REMARKS ON THE LOOP SPACE OF A FIBRATION

By Guy Allaud

Introduction. The fact that the existence of a cross section for a Hurewicz fibration $\mathcal{E} = (E, p, B)$ implies that the loop space of E splits in a natural way is well known and has appeared in many variants. On the other hand, the converse is certainly false for let $f:X \to Y$ be a map such that $\Omega(f):\Omega(X, x_0) \to \Omega(Y, y_0)$ is inessential but f is not, then, if $\mathcal{E}(Y)$ is the fibration of paths in Y based at y_0 the fiber space $f^{-1}(\mathcal{E}(Y))$ has no cross section but its loop space splits as a product (at least if the spaces involved are CW complexes). The purpose of this note is to point out that if we consider the loop space $\Omega(E, x_0)$ as a fibration over $\Omega(B, b_0)$ with fiber $\Omega(F, x_0)$, $F = p^{-1}(b_0)$, $x_0 \in F$, then its fiber homotopy type is essentially determined (See Corollary 2.1 for a precise statement) by the homotopy class of the map ϕ: $\Omega(B, b_0) \to F$ induced by a lifting function. (ϕ gives rise to the boundary homomorphism in the exact sequence of $\mathcal{E}$.) For instance, in the example above the class of ϕ is obviously zero. Our result contains the standard case for fibrations with cross sections but, in addition, it applies to situations where cross sections do not exist e.g., the generalized Whitney sum (§3).

Some remarks about notation and conventions. A fiber space means a triple $\mathcal{E} = (E, p, B)$, $p:E \to B$ continuous, which has the covering homotopy property (CHP). $\Omega(X, x_0)$ denotes the space of loops in X based at x_0, and x^* stands for the constant path at x all path spaces being given the C–0 topology. The word "map" will mean continuous map, and all spaces are assumed to be T_2. In so far as has been possible no restrictions have been imposed on the spaces involved and this, of course, has complicated some of the arguments e.g., in proving that a fiber map is a fiber homotopy equivalence we cannot appeal to the fact that it is a homotopy equivalence on fibers because the base space is not assumed to have any "nice" local properties.

I would like to thank the referee for several valuable comments and, in particular, for suggesting Definition 2.1.

1. Fibrations associated to a map. Associated to a map $f : (X, x_0) \to (Y, y_0)$ between spaces with base points we consider the three fibrations below

$$(1) \qquad \mathcal{E}(f) = (E(f), p_f, Y)$$

$$E(f) = \{(x, w) \mid x \in X, w:I \to Y, f(x) = w(0)\}$$

$$P_f(x, w) = w(1)$$

Received June 6, 1969. This work was supported in part by the National Science Foundation under Grant NSF GP-8896.

$$(2)\qquad \begin{aligned} \mathfrak{A}(f) &= (A(f), \pi, X) \\ A(f) &= p_f^{-1}(y_0) = \{(x, w) \in E(f) \mid w(1) = y_0\} \\ \pi(x, w) &= x \end{aligned}$$

$$(3)\qquad \begin{aligned} \Gamma(f) &= (\Gamma(X), q, F)F = f^{-1}(y_0) \\ \Gamma(X) &= \{w: I \to X \mid w(0) = x_0, w(1) \in F\} \\ q(w) &= w(1). \end{aligned}$$

Note that $A(f)$ and π are, essentially, the third space and second map respectively in the mapping sequence of f (the dual of the Puppe sequence) e.g., [7; 117].

The projection $\bar{\pi}: E(f) \to X$ (the obvious extension of π) is a homotopy equivalence a homotopy inverse being provided by the canonical inclusion $X \to E(f)$ which sends x into $(x, f(x)^*)$, and if f is itself a fiber map we have the following proposition which is a special case of Proposition 1 of [4].

