow[d, "p"] & E_\infty \arrow[d, "p_\infty"] \\ S(X) \arrow[r, "f"] & B_\infty \end{tikzcd}$$

such that $\bar{f}$ is a homotopy equivalence on each fiber and

(7) $$\bar{f}\, g \sim g_\infty, \qquad f(x_2) = x_\infty .$$

If $\bar{k}$: $F \cup X \times F \times I \cup F_\infty \to Z_{\pi, \bar{\varphi}}$ is the quotient map this means that $\bar{f}\,\mu\,\bar{k}$ restricted to $X \times F \times I$ is a map into E_∞ and

(8) $$\bar{f}\,\mu\,\bar{k} \,|\, X \times F \times \{0\} = \bar{f}\,\mu\,\bar{i}_1\,\pi ,$$

(9) $$\bar{f}\,\mu\,\bar{k} \,|\, X \times F \times \{1\} = \bar{f}\,\mu\,\bar{i}_2\,\bar{\varphi} .$$

$\bar{i}_1$ and $\bar{i}_2$ being the inclusions of F and F_∞ into $Z_{\pi, \bar{\varphi}}$. Since all maps involved are homotopy equivalences on fibers this is equivalent to a homotopy H_t: $X \to E_\infty^F$ such that:

(10) $$H_0(x) = \bar{f}\,\mu\,\bar{i}_1,$$

(11) $$H_1(x) = (\bar{f}\,\mu\,\bar{i}_2) \cdot \varphi(x) .$$

By (7) and (5)

(12) $$\bar{f}\,\mu\,\bar{i}_2 \sim \bar{f}\,\mu\,\bar{i}_2\, g_\infty\, \bar{g}_\infty \sim (\bar{f}\, g)\, \bar{g}_\infty \sim g_\infty\, \bar{g}_\infty \sim 1 .$$

$\bar{g}_\infty$ being an inverse to g_∞. It follows that φ and the constant map $X \to \bar{f}\,\mu\,\bar{i}_1$ are homotopic as maps into E_∞^F. But this implies that any map φ is homotopic to the constant map $X \to g_\infty$, proving the lemma and hence Theorem 4.1.

Corollary 4.1. $\pi_i(B_\infty, b_\infty) \approx \pi_{i-1}(F_\infty^F, g_\infty)$.

Corollary 4.2. *If F is compact E_∞^F is contractible.*

Proof. By Corollary 2 of [*12*] F_∞^F has the homotopy type of a CW complex and therefore E_∞^F does also. But if X is a CW complex any map φ: $X \to E_\infty^F$ is trivial.

V. A Special Case

Let F and F_∞ be as in Section IV. Let x_0 be a base point for F and let $x_\infty = g_\infty(x_0)$. By [*12*], Theorem p. 279, we can assume that there is a CW complex Z with base point z_0 such that the pairs (F, x_0) and (Z, z_0) are of the same homotopy type. In fact, Z is a simplicial complex in the weak topology and F is simply Z with the strong topology. Denote by H the set of homotopy equivalences from F into F_∞[5] and by H_0 the subset of H consisting of equivalences g: $F \to F_\infty$ such that $g(x_0) = x_\infty$.

Lemma 5.1. *Let v: $H \to F_\infty$ be the evaluation map at x_0, i.e. $v(g) = g(x_0)$. Then the triple (H, v, F_∞) is a quasifibration and $v^{-1}(x_\infty) = H_0$.*

Proof. Let P be a finite polyhedron and suppose we have a diagram:

(1) $$\begin{array}{ccc} & & F_\infty^F \\ & \overset{f}{\nearrow} & \downarrow v \\ P \times \{0\} \subset P \times I & \xrightarrow{K} & F_\infty \end{array} \qquad v\, f = K .$$

[5] That is $H = \tilde{p}_\infty^{-1}(x_\infty) =$ fiber over x_∞ of $\mathscr{E}_\infty^F$.

