Relatives of the quotient of the complex projective plane by the complex cojugation

V.I.Arnold *

Abstract

It is proved, that the quotient space of the four-dimensional quaternionic projective space by the automorphism group of the quaternionic algebra becomes the 13-dimensional sphere while quotioned the the quaternionic conjugation.

This fact and its various generalisations are proved using the results of the theory of the hyperbolic partial differential equations, providing also the proof of the theorem (which was, it seems, known to L.S.Pontriagin already in the thirties) claiming that the quotient of the complex projective plane by the complex conjugation is the 4-sphere.

1 Introduction

In the paper [1] on the topology of the real algebraic curves, written in 1971, I have used the fact, that the quotient space of the complex projective plane by complex conjugation is the four-dimensional sphere. The attempts to find a reference for this fact in the literature were not successful at this time [1]. However V.A.Rokhlin told me that this result had been known to L.S.Pontriagin already in the thirties.

I do not know how had Pontriagin proved this theorem. My proof (published later in[2]) was based on the theory of the hyperbolic partial differential equations. Thinking on this proof recently once more from the view point of the systematical complexification and quaternionisation of the results of [3] (related to the attempts to find the quaternionic version of the Berry phase and on the quantum Hall effect, see [4, 5]) I have realised that this proof, based on the complex numbers, has quaternionic and real versions, leading to the following strange trinity of theorems with a common proof (of which the first theorem is trivial while the third one seems to be new):

Theorem R. *The real one-dimensional projective space is homeomorphic to the circle:*

$$\mathbf{R}\mathrm{P}^1 \approx S^1.$$

*Partially supported by RBRF 96-01-01104 and by the Institut universitaire de France

[1] among others I had asked N.Kuiper whether he knew such a reference

Theorem C. *The quotient space of the complex projective plane by the complex conjugation is homeomorphic to the four-dimensional sphere:*

$$\mathbf{C}\mathrm{P}^2/\mathrm{conj} \approx S^4.$$

Theorem H. *The quotient space of the four-dimensional quaternionic space by the automorphisms of the algebra of the quaternions becomes, after a factorisation by the quaternionic involution, a* 13*-dimensional sphere:*

$$(\mathbf{H}\mathrm{P}^4/\mathrm{Aut}\mathbf{H})/\mathrm{Conj} \approx S^{13}.$$

Remark. I have no doubt that the homeomorphisms constructed below become diffeomorphisms, if the quotients are equiped with their natural smooth structures. However the description of these structures is too long and so the diffeomorphisms are not proved below (see, however, [6]-[9]).

2 Geometry of hyperbolic cones

Definition. A real algebraic hypersurface in the real projective space is called *hyperbolic* with respect to some point P of its complement, if every real straight line containing P intersects the hypersurface only in real points (whose number is equal to the degree of the hypersurface provided that they are counted taking multiplicities into account).

The point P is then called a *time-like point*. The hyperbolic hypersurface of the projective space $\mathbf{R}\mathrm{P}^{N-1}$ are represented in the vector-space $\mathbf{R}^N$(whose vectors define the homogeneous coordinates in the projective space) by the algebraic hyperbolic cones (with vertices at the origin). The time-like points are represented by the *time-like vectors* of this space. The hyperbolicity condition takes in the vector-space the following form: every real straight line in $\mathbf{R}^N$, parallel to the time-like vector, intersects the hyperbolic cone at the real point, whose number is equal to the degree of the cone.

Example 1. Consider the space of the quadratic forms in $\mathbf{R}^n$. This real vector-space of dimension $N = n(n+1)/2$ contains the cone of the degenerate forms. This cone is hyperbolic with respect to the time-like vector, representing any positive-definite quadratic form (i.e. any Euclidean structure in $\mathbf{R}^n$). This property of the cone is essentially the theory of the eigenvalues of the quadratic forms in the Euclidean space. The singular points of the cone form an algebraic variety of codimension 3 in the ambient space of forms $\mathbf{R}^N$.