PROPOSITION 1.1. *Suppose* $\mathcal{E} = (X, f, Y)$ *is a fibration and let* $\tilde{f}: \Gamma(X) \to \Omega(Y, y_0)$ *be the map induced by* f. *Then* $\tilde{f}$ *is a homotopy equivalence. In fact, if* Λ *is a lifting function for* $\mathcal{E}$ *a homotopy inverse for* $\tilde{f}$ *is given by the map*

$$\alpha \to \Lambda(x_0, \alpha), \alpha \in \Omega(Y, y_0).$$

2. Regular points of fibrations.

DEFINITION 2.1. Let $\mathcal{E} = (E, p, B)$ be a Hurewicz fibration. A point $x_0 \in E$ is said to be a *regular point* of $\mathcal{E}$ if there exists a lifting function Λ for $\mathcal{E}$ such that

$$\Lambda(x_0, b_0^*) = x_0^*, b_0 = p(x_0).$$

This is equivalent to saying that the triple $((E, x_0), p, (B, b_0))$ is a fibration in the category of topological spaces with base points and maps preserving base points. Given $\mathcal{E}$ a preassigned x_0 may fail to be a regular point, the non-regular (in the usual sense) fibration of Tulley [12] being such an example, but I do not know if there exists an $\mathcal{E}$ with no regular points. In any case, here is a simple sufficient condition for a point to be regular.

PROPOSITION 2.1. *Suppose that* $b_0 = f^{-1}(0)$ *for some map* $f: B \to I$ (e.g., B = *CW complex*). *Then for any fibration* $\mathcal{E}$ *over* B *every point of* $p^{-1}(b_0)$ *is a regular point.*

Proof. Define a map $\gamma: B^I \to I$

$$(1)\qquad \gamma(\alpha) = \sup_{0 \le t \le 1} f(\alpha(t)).$$

γ is continuous, and furthermore $\gamma(\alpha) = 0$ iff $\alpha = b_0^*$. The argument used in [12; 127] will give a lifting function with the desired property.

Remarks. If the pair (B, b_0) has the *HEP*, a map f as above can always be found [8; 82]. It should also be pointed out that if B is a *CW* complex, any fibration over B turns out to be a *regular* fibration. This follows from the fact that while $B \times B$ need not be a *CW* complex, it is nevertheless a perfectly normal space [1; 121] so that the statement on page 133 of [12], whose proof is straightforward, is applicable.

PROPOSITION 2.2. *If $\mathcal{E}$ is a fibration with regular point x_0 the triple $\Omega(\mathcal{E}) = (\Omega(E, x_0), \Omega(p), \Omega(B, b_0))$ is a fibration with regular point x_0^*.*

Proof. Let Λ be a lifting function for $\mathcal{E}$ such that $\Lambda(x_0, b_0^*) = x_0^*$. Given $\alpha \in \Omega(E, x_0)$ and $w \in \Omega(B, b_0)^{[0,1]}$ such that $\Omega(p)(\alpha) = w(0)$ we need to find a path in $\Omega(E, x_0)$ starting at α and lying over w (and of course varying continuously with respect to α and w). If $s \in [0, 1]$, let $w_s : [0, 1] \to B$ be the path

$$w_s(t) = (w(t))(s); \tag{2}$$

i.e., the value of w_s at t is the loop $w(t)$ evaluated at s. A lifting function $\tilde{\Lambda}$ for $\Omega(\mathcal{E})$ is then obtained by letting $\tilde{\Lambda}(\alpha, w)(t)$ be the loop in E whose value at s is $\Lambda(\alpha(s), w_s)(t)$ or more formally

$$[\tilde{\Lambda}(\alpha, w)(t)](s) = \Lambda(\alpha(s), w_s)(t). \tag{3}$$

Obviously all that is involved is the fact that an element of $\Omega(B, b_0)^{[0,1]}$ can be considered as an element of $\Omega(B^{[0,1]}, b_0^*)$, or in other words "a path of loops is a loop of paths."

Remark. Actually, as pointed out by the referee, the existence of a regular point is not necessary to show that $\Omega(\mathcal{E})$ is a fibration.

From now on we work in the category of based spaces; i.e., given a fibration $\mathcal{E}$ we assume that $\mathcal{E}$ has a regular point x_0 which is the base point of E, and $b_0 = p(x_0)$ is the base point of B. We let $F = p^{-1}(b_0)$; and when working with two fibrations we use subscripts $i = 1$ or 2. Unless stated, lifting functions, homotopies, and so on all preserve base points. The one exception is that in constructing fiber homotopy equivalences we do not insist that base points be preserved, the reason being that we want to make no restriction on the spaces involved; e.g., we do not assume that x_0 is "nicely" imbedded in E.