P and F being first axiom this is equivalent to having a map $f^*: P\times F\times\{0\}\to F_\infty$ such that $f^*(p, x_0, 0)=x_\infty$, $p\in P$. Now let α and β be maps α: $(F, x_0)\rightleftarrows(Z, z_0)$: β such that $\alpha\beta\sim 1$, $\beta\alpha\sim 1$, preserving base points. Consider the map $f^*(1\times\beta\times 1)$: $P\times Z\times\{0\}\to F_\infty$. Define G: $P\times Z\times\{0\}\cup P\times z_0\times I\to F_\infty$ by:

$$\text{(2)}\qquad \begin{cases} G(p, z, 0)=f(1\times\beta\times 1)\\ G(p, z_0\, t)=K(p, t). \end{cases}$$

$(P\times Z, P)$ as a CW complex pair has the homotopy extension property and G can be extended to all of $P\times Z\times I$. Let $\bar{G}$ be the extension and consider $\bar{G}(1\times\alpha\times 1)$: $P\times F\times I\to F_\infty$. Since $\bar{G}(p, z_0, t)=G(p, z_0, t)$ we see that if G^*: $P\times I\to F_\infty^F$ is the map induced by $\bar{G}(1\times\alpha\times 1)$ then $v(G^*(p, t))=G^*(p, t)(x_0)$ $=K(p, t)$, i.e. G^* is a homotopy over K. Also $G^*(p, 0)(x)=f\beta\alpha(x)$ and since $\beta\alpha\sim 1$ keeping x_0 fixed we see that the maps f and $G^*|P\times\{0\}$ are vertically homotopic. This means that the triple (H, v, F_∞) has the weak covering homotopy property with respect to maps from finite polyhedra [3], Theorem 5.13. This is sufficient to imply the exactness of the homotopy sequence as remarked by Dold [3], p. 238[6].

Assume now that F has only a finite number of non zero homotopy groups, i.e., there are integers p and q such that

$$\text{(3)}\qquad \pi_i(F, x_0)=0 \qquad i<q,\ i>p+q-2,\ p\geqq 2,\ q\geqq 2.$$

Theorem 5.1. *v^*: $\pi_i(H, g_\infty)\to\pi_i(F_\infty, x_\infty)$ is an isomorphism if $i\geqq p$ and a monomorphism if $i=p-1$.*

Proof. By Lemma 5.1 it is sufficient to show that $\pi_i(H_0, g_\infty)=0$ if $i\geqq p-1$. Letting y_0 be a base point for the i-th sphere S^i we have to consider homotopy classes of maps $S^i\times F\to F_\infty$ with preassigned values on $(S^i\times x_0)\cup(y_0\times F)=S^i\vee F$. Replacing F by the CW complex Z, we have the problem of finding the homotopy classes of maps of $S^i\times Z$ into F_∞, relative to $S^i\vee Z$, which extend f_0: $S^i\vee Z\to F_\infty$ given by:

$$\text{(4)}\qquad \begin{cases} f_0(y)=x_\infty & y\in S^i\\ f_0(z)=g_\infty\beta(z) & z\in Z \end{cases}$$

where β is the homotopy equivalence $(Z, z_0)\to(F, x_0)$. Consider the cohomology groups:

$$\text{(5)}\qquad H^n=H^n(S^i\times Z, S^i\vee Z; \pi_n(F_\infty)).$$

By [14], Lemma 1.6, this is equivalent to considering the group of the reduced product $(S^i ⨳ Z, y_0 ⨳ z_0)$. Since $S^i ⨳ Z$ is the i-th suspension of Z this means that:

$$\text{(6)}\qquad H^n=H^{n-i}(Z, z_0; \pi_n(F_\infty)).$$

[6] This was also pointed out to me by Dr. Martin Fuchs.

If $n>p+q-2$ then $H^n=0$. Now suppose that $i\geqq p-1$. Then if $n\leqq p+q-2$, $n-i\leqq q-1$ and $H^n=0$. In particular $H^q=0$ and by a standard result of obstruction theory, any two maps f and g such that $f|S^i\vee Z=g|S^i\vee Z=f_0$ are homotopic relative to $S^i\vee Z$, i.e., $\pi_i(H_0, g_\infty)=0$ if $i\geqq p-1$.

