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Spectral asymmetry and Riemannian Geometry. I

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1. *Introduction.* The main purpose of this paper is to present a generalization of Hirzebruch's signature theorem for the case of manifolds with boundary. Our result is in the framework of Riemannian geometry and can be viewed as analogous to the Gauss–Bonnet theorem for manifolds with boundary, although there is a very significant difference between the two cases which is, in a sense, the central topic of the paper. To explain this difference let us begin by recalling that the classical Gauss–Bonnet theorem for a surface X with boundary Y asserts that the Euler characteristic $E(X)$ is given by a formula:

$$E(X) = \frac{1}{2\pi}\left\{\int_X K + \int_Y \sigma\right\}, \tag{1·1}$$

where K is the Gauss curvature of X and σ is the geodesic curvature of Y in X. In particular if, near the boundary, X is isometric to the product $Y \times R^+$, the boundary integral in (1·1) vanishes and the formula is the same as for closed surfaces. Similar remarks hold in higher dimensions. Now if X is a closed oriented Riemannian manifold of dimension 4, there is another formula relating cohomological invariants with curvature in addition to the Gauss–Bonnet formula. This expresses the signature† of the quadratic form on $H^2(X, \mathbf{R})$ by an integral formula

$$\operatorname{sign}(X) = \frac{1}{3}\int_X p_1, \tag{1·2}$$

where p_1 is the differential 4-form representing the first Pontrjagin class and is given in terms of the curvature matrix R by $p_1 = (2\pi)^{-2}\,Tr\,R^2$. It is natural to ask if (1·2) continues to hold for manifolds with boundary, provided the metric is a product near the boundary. Simple examples show that this is false, so that in general

$$\operatorname{sign} X - \frac{1}{3}\int_X p_1 = f(Y) \neq 0. \tag{1·3}$$

The notation in (1·3) is meant to emphasize that $f(Y)$ depends only on Y (as an oriented Riemannian manifold) and not on X. In other words, if X' is another manifold with boundary Y, then

$$\operatorname{sign} X - \frac{1}{3}\int_X p_1 = \operatorname{sign} X' - \frac{1}{3}\int_{X'} p_1'.$$

† The signature of the quadratic form $\sum_1^p x_i^2 - \sum_{p+1}^{p+q} x_i^2$ is $p-q$.

This is an easy consequence of (1·2) applied to the closed manifold $X \cup (-X')$ obtained by glueing X' (with orientation reversed) to X along Y. We are therefore left with the question of understanding how $f(Y)$ depends on the metric on Y. The following properties are clear from (1·3):

$$f \text{ is a continuous function of the metric,} \tag{1·4}$$

$$f(-Y) = -f(Y), \tag{1·5}$$

where $-Y$ is Y with the same metric but with opposite orientation. A natural conjecture, consistent with (i) and (ii) would be that f is given by some integral expression $f(Y) = \int_Y \theta$, where θ is a 3-form on Y canonically constructed out of the metric. However, if this were the case, it would imply that f was multiplicative for finite coverings, namely $f(\tilde{Y}) = df(Y)$, whenever $\tilde{Y}$ is a d-fold covering of Y. Explicit examples show that this is false: in fact, for $\tilde{Y}$ = 3-sphere, $f(\tilde{Y}) = 0$ but $f(Y) \neq 0$ for a suitable lens space Y (a quotient of the 3-sphere by the cyclic group of order d)†. Thus f must be a global invariant of the metric. Rather surprisingly, and this is our main result, it turns out to be a *spectral* invariant.

We recall that the eigenvalues λ of the Laplace operator of the metric are conveniently studied via the Zeta-function:‡

$$\zeta(s) = \sum_{\lambda \neq 0} \lambda^{-s}. \tag{1·6}$$

Real-valued invariants of the metric satisfying (1·4) can then be obtained by evaluating $\zeta(s)$ at some point where it is known to be finite. Invariants obtained in this way are however insensitive to the choice of orientation so that they will not satisfy (1·5).

To find a function of the eigenvalues which satisfies (1·5) we have to do something more subtle. To begin with, we should use the Laplace operator on forms as well as on scalar functions. Next we observe that this total Laplace operator Δ is in fact the square of a self-adjoint first order operator§ $B = \pm(d* - *d)$. Thus the eigenvalues of Δ are of the form λ^2, where λ is an eigenvalue of B. Now, unlike Δ, the operator B is not positive so its eigenvalues can be positive or negative. Taking this sign into account, we can therefore refine (1·6) to define a new function

$$\eta(s) = \sum_{\lambda \neq 0} (\operatorname{sign} \lambda)\,|\lambda|^{-s}. \tag{1·7}$$

If we regard (1·6) as a generalization of the Riemann Zeta-function, then (1·7) can be regarded as analogous to the Dirichlet L-functions. Since B involves the $*$ operator (linearly), it follows that reversing the orientation of Y changes B into $-B$ and hence $\eta(s)$ into $-\eta(s)$. If we now evaluate $\eta(s)$ at a suitable value of s, we will therefore obtain an invariant satisfying (1·5).

To decide what value of s is a reasonable candidate, we consider the behaviour of our function f with respect to scale changes of the metric. Since a scale change ($g_{ij} \mapsto k^2 g_{ij}$, k a positive constant) does not affect the curvature, it does not alter the

† These examples will be treated in Part II.

‡ An eigenvalue is repeated in (1·6) according to its multiplicity.

§ Signs are technically most important in what follows, but to avoid complications at this stage we simply write $\pm$. The correct signs will be found in section 4.

value of f. On the other hand an easy check (see ((2); section 5)) shows that the eigenvalues λ of B become $k^{-1}\lambda$ and so $\eta(s)$ becomes $k^{-s}\eta(s)$. The only value of s for which $\eta(s)$ is unaltered is therefore $s = 0$.

To sum up, we see that the simplest spectral invariant of Y that behaves like $f(Y)$ is $\eta(0)$. In fact our main result (Theorem (4·14)) asserts that $f(Y) = \frac{1}{2}\eta(0)$. The factor $\frac{1}{2}$ has an obvious explanation, because B is clearly the direct sum of two operators B^{ev}, B^{odd} acting on forms of even and odd degrees respectively and $*$ switches the two so that the η-invariant of B^{ev} is half the η-invariant of B. Thus we can remove the factor $\frac{1}{2}$ if we replace the operator B by the operator B^{ev}. Note that in the classical notation for flat space B^{ev} is essentially the operator

$$\begin{pmatrix} 0 & \mathrm{div} \\ \mathrm{grad} & \mathrm{curl} \end{pmatrix}$$

acting on the 4-vector consisting of a function and a vector-field (which may be identified with a 2-form).

In the above discussion, we restricted ourselves to 4-dimensional manifolds X (with 3-dimensional boundary Y) only for simplicity. In fact everything goes through for $4k$-dimensional manifolds (as stated in Theorem (4·14)), the main difference being that the simple integrand $\frac{1}{3}p_1$ in (1·3) has to be replaced by the Hirzebruch L-polynomial in the Pontrjagin forms in the general case. Moreover, we show, in Proposition (4·20), that in computing $\eta(s)$ the only relevant part of the operator B is $d*$ acting on $d\Omega^{2k-1}$, which is roughly in the middle dimension. Thus, in the classical situation discussed in the previous paragraph, only curl contributes to the computation of $\eta(s)$.

For positive self-adjoint elliptic operators the Zeta-functions given by (1·6) have finite values at $s = 0$, which are given by explicit integral formulae. In view of Theorem (4·14) and of the non-local character of f, the η-functions must behave in a different way. The explanation for this difference between η and ζ is that η involves the separation of Spec B into positive and negative. This is a global operation and it accounts for the non-local character of η.

Of course it is implicit in what we have said that $\eta(s)$ has a finite value at $s = 0$. Now, just as for $\zeta(s)$, this value is defined by analytic continuation in s from the half-plane of absolute convergence (Re s large). The behaviour near $s = 0$ of the two functions is however somewhat different. Whereas $\zeta(0)$ is finite and explicitly computable as an integral, $\eta(s)$ has at first sight a *simple pole* at $s = 0$ with an explicitly computable *residue*. It is a remarkable fact that, for the operator B above, the integral formula for this residue *vanishes identically* so that $\eta(0)$ is finite. This situation has some close analogies with the analytic R-torsion studied in (14) and (15). There the appropriate ζ-function turns out to *vanish* at $s = 0$ and one then proceeds to consider the next term in the Taylor expansion, namely $\zeta'(0)$. This turns out to have a non-local character which is related to the global topological invariant studied in Reidemeister torsion. From yet another point of view, this paper has something in common with (14) and (15). Each of these three papers introduces an analytic invariant for manifolds which is related to a classical index invariant: namely the signature, Euler characteristic and arithmetic genus. However, the differences between the three cases are substantial and we do not see at present how to unify them under any common generalization.

The finiteness of $\eta(0)$ turns out to be a general phenomenon valid for all self-adjoint elliptic operators.† For certain other operators arising in Riemannian geometry (operators of 'Dirac type'), this finiteness can be proved in exactly the same manner as for the particular operator B. That is, one computes the residue as an explicit integral and then shows it vanishes identically, or alternatively one deduces it fom the analogue of Theorem (4·14). For general self-adjoint operators, such a direct argument seems difficult and instead we shall deduce it (in Part III) from the special Riemannian cases by topological methods.

We now return to explain the method of proof of our main theorem. Recall first that the Hirzebruch signature theorem for closed manifolds is a special case of the general index theorem for elliptic operators. In fact there is an operator (the 'signature operator') acting on certain spaces of differential forms whose index can be identified via Hodge theory with the signature of the manifold. It is natural therefore, for a manifold X with boundary Y, to try to set up a suitable elliptic boundary value problem for the signature operator whose index will be the signature of X. The difficulty with this programme is that there are topological obstructions to the existence of such boundary conditions (1) and these obstructions are non-zero for the signature operator. This is on the assumption that we are looking for classical or *local* boundary conditions as for example in the Dirichlet or Neumann problems. If we enlarge our point of view however and permit *global* boundary conditions, then it turns out that we can indeed set up an appropriate boundary value problem. Near the boundary, the signature operator takes the form

$$\sigma\left(\frac{\partial}{\partial u}+B\right), \tag{1·8}$$

where u is the normal coordinate, σ is a bundle isomorphism and B is the self-adjoint operator on Y described earlier. For our boundary condition, we require that *the boundary value $\phi|Y$ should lie in the subspace spanned by the eigenfunctions ϕ_λ of B with* $\lambda < 0$. If P denotes the orthogonal projection onto the space spanned by the eigenfunctions with $\lambda \geqslant 0$, the boundary condition is $P(\phi|Y) = 0$. The operator P is pseudo-differential and its symbol $p = p(y, \xi)$ is an idempotent $m \times m$ matrix of rank $\frac{1}{2}m$ defined on the cotangent sphere bundle of Y. It turns out that p is not deformable (through idempotents of rank $\frac{1}{2}m$) to a matrix function of y alone, i.e. to the symbol of a multiplication operator. This is the topological obstruction referred to above which shows that there is no (elliptic) boundary condition of *local* type for the signature operator. For our global boundary condition $P(\phi|Y) = 0$ however, we have a good elliptic theory and, in particular, a finite index. This index can be identified with the signature of X so that Theorem (4·14) appears as an index formula; the index being expressed as the sum of two terms, one an integral over the interior X and the other $\eta(0)$ coming from the boundary Y.