Example 2. Consider the space of the *Hermitian quadratic forms* in $\mathbf{C}^n$ (of the real quadratic forms in the corresponding space $\mathbf{R}^{2n}$, which are invariant under the action of the group S^1 of complex numbers of modulus one, acting as the multiplication of vectors of $\mathbf{C}^n$).

This real vector-space of dimension $N = n^2$ contains the cone of the degenerate forms. This cone is hyperbolic with respect to the time-like vector, representing any positive-definite Hermitian form (i.e. any Hermitian structure in $\mathbf{C}^n$). This property of the cone is essentially the theory of the eigenvalues of the Hermitian forms in the Hermitian space. The singular points of the cone form an algebraic variety of codimension 4 in the ambient space of forms, $\mathbf{R}^N$.

Example 3. Consider the space of the hyperhermitian quadratic forms in the quaternionic n-dimensional vector-space $\mathbf{H}^n$. These are real quadratic forms in the corresponding space $\mathbf{R}^{4n}$, which are invariant under the action of the group S^3 of unitary quaternions (acting on the vectors multiplying each component by the quaternion from some side, chosen once forever, say from the left).

This real vector-space of dimension $N = n(2n-1)$ contains the cone of the degenerate forms. This cone is hyperbolic with respect to the time-like vector, representing any positive-definite hyperhermitian form (i.e. any hyperhermitean structure in $\mathbf{H}^n$). This property of the cone is essentially the theory of the eigenvalues of the hyperhermitian forms. The singular points of the cone form an algebraic variety of codimension 6 in the ambient space $\mathbf{R}^N$ of forms.

The main fact of the geometry of the hyperbolic cones that we shall use is the following theorem (see [2]).

Consider the intersection of an hyperbolic cone of degree k in $\mathbf{R}^N$ with a sphere S^{N-1} centered at the origin. Fix a time-like point P (a pole) on the sphere and connect it with $-P$ by the meridians. Consider on each meridian the first intersection point with the cone (counting from P), the second one, ..., the k-th one.

Theorem. *The i-th points situated on all the meridians, form a semialgebraic variety V_i, homeomorphic to the sphere S^{N-2}.*

Indeed, the homeomorphism of V_i onto the equator is defined by the projection along the meridians.

The proofs of the results formulated in the introduction are deduced below from the general theorem above, applied to the variety V_1 of the nonnegative degenerate quadratic forms of a suitable number of variables.

3 The Gram matrix of a system of vectors.

Consider n vectors $v_1, \dots, v_n$ in an Euclidean space. The *Gram matrix* of this system of vectors is the matrix $a_{i,j} = (v_i, v_j)$ formed by their scalar products.

Lemma. *The Gram matrix of a system of vectors is the matrix of a nonnegative quadratic form, whose rank is equal to the dimension of the linear hull of the vectors.*

Each matrix of a nonnegative quadratic form of rank r is realizable as the Gram matrix of a system of n vectors in an r-dimensional Euclidean space.

The ordered system of n vectors is defined by its Gram matrix up to the orthogonal transformation of the ambient space $\mathbf{R}^r$.

The proof of this well-known fact is based on the representation of the orthogonalization of a given basis of an Euclidean space in terms of its Gram matrix. The quadratic form, corresponding to the given system of vectors, is provided by the formula $f(c) = (\sum c_i v_i, \sum c_i v_i)$. The Lemma implies the

Corollary. *The variety V_1 of the nonnegative degenerate quadratic forms in $\mathbf{R}^n$ (normed by the condition "the sum of the squares of the components of the matrix of the form in a fixed basis is equal to 1") is homeomorphic to the variety of the ordered sets of n vectors in the Euclidean space $\mathbf{R}^{n-1}$ (normed by the condition "the sum of the squares of the scalar products of the vectors is equal to 1"), considered up to the orthogonal transformations of the space $\mathbf{R}^{n-1}$.*

Applying to V_1 the Theorem of Section 2, we get the following

Theorem. *The variety of the ordered normed sets of n vectors in $\mathbf{R}^{n-1}$, considered up to the orthogonal transformations of this Euclidean space, is naturally identified with the sphere S^M, $M = n(n+1)/2 - 2$.*

The Theorems of Section 1 are deduced from this fact, n being $2, 3$ or 5 ($\mathbf{R}^{n-1}$ beeing, correspondingly, $\mathbf{R}, \mathbf{C}, \mathbf{H}$).

