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Finitely presented modules over Leavitt algebras

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Abstract

Given a field k and a positive integer n, we study the structure of the finitely presented modules over the Leavitt k-algebra L of type $(1,n)$, which is the k-algebra with a universal isomorphism $i:L \to L^{n+1}$. The abelian category of finitely presented left L-modules of finite length is shown to be equivalent to a certain subcategory of finitely presented modules over the free algebra of rank $n+1$, and also to a quotient category of the category of finite dimensional (over k) modules over a free algebra of rank $n+1$, modulo a Serre subcategory generated by a single module. This allows us to use Schofield's exact sequence for universal localization to compute the K_1 group of a certain von Neumann regular algebra of fractions of L.

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0. Introduction

Given a field k and a nonnegative integer n, the *Leavitt k-algebra L* of type $(1,n)$ is the algebra with generators x_i, y_j, $0 \leqslant i,j \leqslant n$, and defining relations which, in matrix form, can be written as

$$(x_0,\dots,x_n)(y_0,\dots,y_n)^{\mathrm{T}} = 1, \quad (y_0,\dots,y_n)^{\mathrm{T}}(x_0,\dots,x_n) = I_{n+1},$$

where I_r denotes the identity matrix of size $r \times r$. Here the Leavitt type [19] of a nonzero ring R is the pair of numbers (r,s) which is defined as follows. If R has

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invariant basis number (IBN), meaning that $R^a \cong R^b \Rightarrow a = b$ for all $a, b \in \mathbb{N}$, then we set $(r,s) = (1,0)$. If R does not have IBN, then r is the least positive integer such that $R^r \cong R^{r+i}$ for some positive integer i, and then s is the least positive integer with $R^r \cong R^{r+s}$. For each Leavitt type (r,s), Leavitt constructed k-algebras $V_{r,s}$ having a universal isomorphism $i : (V_{r,s})^r \to (V_{r,s})^{r+s}$, see [18,19], (the ones of type $(1,n)$ are those described above). Observe that the Leavitt algebra of type $(1,0)$ is precisely $k[x_0, x_0^{-1}]$, the algebra of Laurent polynomials in the indeterminate x_0. It was shown by Bergman [4, Theorem 6.1] that all finitely generated projective L-modules are free, of rank uniquely determined modulo n.

In this paper, we will study the structure of the abelian category $\mathrm{fp}(L)$ of *finitely presented* left L-modules, obtaining it in an explicit way as a quotient category of the category of finitely presented modules over the free algebra $k\langle Y\rangle := k\langle y_0, y_1, \ldots, y_n\rangle$ (Theorem 5.1). (We use $X = \{x_0, x_1, \ldots, x_n\}$ and $Y = \{y_0, y_1, \ldots, y_n\}$ to denote the two sets of $n+1$ variables appearing in the above definition of the Leavitt algebra L.) A crucial fact is that the algebra L fits in a commutative diagram of algebra embeddings

$$\begin{tikzcd}
k \arrow[r] \arrow[d] & k\langle X\rangle \arrow[r] \arrow[d] & k_{\mathrm{rat}}\langle X\rangle \arrow[r] \arrow[d] & k\langle\langle X\rangle\rangle \arrow[d] \\
k\langle Y\rangle \arrow[r] & L \arrow[r] & Q \arrow[r] & T
\end{tikzcd}$$

For $n = 0$, this diagram reduces to the commutative diagram:

$$\begin{tikzcd}
k \arrow[r] \arrow[d] & k[x_0] \arrow[r] \arrow[d] & k_{\mathrm{rat}}[x_0] \arrow[r] \arrow[d] & k[[x_0]] \arrow[d] \\
k[x_0^{-1}] \arrow[r] & k[x_0, x_0^{-1}] \arrow[r] & k(x_0) = k(x_0^{-1}) \arrow[r] & k((x_0))
\end{tikzcd}$$

where $k[[x_0]]$ is the power series algebra, $k_{\mathrm{rat}}[x_0]$ is the algebra of rational series, $k(x_0) = k(x_0^{-1})$ is the algebra of rational functions (the quotient field of $k[x_0]$), and $k((x_0))$ the Laurent power series field.

The ring Q appearing in our diagram has been studied in [3] in connection with the construction of purely infinite simple von Neumann regular rings. Recall that a simple ring R is called *purely infinite* in case R is not a division ring and, for each nonzero $a \in R$ there are $z, t \in R$ such that $zat = 1$. The rings satisfying the latter property have been termed 1-*simple rings* by Cohn [10]. The name "purely infinite" comes from the characterization of these rings in [3, Proposition 1.5] as those such that each nonzero right ideal contains an infinite idempotent. We remark that purely infinite simple C^*-*algebras* play a central role in the ongoing Elliott's classification programme for nuclear C^*-algebras, see for example [23]. By Ara et al. [3, Sections 5–7], the ring Q is a purely infinite simple, von Neumann regular ring which is a universal localization of both $k\langle X\rangle$ and $k\langle Y\rangle$. Moreover, it coincides with two apparently different constructions of localizations of the free algebra, obtained independently by Schofield [28] and Rosenmann and Rosset [25]. By the above diagram, Q can be thought of as a higher dimensional analogue of the field of rational functions in one variable. Of course, both $k\langle X\rangle$ and $k_{\text{rat}}\langle X\rangle$ can also be embedded in the *universal field of fractions* of $k\langle X\rangle$. This is a much more classical approach, see [8,9]. The ring T in our diagram also appears in [3], in connection with some constructions of Tyukavkin [33].

A key subcategory of $\text{fp}(L)$ is the category $\text{fp}(L)_{\text{fl}}$ of finitely presented left L-modules of finite length. It turns out that $\text{fp}(L)_{\text{fl}}$ is the category of torsion modules with respect to the localization $L \to Q$. We obtain a category equivalence between $\text{fp}(L)_{\text{fl}}$ and a quotient of an abelian category of finitely presented left $k\langle Y\rangle$-modules of negative Euler characteristic, in fact of characteristic lying in $-n\mathbb{N}$, modulo the Serre subcategory generated by a single module M_0. (The *Euler characteristic* of a finitely presented module M over $R = k\langle Y\rangle$ is defined to be $r - s$, where $0 \to R^s \to R^r \to M \to 0$ is a presentation of M.) At the same time, the category $\text{fp}(L)_{\text{fl}}$ is equivalent to a category of finitely presented left $k\langle X\rangle$-modules of Euler characteristic 0. All this is summarized in Theorem 6.2. This provides a useful link between these two classes of modules. Some of the ideas used here come from the papers [14,16], where Farber and Vogel deal with the group ring $\mathbb{Z}[F]$ of the free group F. In fact, one can construct, over any ring k, an analogue of the Leavitt algebra by using the universal localization $k[F]_g$, where $k[F]$ is embedded in $k\langle\langle X\rangle\rangle$ via the Magnus embedding, and $k[F]_g$ is the universal localization with respect to the map given by right multiplication by the row $(x_0, x_1, \ldots, x_n)$, as in Section 1. As pointed out to me by Desmond Sheiham, the representation theory of $k[F]_g$ seems to be closely related to the structure of the *link modules* appearing in [14–16,31,32]. We believe that these connections deserve further investigation.

As an application of our techniques, we compute $K_1(Q)$, obtaining in Theorem 6.5 a formula which is analogous to the one known for K_1 of the universal field of fractions of $k\langle X\rangle$; see [23,7] and [8, Section **7**.9]. We also establish a new property of the free algebra (Theorem 7.3), which is obtained from the exchange property of the Leavitt algebras of type $(1, n)$, recently established in [1, Theorem 2.1].

We briefly outline the contents of the paper. In Section 1 we fix some basic notation that will be used throughout the paper, and we recall several basic facts which we will need later. In Section 2, we show that the Leavitt algebra is flat as a right module over the free algebra $k\langle Y\rangle$, a result which we will use repeatedly in the sequel. Section 3 gives a first approximation to the structure of the category $\text{fp}(L)$ of finitely presented modules over the Leavitt algebra. In particular, we establish the relationship with the finite dimensional left $k\langle Y\rangle$-modules. We define the notion of *lattice* for a finitely presented L-module of finite length in Section 4, and we use it to get a right inverse of the tensor product functor on the category of finite dimensional $k\langle Y\rangle$-modules. In Section 5 we obtain one of the main results in the paper, namely a description of $\text{fp}(L)$ as a suitable quotient category of the category $\text{fp}(k\langle Y\rangle)$. We give in Section 6 a further characterization of the category $\text{fp}(L)_{\text{fl}}$ of finitely presented L-modules of finite length, and we obtain the formula $K_1(Q) = k^\times/(k^\times)^n \times D_\varepsilon$, where D_ε is the reduced divisor group of $k\langle X\rangle$. Finally, Section 7 contains an interpretation of the exchange property of L in terms of $k\langle Y\rangle$.