Corollary 5.1. *Let $\mathscr{E}_\infty=(E_\infty, p_\infty, B_\infty)$ be the universal fiber space for F. Then the boundary homomorphism δ_∞: $\pi_{i+1}(B_\infty, b_\infty)\to\pi_i(F_\infty, x_\infty)$ is an isomorphism if $i\geqq p$ and a monomorphism if $i=p-1$.*

Proof. If we consider the associated principal fiber space $\mathscr{E}_\infty^F$ and let $\tilde{\delta}$ be the corresponding boundary homomorphism we have a commutative diagram:

(7)
$$\begin{tikzcd}
\pi_{i+1}(B_\infty, b_\infty) \arrow[rr, "\delta_\infty"] \arrow[dr, "\tilde{\delta}"'] & & \pi_i(F_\infty, x_\infty) \arrow[dl, "v^*"] \\
& \pi_i(H, g_\infty) &
\end{tikzcd}$$

By the theorem, δ_∞ will have the required property if $\tilde{\delta}$ is an isomorphism for $i\geqq p-1$, but by Corollary 4.1 this is the case for all dimensions. As a consequence we have the following theorem, GANEA [8] and also [9]:

Theorem 5.2. *Let X have the homotopy type of a CW complex and suppose that $\pi_i(X)=0$ unless perhaps $q+1\leqq i\leqq p+q-1$ $p\geqq 2$, $q\geqq 2$. Then $\mathscr{H}(\ ,\Omega(X))$ and $[\ ,X]_0$ are naturally equivalent on the category of $(p-1)$-connected CW complexes.*

Proof. Let $\mathscr{E}=(P(X, x_0), p, X)$ and choose a suitable F and a homotopy equivalence g: $F\to\Omega(X, x_0)$. If $(\mathscr{E}_\infty, g_\infty)$ is the classifying fiber space for F there is a map f: $X\to B_\infty$ such that $f^{-1}(\mathscr{E}_\infty, g_\infty)\sim(\mathscr{E}, g)$ i.e., there is a commuting diagram:

(8)
$$\begin{tikzcd}
P(X, x_0) \arrow[r, "\bar{f}"] \arrow[d, "p"'] & E_\infty \arrow[d, "p_\infty"] \\
X \arrow[r, "f"] & B_\infty
\end{tikzcd}$$

such that $f_0=\bar{f}\,|\,p^{-1}(x_0)$ is a homotopy equivalence (in fact, $f_0\,g\sim g_\infty$). Corollary 5.1 and the contractibility of $P(X, x_0)$ imply that f^*: $\pi_i(X, x_0)\to\pi_i(B_\infty, b_\infty)$ is an isomorphism if $i\geqq p$. A slight modification of Theorem 16.3 of [13] then implies that f^*: $[Z, X]_0\approx[Z, B_\infty]_0$ for all $(p-1)$-connected CW complexes. Now let $\tilde{g}$ be a homotopy inverse for g. If we define g^*: $\mathscr{H}(\ ,F)\to\mathscr{H}(\ ,\Omega(X))$ by:

(9) $$g^*\{(\mathscr{E}, g')\}=\{(\mathscr{E}, g'\,\tilde{g})\},$$

we see that g^* is one to one and onto. Defining T: $[Z, X]_0\to\mathscr{H}(Z, F)$ by:

(10) $$T[h]=\{(f\,h)^{-1}(\mathscr{E}_\infty, g_\infty)\},\quad [h]\in[Z, X]_0$$

the required equivalence is the composition $g^*\circ T$.

As a final remark suppose that $\pi_i(F)=0$ unless perhaps $p \leqq i \leqq 2p-2$, i.e., $p=q$. Corollary 5.1 implies that $\pi_i(E_\infty)=0$ if $i \geqq p$. On the other hand there exists a CW complex $\bar{B}_\infty$ and a map $f: \bar{B}_\infty \to B_\infty$ such that

$$\pi_i(\bar{B}_\infty)=0, \qquad i \leqq p-1, \tag{9}$$

$$f^*: \pi_i(\bar{B}_\infty) \approx \pi_i(B_\infty), \qquad i \geqq p. \tag{10}$$

If we consider the induced fiber space $f^{-1}(\mathscr{E}_\infty)$, we see that the homotopy groups of $f^{-1}(E_\infty)$ all vanish. Since $f^{-1}(E_\infty)$ has the homotopy type of a CW complex, [*15*] Proposition 12, it follows that it is contractible, and this in turn implies that F has the homotopy type of a loop space, namely, $\Omega(\bar{B}_\infty)$. As pointed out by the referee, this is a weaker version of a known theorem about spaces whose homotopy groups vanish except possibly in the range p to $2p-2$.

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