4 The proof of the theorem on the quotient of the quaternionic projective space.

Starting from five quaternions $q_0, \ldots, q_4 \in \mathbf{H}^5$, we construct a nonnegative degenerate quadratic form of five real variables, $f(c) = (\sum c_i q_i, \sum c_i q_i)$ — the scalar square of the linear combination of the five quaternions.

By the Theorem of Section 3, the variety of the normed (and hence nonzero) quintuples, quotioned by the action of the orthogonal group $O(4)$ on $\mathbf{H}$, is the sphere S^{13}. The quotient space may be constructed in three steps.

Lemma. *The quotient space $S^{4m+3}/O(4)$ of the sphere in $\mathbf{H}^{m+1}$ by the action of the group $O(4)$ on $\mathbf{H}$ can be obtained from the quaternionic projective space $\mathbf{H}\mathrm{P}^m$ taking first the quotient by the action of the automorphisms of the quaternions algebra and than the quotient of the resulting space by the action of the quaternionic conjugation involution on it:*

$$S^{4m+3}/O(4) \approx (\mathbf{H}\mathrm{P}^m/\mathrm{Aut}\mathbf{H})/\mathrm{Conj}.$$

Proof. Denote by $S^3 (= SU(2))$ the group of the unitary quaternions. The direct product $S^3 \times S^3$ acts orthogonally on the space $\mathbf{H}$ of the quaternions by the left and right multiplications, $q \mapsto uqv^{-1}$. We have constructed a homomorphism $S^3 \times S^3 \to SO(4)$ which is the standard universal two-fold covering $Spin(4) \to SO(4)$, whose kernel consists of two elements $(1,1)$ and $(-1,-1)$.

The quaternionic projective space $\mathbf{H}\mathrm{P}^m$ consists of the equivalence classes of the (nonzero) $m+1$-tuples of the quaternons $(uq_0, \dots, uq_m) \in \mathbf{H}^{m+1} \backslash 0$. The automorphisms of the algebra of quaternions are interior automorphisms (of the form $q \mapsto uqu^{-1}, u \in S^3$). Such an automorphism sends the equivalence class of a vector $q \in \mathbf{H}^{m+1} \setminus 0$ in an equivalence class (of the vector qu^{-1}), since the left and the right multiplications commute. Thus, $S^{4m+3}/SO(4) \approx \mathbf{H}\mathrm{P}^m/\mathrm{Aut}\mathbf{H}$.

The quaternionic conjugation Conj does *not* act on $\mathbf{H}P^m$, but does act on the space of the two-sided equivalence classes that we have constructed: the element $\mathrm{Conj}(uqv^{-1}) = \mathrm{Conj}(v^{-1})\mathrm{Conj}(q)\mathrm{Conj}(u)$ belongs to the class of $\mathrm{Conj}(q)$.

The orthogonal transformation $\mathrm{Conj} : \mathbf{H} \to \mathbf{H}$ changes the orientation of this real 4-space. Hence we find that

$$(\mathbf{H}\mathrm{P}^m/\mathrm{Aut}\mathbf{H})/\mathrm{Conj} \approx S^{4m+3}/O(4),$$

which proves the lemma.

Applying the Lemma to the case $m = 4$, we find from the Theorem of Section 3, that

$$S^{13} \approx S^{19}/O(4) \approx (\mathbf{H}\mathrm{P}^4/\mathrm{Aut}\mathbf{H})/\mathrm{Conj},$$

which is the Theorem **H** of the Introduction.

Theorems **R** and **C** are proved by the same reasoning, one should only replace the quaternions by the real or by the complex numbers and replace the projective four-space by the one-space or by the two-space accordingly.

The quotient space by the action of $O(4)$ will be replaced by the quotient space by the action of $O(1)$ and of $O(2)$ respectively. The real (complex) projective space that one gets at the first step should be quotient by the action of $O(1)/S^0 = 1$ in the real case and of $O(2)/S^1 = S^0$ in the complex case (the nontrivial element of the last group of two elements acts as the complex conjugation on $\mathbf{C}\mathrm{P}^2$).