1. Notation and preliminary results

We start by fixing some convenient notation, which will be coherently used throughout the paper.

Notation 1.1. Let us fix a field k and a positive integer n. Let $X=\{x_0,x_1,\dots,x_n\}$ and $Y=\{y_0,y_1,\dots,y_n\}$ be two sets of $n+1$ independent variables, and denote by $\Lambda=k\langle X\rangle$ the free k-algebra on X and by $R=k\langle Y\rangle$ the free k-algebra on Y. We will denote by X^* the free monoid on X, with identity element (i.e. the empty word) denoted 1. Label words in X^* in the form $x_I=x_{i_1}x_{i_2}\cdots x_{i_t}$ for finite sequences $I=(i_1,\dots,i_t)$ of indices from $\{0,1,\dots,n\}$, with the convention $x_\emptyset=1$. (Similar conventions apply to Y^*.) If $I=(i_1,\dots,i_t)$ is a multi-index as before, we will use I^* to denote the multi-index $(i_t,i_{t-1},\dots,i_1)$. Note that $k\langle X\rangle$ is the monoid algebra on X^*, and similarly for $k\langle Y\rangle$. We will denote by L the Leavitt k-algebra of type $(1,n)$ in the variables $x_0,x_1,\dots,x_n,y_0,y_1,\dots,y_n$, that is

$$L=k\left\langle x_0,x_1,\dots,x_n,y_0,y_1,\dots,y_n \mid y_ix_j=\delta_{ij} \text{ all } i,j,\sum_{i=0}^n x_iy_i=1\right\rangle.$$

There are two canonical maps $\Lambda=k\langle X\rangle\to L$ and $R=k\langle Y\rangle\to L$. It is well-known that these maps are injective, cf. [3]. □

Denote by $k\langle\langle X\rangle\rangle$ the algebra of noncommutative power series on X. For $i=0,1,\dots,n$, the left transductions $\partial_i: k\langle\langle X\rangle\rangle\to k\langle\langle X\rangle\rangle$ are defined by

$$\partial_i\left(\sum_{w\in X^*}\lambda_w w\right)=\sum_{w\in X^*}\lambda_{x_iw}w.$$

The *augmentation* on $k\langle\langle X\rangle\rangle$ is the algebra homomorphism $\varepsilon: k\langle\langle X\rangle\rangle\to k$ sending $\sum_{w\in X^*}\lambda_w w$ to λ_1. The maps ∂_i are left ε-derivations, i.e.

$$\partial_i(\alpha\beta)=\partial_i(\alpha)\beta+\varepsilon(\alpha)\partial_i(\beta)$$

for $\alpha,\beta\in k\langle\langle X\rangle\rangle$.

We will use the theory of universal localization for arbitrary maps between finitely generated projective modules, as developed in [27]. If T is a ring and Σ is a set of maps between finitely generated projective left T-modules, we will denote by T_Σ the universal localization of T with respect to Σ. There is a canonical localization map $\imath_\Sigma: T\to T_\Sigma$ which is universal Σ-inverting, see [27, Theorem 4.1]. When $\Sigma=\{h\}$ for a single map h, we will write T_h instead of $T_{\{h\}}$.

For any k-algebra T such that $k\langle X\rangle\subseteq T\subseteq k\langle\langle X\rangle\rangle$, we consider the map $g: T\to T^{n+1}$ given by right multiplication by the row $(x_0,x_1,\dots,x_n)$. Observe that the canonical map $k\langle X\rangle\to L$ is the localization map $\imath_g: k\langle X\rangle\to k\langle X\rangle_g$, and similarly the canonical map $k\langle Y\rangle\to L$ is the localization map $\imath_f: k\langle Y\rangle\to k\langle Y\rangle_f$, where f is the map $k\langle Y\rangle^{n+1}\to k\langle Y\rangle$ given by right multiplication by the column $(y_0,y_1,\dots y_n)^{\mathrm{T}}$.

Assume now that T is any k-algebra with $k\langle X\rangle \subseteq T \subseteq k\langle\langle X\rangle\rangle$ such that T is closed under all left transductions ∂_i. Note that, for $r \in T$, we can write

$$r = \varepsilon(r) + \sum_{i=0}^{n} x_i \partial_i(r)$$

with $\partial_i(r) \in T$ because of the closure condition on T. In case T is in addition a *local* algebra (so that all elements in T with nonzero constant term are invertible in T), we have the following result from [3].

Theorem 1.2 (Ara, Goodearl and Pardo [3, Theorems 5.4 and 5.6]). *Let T be a local subalgebra of $k\langle\langle X\rangle\rangle$ containing $k\langle X\rangle$, and let $g: T \to T^{n+1}$ be the map given by right multiplication by $(x_0, x_1, \ldots, x_n)$. Then the universal localization T_g is a purely infinite simple, von Neumann regular ring. Moreover $K_0(T_g)$ is a cyclic group of order n, generated by $[T_g]$.*

This obviously applies to the case where $T = k\langle\langle X\rangle\rangle$. More important for us is the case where T is the algebra of *rational series* $k_{\mathrm{rat}}\langle X\rangle$, which has been studied in some detail in [3, Sections 6 and 7]. Recall that $k_{\mathrm{rat}}\langle X\rangle$ is the *division closure* of $k\langle X\rangle$ in $k\langle\langle X\rangle\rangle$. (This algebra plays an important role in formal language theory and the theory of codes, see [6].) By Cohn [8, p. 135], the algebra of rational series is closed under all left transductions. Since it is obviously local, Theorem 1.2 applies to it. It is also known that $k_{\mathrm{rat}}\langle X\rangle$ is a universal localization of $k\langle X\rangle$. We record this fact here for further reference.

Proposition 1.3 (Cohn and Dicks [11, p. 416]). *Let Σ' be the set of those square matrices over $k\langle X\rangle$ which become invertible over $k\langle\langle X\rangle\rangle$. Then $k_{\mathrm{rat}}\langle X\rangle$ is the universal localization of $k\langle X\rangle$ with respect to Σ'.*

In fact, the same result holds when k is just a principal ideal domain by Dicks and Sontag [13, Theorem 24] (see also [16, Theorem 5.1]), or even a Bezout domain [12], but not for a general noncommutative coefficient ring k [29]. It seems to be an open question whether it holds for a general commutative coefficient ring k. It follows from Proposition 1.3 that the ring $Q = k_{\mathrm{rat}}\langle X\rangle_g$, which is a purely infinite simple von Neumann regular ring by Theorem 1.2, is also a universal localization of the Leavitt algebra L, namely $Q = L_{\Sigma'}$. In this paper, we shall study in detail the structure of the "torsion modules" with respect to the latter localization, and we will find a formula for $K_1(Q)$ in close analogy to the formula for K_1 of the universal field of fractions of $k\langle X\rangle$, found in [22,7] (see also [8, Section **7**.9]).

We have the following diagram, in which all maps are inclusions:

$$\begin{tikzcd}
k \arrow[r] \arrow[d] & k\langle X\rangle \arrow[r, "\imath_{\Sigma'}"] \arrow[d, "\imath_g"] & k_{\mathrm{rat}}\langle X\rangle \arrow[r] \arrow[d, "\imath_g"] & k\langle\langle X\rangle\rangle \arrow[d, "\imath_g"] \\
k\langle Y\rangle \arrow[r, "\imath_f"] & L \arrow[r, "\imath_{\Sigma'}"] & Q \arrow[r] & k\langle\langle X\rangle\rangle_g
\end{tikzcd}$$

It was proved in [3, Theorem 7.5] that the map $R = k\langle Y\rangle \to Q$ is also a universal localization, namely $Q = R_\Sigma$, where Σ is the set of monomorphisms with finite-dimensional cokernel between finitely generated projective left R-modules. The universal localization R_Σ was first considered by Schofield [28].

We shall need some results on K_1 from [2]. We denote by $T^\times$ the group of invertible elements of a ring T.