It is clear that the right context in which to view Theorem (4·14) is therefore that of index problems for such global boundary conditions. In section 3, we derive a general

† At least for odd-dimensional manifolds.

analytical result, Theorem (3·10), giving such an index formula for first-order systems which are of the simple type (1·8) near the boundary (with B now being any self-adjoint first-order elliptic differential operator on Y). In Theorem (3·10), the interior integral is only given implicitly since it depends on the asymptotic behaviour of the corresponding heat kernel. If however our interior operator is one of the classical 'Dirac type' operators arising in Riemannian geometry then the main results of (2) enable us to identify the integral with an explicit expression in the curvature. This is done in section 4 and leads to results such as Theorem (4·2) for the Dirac operator on spinors. As a simple illustration, we consider in some detail the case of Riemann surfaces with boundary. In the second half of section 4 we show how the special case of Theorem (4·14) fits into the general Riemannian framework. The main point here is to carry out the analogue of the Hodge theory for manifolds with boundary. Now there is already such a treatment (8), used successfully in (14), in which the absolute and relative cohomology groups of X and (X, Y) are computed in terms of harmonic forms satisfying appropriate *local* boundary conditions. Unfortunately this is not what we need here, and in fact the attempt to use these classical boundary conditions held up progress for a long time. Instead we have to connect cohomology to the null space $\mathcal{N}$ of the signature operator with its global boundary condition. To explain our result, it is convenient to introduce the non-compact 'elongation' $\hat{X}$ of X obtained by attaching the semi-infinite cylinder $Y \times \mathbf{R}^-$ to the boundary Y of X. Note that a form in $\mathcal{N}$ extends to an exponentially decaying harmonic form on $\hat{X}$, and so is certainly square-integrable. The result we need is, then, Proposition (4·9) asserting that *the space of square integrable harmonic forms on* $\hat{X}$ *is naturally isomorphic to the image of the relative cohomology* $H^*(X, Y)$ *in the absolute cohomology* $H^*(X)$. This image is so to speak the part of the cohomology that lies half-way between the absolute and the relative, and when represented by harmonic L^2-forms on $\hat{X}$, it is naturally acted on by the duality operator $*$.

There are many interesting generalizations and applications of our results, some of which have been mentioned in ((5); section 2), and we shall treat these in detail in Part II of this paper. In particular, we shall study the relation with coverings, exploiting the non-multiplicative character of $\eta(0)$ alluded to earlier. We shall also discuss the relation of our invariant with the recent work of Chern and Simons (6).

In defining our η-function by (1·7), we mentioned the analogy with the L-functions of Number Theory. In fact, there are some 3-manifolds arising in the theory of real quadratic fields for which our invariant $\eta(0)$ turns out to coincide essentially with the value $L(0)$ of a certain L-function of the field. Modulo this identification Theorem (4·14) for these 3-manifolds reduces to a result of Hirzebruch (10). One of the primary motivations of our present work was in fact an attempt to understand the significance of Hirzebruch's result in the wider context of Riemannian geometry. Conversely we hope that our results may have Number theoretical significance, particularly for totally real number fields. We shall return to this question in a future publication.

In the search to identify the invariant $f(Y)$, defined by (1·3), another important clue was provided by a joint (unpublished) study of the first author and G. Lusztig. This study was concerned with a periodic family of self-adjoint elliptic operators and an integer invariant called the *spectral flow* of the family. Roughly speaking the spectral flow is the net number of eigenvalues that change sign (from $-$ to $+$) while the parameter of the family is completing a period. Since it involves contrasting the positive and negative eigenvalues, this special flow clearly has something in common with our η-invariant. The precise relation will be explained in Part III.

The main technical and computational part of the paper is in section 2 where we consider our boundary value problem on the semi-infinite cylinder $Y \times \mathbf{R}^+$ instead of X. This can be treated quite explicitly assuming only well-known results concerning elliptic operators on the closed manifold Y. The fundamental solution of this cylinder problem then provides us in section 3 with a parametrix near the boundary for our boundary value problem on the compact manifold X. Combined in the usual way with an interior parametrix, this leads to the usual regularity and finiteness results, showing that our problem is 'elliptic'. In fact our problem can easily be fitted into the general class of elliptic problems† studied for example in (11), but we have preferred to give a direct elementary treatment, partly because we need the explicit formulae for the heat equation.

The basis of our general index formula (3·10) is in principle similar to that of (2) in that it uses the asymptotics of the heat equation. The only difference now is that our heat operator must take into account the boundary condition and this produces a boundary contribution in the asymptotics. The computation of this boundary contribution can be made on the cylinder and much of section 2 is concerned with this calculation, ending up with formula (2·25).

The main results of this paper were announced without proof in (5).

2. *Computations on the cylinder.* In this section, we shall make some explicit calculations which will be basic to the rest of the paper.

Let Y be a closed manifold, E a vector bundle over Y and $A: C^\infty(Y,E) \to C^\infty(Y,E)$ a self-adjoint elliptic first order differential operator.‡ Then A has a discrete spectrum with real eigenvalues λ and eigenfunctions ϕ_λ. Let P denote the projection of $C^\infty(Y,E)$ onto the space spanned by the ϕ_λ for $\lambda \geqslant 0$. Then P is a pseudo-differential operator. To see this, put $B = A + H$ where H denotes projection onto the null-space of A, then B is invertible and so, by the results of Seeley (16), $|B|$ the positive square root of B^2 is pseudo-differential. Clearly $P = \frac{1}{2}B^{-1}(B+|B|)$.

We now form the product $Y \times \mathbf{R}^+$ of Y with the half-line $u \geqslant 0$ and consider the operator

$$D = \frac{\partial}{\partial u} + A \tag{2·1}$$

acting on sections $f(y,u)$ of E lifted to $Y \times \mathbf{R}^+$ (which we still denote by E). Clearly D is elliptic and its formal adjoint is

$$D^* = -\frac{\partial}{\partial u} + A. \tag{2·2}$$

† In the framework of (11) our problem would be an *over-determined* elliptic system. This is quite adequate for finiteness and regularity theorems but it does not deal with the index which must refer to a determined system.

‡ E is assumed to have a C^∞ Hermitian inner product and Y a C^∞ measure dy, so that the inner product for sections of E is given as usual by $\int_Y (f(y), g(y))\, dy$.

We now impose the following boundary condition for D

$$Pf(\cdot, 0) = 0. \tag{2·3}$$

Note that this is a *global* condition for the boundary value $f(\cdot, 0)$: in fact it is equivalent to

$$\int_Y (f(y,0), \phi_\lambda(y)) = 0 \quad \text{for all} \quad \lambda \geqslant 0.$$

The adjoint boundary condition to (2·3) is clearly

$$(1-P)f(\cdot, 0) = 0. \tag{2·4}$$

The space of all C^∞ sections satisfying (2·3) will be denoted by $C^\infty(Y \times \mathbf{R}^+, E; P)$ and C^∞_{comp} will denote sections with compact support (i.e. vanishing for $u \geqslant C$). Also H^l (for $l \geqslant 0$) will denote the Sobolev space of sections with derivatives up to order l in L^2 and H^l_{loc} the space of sections which are locally in H^l. More precisely $\phi \in H^l$ if $B(\partial/\partial u)^j \phi \in L^2$ for all j and for all differential operators B on Y of order $l-j$. Then the following proposition asserts that (2·1) with the boundary condition (2·3) has a good fundamental solution Q.

PROPOSITION (2·5). *There is a linear operator*

$$Q\colon C^\infty_{\text{comp}}(Y \times \mathbf{R}^+, E) \to C^\infty(Y \times \mathbf{R}^+, E; P)$$

such that

(i) *$DQg = g$ for all $g \in C^\infty_{\text{comp}}(Y \times \mathbf{R}^+, E)$*
(ii) *$QDf = f$ for all $f \in C^\infty_{\text{comp}}(Y \times \mathbf{R}^+, E; P)$*
(iii) *The kernel $Q(y, u; z, v)$ of Q is C^∞ for $u \neq v$ (here $y, z \in Y$ and $u, v \in \mathbf{R}^+$)*
(iv) *Q extends to a continuous map $H^{l-1} \to H^l_{\text{loc}}$ for all integral $l \geqslant 1$.*

Proof. To solve $Df = g$, we expand f and g in terms of an orthonormal basis of eigenfunctions of A:

$$f(y,u) = \Sigma f_\lambda(u)\,\phi_\lambda(y), \quad g(y,u) = \Sigma g_\lambda(u)\,\phi_\lambda(y).$$

We must now solve

$$\left(\frac{d}{du} + \lambda\right) f_\lambda = g_\lambda \text{ with } f_\lambda(0) = 0 \quad \text{for} \quad \lambda \geqslant 0. \tag{2·6}$$

We take the explicit solutions

$$\begin{aligned} f_\lambda(u) &= \int_0^u e^{\lambda(v-u)} g_\lambda(v)\,dv \quad \text{for} \quad \lambda \geqslant 0 \\ &= -\int_u^\infty e^{\lambda(v-u)} g_\lambda(v)\,dv \quad \text{for} \quad \lambda < 0 \end{aligned} \tag{2·7}$$

to define Q_λ. Formally $Q = \Sigma Q_\lambda$ satisfies (i) and (ii) and to see that this formal solution for Q converges, it will be sufficient to prove (iv) by estimating the Sobolev norms. To do this, we rewrite the equations (2·7) in terms of the Laplace transform

$$\tilde{f}(\xi) = \int_0^\infty e^{-iu\xi} f(u)\,du$$

and we get

$$\tilde{f}_\lambda(\xi) = \frac{\tilde{g}_\lambda(\xi) + f_\lambda(0)}{\lambda + i\xi}, \tag{2·8}$$

where $f_\lambda(0) = 0$ for $\lambda \geqslant 0$ and

$$f_\lambda(0) = -\int_0^\infty e^{\lambda v} g_\lambda(v)\, dv \quad \text{for} \quad \lambda < 0.$$

Computing L^2-norms from (2·8), we have

$$\begin{aligned}\lambda^2 \|\tilde{f}_\lambda\|^2 &\leqslant \|\tilde{g}_\lambda\|^2 \quad \text{for} \quad \lambda \geqslant 0\\ &\leqslant 2\left\{\|\tilde{g}_\lambda\|^2 + \lambda^2 |f_\lambda(0)|^2 \int_{-\infty}^{\infty} \frac{d\xi}{\lambda^2+\xi^2}\right\}\\ &\leqslant 2\left\{\|\tilde{g}_\lambda\|^2 + \lambda^2 \cdot \frac{1}{2|\lambda|}\|g_\lambda\|^2 \cdot \frac{\pi}{|\lambda|}\right\} \quad \text{for} \quad \lambda < 0.\end{aligned}$$