5 The hyperhermitian geometry

Replacing the quadratic forms by the Hermitian or by the hyperhermitian ones in the Theorem of Section 3, one gets new generalisation of the theorems proved above.

Definition. An *Hermitian form* in a complex vector-space is a real quadratic form on it, which is invariant under the multiplication of the vector of this complex space by the complex numbers whose modulus is equal to one. The dimension of the real vector-space of the Hermitian forms in $\mathbf{C}^m$ is equal to m^2.

A positive-definite Hermitian form is called an *Hermitian structure.* The complex linear transformations, preserving the Hermitian structure, form the *unitary group* $U(m)$ of dimension m^2. A complex vector-space equiped with an Hermitian structure is called the*Hermitian space.*

Theorem. *The variety of the ordered normed n-tuples of vectors in the Hermitian space* $\mathbf{C}^{n-1}$*, considered up to the unitary transformations of this space, is naturally identified with the sphere* $S^M, M = n^2 - 2$.

Example. For $n = 2$ we get the Hopf fibration $S^2 = S^3/S^1$. For $n = 3$ we obtain the quotient space (which is no longer the base of a fibration)

$$S^7 \approx S^{11}/U(2) = \mathbf{C}\mathrm{P}^5/SU(2) = \mathbf{H}\mathrm{P}^2/S^1.$$

Definition. A *hyperhermitian form* in the quaternionic vector-space is a real quadratic form, which is invariant under the multiplication of all the quaternionic components of the vector from the left by any unitary quaternion.

Remark. It does *not* imply in general the invariance under the multiplication from the right.

The positive-definite hyperhermitian form are called *hyperhermitian structures,* and the quaternionic space equiped with a hyperhermitian structure is called the *hyperhermitian space.*

Example. The function $F(q) = q_1\bar{q}_1 + \ldots + q_m\bar{q}_m$ defines in $\mathbf{H}^m$ *the standard hyperhermitian structure.* Each hyperhermitian structure can be written in such a form in terms of the components with respect to a suitable basis. Fixing the basis, we get for the hyperhermitian forms the following description.

Theorem. *Every hyperhermitian form in* $\mathbf{H}^m$ *can be represented using a (uniquely defined) quaternionic matrix* $f_{i,j}$*, verifying the hyperhermiticy condition*
$f_{j,i} = \bar{f}_{i,j}$*, by the formula*

$$f(q) = \sum_{i,j=1}^{m} q_i F_{i,j} \bar{q}_j. \qquad (1)$$

Proof. 1^o. *This form is real.* Indeed, $\bar{f}(q) = \sum q_j \bar{f}_{i,j} \bar{q}_i = f(q)$.
2^o. *This form is invariants under the multiplication of* q *from the left by a unitary quaternion* u. Indeed,

$$f(uq) = \sum uq_i f_{i,j} \bar{q}_j \bar{u} = uf(q)\bar{u} = f(q),$$

since the real number $f(q)$ commutes with u.
3^o. Thus formula (1) defines a hyperhermitian form. We shall prove now, that *every hyperhermitian form can be represented by formula* (1).
To prove it equip with the usual Euclidean scalar products $(p, q) = \mathrm{Re}(p\bar{q})$ the spaces $\mathbf{H}$ and $\mathbf{H}^m = \oplus \mathbf{H}^1_i$. Our form can be written as

$$f(q) = \sum (A_{i,j}(q_i), q_j), \qquad A^*_{i,j} = A_{j,i} \ ,$$

where $A_{i,j} : \mathbf{H} \to \mathbf{H}$ is a real-linear operator, the sign $*$ being Euclidean conjugation. The invariancy condition for f takes the form

$$(A_{i,j}(ux), uy) = (A_{i,j}(x), y)$$

for any quaternions x and y. Hence $A_{i,j}$ should verify the relation $u^{-1}A_{i,j}u = A_{i,j}$ (since $u^* = u^{-1}$). Thus the operator $A_{i,j}$ should commute with the multiplication of quaternions by u from the left.