Given a fibration $\mathcal{E} = (E, p, B)$ a choice of lifting function Λ will give rise to a map ϕ: $\Omega(B, b_0) \to F$ by setting

$$\phi(\beta) = \Lambda(x_0, \beta)(1). \tag{4}$$

If Λ' is another lifting function, it is easy to see that ϕ and ϕ' are homotopic relative b_0^*; and in this way $\mathcal{E}$ gives rise to a well defined homotopy class in $[\Omega(B, b_0), b_0^*; F, x_0]$. ([;] denotes the set of homotopy classes).

DEFINITION 2.2. The homotopy class of ϕ is called the loop class of $\mathcal{E}$ and is denoted by $[\mathcal{E}]$.

Remark. With no restriction on the spaces involved it does not seem possible to guarantee that any representative of the loop class is representable as the restriction of a lifting function.

PROPOSITION 2.3. *$[\Omega(\mathcal{E})]$ and $\Omega([\mathcal{E}])$ are inverses of each other. (Recall that homotopy classes of maps into a loop space form a group.)*

Proof. Considering the unit square $I \times I$ we first find a homotopy $H_u : I \times I \to I \times I$, $0 \leq u \leq 1$, so that

$$H_0(s, t) = (s, t), H_1(s, t) = (1 - t, s), \quad 0 \leq s \leq 1, \quad 0 \leq t \leq 1$$
$$H_u(I \times \dot{I} \cup \dot{I} \times I) \subset I \times \dot{I} \cup \dot{I} \times I. \tag{5}$$

(Note that H_1 is just a 90° counterclockwise rotation around (1/2, 1/2).) This means that the involution of $\Omega(\Omega(B, b_0), b_0^*)$ which sends a loop $w: I \to \Omega(B, b_0)$ into the loop $\bar{w}: I \to \Omega(B, b_0)$ with equation

$$(\bar{w}(s))(t) = (w(1 - t))(s) \tag{6}$$

is homotopic to the identity.

Let $\tilde{\phi}: \Omega(\Omega(B, b_0), b_0^*) \to \Omega(F, x_0)$ be the map induced by $\tilde{\Lambda}$ (See (3)). Comparing $\tilde{\phi}(w)$ and $\Omega(\phi)(w)$ for any $w: I \to \Omega(B, b_0)$ we see that

$$(\tilde{\phi}(w))(s) = \phi(w_s)$$
$$[\Omega(\phi)(w)](s) = \phi(w(s)) \tag{7}$$

where $w_s(t) = (w(t))(s)$.

Now consider $\overline{w^{-1}}: I \to \Omega(B, b_0)$. From (6)

$$(\overline{w^{-1}}(s))(t) = (w^{-1}(1 - t))(s) = (w(t))(s) = w_s(t) \tag{8}$$

i.e., $\overline{w^{-1}}(s) = w_s$. Substituting in (7)

$$(\tilde{\phi}(w))(s) = \phi(w_s) = \phi(\overline{w^{-1}}(s)) = [\Omega(\phi)(\overline{w^{-1}})](s) \tag{9}$$

i.e., $\tilde{\phi}(w) = \Omega(\phi)(\overline{w^{-1}})$. But this means that $\tilde{\phi}$ is homotopic to the map sending w into $\Omega(\phi)(w^{-1})$, and this is exactly what we wished to show since $\Omega(\phi)(w^{-1}) = [\Omega(\phi)(w)]^{-1}$.