Hence, using Parseval's formula, we deduce for all λ

$$|\lambda|\, \|f_\lambda\| \leqslant 2\|g_\lambda\|. \tag{2·9}$$

Combined with (2·6), this gives

$$\left\|\frac{df_\lambda}{du}\right\| \leqslant 3\|g_\lambda\|. \tag{2·10}$$

Since A is a first-order elliptic operator, the Sobolev norm $\|\quad\|_1$ for H^1 may be defined by

$$\begin{aligned}\|f\|_1^2 &= \|f\|^2 + \left\|\frac{\partial f}{\partial u}\right\|^2 + \|Af\|^2\\ &= \sum_\lambda (1+\lambda^2)\, \|f_\lambda\|^2 + \left\|\frac{\partial f_\lambda}{\partial u}\right\|^2.\end{aligned}$$

Except for $\lambda = 0$, (2·9) and (2·10) therefore give continuity $H^0 \to H^1$. For $\lambda = 0$, $f_0(u) = \int_0^u g_0(v)\, dv$ and this, together with (2·10), gives continuity $H^0 \to H^1_{\text{loc}}$ establishing (iv) for $l = 1$. More generally, multiplying (2·9) by powers of λ and differentiating (2·6) with respect to u, we deduce

$$|\lambda|^p \left\|\frac{\partial^q f_\lambda}{\partial u^q}\right\| \leqslant 3|\lambda|^{p+q-1}\|g_\lambda\| + \sum_{r=1}^{q-1} |\lambda|^{p+q-r-1} \left\|\frac{\partial^r g_\lambda}{\partial u^r}\right\|$$

which establishes (iv) for general l. It remains to verify (iii). Now Q is given by convolution in the u-variable with

$$K(t) = \epsilon(t)\, e^{-t|A|} P - \epsilon(-t)\, e^{t|A|}(1-P), \tag{2·11}$$

where $\epsilon(t)$ is the characteristic function of the non-negative real line and

$$|A| = AP - A(1-P).$$

For $t > 0$, $e^{-t|A|}$ is the 'heat operator' associated to the pseudo-differential operator $|A|$ and so its kernel $E(y,z,t)$ is a C^∞ function on $Y \times Y \times \mathbf{R}^+$. Alternatively, this can be deduced as follows. For $0 < t_0 < t_1 < \infty$, we have

$$\begin{aligned}&\int_{t_0}^{t_1}\int_{Y\times Y} \left| A^j \left(\frac{\partial}{\partial t}\right)^k E(y,z,t)\right|^2 dy\, dz\, dt\\ &\qquad = \int_{t_0}^{t_1} \left(\sum_\lambda \lambda^{2j+2k} e^{-2t|\lambda|}\right) dt < (t_1 - t_0) \sum_\lambda \lambda^{2j+2k} e^{-2t_0|\lambda|} < \infty\end{aligned}$$

(since $\Sigma\lambda^{2m}e^{-2t_0|\lambda|} < C\,\Sigma\lambda^{-2N}$ which converges for large N). Since this holds for all j, k the Sobolev Lemma implies that E is C^∞. By (2·11), the same is therefore true of $K(t)$ for $t \neq 0$. Since $Q(y,u;z,v) = K(y,z,u-v)$, where $K(y,z,t)$ denotes the kernel of $K(t)$, Q has a C^∞ kernel for $u \neq v$.

If we replace D by D^* and the boundary condition (2·3) by (2·4), the only difference in the above proof occurs for $\lambda = 0$ in which case we put $f_0(u) = -\int_u^\infty g_0(v)\,dv$. Thus D^* also has a fundamental solution with the properties in (2·5). If we regard D and D^* as unbounded operators on L^2 we have

PROPOSITION (2·12). *The closure of the operators on L^2 defined by D and D^* with domains given by* (2·3) *and* (2·4) *respectively are adjoints of each other.*

Proof. If we decompose the Hilbert space $L^2(Y\times\mathbf{R}^+,E)$ into two parts: $H = H'\oplus H''$, H'' involving the zero-eigenvalue of A and H' all the non-zero eigenvalues, then D and D^* decompose accordingly. On H'', $D = \partial/\partial u$, $D^* = -\partial/\partial u$ and the adjointness is clear. On H' the fundamental solution Q of (2·5) gives a bounded inverse Q' for D and similarly we get a bounded inverse R' for D^*. Then $R' = (Q')^*$ follows by continuity from the fact that $\langle Df,g\rangle = \langle f,D^*g\rangle$ for $f,g\in C^\infty_{\text{comp}}$ and satisfying (2·3) and (2·4) respectively. Since adjoints commute with inverses, the proposition is established.

Let $\mathscr{D}$ denote the closure of the operator D on L^2 with domain given by (2·3). The domain of $\mathscr{D}$ is clearly contained in the closed subspace W, which is the kernel of the composite continuous map

$$H^1(Y\times\mathbf{R}^+,E) \xrightarrow{r} L^2(Y,E) \xrightarrow{P} L^2(Y,E),$$

where r is restriction to the boundary.† A similar remark applies with D replaced by D^* and (2·3) replaced by (2·4). Applying Proposition (2·12) we then see that equality must actually hold, namely Domain $\mathscr{D} = W$ and similarly for Domain $\mathscr{D}^*$.

Now let us form the two self-adjoint operators

$$\Delta_1 = \mathscr{D}^*\mathscr{D},\quad \Delta_2 = \mathscr{D}\mathscr{D}^*.$$

For $t > 0$, we can then consider the bounded operators $e^{-t\Delta_1}$ and $e^{-t\Delta_2}$. We shall give the explicit kernels of these operators in terms of the eigenfunctions ϕ_λ of A. Consider first Δ_1, which is the operator given by $-\partial^2/\partial u^2 + A^2$ with the boundary condition

$$Pf(\cdot,0) = 0 \quad\text{and}\quad (1-P)\left\{\left(\frac{\partial f}{\partial u}+Af\right)_{u=0}\right\} = 0. \tag{2·13}$$

Expanding in terms of the ϕ_λ, so that $f(y,u) = \Sigma f_\lambda(u)\,\phi_\lambda(y)$, we see that for each λ we must study the operator $-d^2/du^2+\lambda^2$ on $u \geqslant 0$ with the boundary conditions

$$f_\lambda(0) = 0 \quad\text{if}\quad \lambda \geqslant 0 \tag{2·14}$$

$$\left(\frac{df_\lambda}{du}+\lambda f_\lambda\right)_{u=0} \quad\text{if}\quad \lambda < 0. \tag{2·15}$$

† Note that restriction is even continuous $H^1(Y\times\mathbf{R}^+) \to H^{\frac{1}{2}}(Y)$.

Now the fundamental solution for $\partial/\partial t - \partial^2/\partial u^2 + \lambda^2$ with the boundary condition (2·14) is easily seen to be

$$\frac{e^{-\lambda^2 t}}{\sqrt{4\pi t}}\left\{\exp\left(\frac{-(u-v)^2}{4t}\right) - \exp\left(\frac{-(u+v)^2}{4t}\right)\right\}, \tag{2·16}$$

while for the boundary condition (2·15), standard Laplace transform methods (see ((6); section 14·2)) give

$$\frac{e^{-\lambda^2 t}}{\sqrt{4\pi t}}\left\{\exp\left(\frac{-(u-v)^2}{4t}\right) + \exp\left(\frac{-(u+v)^2}{4t}\right)\right\} + \lambda e^{-\lambda(u+v)}\operatorname{erfc}\left\{\frac{u+v}{2\sqrt{t}} - \lambda\sqrt{t}\right\}, \tag{2·17}$$

where erfc is the complementary error function defined by

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-\xi^2} d\xi.$$

The kernel of $e^{-t\Delta_1}$ at a point $(y, u; z, v)$ is therefore given by multiplying (2·16) or (2·17) (according as $\lambda \geqslant 0$ or $\lambda < 0$) by $\phi_\lambda(y)\overline{\phi_\lambda(z)}$ and summing over all λ.

For the operator Δ_2, the boundary conditions for each λ are

$$f_\lambda(0) = 0 \quad \text{if} \quad \lambda < 0 \tag{2·18}$$

$$\left(\frac{-df_\lambda}{du} + \lambda f_\lambda\right)_{u=0} = 0 \quad \text{if} \quad \lambda \geqslant 0. \tag{2·19}$$

The fundamental solution of $\partial/\partial t - \partial^2/\partial u^2 + \lambda^2$ for (2·18) is again given by (2·16) while for (2·19) we must use (2·17) with λ changed to $-\lambda$.

Since $\int_x^\infty e^{-\xi^2} d\xi < e^{-x^2}$ we see that (2·16) and (2·17) are both bounded by

$$\left\{\frac{e^{-\lambda^2 t}}{\sqrt{\pi t}} + \frac{2|\lambda| e^{-\lambda^2 t}}{\sqrt{\pi}}\right\}\exp\left(-\frac{(u-v)^2}{4t}\right).$$

Using the inequality $x \leqslant e^{x^2/2}$, it follows that the kernel $e_1(t; y, u; z, v)$ of $e^{-t\Delta_1}$ is bounded by

$$\frac{3}{2\sqrt{\pi t}}\{\sum_\lambda e^{-\lambda^2 t/2}(|\phi_\lambda(y)|^2 + |\phi_\lambda(z)|^2)\}\exp\left(\frac{-(u-v)^2}{4t}\right). \tag{2·20}$$

Since the kernel of e^{-tA^2} on the diagonal of $Y \times Y$ is bounded by $Ct^{-n/2}$ (see for example ((2); section 4)) we deduce

PROPOSITION (2·21). *The kernels of $e^{-t\Delta_1}$ and $e^{-t\Delta_2}$ are exponentially small in t as $t \to 0$ for $u \neq v$. More precisely they are bounded by*

$$Ct^{-\frac{1}{2}(n+1)}\exp\left(\frac{-(u-v)^2}{4t}\right)$$

for some constant C as $t \to 0$.