Lemma. *Any real-linear operator A in $\mathbf{H}$, which commutes with the multiplication of the quaternions (from the left) by the unitary quaternion u, is the operator of the multiplication by some quaternion from the right.*

Proof. The commutativity implies that $A(u) = A(u \cdot 1) = uA(1)$. Hence A acts on any quaternion q as the multiplication from the right, $A(q) = qa$, where $a = A(1)$.

It follows from the Lemma that we have thus proved that $A_{i,j}(q_i) = q_i f_{i,j}$, where $f_{i,j}$ is a quaternion. Since A is symmetric, the resulting quaternionic matrix $f_{i,j}$ is hyperhermitian, $f_{j,i} = \bar{f_{i,j}}$. In particular the diagonal elements are real. We thus get formula (1):

$$f(q) = \mathrm{Re} \sum q_i f_{i,j} \bar{q}_j = \sum q_i f_{i,j} \bar{q}_j.$$

4^o. *The huperhermitian matrix is unambiguosly defined by the form f.* Indeed, if $f = 0$, its restriction to $\mathbf{H}_i^1 \oplus \mathbf{H}_j^1$ also vanishes. Hence the operators $A_{i,j}$ (and thus also the quaternions $f_{i,j}$) vanish. The theorem is thus proved.

Corollary. *The dimension of the real vector-space of the hyperhermitian forms in $\mathbf{H}^m$ equals $m + 4m(m-1)/2 = m(2m-1)$.*

Definition. A *hyperunitary transformation* of a hyperhermitian space is a quaternionic-linear transformation, preserving the hyperhermitian structure. (A quaternionic-linear operator is a linear operator A, for which $A(qx) = qA(x)$ for any quaternion q).

The dimension of the hyperunitary group of the hyperhermitian space $\mathbf{H}^m$ equals $m(2m+1)$. This group is denoted (unfortunatly) $Sp(m)$.

The hyperunitary transformations are defined by the matricies, whose elements should be written *from the right* of the components of the vectors, like

$$A(q_1, q_2) = (q_1 a_{1,1} + q_2 a_{2,1}, q_1 a_{1,2} + q_2 a_{2,2}).$$

The condition that transformation is hyperunitary (with respect to the standard hyperhermitian structure $F(q) = \sum q_i \bar{q}_i$ in $\mathbf{H}$) takes for the matrix the form

$$\sum_j a_{i,j} \bar{a_{k,j}} = \delta_{i,k}.$$

Hence, the hyperhermitian scalar product in $\mathbf{H}^m$ (with quaternionic values) defined by the formula

$$\langle q, r\rangle = \sum q_i \bar{r}_i$$

is invariant under the hyperhermitians transformations:

$$\langle Aq, Ar\rangle = \langle q, r\rangle.$$

Remark. This follows also, of course, from the quaternionic polarization formula, which is however too long to be reproduced here.

Theorem. *The variety of the ordered n-tuples of vectors in the hyperhermitian space $\mathbf{H}^{n-1}$, considered up to hyperunitary transformations of this space, is the sphere $S^M, M = n(2n-1)-2$.*

Example. For $n = 2$ we get the quaternionic Hopf bundle, $S^4 = S^7/S^3$. For $n = 3$ we get $S^{13} = S^{23}/Sp(2)$. This suggest some relation to the Caley projective plane.

The proof of the theorem, as well as that of its Hermitian version, is based on the same reduction to the theory of the hyperbolic cones which was used in Section 3 to prove the Euclidean version of this theorem.

Definition. The *Gram matrix* of a system of n vectors q_i in the standard hyperhermitian (hermitian) space $\mathbf{H}^m$ ($\mathbf{C}^m$) is the matrix of the hyperhermitian (Hermitian) form

$$f(c) = F(c_1 q_1 + \ldots + c_n q_n), \qquad c \in \mathbf{H}^m(\mathbf{C}^m).$$

This form is hyperhermitian (Hermitian) since the argument of F is multiplied by u from the left when all the q_i are multiplied by u from the left. By the definition of the standard hyperhermitian (Hermitian) structure F, we get

$$f(s) = \sum c_i \langle q_i, q_j\rangle \bar{c}_j, \quad \text{whence} \quad f_{i,j} = \langle q_i, q_j\rangle.$$

Thus the form f is invariant under the hyperunitary (unitary) transformations of the space $\mathbf{H}^m$ ($\mathbf{C}^m$).