Theorem 1.4 (Ara and Brustunga [2]). *Let L and Q be the rings described before. Then we have*:

(a) *The canonical map $k^\times \to K_1(L)$ is surjective with kernel $(k^\times)^n$, so that $K_1(L) \cong k^\times/(k^\times)^n$.*
(b) *The canonical map $K_1(L) \to K_1(Q)$ is injective.*

2. Flatness

In this section, we prove that L is flat as a *right* R-module. This will play an important role in the sequel.

We will use the properties of the construction of skew polynomial rings with freely independent indeterminates, as in [3, Section 3], in the special case where the coefficient ring is $k\langle X\rangle$. Then the ring $S = (k\langle X\rangle)\langle Y; \varepsilon, \partial\rangle$ coincides with the k-algebra $k\langle X, Y \mid y_i x_j = \delta_{ij} \text{ all } i,j\rangle$, so that $L = S/I$, where I is the ideal of S generated by $1 - \sum_{i=0}^n x_i y_i$, see [3, Section 4]. Elements in S can be uniquely written as $\sum_I p_I y_I$ with $p_I \in k\langle X\rangle$ and the multiplication rule in S is determined by

$$y_i p = \varepsilon(p) y_i + \partial_i(p)$$

for $p \in k\langle X\rangle$ and $i = 0, 1, \ldots, n$. (Note that we are dealing here with *left* ε-derivations, and so coefficients appear in the opposite side of [3].)

The ideal $I = S(1 - \sum_{i=0}^n x_i y_i)S$ coincides with the socle of S, denoted $\operatorname{Soc} S$. In fact, it is easy to check that $e := 1 - \sum_{i=0}^n x_i y_i$ is a minimal idempotent and that $(x_I e y_{J^*} \mid I, J)$ is an infinite system of matrix units generating I as a k-vector space.

Proposition 2.1. *Let $R = k\langle Y\rangle$ be the free algebra on Y and let L be the Leavitt algebra. Then L is flat as a right R-module.*

Proof. In order to prove that L_R is flat, it suffices to show that $\operatorname{Tor}_1^R(L, M) = 0$ for every left R-module M. Note that $S = (k\langle X\rangle)\langle Y; \varepsilon, \partial\rangle$ is a free right R-module with free R-basis $(x_I)_I$. It follows from the description of $\operatorname{Soc} S$ given above that $\operatorname{Soc} S$ is also a free right R-module with free R-basis $(x_I e)_I$, where $e = 1 - \sum_{i=0}^n x_i y_i$. We have a short exact sequence of right R-modules

$$0 \to \operatorname{Soc} S \to S \to L \to 0$$

and to see that $\mathrm{Tor}_1^R(L,M)=0$ for a given left R-module M, it suffices to check that the induced map

$$\tau\colon \bigoplus_I x_I e \otimes_R M \cong \mathrm{Soc}\, S \otimes_R M \to S \otimes_R M \cong \bigoplus_I x_I \otimes_R M$$

is injective. Note that

$$\tau\left(\sum_I x_I e \otimes m_I\right) = \sum_I x_I \otimes m_I - \sum_{i=0}^{n} \sum_I x_I x_i \otimes y_i m_I.$$

If $\sum_I x_I e \otimes m_I \in \oplus_I x_I e \otimes_R M$ is nonzero, take an index I_0 of minimal length such that $m_{I_0} \neq 0$. Since $x_{I_0} \otimes_R M \cong M$, we get $x_{I_0} \otimes m_{I_0} \neq 0$. Note also that the term $x_{I_0} \otimes m_{I_0}$ cannot be cancelled in the sum $\sum_I x_I \otimes m_I - \sum_{i=0}^n \sum_I x_I x_i \otimes y_i m_I$, because for each of the nonzero terms $x_I x_i \otimes y_i m_I$ appearing in that expression, the length of $I \cup \{i\}$ is strictly larger than the length of I_0, and the sum $\oplus_I x_I \otimes_R M$ is a direct sum. It follows that τ is injective and so $\mathrm{Tor}_1^R(L,M)=0$, as desired. □

3. Finitely presented modules over the Leavitt algebra L

Recall that for every left semihereditary ring S, the category of finitely presented left S-modules $\mathrm{fp}(S)$ is an abelian category. (Here, we are looking at $\mathrm{fp}(S)$ as a full subcategory of the category S-Mod of all left S-modules. The fact that S is left semihereditary implies that the kernel, image and cokernel of every map between finitely presented modules are also finitely presented).

Let $R=k\langle Y\rangle$ be the free algebra on $Y=\{y_0,\ldots,y_n\}$, viewed as a subalgebra of L. We also view L as a subalgebra of Q, the universal localization of R obtained by inverting the set Σ of all the monomorphisms between finitely generated free left R-modules with finite-dimensional cokernel, see [3, Sections 6 and 7] and Section 1. By a result of Bergman and Dicks (see [5] or [27, Theorem 4.9]), every universal localization of a hereditary ring is also hereditary. Since the free algebra $k\langle Y\rangle$ is hereditary (in fact, it is a fir [8, Corollary **2**.4.3]), we see that L and Q are both hereditary algebras. Note that Q is a von Neumann regular ring by Theorem 1.2, and so every finitely presented left Q-module is projective.

Let $\mathscr{T}$ be the full subcategory of R-Mod consisting of all the left R-modules of finite dimension over k. This category is obviously an abelian category, and we will show below that it is the category of objects with finite length in the category $\mathrm{fp}(R)$. First of all, note that every finite dimensional R-module is finitely presented, since a submodule of finite codimension in a finitely generated free R-module is also finitely generated [20, Theorem 4].

Proposition 3.1. *The category $\mathscr{T}$ of finite-dimensional R-modules coincides with the category $\mathrm{fp}(R)_{\mathrm{fl}}$ of modules with finite length in $\mathrm{fp}(R)$.*

Proof. Clearly all the objects in $\mathscr{T}$ are objects of finite length in $\mathrm{fp}(R)$. It remains to observe that a simple object in $\mathrm{fp}(R)$ must be finite dimensional. Let M be a simple object in $\mathrm{fp}(R)$. By Lewin [20, Theorem 2], there is a finitely generated free R-module

P such that $P \leqslant M$ and M/P is finite dimensional. Since M is simple in $\mathrm{fp}(R)$, we must have $P = 0$ and thus M is finite dimensional. □

We can now compute the Grothendieck group of the abelian categories $\mathrm{fp}(R)$ and $\mathscr{T} = \mathrm{fp}(R)_{\mathrm{fl}}$.

Proposition 3.2. *Let $\mathscr{T}$ be the category of finite-dimensional left R-modules, where $R = k\langle Y\rangle$ is the free algebra of rank $n+1$. Then the following properties hold:*

(1) *$K_0(\mathscr{T})$ is a free abelian group over the set of isomorphism classes of simple, finite-dimensional left R-modules.*
(2) *The canonical map $\imath : K_0(R) \to K_0(\mathrm{fp}(R))$ is an isomorphism, so that $K_0(\mathrm{fp}(R))$ is a cyclic group generated by $[R]$.*
(3) *The map $K_0(\mathscr{T}) \to K_0(\mathrm{fp}(R))$ sends $K_0(\mathscr{T})$ onto the subgroup of $K_0(\mathrm{fp}(R))$ generated by $n[R]$.*

Proof. (1) Since the category $\mathscr{T}$ coincides with $\mathrm{fp}(R)_{\mathrm{fl}}$ by Proposition 3.1, the result follows from [24, Theorem 3.1.8(1)].

(2) This is a standard argument. Since R is a fir, every finitely presented module M admits a presentation

$$0 \to P_1 \to P_0 \to M \to 0$$

with P_0 and P_1 free R-modules. By standard computations, the map $\chi : K_0(\mathrm{fp}(R)) \to K_0(R)$ given by $\chi([M]) = [P_0] - [P_1]$ is the inverse of the canonical map $\imath : K_0(R) \to K_0(\mathrm{fp}(R))$.