Proposition (2·21) asserts that, as usual, the contribution outside the diagonal ($u = v$) is asymptotically negligible. What we are primarily interested in is of course the contribution from the diagonal. Moreover we are interested in the difference between

$e^{-t\Delta_1}$ and $e^{-t\Delta_2}$, so let $K(t,y,u)$ denote the kernel of $e^{-t\Delta_1}-e^{-t\Delta_2}$ at the point $(y,u;\,y,u)$ of $(Y\times\mathbf{R}^+)\times(Y\times\mathbf{R}^+)$. Formulae (2·16), (2·17) and their counterparts for Δ_2 then give

$$\begin{aligned}K(t,y,u) &= \sum_\lambda \operatorname{sign}\lambda\left\{-\frac{e^{-\lambda^2 t}e^{-u^2/t}}{\sqrt{\pi t}}+|\lambda|\,e^{2|\lambda|u}\operatorname{erfc}\left(\frac{u}{\sqrt{t}}+|\lambda|\sqrt{t}\right)\right\}|\phi_\lambda(y)|^2\\ &= \sum_\lambda \operatorname{sign}\lambda\frac{\partial}{\partial u}\left\{\tfrac{1}{2}e^{2|\lambda|u}\operatorname{erfc}\left(\frac{u}{\sqrt{t}}+|\lambda|\sqrt{t}\right)\right\}|\phi_\lambda(y)|^2, \qquad (2\cdot22)\end{aligned}$$

where, for convenience, we have defined $\operatorname{sign}\lambda=+1$ if $\lambda=0$. Integrating over $Y\times\mathbf{R}^+$, we obtain

$$K(t)=\int_0^\infty\int_Y K(t,y,u)\,dy\,du=-\sum_\lambda\frac{\operatorname{sign}\lambda}{2}\operatorname{erfc}(|\lambda|\sqrt{t}) \qquad (2\cdot23)$$

and hence, differentiating with respect to t,

$$K'(t)=\frac{1}{\sqrt{4\pi t}}\sum_\lambda\lambda e^{-\lambda^2 t}. \qquad (2\cdot24)$$

Note that, as $t\to\infty$ in (2·23), $K(t)\to-\tfrac{1}{2}h$, where $h=\dim\operatorname{Ker}A$ is the multiplicity of the 0-eigenvalue. Moreover, $K(t)+\tfrac{1}{2}h\to0$ exponentially as $t\to\infty$. Also (2·23) shows that $|K(t)|\leqslant 1/\sqrt{\pi}\Sigma e^{-\lambda^2 t}<Ct^{-\frac{1}{2}n}$ as $t\to0$. Hence for $Re(s)$ large

$$\int_0^\infty (K(t)+\tfrac{1}{2}h)\,t^{s-1}dt$$

converges. Integrating by parts and using (2·24), we get

$$\begin{aligned}\int_0^\infty (K(t)+\tfrac{1}{2}h)\,t^{s-1}dt &= -\frac{\Gamma(s+\frac{1}{2})}{2s\sqrt{\pi}}\sum_{\lambda\neq0}\frac{\operatorname{sign}\lambda}{|\lambda|^{2s}}\\ &= -\frac{\Gamma(s+\frac{1}{2})}{2s\sqrt{\pi}}\eta(2s) \qquad (2\cdot25)\end{aligned}$$

by definition of $\eta(s)$. This is the final formula which we shall be applying in the next section. In particular if we *assume* that $K(t)$ has an asymptotic expansion

$$K(t)\sim\sum_{k\geqslant -n}a_k t^{\frac{1}{2}k}\ \text{as}\ t\to0,$$

then (2·25) yields

$$\eta(2s)=-\frac{2s\sqrt{\pi}}{\Gamma(s+\frac{1}{2})}\left\{\frac{h}{2s}+\sum_{k=-n}^{N}\frac{a_k}{\frac{1}{2}k+s}+\theta_N(s)\right\}, \qquad (2\cdot26)$$

where $\theta_N(s)$ is holomorphic for $Re(s)>-\tfrac{1}{2}(N+1)$. Thus (2·26) gives the analytic continuation of $\eta(2s)$ to the whole s-plane. In particular, $\eta(s)$ is holomorphic near $s=0$ and its value at $s=0$ is given by

$$\eta(0)=-(2a_0+h). \qquad (2\cdot27)$$

Finally, we note that if, instead of defining $K(t)$ by integrating over $Y\times\mathbf{R}^+$, we integrate only over $Y\times[0,\delta]$ for some $\delta>0$, the asymptotic expansion is unchanged. In fact, the difference is

$$\sum_\lambda\operatorname{sign}\lambda\tfrac{1}{2}e^{2|\lambda|\delta}\operatorname{erfc}(\delta/\sqrt{t}+|\lambda|\sqrt{t})$$

which is bounded by

$$\frac{e^{-\delta^2/t}}{\sqrt{\pi}} \Sigma e^{-\lambda^2 t} < C e^{-\delta^2/t} t^{-\frac{1}{2}n}$$

and hence is exponentially small.

3. *The index formula.* In this section, we shall consider a boundary value problem on a compact manifold X with boundary Y which coincides near the boundary with the problem studied in section 2. The computations in section 2 will then be used to derive an explicit formula for the index of our problem. The main result is stated in Theorem (3·10).

Let $D: C^\infty(X, E) \to C^\infty(X, F)$ be a first order elliptic differential operator. In a neighbourhood $Y \times I$ of the boundary, it is then of the form

$$D = \sigma\left(\frac{\partial}{\partial u} + A_u\right), \tag{3·1}$$

where $u \in I$ is the normal coordinate, $\sigma = \sigma_D(du)$ is the bundle isomorphism $E \to F$ given by the symbol σ_D of D and A_u: $C^\infty(Y, E_u) \to C^\infty(Y, E_u)$ is an elliptic operator on Y depending on the parameter u (and $E_u = E | Y \times \{u\}$). We now *assume* that A_u is *independent* of u. More precisely, this means that a suitable isomorphism of $E \cong \pi^*(E_0)$ identifies all A_u with $A = A_0$ (here π denotes the projection $Y \times I \to Y$). With this assumption, we shall just write our operator in the form

$$D = \sigma\left(\frac{\partial}{\partial u} + A\right). \tag{3·1}$$

As in section 2, we assume A is self-adjoint, with respect to given C^∞ hermitian inner products on $E | Y$ and a C^∞ measure dy on Y, and we shall fix inner products on E, F extending this inner product from $Y \times I$ to X and a C^∞ measure dx on X extending $dy\,du$ on the collar. In particular, σ is taken to be an isometry.

We consider the operator D with the boundary condition (2·3) and we construct a parametrix R by patching together the fundamental solution Q of (2·5) with an interior parametrix Q_2. More precisely, let $\rho(a, b)$ denote an increasing C^∞ function of the real variable u, such that

$$\rho = 0 \quad \text{for} \quad u \leqslant a \quad \text{and} \quad \rho = 1 \quad \text{for} \quad u \geqslant b,$$

and define four C^∞ functions ϕ_1, ϕ_2, ψ_1, ψ_2 by

$$\phi_2 = \rho(\tfrac{1}{4}, \tfrac{1}{2}), \qquad \psi_2 = \rho(\tfrac{1}{2}, \tfrac{3}{4}),$$
$$\phi_1 = 1 - \rho(\tfrac{3}{4}, 1), \qquad \psi_1 = 1 - \psi_2.$$

Note that $\phi_i = 1$ on the support of ψ_i. We regard these functions of u as functions on the cylinder $Y \times I$ and then extend them to X in the obvious way: ϕ_1, ψ_1 being extended by 0 and ϕ_2, ψ_2 being extended by 1. Finally, we put $Q_1 = Q\sigma^{-1}$ and (considering ϕ_i, ψ_i as multiplication operators)

$$R = \phi_1 Q_1 \psi_1 + \phi_2 Q_2 \psi_2.$$

Here Q_2 can, for definiteness, be taken as the Green's operator for D on the double of X (note that, because of the form (3·1), there is a natural double of D on the double of X, the roles of E and F being switched on the two halves). R is a linear operator

$$C^\infty(X, F) \to C^\infty(X, E; P),$$

where $C^\infty(X, E; P)$ is the space of sections of E which satisfy the boundary condition (2·3).

Proposition (2·5) shows that R is a right parametrix, that is $DR - 1$ has a C^∞-kernel. Switching the roles of ϕ_i, ψ_i gives a left parametrix, hence R is, in fact, a 2-sided parametrix. Proposition (2·5) also shows that R is continuous from $H^{l-1} \to H^l$ (for $l \geqslant 1$). It now follows that

$$D\colon C^\infty(X, E; P) \to C^\infty(X, F)$$

and

$$D_l\colon H^l(X, E; P) \to H^{l-1}(X, F) \quad (\text{for } l \geqslant 1)$$

are Fredholm operators with the same null-space. Similar statements hold for the operator D^* with the adjoint boundary condition (2·4) and by essentially the same argument as in (2·12), it follows that this gives the Hilbert space adjoint of D (as a closed operator on L^2). This implies that an L^2-section of F, orthogonal to the image of D, is necessarily C^∞ (being in $\operatorname{Ker} D_1^*$). Thus D has a well-defined index, computed either in C^∞ or in L^2, and

$$\operatorname{index} D = \dim \operatorname{Ker} \mathscr{D} - \dim \operatorname{Ker} \mathscr{D}^*, \tag{3·2}$$

where $\mathscr{D}$ is the closed operator defined by D (with domain $H^1(X, E; P)$).

The operator $\mathscr{D}^*\mathscr{D}$ is then a self-adjoint operator and $e^{-t\mathscr{D}^*\mathscr{D}}$ is the fundamental solution of $\partial/\partial t + D^*D$ with the boundary condition (2·13). An approximate fundamental solution can be constructed from the fundamental solution e_1, constructed for the corresponding operator on the cylinder in section 2, and the fundamental solution† e_2 of $\partial/\partial t + D^*D$ on the double of X. Using the functions ϕ_i, ψ_i defined above, we put

$$f = \phi_1 e_1 \psi_1 + \phi_2 e_2 \psi_2. \tag{3·3}$$

Proposition (2·21) and the corresponding result for closed manifolds show that $(\partial/\partial t + D^*D)f$ is exponentially small as $t \to 0$. From this, and the fact that $f \to I$ as $t \to 0$, it follows (see for example(13)) that our fundamental solution $e = e^{-t\mathscr{D}^*\mathscr{D}}$ is given by a convergent series of the form

$$e = f + \sum_{m \geqslant 1}^{\infty} (-1)^m c_m * f,$$

where $*$ denotes convolution in t and composition of operators. Here $c_1 = (\partial/\partial t + D^*D)f$ and $c_m = c_{m-1} * c_1$. In particular, for $t > 0$, e has a C^∞ kernel which differs from the kernel of f by an exponentially small term as $t \to 0$. Thus, in computing the asymptotic behaviour of $\operatorname{Tr} e^{-t\mathscr{D}^*\mathscr{D}}$, we can replace e by f. Applying the same remarks to $e^{-t\mathscr{D}\mathscr{D}^*}$ and subtracting, we obtain asymptotically

$$\operatorname{Tr} e^{-t\mathscr{D}^*\mathscr{D}} - \operatorname{Tr} e^{-t\mathscr{D}\mathscr{D}^*} \sim \int_0^1 \int_Y K(t, y, u)\, \psi_1(u)\, dy\, du + \int_X F(t, x)\, \psi_2(x)\, dx, \tag{3·4}$$

† Note that this is not the e_2 of section 2 which was the counterpart of e_1 but for the operator $\mathscr{D}\mathscr{D}^*$.

where K is defined by (2·22) and $F(t,x)$ is obtained from the kernels of e^{-tD^*D} and e^{-tDD^*} on the double of X. More precisely, $F(t,x) = F_1(t,x) - F_2(t,x)$, where

$$F_i(t,x) = \sum_\mu e^{-\mu t}|\phi_\mu(x)|^2$$

μ runs over the eigenvalues of D^*D (for $i = 1$) or DD^* (for $i = 2$) and ϕ_μ are the corresponding eigenfunctions.