The hyperhermitian (Hermitian) matrix $f_{i,j}$ defines the orthogonalization of the n-tuple q_i. Hence this matrix determines this n-tuple up to a hyperunitary (unitary) transformations of the space $\mathbf{H}^m$ ($\mathbf{C}^m$).

6 The equivariant Neuman-Wigner theorem on the nonintersections of electronic levels.

As we have seen, the eigenvalues theories for quadratic, Hermitian and hyperhermitian forms are theories of the hyperbolic cones in the space of forms which are invariant under some representation (of group $U(1)$ in the Hermitian case and of group $SU(2)$ in the the hyperhermitian case), namely under a representation which is a multiple of an irreducible one. The corresponding generalized von Neuman-Wigner theorems (see [6], [3]) claim in our present terminology that the codimensions of the varieties of the singular points on the cones of the degenerate points are equal, in the real, complex and quaternionic cases, to 2, 3 and 5 (these numbers are the codimensions of the one-dimensional spaces of the diagonal forms of two variables in the spaces of quadratic, Hermitian and hyperhermitian forms of two variables).

One can replace here the standard irreducible representations of groups $U(1)$ and $SU(2)$ by any irreducible representation of any compact Lie group. The results, describe below, show, that such a generalization *provides no new hyperbolic cones:* all the spaces of quadratic forms, invariant under the multiples of the irreducible real representations of compact Lie groups, are naturally isomorphic either to the space of all real quadratic forms, or to the space of the Hermitian forms, or to the space of the hyperermitian forms.

Fix a real irreducible representations of a compact Lie group G by orthogonal transformations of the Euclidean space $\mathbf{R}^n$.

Definition. A *symmetry* of the representation is a real linear operator, commuting with all the operators of the representation: $Ag = gA$ (in other terms it is an operator interwinning the representation with itself).

The symmetries of a given representation form an (associative) algebra (a subalgebra of the algebra of all the operators from $\mathbf{R}^n$ to itself). We shall use the following real version of the Shur's lemma:

Theorem. *The symmetry algebra of any real irreducible representation of a compact Lie group is isomorphic (as an algebra of linear transformations of the Euclidean space) to one of the following three algebras:*

1) *algebra $\mathbf{R}$ of real numbers, acting in $\mathbf{R}^n$ as the scalar matricies;*

2) *algebra $\mathbf{C}$ of complex numbers, acting in $\mathbf{C}^m$ $(n = 2m)$ as the scalar matricies;*

3) *algebra $\mathbf{H}$ of quaternions, acting in $\mathbf{H}^k$ $(n = 4k)$ by the multiplication of the component of a vector by a quaternion from the right.*

Remark. This classical theorem provides the royal way to quaternions, which appear here not as an uncomplete axiomatic generalization of complex numbers, but as the solution of a natural problem in real Euclidean geometry. All the "axioms" of quaternions are simply the necessary *properties* of the solutions of this natural problem, and to discover them one is not forced to use the spirits as employed by Hamilton. Simultaneously one gets the classification of the associative algebras with division (not only as of abstract objects but also as operators algebras). The proof is so simple that I shall give it below.

Proof. Complexify $\mathbf{R}^n$ to get $\mathbf{C}^n \approx \mathbf{R}^{2n}$. The complexified representation operators g and symmetries A act on $\mathbf{C}^n$ as complex linear operators. The representation of G

in $\mathbf{R}^{2n}$ that we obtain is reducible: it is the direct sum of two copies of the original representation.