(3) Let M be a finite-dimensional R-module and take a free resolution

$$0 \to R^m \to R^s \to M \to 0.$$

By Lewin's Theorem [20, Theorem 4], we have $m = nr + s$, where r is the k-dimension of M. It follows that $[M] = -nr[R]$ in $K_0(\mathrm{fp}(R))$. Since there are one-dimensional R-modules (e.g. the module M_0 defined below), we see that the image of the map $K_0(\mathscr{T}) \to K_0(\mathrm{fp}(R))$ is exactly the subgroup generated by $n[R]$. □

Now we recall the exact sequence in universal localization constructed by Schofield. Let $R \to R_\Sigma$ be an injective universal localization. Let $\bar{\Sigma}$ be a set of maps between finitely generated projective modules such that each map which becomes invertible over R_Σ is associated to a map in $\bar{\Sigma}$. Let $\mathscr{G}$ be the full subcategory of the category of f.p. modules with objects the cokernels of the maps in $\bar{\Sigma}$. Then we have an exact sequence [27, Theorem 4.12]

$$K_1(R) \to K_1(R_\Sigma) \to K_0(\mathscr{G}) \to K_0(R) \to K_0(R_\Sigma).$$

We specialize again to the case where $R = k\langle y_0, \ldots, y_n\rangle$. Let $R \to R_\Sigma$ be an injective universal localization of R such that R_Σ is flat as a right R-module. Then we have an exact functor $F : \mathrm{fp}(R) \to \mathrm{fp}(R_\Sigma)$ given by $F(M) = R_\Sigma \otimes_R M$, and it follows easily that the kernel of this functor is precisely $\mathscr{G}$. For example, in the case where Σ is the family of all the monomorphisms having finite-dimensional cokernel, we get a universal localization Q which is flat as a right R-module by [3, p. 89], and the corresponding category $\mathscr{T}$ of torsion modules coincides with the category of finite-dimensional left R-modules. In order to understand what happens with the Leavitt algebra L, we consider the map $f: R^{n+1} \to R$ defined by

$$f(r_0, r_1, \ldots, r_n) = r_0 y_0 + r_1 y_1 + \cdots + r_n y_n.$$

The cokernel M_0 of this map will play an important role in what follows. Note that M_0 is a one-dimensional module, $M_0 = k$, and R acts on M_0 via the augmentation map:

$$y_i \alpha = 0 \quad \text{for all } i \in \{0, 1, \ldots, n\} \text{ and all } \alpha \in M_0.$$

Now L is a flat right R-module by Proposition 2.1. The exact sequence in universal localization gives us in this case

$$K_1(R) \to K_1(L) \to K_0(\mathscr{T}') \to \mathbb{Z} \to \mathbb{Z}_n \to 0.$$

Here $\mathscr{T}'$ is the kernel of the exact functor $\mathrm{fp}(R) \to \mathrm{fp}(L)$. Note that $\mathscr{T}' \subseteq \mathscr{T}$. By Theorem 1.4(a), the map $K_1(R) = k^\times \to K_1(L)$ is surjective, so we get $K_0(\mathscr{T}') \cong \mathbb{Z}$. This implies that $[M_0]$ must be a generator of $K_0(\mathscr{T}')$.

Our next result gives a first description of the structure of the finitely presented L-modules of finite length.

Proposition 3.3. *Let L be the Leavitt algebra of type $(1, n)$. Then the following holds*:

(1) *Let N be a finite-dimensional left R-module with a composition series of length k*:

$$0 < N_1 < N_2 < \cdots < N_k = N.$$

Assume that exactly r composition factors are isomorphic to M_0. Then $L \otimes_R N$ is an L-module of finite length and its length is exactly $k - r$.

(2) *Let M be a finitely presented left L-module. Then there is a finitely generated free module P such that $P \leqslant M$ and M/P is a module of finite length.*

(3) *Every finitely presented left L-module M of finite length is isomorphic to a module of the form $L \otimes_R N$, where N is a finite-dimensional left R-module.*

Proof. We begin with the crucial observation that, for a simple, finite-dimensional R-module N, we have $L \otimes_R N = 0$ if and only if $N \cong M_0$. Indeed, if $L \otimes_R N = 0$ and $N \ncong M_0$, then $[N]\mathbb{Z} \oplus [M_0]\mathbb{Z} \subseteq K_0(\mathscr{T}') \subseteq K_0(\mathscr{T})$, a contradiction since $K_0(\mathscr{T}')$ is a cyclic group.

Next, we show that $L \otimes_R N$ is a simple module for every simple, finite-dimensional R-module N such that $N \ncong M_0$. By the above paragraph, we have $L \otimes_R N \neq 0$. Let $\alpha = \sum_I x_I \otimes \alpha_I$ be a nonzero element in $L \otimes N$, where $\alpha_I \in N$. Since $\sum_{i=0}^n x_i y_i = 1$, there is j such that $y_j \alpha \neq 0$, and we see inductively that we can find J such that $y_J \alpha \neq 0$ and $y_J \alpha \in 1 \otimes N$. Now the simplicity of ${}_R N$ gives us $R(y_J \alpha) = 1 \otimes N$, showing the simplicity of $L \otimes_R N$.

(1) It follows easily from the above observations and the fact that L is flat as a left R-module (Proposition 2.1).

(2) Let M be a finitely presented left L-module. By [27, Corollary 4.5], there is a finitely presented left R-module N such that $L\otimes_R N\cong M$. Now by [20, Theorem 2], there is a finitely generated free R-module P such that $P\leqslant N$ and N/P is finite-dimensional. Since L_R is flat, we have that $M\cong L\otimes_R N$ contains the f.g. free L-module $L\otimes_R P$. Moreover, $(L\otimes_R N)/(L\otimes_R P)\cong L\otimes_R(N/P)$ is an L-module of finite length by (1).

(3) Let M be a finitely presented left L-module of finite length. It is clear that M cannot contain any nonzero free module. Hence, the result follows from (2). □

4. Modules of type L for $k\langle X\rangle$

In this section, we introduce another point of view for the modules over the Leavitt algebra L. This has been inspired by the papers [14–16], so that we (temporarily) borrow some of the terminology used in these papers.

A left $\Lambda=k\langle X\rangle$-module M is said to have the *Sato property* if $\mathrm{Tor}_q^{\Lambda}(k,M)=0$ for all q, where k is viewed as a right Λ-module with trivial action via the augmentation map. It is not hard to show that a left Λ-module M has the Sato property if and only if the map $\mu:M^{n+1}\to M$ given by

$$\mu((m_0,m_1,\dots,m_n))=\sum_{i=0}^{n}x_im_i$$

is an isomorphism, cf. [26, Proposition 2.3]. From this characterization we easily see that a left Λ-module M has the Sato property if and only if it is a left L-module. In fact, if $\mu:M^{n+1}\to M$ is an isomorphism then we can define an action of $k\langle Y\rangle$ on M by the rule $\mu^{-1}(m)=(y_0m,y_1m,\dots,y_nm)$ for all $m\in M$. This action combines with the Λ-action to provide the structure of L-module on M.

A *module of type* L is a left finitely generated Λ-module with the Sato property. By the above observation, a module of type L is a left L-module which is finitely generated as Λ-module.

Let M be a Λ-module of type L. A *lattice* in M is an R-submodule $A\subset M$ such that A is finite dimensional over k and $M=\Lambda A$. (Recall our notation: $R:=k\langle Y\rangle$.)

Proposition 4.1. (1) *Let M be a left L-module. Then M is a module of type L if and only if M is a finitely presented L-module of finite length.*

(2) *Let M be a module of type L. Then M contains a lattice. Moreover, an R-submodule A of M is a lattice if and only if A is finite dimensional and the natural map $L\otimes_R A\to M$ is an isomorphism.*

(3) *Every module of type L contains an smallest lattice.*

Proof. (1) If M is a finitely presented L-module of finite length then by Proposition 3.3(3) there is a finite-dimensional left R-module N such that $L\otimes_R N\cong M$. Then clearly M is finitely generated as a Λ-module.

The converse follows from (2).

(2) Assume that M is a left L-module which is finitely generated as Λ-module. Let $a_1,\dots,a_r$ be generators of M as a left Λ-module. Then

$$y_i a_j = \sum_k \gamma_{kij} a_k,$$

where $\gamma_{kij} \in \Lambda$. Let r be a bound for the degrees of the polynomials γ_{kij}. Let A be the k-space generated by $x_I a_j$, where $|I| \leqslant r$. Then $y_i A \subseteq A$ for all i, and clearly A is a lattice for M.

If $A \subset M$ is finite dimensional and the natural map $L \otimes_R A \to M$ is an isomorphism, then $M = \Lambda A$ and thus A is a lattice in M. Conversely assume that A is a lattice in M. Since L is flat as a right R-module, the map $L \otimes_R A \to L \otimes_R M$ is injective. Now the natural map $L \otimes_R M \to M$ is an isomorphism, because the inclusion $R \to L$ is a ring epimorphism. It follows that the map $L \otimes_R A \to M$ is injective. Since A is a lattice this map is clearly surjective.