Because of the estimates at the end of section 2, it follows that, in the first integral of (3·4), we can replace ψ_1 by 1 and $\int_0^1$ by $\int_0^\infty$ so that we end up with the function $K(t)$ given by (2·23). On the closed manifold (double of X), we know (see ((2); section 4)) that we have an asymptotic expansion

$$F(t,x) \sim \sum_{k\geqslant -n} \alpha_k(x)\, t^{\frac{1}{2}k}, \tag{3·5}$$

where the coefficients $\alpha_k(x)$ are explicit local functions of the operators DD^* and D^*D. In the collar neighbourhood of Y, the operators DD^* and D^*D are isomorphic, via σ; and so $F(t,x) \sim 0$ for x in the collar. Hence, $F(t,x)\psi_2(x) \sim F(t,x)$ and so (3·4) becomes

$$\operatorname{Tr} e^{-t\mathscr{D}^*\mathscr{D}} - \operatorname{Tr} e^{-t\mathscr{D}\mathscr{D}^*} \sim K(t) + \sum_{k\geqslant -n}\int_X \alpha_k(x)\,dx\, t^{\frac{1}{2}k}. \tag{3·6}$$

Now the operators $\mathscr{D}^*\mathscr{D}$ and $\mathscr{D}\mathscr{D}^*$ have discrete spectrum with finite multiplicities and their non-zero eigenvalues coincide. In fact, if

$$\mathscr{D}^*\mathscr{D}\phi = \mu\phi, \quad \text{then} \quad \mathscr{D}\mathscr{D}^*(\mathscr{D}\phi) = \mu\mathscr{D}\phi$$

so $\phi \mapsto \mathscr{D}\phi$ defines an isomorphism of the μ-eigenspace of $\mathscr{D}^*\mathscr{D}$ onto that of $\mathscr{D}\mathscr{D}^*$ (with inverse $\psi \mapsto \mu^{-1}\mathscr{D}^*\psi$). Also the null-space of $\mathscr{D}^*\mathscr{D}$ coincides with that of $\mathscr{D}$, while the null-space of $\mathscr{D}\mathscr{D}^*$ coincides with that of $\mathscr{D}^*$. Hence by (3·2)

$$\operatorname{index} D = \operatorname{Tr} e^{-t\mathscr{D}^*\mathscr{D}} - \operatorname{Tr} e^{-t\mathscr{D}\mathscr{D}^*}.$$

Combined with (3·6), this gives an asymptotic expansion for $K(t)$:

$$K(t) \sim \operatorname{index} D - \sum_{k\geqslant -n}\int_X \alpha_k(x)\,dx\, t^{\frac{1}{2}k}. \tag{3·7}$$

Applying (2·26) we deduce that

$$\eta(2s) = -\frac{2s\sqrt{\pi}}{\Gamma(s+\frac{1}{2})}\left\{\frac{\frac{1}{2}h + \operatorname{index} D}{s} - \sum_{k=-n}^{N}\int_X \frac{\alpha_k(x)}{\frac{1}{2}k+s}\,dx + \theta_N(s)\right\}, \tag{3·8}$$

where $\qquad \theta_N(s)$ is holomorphic for $Re(s) > -\dfrac{N+1}{2}$.

Thus $\eta(s)$ is extended meromorphically to the whole plane and in particular

$$\eta(0) = 2\int_X \alpha_0(x)\,dx - (h + 2\operatorname{index} D)$$

or

$$\operatorname{index} D = \int_X \alpha_0(x)\,dx - \frac{h+\eta(0)}{2}. \tag{3·9}$$

This formula is what we have been aiming at and we therefore summarize our results so far in the following theorem:

THEOREM (3·10). *Let X be a compact manifold with boundary Y and let*

$$D\colon C^\infty(X,E) \to C^\infty(X,F)$$

be a linear first order elliptic differential operator on X (acting from the vector bundle E to the vector bundle F). We assume that, in a neighbourhood $Y \times I$ of the boundary, D takes the special form

$$D = \sigma\left(\frac{\partial}{\partial u}+A\right)$$

where u is the inward normal coordinate, σ is a bundle isomorphism $E|Y \to F|Y$ and A is a self-adjoint elliptic operator on Y. Let $C^\infty(X,E;P)$ denote the space of sections f of E satisfying the boundary condition

$$Pf(\cdot,0) = 0$$

where P is the spectral projection of A corresponding to eigenvalues $\geqslant 0$. Then

$$D\colon C^\infty(X,E;P) \to C^\infty(X,F)$$

has a finite index given by

$$\operatorname{index} D = \int_X \alpha_0(x)\,dx - \frac{h+\eta(0)}{2},$$

where α_0, h, η are defined as follows:

(i) *$\alpha_0(x)$ is the constant term in the asymptotic expansion (as $t \to 0$) of*

$$\Sigma e^{-t\mu'}|\phi'_\mu(x)|^2 - \Sigma e^{-t\mu''}|\phi''_\mu(x)|^2,$$

where μ', ϕ'_μ denote the eigenvalues and eigenfunctions of D^*D on the double of X, and μ'', ϕ''_μ are the corresponding objects for DD^*.

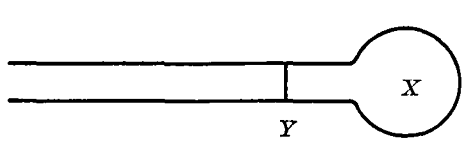
(ii) *$h = \dim \operatorname{Ker} A$ = multiplicity of 0-eigenvalue of A*

(iii) *$\eta(s) = \sum_{\lambda \neq 0} \operatorname{sign}\lambda\,|\lambda|^{-s}$, where λ runs over the eigenvalues of A.*

In (iii) *the series converges absolutely for $Re(s)$ large and then $\eta(s)$ extends to a meromorphic function on the whole s-plane with a finite value at $s = 0$. Moreover, if the asymptotic expansion in* (i) *has no negative power of t then $\eta(s)$ is holomorphic for $Re(s) > -\frac{1}{2}$.*

Remark. This theorem can be viewed as providing information about the function $\eta(s)$, associated to the operator A on Y; in particular that $\eta(0)$ is finite. The proof depends on being able to find X with $\partial X = Y$ and an elliptic operator D on X extending $\partial/\partial u + A$. As we shall see in the next section, the natural operators A arising in Riemannian geometry tend to be extendable in this sense. For more general operators, one has to work somewhat harder and this will be discussed in Part III.

The index of the boundary value problem in Theorem (3·10) can be given an alternative description by introducing the non-compact manifold $\hat{X} = X \cup \{Y \times [0, -\infty)\}$ illustrated in the figure.



We extend the bundles E, F (together with their inner products) to $\hat{X}$ in the obvious way and we take the measure $dx = dy\,du$ on $Y \times [0, -\infty)$. We shall be primarily interested in the L^2-sections of E and F over $\hat{X}$, but we shall also consider a slightly larger space. By an *extended* L^2-section of E, we shall mean a section f which is locally in L^2 and such that, for large negative u,

$$f(y,u) = g(y,u) + f_\infty(y),$$

where g is in L^2 and $f_\infty \in \operatorname{Ker} A$. Thus f has f_∞ as an asymptotic or limiting value as $u \to -\infty$. A similar definition holds for extended L^2-sections of F (using the isomorphism $\sigma: E \to F$ in the cylinder $Y \times [0, -\infty)$).

PROPOSITION (3·11). *Let $D: C^\infty(X, E, P) \to C^\infty(X, F)$ be the operator in* (3·10) *and $D^*: C^\infty(X, F, 1-P) \to C^\infty(X, E)$ the adjoint operator. Then*

(i) $\operatorname{Ker} D$ *is isomorphic to the space of L^2-solutions of $Df = 0$ on $\hat{X}$.*

(ii) $\operatorname{Ker} D^*$ *is isomorphic to the space of extended L^2-solutions of $D^*f = 0$ on $\hat{X}$.*

Proof. (i) Let $f \in \operatorname{Ker} D$ and expand it near the boundary in the form

$$f(y,u) = \Sigma f_\lambda(u)\,\phi_\lambda(y),$$

where the ϕ_λ are an orthonormal base of eigenfunctions of A. Since $Df = 0$, we have $\partial f/\partial u + Af = 0$, so that $f_\lambda(u) = e^{-\lambda u} f_\lambda(0)$. Since $Pf(\cdot, 0) = 0$, we have $f_\lambda(0) = 0$ for $\lambda \geqslant 0$ and so

$$f(y,u) = \sum_{\lambda<0} e^{-\lambda u} f_\lambda(0)\,\phi_\lambda(y). \tag{3·12}$$

This shows that f extends to a section $\hat{f}$ on $\hat{X}$ which satisfies $D\hat{f} = 0$ and is exponentially decaying as $u \to -\infty$, hence certainly in L^2. Conversely a solution of $Df = 0$ on $\hat{X}$, which is in L^2, must be of the form (3·12), because terms involving $e^{-\lambda u}$ with $\lambda \geqslant 0$ are not in L^2. Thus $f \mapsto \hat{f}$ gives the required isomorphism.

(ii) Let $f \in \operatorname{Ker} D^*$, then instead of (3·12), we get

$$f(y,u) = \sum_{\lambda \geqslant 0} e^{\lambda u} f_\lambda(0)\,\phi_\lambda(y)$$

and $\hat{f}$ is now an extended L^2-section of F (as defined above) with limiting value

$$f_\infty(y) = \sum_{\lambda=0} f_\lambda(0)\,\phi_\lambda(y). \tag{3·13}$$

Conversely, an extended L^2-section f of F satisfying $D^*f = 0$ is necessarily of the form (3·13) so that $f \mapsto \hat{f}$ gives an isomorphism as required.

In view of (3·2) we therefore deduce

COROLLARY (3·14) $\operatorname{index} D = h(E) - h(F) - h_\infty(F)$ *where $h(E)$ is the dimension of the space of L^2 solutions of $Df = 0$ on $\hat{X}$, $h(F)$ the corresponding dimension for D^* and $h_\infty(F)$ is the dimension of the subspace of $\operatorname{Ker} A$ consisting of limiting values of extended L^2-sections f of F satisfying $D^*f = 0$.*

On the compact manifold X we have already seen that $\operatorname{Ker} D$ coincides with $\operatorname{Ker} D^*D$ (with the appropriate boundary conditions). We shall now show that similar results hold for L^2-sections on $\hat{X}$.