Lemma 1. *The complexified symmetry operator A either is the multiplication by a real number, or has two complex-conjugate eigenvalues $\lambda = \alpha \pm \omega$ of multiplicity $m = n/2$ each (the dimension of the original real representation space being even).*

Proof. Otherwize A would have a nontrivial complex eigenspace whose complex dimension either smaller or greater than m. The first is impossible, since this invariant space of the representation would have a nontrivial intersection with $\mathbf{R}^n$, in contradiction with the irreducibility. In the second case the orthogonal complement to the eigenspace (which is also invariant under the representation) would have a nontrivial intersection with $\mathbf{R}^n$, in contradiction with the irreducibility.

Denote $I = (A - \alpha\mathbf{1})/\omega$. One obviously get the following

Lemma 2. *Symmetry operator I is a complex structure in $\mathbf{R}^n$, i.e. $I^2 = -\mathbf{1}$.*

Lemma 3. *Each complex structure, which is a symmetry of an irreducible orthogonal representation in $\mathbf{R}^n$, preserves the Euclidean structure.*

Indeed, the nonnegative form (Ix, Ix) is invariant, since $(Igx, Igx) = (gIx, gIx) = (Ix, Ix)$, the form (y, y) being g-invariant. Hence one gets $(Ix, Ix) = \lambda(x, x)$ (othervize the eigenspace of this form would be a nontrivial G-invariant subspace in $\mathbf{R}^n$). Therefore we find, that $(x, x) = (I^2x, I^2x) = \lambda^2(x, x), \lambda^2 = 1$. Since the form (Ix, Ix) is nonnegative, $\lambda = 1$. Thus every symmetry which is a complex structure is orthogonal.

If the real linear combinations of $\mathbf{1}$ and I exhaust the symmetry algebra, we get the case 2 of the theorem. Suppose there is one more symmetry in the algebra. Replacing it by its linear combination with the identity $\mathbf{1}$, we construct a complex structure B. The symmetry operator $IB + BI$ is symmetric. The irreducibility of the original representation implies that any symmetric symmetry operator is a scalar: $IB + BI = 2\varepsilon \in \mathbf{R}$. Denote by J the operator $J = (B + \varepsilon I)/\sqrt{1 - \varepsilon^2}$. Here $\varepsilon^2 \leq 1$, since the orthogonal operator $C = IB$ is different from $\pm\mathbf{1}$, B being independent of I.

Lemma 4. *Operator J is a complex structure, anticommuting with I.*

Indeed,

$$J^2 = (B + \varepsilon I)^2/(1 - \varepsilon^2) = -\mathbf{1}, \;\; IJ + JI = (IB + BI + 2\varepsilon I^2)/\sqrt{1 - \varepsilon^2} = 0.$$

Lemma 5. *Operator $K = IJ$ is also a complex structure; all the three complex structures (I,J,K) anticommute.*

Indeed,

$$K^2 = (IJ)(IJ) = IJ(-JI) = I^2 = -\mathbf{1},$$

$$IK = I^2J, KI = -JI^2, JK = -J^2I, KJ = IJ^2.$$

Therefore the vectors (a, Ia, Ja, Ka), where a is any unite vector, are mutually orthogonal and generate $\mathbf{R}^4$ on which the operators (I, J, K) are acting as the quaternions (i, j, k). In the orthogonal complement to $\mathbf{R}^4$ we choose one more unite vector and construct one more $\mathbf{R}^4$. Continuing this way, we identify at the end the original Euclidean space $\mathbf{R}^n$ with $\mathbf{H}^k, n = 4k$.

Lemma 6. *Any symmetry of the irreducible representation is a real linear combination of the four operators* $(\mathbf{1}, I, J, K)$ *that we have constructed.*

If there were one more symmetry, we would construct, as above, its linear combination L with $\mathbf{1}$ and I which would be, as J, a complex structure, anticommuting with $I : IL + LJ = 0, L^* = -L$.

Operator $JL = LJ$ is symmetric. The irreducibility implies that $JL = LJ = 2\xi\mathbf{1}$ is a real number. Operator $X = K(L + \xi J)$ is symmetric

$$X^* = LK + \xi JK = LIJ + \xi I = -ILJ + \xi I = -I(2\xi - JL) + \xi I = -\xi I + KL.$$

The irreducibility implies that $X = \mu \in \mathbf{R}$, whence $L = -\xi J - \mu K$. This proves the Lemma and hence the Theorem.