(3) This follows from the arguments in [14, Lemma 1.6]. We include a short proof for completeness. We first show that $A_1 \cap A_2$ is a lattice for any two lattices A_1 and A_2. Namely assume $a \in A_1$. Then, since $\Lambda A_2 = M$, we can write $a = \sum_{|I|=k} x_I b_I$, for some k, where $b_I \in A_2$ for all I. For a fixed J with $|J| = k$, we have

$$b_J = y_{J^*} a \in A_1$$

which shows that $b_J \in A_1 \cap A_2$. Therefore, $A_1 \subset \Lambda(A_1 \cap A_2)$ and so $M = \Lambda A_1 = \Lambda(A_1 \cap A_2)$. This proves that $A_1 \cap A_2$ is a lattice.

Now let A be a lattice in M of minimal k-dimension. By the above, it is clear that A is the smallest lattice in M. □

Proposition 4.2 (Farber [14, Lemma 2.6]). *Let A be a lattice in a module M of type L, and B be any L-module. Then any R-homomorphism $A \to B$ can be uniquely extended to an L-homomorphism $M \to B$. Thus* $\mathrm{Hom}_L(M,B) = \mathrm{Hom}_R(A,B)$. *If $f : M_1 \to M_2$ is a surjective L-homomorphism between modules of type L, then f restricts to a surjection from A_1 onto A_2, where A_i is the minimal lattice of M_i.*

Proof. Let $f : A \to B$ be an R-homomorphism. We have $L \otimes_R A \cong M$ and also $L \otimes_R B \cong B$, so the extension is just $\mathrm{id} \otimes f$.

If $f : M_1 \to M_2$ is surjective, then $f(A_1)$ is a lattice in M_2 and so $A_2 \subseteq f(A_1)$. Let $C = f^{-1}(A_2) \cap A_1$. Then C is a lattice in M_1 contained in the minimal lattice A_1. So $C = A_1$ and $f(A_1) = A_2$. (To show that C is a lattice, take $a \in A_1$. Then $f(a) \in M_2$ so that we can write $f(a) = \sum \lambda_i b_i$ for $\lambda_i \in \Lambda$ and $b_i \in A_2$. Now $b_i \in f(A_1)$ so there are $a_i \in A_1$ such that $f(a_i) = b_i$. But then $a_i \in A_1 \cap f^{-1}(A_2) = C$. Observe that $f(a - \sum \lambda_i a_i) = 0$, so that $c := a - \sum \lambda_i a_i \in C$. It follows that $a \in \Lambda C$, as desired.) □

Proposition 4.3. *Let $f : M_1 \to M_2$ be an L-homomorphism between modules of type L. Let A_i be the minimal lattice in M_i, for $i = 1, 2$. Then $f(A_1) \subseteq A_2$.*

Proof. By Proposition 4.2, we have $f(A_1) = B$, where B is the minimal lattice in $f(M_1)$. So it suffices to prove that $B \subseteq A_2$. To show this, it is enough to prove that

$A_2 \cap f(M_1)$ is a lattice in $f(M_1)$. For $a \in f(M_1)$, we can write $a = \sum_{|I|=k} x_I a_I$, for some k, where $a_I \in A_2$. For a fixed J with $|J|=k$, we have $y_{J^*} a = a_J$, and so $a_J \in A_2 \cap f(M_1)$. We conclude that $A_2 \cap f(M_1)$ is a lattice in $f(M_1)$. □

Recall from Section 3 that $\mathscr{T}$ denotes the category of finite-dimensional left R-modules. Let $\mathscr{S}$ denote the category of finitely presented left L-modules of finite length. By Proposition 3.3, we have a well-defined functor $F : \mathscr{T} \to \mathscr{S}$ given by $F(N) = L \otimes_R N$. We are going to define a right inverse of F. For an object M in $\mathscr{S}$, set $G(M) = A$, where A is the minimal lattice in M. If $f : M_1 \to M_2$ is a map in $\mathscr{S}$, then $f(G(M_1)) \subseteq G(M_2)$ by Proposition 4.3. Hence, we can define $G(f) : G(M_1) \to G(M_2)$ as the restriction of f to $G(M_1)$.

Clearly, this assignment defines a functor $G : \mathscr{S} \to \mathscr{T}$.

Theorem 4.4. *With the above notation, we have* $FG \cong I_{\mathscr{S}}$.

Proof. Let M be an object in $\mathscr{S}$. By Proposition 4.1(2), the natural map

$$F(G(M)) = L \otimes_R G(M) \to M$$

is an isomorphism. It is clear that this isomorphism is natural, so the result is proved. □

5. The category fp(L) as a quotient category

Let M_0 be the one-dimensional $R = k\langle Y \rangle$-module defined in Section 3. We will show in this section that fp(L) is equivalent to the quotient category fp(R)$/\mathscr{M}_0$, where $\mathscr{M}_0$ is the Serre subcategory of fp(R) generated by M_0.

We first recall some basic definitions on abelian categories. Let $\mathscr{A}$ be an abelian category. A *Serre subcategory* of $\mathscr{A}$ is an abelian subcategory $\mathscr{B}$ which is closed under subobjects, quotients and extensions. Note that the Serre subcategory $\mathscr{M}_0$ of fp(R) defined above is the full subcategory of fp(R) whose objects are the finite-dimensional left R-modules having all composition factors isomorphic to M_0.

Given a Serre subcategory $\mathscr{B}$ of an small abelian category $\mathscr{A}$, one can form a quotient abelian category $\mathscr{A}/\mathscr{B}$ and an exact functor $T : \mathscr{A} \to \mathscr{A}/\mathscr{B}$ with the following universal property: given an exact functor $S : \mathscr{A} \to \mathscr{C}$ from $\mathscr{A}$ to an abelian category $\mathscr{C}$ such that $S(B) \cong 0$ for every object B of $\mathscr{B}$, there exists a unique exact functor $S' : \mathscr{A}/\mathscr{B} \to \mathscr{C}$ such that $S = S' \circ T$; see [36, Chapter 2].

We denote by F the functor fp(R) $\to$ fp(L) induced by the tensor product $L \otimes_R -$. Note that F is an exact functor because L is a flat right R-module (Proposition 2.1).

Theorem 5.1. *Let $\mathscr{M}_0$ be the Serre subcategory of* fp(R) *generated by the simple left R-module M_0. Then the abelian categories* fp(R)$/\mathscr{M}_0$ *and* fp(L) *are equivalent.*

Proof. Clearly the category $\mathscr{M}_0$ consists of all the finite-dimensional left R-modules whose composition factors are all isomorphic to M_0.

Consider the exact functor $F: \mathrm{fp}(R) \to \mathrm{fp}(L)$ such that $F(P) = L \otimes_R P$ for every object P in $\mathrm{fp}(R)$. By Proposition 3.3(1), $F(N) \cong 0$ for all objects N in $\mathscr{M}_0$. By the universal property of $\mathrm{fp}(R)/\mathscr{M}_0$, we get an exact functor $F': \mathrm{fp}(R)/\mathscr{M}_0 \to \mathrm{fp}(L)$ such that $F = F' \circ T$, where $T: \mathrm{fp}(R) \to \mathrm{fp}(R)/\mathscr{M}_0$ is the localization functor.

To show that F' is an equivalence, we have to show that every object in $\mathrm{fp}(L)$ is isomorphic to an object of the form $F(P)$ for some object P in $\mathrm{fp}(R)$ and that F' is fully faithful. The former assertion follows from [27, Corollary 4.5]. To see that F' is fully faithful we have to check that the canonical map

$$\mathrm{Hom}_{\mathscr{C}}(P_1, P_2) \to \mathrm{Hom}_L(F(P_1), F(P_2))$$

is an isomorphism for all finitely presented left R-modules P_1 and P_2, where $\mathscr{C}$ denotes the abelian category $\mathrm{fp}(R)/\mathscr{M}_0$.

The maps in $\mathscr{C}$ are equivalence classes $[(f,g)]$ of diagrams in $\mathrm{fp}(R)$,

$$P_1 \xleftarrow{f} P \xrightarrow{g} P_2,$$

where the kernel and the cokernel of f are objects in $\mathscr{M}_0$. For such a pair, we have $F'([(f,g)]) = (1 \otimes g) \circ (1 \otimes f)^{-1}$. Now assume that $(1 \otimes g) \circ (1 \otimes f)^{-1} = 0$. Then $1 \otimes g = 0$. By Proposition 3.3, we have $\mathrm{Im}(g) \in \mathscr{M}_0$, and consequently $[(f,g)] = [(f,0)] = 0$ in $\mathscr{C}$.