PROPOSITION (3·15). *The L^2-solutions of D and D^*D coincide on $\hat{X}$, as do the extended L^2-solutions. The same also holds with D and D^* interchanged.*

Proof. Any solution of $D^*Df = 0$ has an expansion on $Y \times [0, -\infty)$ of the form

$$f(y,u) = \sum_{\lambda} (a_\lambda e^{\lambda u} + b_\lambda e^{-\lambda u})\, \phi_\lambda(y) \tag{3·16}$$

and for this to be in L^2 as $u \to -\infty$, we must have $a_\lambda = 0$ for $\lambda \leqslant 0$ and $b_\lambda = 0$ for $\geqslant 0$. Hence, f is exponentially decaying as $u \to -\infty$. More precisely, we have

$$\|f(\cdot,u)\| < Ce^{\alpha u} \quad \text{with} \quad \alpha > 0, \tag{3·17}$$

where $\| \;\|$ denotes the L^2-norm on Y. Applying D to (3·16), we see that Df also satisfies an estimate of the form (3·17). We now apply Green's formula to the compact manifold $X_U = X \cup [0, U] \subset \hat{X}$ obtaining

$$\int_{X_U} (D^*Df, f)\, dx - \int_{X_U} (Df, Df)\, dx = \int_{Y_U} (Df, \sigma f)\, dy, \tag{3·18}$$

where $Y_U = \partial X_U$ is the copy of Y at $u = U$. In view of the estimates (3·17) for f and Df, it follows that the right-hand side of (3·18) tends to zero as $U \to -\infty$. Since $D^*Df = 0$, (3·18) gives

$$\int_{\hat{X}} (Df, Df)\, dx = 0 \tag{3·19}$$

and hence $Df = 0$. This proves the first part of the Proposition. For an extended L^2-section f satisfying $D^*Df = 0$, the expansion (3·16) may now have non-zero terms corresponding to $\lambda = 0$ so, instead of (3·17), we can only assert that $\|f(\cdot,u)\|$ is bounded as $u \to -\infty$. However applying D removes these terms and so (3·17) holds for Df and this is now enough to show that the right-hand side of (3·18) tends to zero as $U \to -\infty$. As before, we conclude that $Df = 0$, proving the second part of the Proposition. The corresponding statements with D and D^* interchanged are proved in exactly the same manner.

In view of this Proposition, the dimensions in (3·14) can equally well be defined using D^*D and DD^*.

We conclude this section with some further comments on the 0-eigenvalue. If we replace D by D^* from the beginning, we obtain an operator $\tilde{\mathscr{D}}$: $C^\infty(X, F; \tilde{P}) \to C^\infty(X, E)$, where $\tilde{P}$ is essentially the spectral projection of A corresponding to $\lambda \leqslant 0$. This is because near the boundary of X we have

$$D = \sigma\left(\frac{\partial}{\partial u} + A\right), \quad D^* = -\sigma^{-1}\left(\frac{\partial}{\partial u} - A\right). \tag{3·20}$$

Thus $\tilde{\mathscr{D}}$ has a slightly larger domain than D^*: $C^\infty(X, F, 1-P) \to C^\infty(X, E)$ and hence a different index. In fact, applying Corollary (3·14), we get

$$\operatorname{index} \tilde{D} = h(F) - h(E) - h_\infty(E). \tag{3·21}$$

where $h_\infty(E)$ is the dimension of the subspace of $\operatorname{Ker} A$ consisting of limiting values of extended L^2-sections f of E satisfying $Df = 0$. On the other hand, Theorem (3·10) applied to $\tilde{D}$ gives

$$\operatorname{index}\tilde{D} = -\int_X \alpha_0(x)\,dx - \frac{h-\eta(0)}{2}, \tag{3·22}$$

where α_0, h, $\eta(0)$ refer to the operator D (the point being that α_0 and $\eta(0)$ change sign when we replace D by $\tilde{D}$ but $h = \dim\operatorname{Ker} A = \dim\operatorname{Ker}(-A)$ is unaltered). Thus (3·21) and (3·22) yield

$$-\int_X \alpha_0(x)\,dx - \frac{h-\eta(0)}{2} = h(F) - h(E) - h_\infty(E) \tag{3·23}$$

while Theorem (3·10) and Corollary (3·14) applied to D give

$$\int \alpha_0(x)\,dx - \frac{h+\eta(0)}{2} = h(E) - h(F) - h_\infty(F). \tag{3·24}$$

Adding (3·23) and (3·24) we deduce

$$h = h_\infty(E) + h_\infty(F). \tag{3·25}$$

This formula suggests that every section in $\operatorname{Ker} A$ is uniquely expressible as a sum of limiting values coming from E and F respectively. In any case, the dimensions behave as if this were true and we shall use (3·25) in the next section for one of our applications.

4. *The Riemannian case.* We shall now apply the general index formula of section 3 to the cases that naturally arise in Riemannian geometry. The main result of (2) asserts, then, that the integrand $\alpha_0(x)\,dx$ that occurs in Theorem (3·10) can be identified explicitly as a suitable Pontrjagin form and that the asymptotic expansion in (3·10) (i) has no negative powers of t (implying, according to Theorem (3·10), that $\eta(s)$ will be holomorphic for $\operatorname{Re}(s) > -\frac{1}{2}$). The prototype, and in some sense the fundamental case, is that of the Dirac operator and so we begin by describing this.

Let X be an oriented compact manifold with boundary of dimension $2n$, so that $Y = \partial X$ has dimension $2n-1$. We assume, moreover, that X is a spin manifold and we choose a definite spin structure. Finally, we choose a Riemannian metric on X which coincides with a product metric on $Y \times I$ in a neighbourhood of the boundary. Then the *Dirac operator* of X is an elliptic first order differential operator

$$D\colon C^\infty(X, S^+) \to C^\infty(X, S^-),$$

where S^+ and S^- are the two spin bundles (associated to the two half-spin representations of Spin $(2n)$). For its definition and formal properties we refer to ((2); section 6) or ((12); Chapter IV). The restriction of S^+ to Y may be identified with the spin bundle S of Y (associated to the spin representation of Spin $(2n-1)$), and in $Y \times I$ we have

$$D = \sigma\left(\frac{\partial}{\partial u} + A\right), \tag{4·1}$$

where A is the Dirac operator on Y, u is the inward normal coordinate and σ is Clifford multiplication by the unit inward normal. Note that the Dirac operator on Y depends on the choice of orientation (it changes sign if we reverse the orientation). The sign convention we are using may be considered as defined by (4·1). As usual, a solution of $Af = 0$ (or equivalently $A^2f = 0$) is called a *harmonic spinor* on Y.

Combining Theorem (3·10) with the results of ((2); section 6), we obtain

THEOREM (4·2). *The index of the Dirac operator on X with the global boundary condition* (2·3) *is given by*

$$\operatorname{index} D = \int_X \hat{A}(p) - \frac{h+\eta(0)}{2},$$

where h is the dimension of the space of harmonic spinors on Y, $\eta(s)$ is the η-function of the Dirac *operator on Y and $\hat{A}(p)$ is the Hirzebruch $\hat{A}$-polynomial applied to the Pontrjagin forms p_i of the Riemannian metric on X. Moreover $\eta(s)$ is holomorphic for $Re(s) > -\frac{1}{2}$.*

Remarks (1) Using (3·14), we can replace index D above by $h(S^+) - h(S^-) - h_\infty(S^-)$, these integers being computed from the non-compact 'elongation' $\hat{X}$ of X as in section 3. Together with (3·25) this yields the formula given in Theorem 4 of (5).

(2) The condition that X be isometric to a product near Y can clearly be relaxed. On the one hand it is fairly clear that the analytical results of section 3, concerning operators like D, continue to hold: the fundamental solution given in section 2 must now be used as simply the first step in an iterative procedure to construct a parametrix. Thus index D will be defined and will remain constant under continuous variation of the interior metric (while the metric on Y remains fixed). On the other hand the integral in (4·2) remains unaltered, provided the first two normal derivatives of the metric vanish on Y: this is because two such metrics yield a C^2-metric on the double of X, and so we can argue as in section 1. Thus, (4·2) continues to hold as it stands for such metrics. For a more general metric, there will be an extra integral over Y involving the second fundamental form, and this is the analogue of the geodesic curvature in Gauss–Bonnet.

(3) If n is odd, i.e. $\dim X \equiv 2 \bmod 4$, then $\hat{A}(p) \equiv 0$. We now distinguish the two cases $n \equiv 1$ and $n \equiv 3 \bmod 4$, using the particular structure of the Clifford algebra in these dimensions (see (3)).

(*a*) $n \equiv 1 \bmod 4$, so $\dim Y \equiv 1 \bmod 8$, then the Dirac operator A on Y is of the form $A = iB$, where B is real skew-adjoint (cf. the case $Y =$ circle, $B = d/dx$). Hence $\operatorname{Spec} A$ is symmetric about the origin, so $\eta(s) \equiv 0$ and (4·2) reduces to:

$$h = -2 \operatorname{index} D.$$

Now $h \bmod 2$ is directly seen to be an invariant of Y, independent of the metric (see (4)) and so (4·2) gives an analytic proof that this invariant vanishes when Y is a spin boundary, or in other words *$h \bmod 2$ is a spin-cobordism invariant.* This was known before but only as a corollary of the mod 2 index theorem ((4); Theorem (3·1)).

(*b*) $n \equiv 3 \bmod 4$, so $\dim Y \equiv 5 \bmod 8$. In this case, the spin-bundle of Y has a natural quaternionic structure (defined say by j) and $jA = -Aj$. This shows again that $\eta(s) \equiv 0$ and also that the harmonic spinors have a quaternion structure. Thus h is automatically even and (4·2) gives no further information (except to identify $\frac{1}{2}h$ with $-\operatorname{index} D$).

Suppose now ξ is a hermitian vector bundle on X with a unitary connexion and that, near the boundary, the metric and connexion are constant in the normal direction. Then we have a generalized Dirac operator D_ξ, acting on spinors with coefficients in ξ (that is, we tensor the spin bundles with ξ) and near Y we have

$$D_\xi = \sigma\left(\frac{\partial}{\partial u} + A_\xi\right),$$

where A_ξ is the generalized Dirac operator on Y. Using the results of ((2); section 6), (4·2) generalizes to

$$\operatorname{index} D_\xi = \int_X \operatorname{ch}\xi\,\hat{\mathscr{A}}(p) - \frac{h_\xi + \eta_\xi(0)}{2}, \tag{4·3}$$

where h_ξ, $\eta_\xi(s)$ relate to the operator A_ξ and in the integrand $\operatorname{ch}\xi$ denotes the Chern character of ξ (as a differential form), $\hat{\mathscr{A}}$ is the sum of all Hirzebruch polynomials $\hat{A}_k$ and we pick out the form in the product of top dimension.