THEOREM 2.1. *$\Omega(\mathcal{E})$ and $\mathcal{A}(\phi)$ are fiber homotopy equivalent and an equivalence $f: A(\phi) \to \Omega(E, x_0)$ can be chosen so that $f \mid \pi^{-1}(b_0^*)$ is homotopic to the canonical map from $\pi^{-1}(b_0^*)$ into $\Omega(F, x_0)$ which sends (b_0^*, α) into α.*

Proof. Consider the map $\mu: E(\phi) \to \Gamma(E)$

$$\mu(\beta, \alpha) = \Lambda(x_0, \beta) \circ \alpha. \tag{10}$$

$\beta \in \Omega(B, b_0)$, $\alpha \in F^I$, $\phi(\beta) = \alpha(0)$, $\circ$ = path multiplication. This gives a commutative diagram

$$\begin{array}{ccc} E(\phi) & \xrightarrow{\mu} & \Gamma(E) \\ & p_\phi \searrow \quad \swarrow q & \\ & F & \end{array} \tag{11}$$

i.e., μ is a fiber map. Composition of μ with $\tilde{p}$: $\Gamma(E) \to \Omega(B, b_0)$ yields a map sending (β, α) into $\beta \circ b_0^*$, and this means that $\tilde{p}\mu$ is homotopic to the projection $\bar{\pi}$: $E(\phi) \to \Omega(B, b_0)$, $\bar{\pi}(\beta, \alpha) = \beta$; but this in turn implies that μ is a homotopy equivalence since this is the case for $\bar{\pi}$ and $\tilde{p}$ (Proposition 1.1). A result of Dold [2, Theorem 6.1] allows us to conclude that μ is a fiber homotopy equivalence.

Now if $g = \mu \mid A(\phi)$, the diagram

$$\begin{array}{ccc} A(\phi) & \xrightarrow{g} & \Omega(E, x_0) \\ \pi \searrow & & \swarrow \Omega(p) \\ & \Omega(B, b_0) & \end{array} \tag{12}$$

is homotopy commutative since $\Omega(p)(g(\beta, \alpha)) = \beta \circ b_0^*$. ($\beta \in \Omega(B, b_0)$, $\alpha \in F^I$, $\phi(\beta) = \alpha(0)$, $\alpha(1) = x_0$.) Because the map $\beta \to \beta \circ b_0^*$ is homotopic to the identity keeping the base point b_0^* fixed, we can use the CHP and find a fiber map f: $A(\phi) \to \Omega(E, x_0)$, such that $f \mid \pi^{-1}(b_0^*)$ and $g \mid \pi^{-1}(b_0^*)$ are homotopic as maps into $\Omega(F, x_0)$. Since g is a homotopy equivalence, so is f and, again using Dold's result, f is a fiber homotopy equivalence with the required property since $g(b_0^*, \alpha) = x_0^* \circ \alpha$.

DEFINITION 2.3. Two loop classes $[\mathcal{E}_1]$ and $[\mathcal{E}_2]$ are said to be related if there are maps k: $F_1 \to F_2$ and Δ: $\Omega(B_1, b_1) \to \Omega(B_2, b_2)$ such that $k_\#[\mathcal{E}_1] = \Delta_\#[\mathcal{E}_2]$ (Upper and lower # denote the induced maps between homotopy classes.) If, in addition, k and Δ are homotopy equivalences, $[\mathcal{E}_1]$ and $[\mathcal{E}_2]$ are said to be equivalent.

THEOREM 2.2. *Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be fibrations with related loop classes. Then there exists a fiber map f: $\Omega(E_1, x_1) \to \Omega(E_2, x_2)$ over Δ such that $f \mid \Omega(F_1, x_1)$ and $\Omega(k)$ are homotopic.*

Proof. This is a direct consequence of the functorial character of $\mathfrak{A}(\phi)$. By hypothesis there is a homotopy commutative diagram

$$\begin{array}{ccc} \Omega(B_1, b_1) & \xrightarrow{\Delta} & \Omega(B_2, b_2) \\ \downarrow \phi_1 & & \downarrow \phi_2 \\ F_1 & \xrightarrow{k} & F_2 \end{array} \tag{13}$$

and this means that we can construct a commutative diagram

$$\begin{array}{ccc} A(\phi_1) & \xrightarrow{m} & A(\phi_2) \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ \Omega(B_1, b_1) & \xrightarrow{\Delta} & \Omega(B_2, b_2) \end{array} \tag{14}$$

such that m restricted to the fiber over b_1^* is homotopic to the map sending (b_1^*, α) into $(b_2^*, \Omega(k)(\alpha))$. For a proof see [7; 120] keeping in mind that Nomura's definitions do not quite agree with ours, the role of 0 and 1 being interchanged in the definition of lifting function. Now let $f_i : A(\phi_i) \to \Omega(E_i, x_i)$ $i = 1, 2$ be the maps guaranteed by Theorem 2.1 and denote by $\bar{f}_1$ a fiber homotopy inverse for f_1. Setting $f = f_2 m \bar{f}_1$ completes the proof.