To prove surjectivity, we need a somewhat different approach to the universal localization $L = R_f$ where here $f: R^{n+1} \to R$ denotes the map given by right multiplication by the column $(y_0, y_1, \ldots, y_n)^{\mathrm{T}}$. This is in close analogy to Schofield's construction, see [3, Section 6].

We denote by $\mathscr{P}(R)$ the category of finitely generated projective left R-modules. Let $\Phi = \mathrm{Mor}(\mathscr{P}(R))$ denote the class of all homomorphisms between finitely generated projectives. Let Υ denote the class of all *mono*morphisms in Φ whose cokernel belongs to $\mathscr{M}_0$. The same proof as in [3, Proposition 6.2] gives that Υ is a multiplicative set of nonzerodivisor maps in Φ which satisfies the right Ore condition. It is then possible to construct the quotient category $\mathscr{D} = \mathscr{P}(R)\Upsilon^{-1}$, [35, Section 10.3], [3]. Set $H = \mathrm{End}_{\mathscr{D}}(R)$ be the endomorphism ring of the object R in the localized category $\mathscr{D}$. Then H is the universal localization of R with respect to Υ. By Proposition 3.3 all maps in Υ become invertible in L, so we have a unique k-algebra homomorphism $H \to L$ which is the identity on R. Since $f \in \Upsilon$, the above map is an isomorphism, and we can identify L and H. Let $h: L \otimes_R P_1 \to L \otimes_R P_2$ be a homomorphism between induced finitely generated projective left L-modules. Then h is in the localized category $\mathscr{P}(R)\Upsilon^{-1}$ and so there exists a diagram

$$P_1 \xleftarrow{s} P \xrightarrow{g} P_2$$

in $\mathscr{P}(R)$ such that $s \in \Upsilon$ and $h = (1 \otimes g) \circ (1 \otimes s)^{-1}$. Note that s is an $\mathscr{M}_0$-iso and so $F'([s,g]) = h$.

Finally, consider a homomorphism h: $L \otimes_R M_1 \to L \otimes_R M_2$ between induced finitely presented left L-modules. Choose presentations

$$0 \longrightarrow P_1 \xrightarrow{\pi_1} P_0 \xrightarrow{\pi_0} M_1 \longrightarrow 0,$$

$$0 \longrightarrow Q_1 \xrightarrow{\rho_1} Q_0 \xrightarrow{\rho_0} M_2 \longrightarrow 0$$

for M_1 and M_2, respectively. Then there exists $\bar{h}: L\otimes_R P_0 \to L\otimes_R Q_0$ such that $(1\otimes \rho_0)\circ \bar{h} = h\circ(1\otimes\pi_0)$. By the above argument there exists a finitely generated projective R-module P and a diagram

$$P_0 \xleftarrow{s} P \xrightarrow{g} Q_0$$

such that $s\in\Upsilon$ and $\bar{h} = (1\otimes g)\circ(1\otimes s)^{-1}$. We have to verify that the pair $[(s,g)]$ induces a map in $\mathscr{C} = \mathrm{fp}(R)/\mathscr{M}_0$ from M_1 to M_2. To this end, we have to prove that the composition

$$[(1,\rho_0)]\circ[(s,g)]\circ[(1,\pi_1)] \tag{1}$$

is 0 in $\mathscr{C}$. To simplify notation, assume that $P_1\subseteq P_0$ and π_1 is the inclusion map. Then the above composition is represented by the diagram

$$P_1 \xleftarrow{s_{|s^{-1}(P_1)}} s^{-1}(P_1) \xrightarrow{\rho_0\circ g_{|s^{-1}(P_1)}} M_2.$$

For $p\in s^{-1}(P_1)$ we have

$$(1\otimes(\rho_0\circ g))(1\otimes p) = (1\otimes\rho_0)(\bar{h}(1\otimes s(p))) = h(1\otimes\pi_0(s(p))) = 0$$

and so $L\otimes_R \mathrm{Im}(\rho_0\circ g_{|s^{-1}(P_1)}) = 0$. We infer from Proposition 3.3 that $\mathrm{Im}(\rho_0\circ g_{|s^{-1}(P_1)})$ is in $\mathscr{M}_0$. It follows that composition (1) is represented by the diagram

$$P_1 \xleftarrow{s_{|\mathrm{Ker}(\rho_0\circ g_{|s^{-1}(P_1)})}} \mathrm{Ker}(\rho_0\circ g_{|s^{-1}(P_1)}) \xrightarrow{0} M_2$$

and so it is 0 in $\mathscr{C}$.

It follows that there exists a unique map $k: M_1\to M_2$ in $\mathscr{C}$ such that $k\circ[(1,\pi_0)] = [(1,\rho_0)]\circ[(s,g)]$. Now we have

$$h\circ(1\otimes\pi_0) = (1\otimes\rho_0)\circ\bar{h} = F'([(1,\rho_0)]\circ[(s,g)]) = F'(k)\circ(1\otimes\pi_0).$$

We conclude that $h = F'(k)$, which finish the proof of surjectivity. $\square$

We can now get a better picture of the category $\mathscr{S}$ of finitely presented left L-modules of finite length. Recall that $\mathscr{T}$ denotes the abelian category of finite dimensional left R-modules. The exact functor $F:\mathscr{T}\to\mathscr{S}$ induces an exact functor $F_1:\mathscr{T}/\mathscr{M}_0\to\mathscr{S}$. On the other hand, we constructed a functor $G:\mathscr{S}\to\mathscr{T}$ in Section 4 which is a right inverse of the functor $F:\mathscr{T}\to\mathscr{S}$. Let us denote again by T the localization functor $\mathscr{T}\to\mathscr{T}/\mathscr{M}_0$.

Corollary 5.2. *With the above notation, the functors $F_1:\mathscr{T}/\mathscr{M}_0\to\mathscr{S}$ and $T\circ G:\mathscr{S}\to\mathscr{T}/\mathscr{M}_0$ define mutually inverse category equivalences between the abelian categories $\mathscr{T}/\mathscr{M}_0$ and $\mathscr{S}$.*

Proof. The functor $F_1:\mathscr{T}/\mathscr{M}_0\to\mathscr{S}$ is just the restriction to $\mathscr{T}/\mathscr{M}_0$ of the functor $F':\ \mathrm{fp}(R)/\mathscr{M}_0\to\mathrm{fp}(L)$ obtained in the proof of Theorem 5.1. By using Theorem 4.4, we get

$$F_1\circ(T\circ G) = F\circ G \cong I_{\mathscr{S}}.$$

Since F': $\mathrm{fp}(R)/\mathscr{M}_0 \to \mathrm{fp}(L)$ is an equivalence by Theorem 5.1, we conclude that F_1 and $T \circ G$ are mutually inverse category equivalences. □

It is now a simple matter to get the following information on the structure of $\mathscr{S}$.

Corollary 5.3. *Keep notation as above. The following properties hold*:

(1) *$K_0(\mathscr{S})$ is a free abelian group over the set of isomorphism classes of simple, finite-dimensional left R-modules not isomorphic to M_0.*
(2) *The canonical map $\imath : K_0(L) \to K_0(\mathrm{fp}(L))$ is an isomorphism, so that $K_0(\mathrm{fp}(L))$ is a cyclic group of order n generated by $[L]$.*
(3) *The map $K_0(\mathscr{S}) \to K_0(\mathrm{fp}(L))$ is zero.*

6. K_1 of the universal localization of the Leavitt algebra

Recall that for any left semihereditary ring S, the category $\mathrm{fp}(S)$ of finitely presented left S-modules is an abelian category. Assume now that S is a semifir. Given a finitely presented module M we have a presentation

$$0 \longrightarrow S^n \xrightarrow{A} S^m \longrightarrow M \longrightarrow 0$$

and the Euler characteristic $\chi(M) = m - n$ is well defined. Here of course $A \in M_{n\times m}(S)$ acting on the right on S^n. If $B \in M_{s\times t}(S)$ is another matrix defining an injective map, then A and B define the same module (i.e. $\mathrm{coker}(A) \cong \mathrm{coker}(B)$) if and only if A and B are stably associated, meaning that the matrices $A \oplus I_p$ and $B \oplus I_q$ are associated for suitable identity matrices. See [8, Theorem **0**.6.2]. In fact it can be shown that if A and B as above are stably associated then $A \oplus I_t$ is associated to $B \oplus I_m$, [8, Corollary **0**.6.3].