Similar formulae hold for spin^c-manifolds (the condition $\omega_2(X) = 0$, defining a spin-manifold being now relaxed to allow $\omega_2(X)$ to be the image of an *integral* cohomology class). As explained in ((2); section 6), this includes the case of complex Kähler manifolds. As a simple illustration, let us consider the case of Riemann surfaces. Let $\bar{X}$ be a compact Riemann surface (without boundary) and delete from it small disjoint open discs $B_1, \ldots, B_r$ around points $p_1, \ldots, p_r$. Then

$$X = \bar{X} - \bigcup_{i=1}^{r} B_i$$

is a surface, whose boundary Y consists of r circles. If z_i is a local parameter centred at p_i, we choose a metric on $\bar{X} - \bigcup p_i$, which is $|dz_i/z_i|^2$ near p_i. This identifies the elongated manifold $\hat{X}$ *conformally* with the punctured surface $\bar{X} - \bigcup p_i$. Applying now the analogue of (4·3) for the $\bar{\partial}$-operator on X, we obtain

$$\operatorname{index}\bar{\partial} = \int_X \frac{c_1}{2} - \frac{r}{2}. \tag{4·4}$$

Here we have used the fact that the tangential component of $\bar{\partial}$ on each boundary circle is the standard operator $-i(\partial/\partial\theta)$, hence $\eta(s) \equiv 0$ and $h = 1$. Now the Chern form c_1 of a surface coincides with the Euler form and so, by Gauss–Bonnet,

$$\int_X c_1 = 2 - 2g - r,$$

where g is the genus of $\bar{X}$. Thus (4·4) becomes

$$\operatorname{index}\bar{\partial} = 1 - g - r. \tag{4·5}$$

It is interesting to check (4·5) directly using (3·14). Since a holomorphic L^2-function on $\hat{X}$ decays exponentially at ∞, we have $h(E) = 0$ (assuming $r > 0$, so that $\hat{X} \neq \bar{X}$). On the other hand, the space of holomorphic L^2-one-forms is a conformal invariant (the condition $\int \omega \wedge \bar{\omega} < \infty$ does not use the metric) and hence can be computed from the punctured surface $\bar{X} - \bigcup p_i$. One easily checks that a holomorphic L^2 form must then extend holomorphically across the puncture, so that $h(F) = g$. To compute $h_\infty(F)$ we use (3·25) and the fact that $h_\infty(E) = 1$ (extended holomorphic L^2 functions on $\hat{X}$ being just the constants). Applying (3·14) this gives

$$\operatorname{index}\bar{\partial} = 0 - g - (r-1),$$

which checks with (4·5).

Included amongst the case of (4·3) is that of the *signature operator*. This is of particular importance because of its relation with cohomology and so we shall now treat this in detail. Moreover, since the signature operator exists without the spin restriction involved in (4·3), we need to give it an independent treatment.

We recall that, for an oriented Riemannian manifold X of dimension $2l$, the operator $d + d^*$ acts on the space Ω of all differential forms and anti-commutes with the involution τ defined by $\tau\phi = i^{p(p-1)+l} * \phi$ for $\phi \in \Omega^p$. Denoting by Ω_+ and Ω_- the $\pm$ 1-eigenspaces of τ, it follows that $d + d^*$ interchanges Ω_+ and Ω_-, and hence defines by restriction an operator

$$A : \Omega_+ \to \Omega_-$$

which we call the signature operator. For a *closed* manifold X of dimension $4k$, one checks easily, by the Hodge theory, that

$$\operatorname{index} A = \operatorname{sign}(X),$$

where sign (X) denotes the signature of the quadratic form on $H^{2k}(X; \mathbf{R})$ given by the cup-product.

Suppose now X is a manifold with boundary Y and is isometric to a product near the boundary; then near Y, A is of the required form

$$A = \sigma\left(\frac{\partial}{\partial u} + B\right)$$

with B a self-adjoint operator on Y. In fact we may identify the restriction of Ω_+ to Y with the space of all differential forms on Y and a little computation then shows that

$$B\phi = (-1)^{k+p+1}(\epsilon * d - d *)\phi, \tag{4·6}$$

where $*$ now denotes the duality operator on Y, $4k = \dim X$ and ϕ is either a $2p$-form ($\epsilon = 1$) or a $(2p-1)$-form ($\epsilon = -1$). Note that B preserves the parity of forms on Y and commutes with $\phi \mapsto (-1)^p * \phi$, so that $B = B^{\text{ev}} \oplus B^{\text{odd}}$ and B^{ev} is isomorphic to B^{odd}. In particular, the η-function of B is twice the η-function of B^{ev}, and the same holds for the dimension of the null spaces.

Combining Theorem (3·10) with the *local signature theorem* of ((2); section 5) we deduce, as in (4·2) and (4·3),

$$\operatorname{index} A = \int_X L(p) - (h + \eta(0)), \tag{4·7}$$

where h and η refer to the operator B^{ev}, and L is the Hirzebruch L-polynomial. Also by (3·14), we can replace index A by

$$\operatorname{index} A = h^+ - h^- - h_\infty^-, \tag{4·8}$$

where $h^\pm$ are the dimensions of L^2-harmonic forms in $\Omega_\pm$ of the elongated manifold $\hat{X}$ and h_∞^- is the dimension of the space of limiting values of extended L^2 harmonic forms in $\Omega_-(X)$. We proceed now to give topological interpretations to the integers occurring in (4·8). First we prove the following result which is of some independent interest:

PROPOSITION (4·9). *The space $\mathscr{H}(\hat{X})$ of L^2 harmonic forms on $\hat{X}$ is naturally isomorphic to the image $\hat{H}(X)$ of $H^*_{\text{comp}}(\hat{X}) \to H^*(\hat{X})$ (or equivalently $H^*(X, Y) \to H^*(X)$).*

Proof. Because of (3·15) the L^2 harmonic forms on $\hat{X}$ coincide with the L^2 solutions of $(d+d^*)\phi = 0$. Since the Laplace operator preserves degrees of forms the components ϕ^q of a harmonic form $\phi = \Sigma\phi^q$ are also harmonic. It follows that an L^2 harmonic form ϕ also satisfies $d\phi = d^*\phi = 0$. In particular, ϕ defines an element $[\phi]$ of $H^*(\hat{X})$. To see that $[\phi]$ lies in the image of $H^*_{\text{comp}}(\hat{X})$, it is sufficient to check that its restriction to $H^*(Y)$ is zero. Now, since ϕ is closed,

$$\int_\gamma \phi = \int_{\gamma_u} \phi,$$

where γ is a cycle in Y and γ_u the corresponding cycle in $Y_u = Y \times \{u\} \subset \hat{X}$. But ϕ is exponentially decreasing as $u \to -\infty$ (see (3·17)) hence $\int_\gamma \phi = 0$ for any γ and so $[\phi]$ restricts to zero in $H^*(Y)$. Thus $\phi \mapsto [\phi]$ defines a map

$$\alpha: \mathscr{H}(\hat{X}) \to \hat{H}(X)$$

which we shall prove is an isomorphism. First, we deal with surjectivity. A quite general theorem of de Rham–Kodaira ((9); p. 169) asserts that any closed L^2-form ψ can be written as

$$\psi = d\theta + \phi, \tag{4·9}$$

where $\phi \in L^2$, $d\phi = d^*\phi = 0$ and θ is some current (distributional form). In particular, a closed C^∞ form ψ with compact support has such a decomposition. Since every $\xi \in \hat{H}(X)$ is represented by such a ψ and since $H^*(\hat{X})$ can be computed from the complex of currents, (4·9) shows that $\alpha(\phi) = \xi$, so that α is surjective. Finally, to prove injectivity we assume $\alpha(\phi) = 0$, so that $\phi = d\theta$ for some C^∞ form θ on $\hat{X}$. We claim first that θ can be chosen bounded (i.e. as $u \to -\infty$, $\theta = \theta_0 + \theta_1 du$, where θ_0, θ_1 are forms on Y, depending on u, and are bounded). In fact, putting $v = 1/u$ we get a normal coordinate near the boundary for the compactification $\bar{X}$ of $\hat{X}$ (obtained by adjoining a copy of Y 'at ∞'). Since ϕ is exponentially decaying as $v \to 0$, it follows that ϕ extends to a C^∞ form on $\bar{X}$. Hence $\phi = \mathrm{d}\theta$ with θ a C^∞-form on $\bar{X}$. Since $dv = -du/u^2$, it follows that θ_0 and $\theta_1 u^2$ are both bounded. Now apply Green's formula to a compact part X_U of $\hat{X}$ (given by $-u \leqslant -U$), and we get

$$\int_{X_U} (\phi, \mathrm{d}\theta) - \int_{X_U} (d^*\phi, \theta) = \int_{Y_U} (\phi_1, \theta_0), \tag{4·10}$$

where $\phi = \phi_0 + \phi_1 du$, $\theta = \theta_0 + \theta_1 du$ and $Y_U = \partial X_U$ is a copy of Y. Since $\phi \to 0$ exponentially while θ is bounded as $U \to -\infty$, the boundary contribution in (4·10) tends to zero. Since $d^*\phi = 0$ and $\phi = d\theta$, we deduce that

$$\int_{X_U} (\phi, \phi) \to 0$$

as $U \to -\infty$. But

$$\int_{X_U} (\phi, \phi) \to \int_{\hat{X}} (\phi, \phi) = \|\phi\|^2,$$

hence $\phi = 0$ and α is injective as required.

Remark. In (9), it is pointed out that the Hodge theorem, identifying cohomology with harmonic forms on a compact manifold, does not extend to L^2-forms on a non-compact manifold. Proposition (4·9) shows nevertheless that, for the special cylinder-like manifolds $\hat{X}$, there is still an interesting connexion between L^2 harmonic forms and cohomology.

We recall now that the signature of a $4k$-dimensional manifold X with boundary Y is defined to be the signature of the (non-degenerate) quadratic form defined on $\hat{H}^{2k}(X)$. This quadratic form is induced by the degenerate quadratic form on $H^{2k}(X, Y)$ given by the cup-product: Poincaré duality for (X, Y) shows that the radical is precisely the kernel of $H^{2k}(X, Y) \to H^{2k}(X)$. Hence, just as for a compact manifold, (4·9) implies

COROLLARY (4·11). Sign $(X) = h^+ - h^-$, *where* $h^\pm = \dim \mathscr{H}^\pm$ *and*

$$\mathscr{H}^\pm = \mathscr{H}(\hat{X}) \cap \Omega_\pm(\hat{X}).$$

This corollary identifies two of the three terms in formula (4·8). To deal with the third term h_∞^-, we consider the space $\mathscr{K} \supset \mathscr{H}(\hat{X})$ of all extended L^2 harmonic forms on $\hat{X}$, as defined in section 3. Again, because of (3·15), any $\phi \in \mathscr{K}$ satisfies $d\phi = d^*\phi = 0$ and hence we obtain a map β: $\mathscr{K} \to H^*(X)$ extending α. In the cylinder, we can write $\phi = \psi + \theta$, where $\psi = \psi_0 + \psi_1 du$ with ψ_0, ψ_1 harmonic forms on Y (independent of u) and θ an L^2 harmonic form (hence decaying exponentially). The composition

$$\mathscr{K} \to H^*(X) \xrightarrow{j} H^*(Y)$$

is thus given by $\phi \mapsto \psi_0$. Now if $\mathscr{K}^\pm = \mathscr{K} \cap \Omega_\pm$, we see that, for $\phi \in \mathscr{K}^\pm$, we have $\psi_1 du = \pm \tau(\psi_0)$ and so $\psi_0 = 0 \Rightarrow \phi \in \mathscr{H}^\pm$. Thus, if $\beta^\pm: \mathscr{K}^\pm \to H^*(X)$ are the restrictions of β, we have

$$\operatorname{Ker} j\beta^\pm = \mathscr{H}^\pm. \tag{4·12}$$

Now Poincaré duality in the exact sequence

$$H^*(X) \xrightarrow{j} H^*(Y) \xrightarrow{\delta} H^*(X, Y)$$

shows that Imj is dual to its orthogonal complement, so that

$$\dim \operatorname{Im} j = \tfrac{1}{2} \dim H^*(Y) = h$$

(where h is the same as in (4·7)). From (4·12) we therefore deduce

$$h_\infty^\pm \leqslant h, \tag{4·13}$$

where $h_\infty^\pm = \dim \mathscr{K}^\pm / \mathscr{H}^\pm$ in the notation of section 3 and (4·8). But (3·25) asserts that

$$h_\infty^+ + h_\infty^- = 2h$$

(recall that the h in (3·25) is $\dim H^*(Y)$ which is $2h$ of the present section). Hence