COROLLARY 2.1. *If* $[\mathcal{E}_1]$ *and* $[\mathcal{E}_2]$ *are equivalent,* $\Omega(\mathcal{E}_1)$ *and* $\Delta^{-1}\,\Omega(\mathcal{E}_2)$ *are fiber homotopy equivalent.*

Proof. In this case m turns out to be a homotopy equivalence [7, Lemma 6], and the same is true of f. In general, given a diagram of fiber spaces

$$\begin{array}{ccc} E & \xrightarrow{\bar{k}} & E' \\ \downarrow p & & \downarrow p' \\ B & \xrightarrow{k} & B' \end{array},$$

$\bar{k}$ induces $\tilde{k}$: $E \to k^{-1}(E')$, $\tilde{k}(e) = (\bar{k}(e),\ p(e))$; and if j: $k^{-1}(E') \to E'$ is the canonical map $j(e', b) = e'$, then $\bar{k} = j\tilde{k}$. If k is a homotopy equivalence, so is j (e.g., [10; 81]) therefore if $\bar{k}$ is a homotopy equivalence the same is true of $\tilde{k}$ which is then a fiber homotopy equivalence.

COROLLARY 2.2. *If* $[\mathcal{E}] = 0$, $\Omega(\mathcal{E})$ *is fiber homotopy equivalent to* $\Omega(B \times F)$. (0 *denotes the class of the constant map.*)

3. Concluding remarks.

(A) Suppose $\mathcal{E}$ has a cross section σ: $B \to E(\sigma(b_0) = x_0)$. Then $[\mathcal{E}] = 0$. To see this let

$$\beta_s(t) = \beta(s + (1 - s)t), \quad 0 \leq s \leq 1,\ \beta \in \Omega(B, b_0) \tag{1}$$

and consider

$$H(\beta, s) = \Lambda(\sigma(\beta(s)), \beta_s)\ (1). \tag{2}$$

Then $H(\beta, 0) = \phi(\beta)$, $H(\beta, 1) = x_0$, $H(b_0^*, s) = x_0$, and we see that $\Omega(E, x_0)$ is fiber homotopy equivalent to $\Omega(B \times F, (b_0, x_0))$. (cf [6; 104], [3; 50], [7; 126]).

(B) If $\mathcal{E}_1$ and $\mathcal{E}_2$ are fibrations over a common base B, let $\mathcal{E}_1 \oplus \mathcal{E}_2$ denote their Whitney sum as defined in [5] the fiber over b_0, $F_1 * F_2$, being the join (in the strong topology) of the respective fibers. In the notation of [5] let $((1/2)x_1, (1/2)x_2)$ be the base point for $E_1 \oplus E_2$. If we use the lifting function described in [5; 363], the expression for ϕ becomes

$$\phi(\beta) = ((1/2)\phi_1(\beta), \quad (1/2)\phi_2(\beta)). \tag{3}$$

Now, in general, the map from $A \times B$ into $A * B$ sending (a, b) into $((1/2)a, (1/2)b)$ is inessential, but not necessarily relative, to a preassigned base point; i.e., ϕ is freely homotopic to the constant map into $((1/2)x_1, (1/2)x_2)$. To be able to conclude that $[\mathcal{E}_1 \oplus \mathcal{E}_2] = 0$ we may as well assume that the pair (B, b_0) has the HEP. (Actually, it is sufficient to assume that $(\Omega(B, b_0), b_0^*)$ has the HEP, but the condition on (B, b_0) also guarantees that x_1 and x_2 are regular points of $\mathcal{E}_1$ and $\mathcal{E}_2$ respectively for any choice of x_i in $p_i^{-1}(b_0)$ $i = 1, 2$.)