For a semifir S, let $\mathrm{Tor}(S)$ denote the full subcategory of S-Mod consisting of all modules M admitting a presentation

$$0 \longrightarrow S^n \xrightarrow{A} S^n \longrightarrow M \longrightarrow 0,$$

where A is a full matrix over S. By Cohn [8, Theorem **3**.3.3], the category $\mathrm{Tor}(S)$ is an abelian subcategory of the category $\mathrm{fp}(S)$.

If S is a fir, then every full matrix over S admits a complete factorization into atomic factors, and any two such complete factorizations are unique, cf. [8, Theorem **3**.3.7]. This is due to the fact that the category $\mathrm{Tor}(S)$ is an abelian category with objects of finite length, and so the Jordan–Hölder Theorem holds in it. The simple modules in $\mathrm{Tor}(S)$ correspond to the equivalence classes of full atomic matrices under the relation of stable association.

Let $\Lambda = k\langle X\rangle$ be the free algebra on X. Recall that Λ is a fir, so that the above theory applies to it. Let Σ' be the set of all square matrices over Λ which become invertible in $k\langle\langle X\rangle\rangle$. The set Σ' is exactly the set of matrices X such that $\varepsilon(X)$ is invertible, where $\varepsilon : \Lambda \to k$ is the augmentation homomorphism. Obviously all matrices in Σ' are full. Let us consider now the class $\mathscr{X}$ of all left Λ-modules M admitting a presentation

$$0 \longrightarrow \Lambda^n \xrightarrow{A} \Lambda^n \longrightarrow M \longrightarrow 0,$$

where $A \in \Sigma'$. If

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

is an exact sequence of finitely presented left Λ-modules, then we have a relation $C = AB$, where A, B, C are matrices presenting M_1, M_3, M_2, respectively, cf. [8, Proposition **0**.6.1]. Since $\varepsilon(C)$ is invertible if and only if $\varepsilon(A)$ and $\varepsilon(B)$ are both invertible, we conclude that $\mathscr{Z}$ is a Serre subcategory of $\mathrm{Tor}(\Lambda)$. In particular $\mathscr{Z}$ is an abelian category. Recall that the category $\mathscr{S}$ is the category of all finitely presented left L-modules of finite length.

Proposition 6.1 (cf. Farber and Vogel [16, Lemma 4.3]). *There exists an equivalence of categories between $\mathscr{Z}$ and $\mathscr{S}$.*

Proof. We are going to define a functor $C: \mathscr{Z} \to \mathscr{S}$. Let M be a module with a presentation

$$0 \longrightarrow \Lambda^n \xrightarrow{A} \Lambda^n \longrightarrow M \longrightarrow 0,$$

where A is a matrix over Λ such that $\varepsilon(A)$ is invertible. Then $\mathrm{Tor}_*^{\Lambda}(k; M)$ may be computed as the homology of

$$0 \longrightarrow k \otimes_\Lambda \Lambda^n \xrightarrow{1 \otimes A} k \otimes_\Lambda \Lambda^n \longrightarrow 0$$

which coincides with

$$0 \longrightarrow k^n \xrightarrow{\varepsilon(A)} k^n \longrightarrow 0.$$

The last complex is acyclic because $\varepsilon(A)$ is invertible over k. It follows that $\mathrm{Tor}_*^{\Lambda}(k; M) = 0$ and so M is an L-module. Since M is finitely generated as left Λ-module, it follows from Proposition 4.1(1) that M is an object in $\mathscr{S}$. Since the map $\Lambda \to L$ is a ring epimorphism, it follows that any left Λ-homomorphism $f: M_1 \to M_2$ between objects M_1, M_2 in $\mathscr{Z}$ is automatically an L-homomorphism. Thus we get a functor C: $\mathscr{Z} \to \mathscr{S}$.

Let M be a finitely presented left L-module of finite length. Let N be the minimal lattice in M. Consider the Λ-homomorphism

$$u: \Lambda \otimes_k N \to \Lambda \otimes_k N, \quad u(\lambda \otimes z) = \lambda \otimes z - \sum_{i=0}^{n} \lambda x_i \otimes y_i z,$$

where $\lambda \in \Lambda$, $z \in N$. It is clear that $u \in \Sigma'$ and thus $\mathrm{coker}(u) \in \mathscr{Z}$. As in [16, proof of Lemma 4.3], the map g: $\mathrm{coker}(u) \to M$ given by $g([\lambda \otimes z]) = \lambda z$ is an isomorphism of L-modules. It follows that every object in $\mathscr{S}$ is isomorphic to an object of the form $C(M)$ for M in $\mathscr{Z}$. Since C is clearly fully faithful, we get that C is an equivalence of categories. $\square$

Let $\mathscr{T}_0$ be the abelian category of finite-dimensional left R-modules admitting a composition series all of whose composition factors are *not* isomorphic to M_0. As has been pointed out to me by Desmond Sheiham, the natural exact functor $T' : \mathscr{T}_0 \to \mathscr{T}/\mathscr{M}_0$ is not a category equivalence. However, it induces an isomorphism $K_0(\mathscr{T}_0) \to K_0(\mathscr{T}/\mathscr{M}_0)$, as follows. First, note that by Heller's localization theorem [36, Theorem II.6.4], we have $K_0(\mathscr{T}/\mathscr{M}_0) \cong K_0(\mathscr{T})/K_0(\mathscr{M}_0)$. Second, the natural map $K_0(T') : K_0(\mathscr{T}_0) \to K_0(\mathscr{T}/\mathscr{M}_0) = K_0(\mathscr{T})/K_0(\mathscr{M}_0)$ sends a $\mathbb{Z}$-basis of $K_0(\mathscr{T}_0)$ to a $\mathbb{Z}$-basis of $K_0(\mathscr{T})/K_0(\mathscr{M}_0)$, so it is a group isomorphism.

By Corollary 5.2, the functor $T \circ G$: $\mathscr{S} \to \mathscr{T}/\mathscr{M}_0$ is a category equivalence, so that $K_0(\mathscr{S}) \cong K_0(\mathscr{T}/\mathscr{M}_0) \cong K_0(\mathscr{T}_0)$. We can summarize the above observations and Proposition 6.1 as follows:

Theorem 6.2. *The functors $C : \mathscr{L} \to \mathscr{S}$ and $T \circ G : \mathscr{S} \to \mathscr{T}/\mathscr{M}_0$ are category equivalences. The natural exact functor T': $\mathscr{T}_0 \to \mathscr{T}/\mathscr{M}_0$ induces a group isomorphism from $K_0(\mathscr{T}_0)$ onto $K_0(\mathscr{T}/\mathscr{M}_0)$, so that we get $K_0(\mathscr{L}) \cong K_0(\mathscr{S}) \cong K_0(\mathscr{T}_0)$. In particular, there is a bijection between the set of equivalence classes of stably associated full atomic matrices over Λ which become invertible under ε and the set of isomorphism classes of finite-dimensional simple left R-modules not isomorphic to M_0.*

We now introduce the *reduced divisor group D_ε* in analogy with the divisor group of [23,7] and [8, Section **7**.9].

Definition 6.3. Let $\Lambda = k\langle X\rangle$ be the free algebra on X. The *reduced divisor group D_ε* of Λ is the free abelian group on the set of equivalence classes of stably associated full atomic matrices over Λ which become invertible under ε.

We get from Theorem 6.2.

Proposition 6.4. $D_\varepsilon \cong K_0(\mathscr{L}) \cong K_0(\mathscr{S}) \cong K_0(\mathscr{T}_0)$.

Proof. Since $\mathscr{L}$ is an abelian category with objects of finite length, it follows from [24, Theorem 3.1.8(1)] that $K_0(\mathscr{L})$ is a free abelian group on the set of isomorphism classes of simple objects in $\mathscr{L}$. On the other hand, the isomorphism classes of simple objects in $\mathscr{L}$ are in a bijective correspondence with the classes of stably associated atomic full matrices over Λ. We conclude that $K_0(\mathscr{L}) \cong D_\varepsilon$.