$h_\infty^- = h_\infty^+ = h$, and so the integers h, h_∞^- cancel when we combine (4·7) with (4·8). Using (4·11) we therefore derive our final result, the *signature theorem for manifolds with boundary*:

THEOREM (4·14). *Let X be a $4k$-dimensional compact oriented Riemannian manifold with boundary Y and assume that, near Y, it is isometric to a product. Then*

$$\operatorname{sign} X = \int_X L(p) - \eta(0),$$

where these terms have the following meaning

(i) *Sign X is the signature of the non-degenerate quadratic form defined by the cup-product on the image of $H^{2k}(X, Y)$ in $H^{2k}(X)$;*

(ii) *$L(p) = L_k(p_1, \ldots, p_k)$, where L_k is the kth Hirzebruch L-polynomial and the p_i are the Pontrjagin forms of the Riemannian metric;*

(iii) *$\eta(s)$ is the η-function (defined by (1·7)) for the self-adjoint operator on even forms on Y given by*

$$\phi \mapsto (-1)^{k+p+1}(*d - d*)\phi \quad (\phi \in \Omega^{2p})$$

and is holomorphic for $Re(s) > -\frac{1}{2}$.

It is remarkable that the three terms in Theorem (4·14) arise from three different areas: sign X is a topological invariant, $\int L(p)$ is differential-geometric and $\eta(0)$ is a spectral invariant. This interplay gives the theorem special interest and leads one to expect a variety of applications.

In section 1, we mentioned the Gauss–Bonnet theorem as an analogue and motivation for the signature theorem we have just arrived at. However, we also pointed out the significant difference between the two cases, namely that in Gauss–Bonnet there is no term corresponding to $\eta(0)$ in (4·14). We shall now explain this in more detail.

We begin by recalling (see ((2); section 5)) that the signature operator $A\colon \Omega_+ \to \Omega_-$ interchanges even forms and odd forms, so that $A = A^+ \oplus A^-$, where

$$A^+\colon \Omega_+^{\mathrm{ev}} \to \Omega_-^{\mathrm{odd}}, \quad A^-\colon \Omega_+^{\mathrm{odd}} \to \Omega_-^{\mathrm{ev}}.$$

Now let us replace A by A^+ in the proof of (4·14). The tangential operator on Y is, then, just B^{ev} whose η-function is half that of B: thus $\eta(0)$ in (4·14) gets halved. Using the results of ((2); section 6), the integral in (4·14) now becomes $\frac{1}{2}\left\{\int_X L(p) + \int_X e\right\}$, where e is the Euler form of the metric (generalizing $(2\pi)^{-1} \times$ Gauss curvature for surfaces). To find the integer contribution, we must return to the general formula given in (3·10) together with the L^2 interpretation of the index given in (3·14). From these, we see that sign X in (4·14) must be replaced by

$$\dim \mathscr{H}_{\mathrm{ev}}^+ - \dim \mathscr{K}_{\mathrm{odd}}^- + \tfrac{1}{2}\dim H^{\mathrm{ev}}(Y), \tag{4·15}$$

where $H^{\mathrm{ev}}(Y)$ is the even-dimensional cohomology of Y and $\mathscr{H}$, $\mathscr{K}$ refer to the L^2, and extended L^2, harmonic forms on $\hat{X}$ of the appropriate types ($\pm$, even/odd). Using (4·9), we deduce

$$\dim \mathscr{H}_{\mathrm{ev}}^+ - \dim \mathscr{H}_{\mathrm{odd}}^- = \tfrac{1}{2}\{\operatorname{sign} X + E(\operatorname{Ker} j)\}, \tag{4·16}$$

where $E = \dim H^{\mathrm{ev}} - \dim H^{\mathrm{odd}}$ and j is the restriction $H^*(X) \to H^*(Y)$. Moreover, in the course of proving (4·14), we showed that (4·13) was an equality and hence

$$j\beta^-: \mathscr{K}^-/\mathscr{H}^- \to \operatorname{Im} j$$

is an isomorphism. Thus

$$\dim \mathscr{K}^-_{\mathrm{odd}} - \dim \mathscr{H}^-_{\mathrm{odd}} = \dim (\operatorname{Im} j)^{\mathrm{odd}}. \tag{4·17}$$

Using (4·16), (4·17) and the equality

$$\dim H^{\mathrm{ev}}(Y) = \tfrac{1}{2} \dim H^*(Y) = \dim (\operatorname{Im} j)$$

proved earlier, (4·15) becomes

$$\tfrac{1}{2}\{\operatorname{sign} X + E(\operatorname{Ker} j)\} + \tfrac{1}{2} E(\operatorname{Im} j). \tag{4·18}$$

Finally, since $E(\operatorname{Ker} j) + E(\operatorname{Im} j) = E(X)$, the Euler characteristic of X, we deduce the refinement of (4·14) which we want:

$$\tfrac{1}{2}\{\operatorname{sign} X + E(X)\} = \frac{1}{2}\int_X (L(p) + e) - \tfrac{1}{2}\eta(0). \tag{4·19}$$

Replacing A^+ by A^- yields the same formula except that $E(X)$ and e are changed in sign. Adding the two formulae gives (4·14) back again, but subtracting them gives the Gauss–Bonnet theorem

$$E(X) = \int_X e.$$

From this point of view, we derive Gauss–Bonnet as the difference between two index problems in which the non-local boundary contribution $\eta(0)$ cancels out. In fact, the topological obstruction mentioned in section 1 to the existence of a classical local boundary condition for the operator $d + d^*: \Omega^{\mathrm{ev}} \to \Omega^{\mathrm{odd}}$ vanishes and one can derive Gauss–Bonnet from the index theorem as given in (1) in which there is no non-local contribution.

The cohomological term sign (X) in Theorem (4·14) only involves the middle dimension $H^{2k}(X)$. In fact the analytical invariant $\eta(0)$ can also be shown to involve only the $2k$-forms on Y. To see this, we use the decomposition

$$\Omega^{\mathrm{ev}} = H^{\mathrm{ev}} \oplus d\Omega^{\mathrm{odd}} \oplus d^*\Omega^{\mathrm{ev}}$$

and observe that the operator B of (4·6) annihilates H^{ev} and coincides up to sign with $d*$ on $d\Omega^{\mathrm{odd}}$ and with $*d$ on $d^*\Omega^{\mathrm{odd}}$. Now we have

$$d*: d\Omega^{2p-1} \to d\Omega^{4k-2p-1}, \quad *d: d^*\Omega^{2p-1} \to d^*\Omega^{4k-2p+1}$$

and

$$2p - 1 = 4k - 2p - 1 \quad \text{only if} \quad p = k$$
$$2p - 1 \neq 4k - 2p + 1.$$

Hence we can decompose B in the form

$$B = B_0 \oplus B_1 \oplus B_2,$$

where B_0 is the zero operator on H^{ev}, $B_1 = d*$ on $d\Omega^{2k-1}$ and B_2 is an operator of the form

$$B_2 = \begin{pmatrix} 0 & T \\ T^* & 0 \end{pmatrix}.$$

Since B_2 is conjugate to the operator

$$\begin{pmatrix} \sqrt{(TT^*)} & 0 \\ 0 & -\sqrt{(T^*T)} \end{pmatrix}$$

and since $\sqrt{(TT^*)}$ and $\sqrt{(T^*T)}$ have the same spectrum it follows that the spectrum of B_2 is symmetric so that its η-function vanishes identically. Hence the η-function of B is equal to the η-function of B_1. Thus we have established

PROPOSITION (4·20). *The term $\eta(s)$ in Theorem (4·14) is equal to $\eta_1(s)$ where $\eta_1(s)$ is the η-function of the operator $d*$ acting on the space $d\Omega^{2k-1}$.*

This Proposition leads to a suggestive way of looking at $\eta(0)$. Let us define a quadratic form Q on Ω^{2k-1} by

$$Q(\alpha) = \int_Y \alpha \wedge d\alpha. \tag{4·21}$$

In fact the radical of Q is $\operatorname{Ker} d$, so that Q can be viewed as a non-degenerate quadratic form on $d\Omega^{2k-1}$ (note the formal analogy with the quadratic form leading to sign X). If $d\alpha$ is an eigenvector of $d*$ with eigenvalue λ, then

$$\begin{aligned} Q(\alpha) &= \langle \alpha, *d\alpha\rangle = \lambda^{-1}\langle \alpha, *d*d\alpha\rangle = -\lambda^{-1}\langle \alpha, d^*d\alpha\rangle \\ &= -\lambda^{-1}\langle d\alpha, d\alpha\rangle. \end{aligned}$$

Thus $-Q(\alpha)$ has the same sign as λ. In other words, we can formally interpret $-\eta(0)$ as the 'signature' of the quadratic form Q (on the infinite-dimensional space $d\Omega^{2k-1}$). The interest of this reformulation is that (4·21) *does not involve the metric*. However to give proper meaning to its signature, we need a metric and the value of the signature will indeed depend on the metric.

We conclude with a few remarks about the finiteness of $\eta(0)$. Theorem (4·14) implies of course that, for the operator on Y defined in (4·14) (iii), $\eta(0)$ is finite. This *assumes* however that we can find an oriented manifold X with boundary Y. This is not always the case but the main results of Thom's cobordism theory assert that, in all *odd* dimensions, we can always find an X whose boundary consists of *two* copies of Y (with correct orientation). Clearly the η-function on $2Y$ is twice the η-function of Y and so the finiteness at $s = 0$ follows from Theorem (4·14). Similar remarks apply to the η-functions of the Dirac operator and its generalizations as in (4·3). In each case we need to use the appropriate cobordism theory (see (17)) to deduce that, in odd dimensions, everything bounds after multiplication by some integer. In Part III, we shall give an alternative proof of these finiteness results independent of Theorems (4·14), etc. and we shall then deduce the finiteness for *all* self-adjoint elliptic operators on odd dimensional manifolds.

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