THEOREM 3.1. *Let* $\mathcal{E}_1$ *and* $\mathcal{E}_2$ *be fibrations over a common base* B *and suppose* (B, b_0) *has the* HEP. *Then* $\Omega(\mathcal{E}_1 \oplus \mathcal{E}_2)$ *is fiber homotopy equivalent to* $\Omega(B \times (F_1 * F_2))$ *loops being based at* $((1/2)x_1, (1/2)x_2)$ *and* $(b_0, ((1/2)x_1, (1/2)x_2))$ *respectively for any choice of* x_1 *and* x_2 *over* b_0.

In [5; Theorem 4.1] Hall pointed out that the homotopy sequence of any sum breaks up into short exact sequences because all the differentials are zero, but, in fact, Theorem 3.1 implies that the homotopy sequence of a Whitney sum is isomorphic to that of a product; i.e., all the short exact sequences split, for given any fibration $\mathcal{E}$, we get a ladder ($i \geq 1$)

$$(4)\quad \begin{array}{cccccccccc} (\mathcal{E}) & \cdots \xrightarrow{\partial} & \pi_{i+1}(F) & \xrightarrow[①]{} & \pi_{i+1}(E) & \xrightarrow[②]{} & \pi_{i+1}(B) & \xrightarrow[③]{\partial} & \pi_i(F) & \longrightarrow \cdots \\ & & \downarrow\approx & & \downarrow\approx & & \downarrow\approx & & \downarrow\approx & \\ \Omega(\mathcal{E}) & \cdots \xrightarrow{\tilde{\partial}} & \pi_i(\Omega(F)) & \longrightarrow & \pi_i(\Omega(E)) & \longrightarrow & \pi_i(\Omega(B)) & \xrightarrow{\tilde{\partial}} & \pi_{i-1}(\Omega(F)) & \longrightarrow \cdots \end{array}$$

where the vertical arrows represent the canonical isomorphisms, and ① and ② are commutative while ③ is anticommutative (essentially because of Proposition 2.3, see also [9; 18].

(C) In general, $[\mathcal{E}] = 0$ yields no information regarding the existence or non-existence of cross sections. To begin with, there are examples of fibrations with contractible fibers which do not admit cross sections e.g., [4; 8]. Restricting the base to a reasonable type of space, say a *CW* complex, does not improve the situation because of the example mentioned in the introduction. Furthermore Whitney sums need not have cross sections as was pointed out by Hall [5; 366] and also by Svarc who showed [11; 112] that if we start with a space B and let $\mathcal{E}$ be the fibration over B consisting of paths starting at some fixed b_0, then the sum $\mathcal{E} \oplus \mathcal{E}$ has a cross section if and only if cat $B \leq 2$. On the other hand, we have seen that $[\mathcal{E} \oplus \mathcal{E}] = 0$.

References

1. J. G. Ceder, *Some generalizations of metric spaces,* Pacific J. Math., vol. 11(1961), pp. 105–125.
2. A. Dold, *Partitions of unity in the theory of fibrations,* Ann. of Math., (2), vol. 78(1963), pp. 223–255.
3. A. Dold, *Lectures on homotopy theory and half-exact functors,* Notes prepared by F. Oort.
4. E. Fadell, *On fiber spaces,* Trans. Amer. Math. Soc., vol. 90(1959), pp. 1–14.
5. I. M. Hall, *The generalized Whitney sum,* Quart. J. Math., Oxford Series (2), vol. 16(1965), pp. 360–384.
6. P. J. Hilton, *Homotopy Theory and Duality,* New York, 1965.
7. Y. Nomura, *On mapping sequences,* Nagoya Math. J., vol. 17(1960), pp. 111–145.
8. D. Puppe, *Bemerkungen über die Erweiterung von Homotopien,* Arch. Math. (Basel), vol. 18(1967), pp. 81–88.
9. *Séminaire Henri Cartán, 11* e année, 1958–59.
10. M. Spivak, *Spaces satisfying Poincaré duality,* Topology, vol. 6(1967), pp. 77–101.
11. A. S. Svarc, *The genus of a fiber space,* Amer. Math. Soc., Transl. Series 2, vol. 55(1966), pp. 49–140.
12. P. Tulley, *On regularity in Hurewicz fiber spaces,* Trans. Amer. Math. Soc., vol. 116(1965), pp. 126–134.

SUNY at Albany