Finally, it follows from Theorem 6.2 that $K_0(\mathscr{L}) \cong K_0(\mathscr{S}) \cong K_0(\mathscr{T}_0)$. □

Let Q be the universal localization of the Leavitt algebra L obtained by inverting all the homomorphisms between finitely generated projective left L-modules whose cokernel is a left L-module of finite length (i.e. belongs to $\mathscr{S}$). It is clear from Theorem 6.2 that Q is the ring obtained from $R = k\langle Y\rangle$ by universally inverting all maps between finitely generated projective modules with finite-dimensional cokernel, and also that Q is the universal localization of $\Lambda = k\langle X\rangle$ with respect to $\Sigma' \cup \{g\}$, where $g : \Lambda \to \Lambda^{n+1}$ is the map induced by right multiplication by $(x_0, x_1, \dots, x_n)$. This was shown in [3, Theorem 7.5] by other means.

By using Proposition 6.4 and the exact sequence of universal localization, together with the computations in [2], we can now compute $K_1(Q)$.

Theorem 6.5. *Let Q be the ring obtained from the Leavitt algebra L by universally inverting all the homomorphisms between finitely generated projective left L-modules with cokernel in $\mathscr{S}$. Then*

$$K_1(Q) \cong k^{\times}/(k^{\times})^n \times D_{\varepsilon},$$

where $D_{\varepsilon} = K_0(\mathscr{Z}) \cong K_0(\mathscr{S}) \cong K_0(\mathscr{T}_0)$ is the reduced divisor group of Λ.

Proof. We use Schofield's exact sequence [27, Theorem 4.12] for the universal localization $Q = L_{\Sigma'}$ to get

$$K_1(L) \to K_1(Q) \to K_0(\mathscr{S}) \to K_0(L) \to K_0(Q).$$

The map $K_0(\mathscr{S}) \to K_0(L)$ is the zero map by Corollary 5.3(2),(3). (In fact the map $K_0(L) \to K_0(Q)$ is an isomorphism by [3, Theorem 7.6].) Moreover, $K_0(\mathscr{S}) \cong D_{\varepsilon}$ by Proposition 6.4, so we get

$$K_1(Q) = \overline{K_1(L)} \times D_{\varepsilon}, \tag{2}$$

where $\overline{K_1(L)}$ is the image of $K_1(L) \to K_1(Q)$. By Theorem 1.4, we have $\overline{K_1(L)} = k^{\times}/(k^{\times})^n$, so the result follows from (2). $\square$

Note that Q is a purely infinite simple von Neumann regular ring by [3, Theorem 5.4]. In particular, we have $K_1(Q) = (Q^{\times})^{\mathrm{ab}}$, the abelianized group of units of Q by [3, Theorem 2.3]. Therefore, we have a *Dieudonné determinant*

$$GL(Q) = \bigcup_{i \geqslant 1} GL_i(Q) \to (Q^{\times})^{\mathrm{ab}} = k^{\times}/(k^{\times})^n \times D_{\varepsilon}$$

by Theorem 6.5.

7. The exchange property

As an application of the techniques developed in the previous sections and the main result in [1], we establish in this section a new property of the free algebra.

Following Warfield [34], we say that a unital ring S is an *exchange ring* if the regular left S-module ${}_SS$ satisfies the (finite) exchange property. By a result obtained independently by Goodearl [17] and Nicholson [21], S is an exchange ring if and only if for every element a in S there is an idempotent e in S such that $e \in Sa$ and $1 - e \in S(1 - a)$.

It has been proved in [1] that every purely infinite simple ring is an exchange ring. Combining this result with the fact that Leavitt algebras of type $(1, n)$ are purely infinite simple rings [3, Theorem 4.2], we get:

Theorem 7.1 (cf. Ara [1, Theorem 2.1]). *Let L be a Leavitt algebra of type $(1, n)$ over a field k. Then L is an exchange ring.*

We interpret here this result in terms of the free algebra $R = k\langle Y \rangle$. We keep the notation of the previous sections, so that $Y = \{y_0, \ldots, y_n\}$ and $R = k\langle Y \rangle \subseteq L$.

It is useful to extend the notion of a lattice to the case of finitely generated projective left L-modules.

Let P be a finitely generated projective L-module. A *lattice* in P is an R-submodule M of P such that $L\cdot M = P$ and M is finitely generated projective as R-module. Observe that, if M is a lattice in P, then the natural map $L \otimes_R M \to P$ is an isomorphism. This follows from the facts that L_R is flat and that $L \otimes_R P \cong P$ (because the map $R \to L$ is a ring epimorphism).

Proposition 7.2. *Let P be a finitely generated projective left L-module.*

(a) *P admits a lattice.*
(b) *Let M be a free R-submodule of P. Then M is a lattice in P if and only if the natural map $L \otimes_R M \to P$ is an isomorphism.*
(c) *Let M be a lattice in P and let Q be a finitely generated submodule of P. Then there exists a lattice N of Q such that $N \subseteq M$.*

Proof. (a) Every finitely generated projective L-module is free, and so it is isomorphic to an induced module. Therefore there exists a finitely generated free left R-module A and an isomorphism $h : L \otimes_R A \to P$. Let $N = h(1 \otimes A) \subseteq P$. We want to prove that N is a lattice in P. Clearly $LN = P$. Consider the map $h' : A \to N$ given by $h'(a) = h(1 \otimes a)$. Let K denote the kernel of h'. We have an exact sequence

$$0 \longrightarrow L \otimes_R K \longrightarrow L \otimes_R A \xrightarrow{1 \otimes h'} L \otimes_R N \longrightarrow 0.$$

Since $LN = P$, we see that the map $g : L \otimes_R N \to P$ is an isomorphism. Since $1 \otimes h' = g^{-1} \circ h$, we conclude that $1 \otimes h'$ is an isomorphism too, and then we get from the above exact sequence that $L \otimes_R K = 0$. Since R is a fir, K is a free R-module, and so we conclude that $K = 0$. Thus h' is an isomorphism and so N is a free R-module.

(b) We have observed before that the map $L \otimes_R M \to P$ is an isomorphism for every lattice M in P. Conversely, let M be a free R-submodule of P such that the map $L \otimes_R M \to P$ is an isomorphism. Then $L \cdot M = P$, and also M is necessarily finitely generated. We conclude that M is a lattice.

(c) Let M be a lattice in P and let Q be a finitely generated L-submodule of P. Consider $N = M \cap Q$. Then N is an R-submodule of Q. We claim that $LN = Q$. Let $q \in Q$. Then we can write $q = \sum_{|I|=r} x_I m_I$, for some $r \geqslant 1$, where $m_I \in M$, because M is a lattice in P. For a fixed J such that $|J| = r$, we have

$$y_{J^*} q = m_J \in M,$$

but also $y_{J^*} q \in Q$, and so $m_J \in M \cap Q = N$. We have shown that $q \in LN$.

Observe that N is a submodule of the free R-module M, and so N is a free module. Since $LN = Q$, the usual argument gives that the natural map $L \otimes_R N \to Q$ is an isomorphism. It follows from (b) that N is a lattice in Q. $\square$

We say that a submodule N of an R-module M is *co-$\mathscr{M}_0$* in case $M/N \in \mathscr{M}_0$. Note that every co-$\mathscr{M}_0$ submodule of a finitely generated free module is finitely generated.

Theorem 7.3. *Let A and B be finitely generated left submodules of a free left R-module R^k such that $A+B$ is a co-$\mathscr{M}_0$ submodule of R^k. Then there exist submodules $M \subseteq A$ and $N \subseteq B$ such that $M \cap N = 0$ and $M + N$ is a co-$\mathscr{M}_0$ submodule of R^k.*

Proof. Write $C := LA \subseteq L^k$ and $D := LB \subseteq L^k$, and observe that A and B are lattices in C and D, respectively. Since $A+B$ is a co-$\mathscr{M}_0$ submodule of R^k, we get $C+D=L^k$. By Theorem 7.1, L is an exchange ring, and so every finitely generated projective L-module has the finite exchange property. Hence, we can apply [21, Proposition 2.9] to get a decomposition $L^k = P \oplus Q$, with $P \subseteq C$ and $Q \subseteq D$. By Proposition 7.2(c), there exists a lattice M in P such that $M \subseteq A$. Similarly, there exists a lattice N in Q such that $N \subseteq B$. Therefore $M \cap N = 0$, and $L \otimes_R (R^k/(M \oplus N)) = 0$. It follows from Proposition 3.3 that $M \oplus N$ is a co-$\mathscr{M}_0$ submodule of R^k, as desired. □

The author has been unable to find a more direct proof of Theorem 7.3 (i.e. without utilizing Theorem 7.1).

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