Now suppose that the compact Lie group is represented in $\mathbf{R}^{Nn}$ and that this representation is the direct sum of N copies of the irreducible representation in $\mathbf{R}^n$ whose symmetry algebra ($\mathbf{R}, \mathbf{C}$ or $\mathbf{H}$) has the real dimension $d = 1, 2$ or 4.

Theorem. *The quadratic forms in* $\mathbf{R}^{Nn}$ *which are invariant under this representation form a real vector-space of dimension* $N + dN(N-1)/2$*, which is isomorphic to the space of the quadratic (Hermitian, hyperhermitian) forms in* $\mathbf{R}^N$ *(in* $\mathbf{C}^N$*, in* $\mathbf{H}^N$*).*

The determinant of the invariant quadratic form, considered as a polynomial of the coefficients of the form in $\mathbf{R}^{Nn}$*, is equal to the* $\frac{n}{d}$*-th power of the determinant of the corresponding quadratic form in* $\mathbf{R}^N$ *(in* $\mathbf{R}^{2N}$*, in* $\mathbf{R}^{4N}$*).*

Corollary. *The cone of the degenerate invariant quadratic forms on* $\mathbf{R}^{Nn}$ *is sent by the isomorphism mentioned in the theorem onto the cone of the degenerate quadratic forms in* $\mathbf{R}^N$ *(of the degenerate Hermitian forms in* $\mathbf{R}^{2N}$*, of the degenerate hyperhermitian forms in* $\mathbf{R}^{4N}$*).*

For instance, the codimensions of the varieties of the singularities of these cones (and hence the codimensions of the varieties of the forms with eigenvalues of nonminimal multiplicity) do not depend on the irreducible representation nor on the group — they only depend on the symmetry algebra (being equal to $d+1$ on the cone of the degenerate forms and $d+2$ in the space of forms).

Therefore the phenomenon of the repulsion of the eigenvalues, discribed by the von Neuman-Wigner theorem, has only three variants: the real one, the Hermitian one and the hyperhermitian one. *To collide two eigenvalues one needs* $2, 3$ *or* 5 *independent* *parameters, correspondingly, whatever the compact group and its representation, which is the direct sum of some copies of an irreducible real representation one considers.*

Proof. Decompose the representation space into orthogonal irreducible invariant subspaces $\mathbf{R}^n_i$, $i = 1, \dots, N$. The quadratic form can be written, as in Section 5, as the sum of N^2 blocks of the sise $n \times n$:

$$f(x) = \sum (A_{i,j} x_i, x_j), \qquad A^*_{i,j} = A_{j,i}.$$

The invariancy condition under the g of the group takes the form $g^{-1} A_{i,j} g = A_{i,j}$, g being orthogonal and $(A_{i,j}x, y)$ being equal to $(x, A_{j,i}y)$. Hence $A_{i,j}$ commutes with g. From the (real) Shur lemma we know, that operator $A_{i,j} : \mathbf{R}^n \mapsto \mathbf{R}^n$ acts either as the multiplication by a real scalar, or as the multiplication by a complex scalar in $\mathbf{C}^{n/2}$, or as the multiplication (from the right) by a quaternion in $\mathbf{H}^{n/4}$.

Present the space $\mathbf{R}^n(\mathbf{C}^{n/2}, \mathbf{H}^{n/4})$ as the orthogonal sum of the spaces $\mathbf{R}_i(\mathbf{C}_i, \mathbf{H}_i)$. We thus reduce the matrix of our quadratic form to the block-diagonal form from $n(n/2, n/4)$ identical blocks, each block being the matrix of a quadratic (Hermitian, hyperhermitian) form in the space $\mathbf{R}^N(\mathbf{C}^N, \mathbf{H}^N)$. The Theorem follows.

Remark. The characteristic polynomials of our invariant form in $\mathbf{R}^{Nn}$ are the $\frac{n}{d}$-th power of the polynomials which, generically, have no multiple roots. The forms for which the multiple root occur, form an algebraic submanifold of real codimension $d+1 = 2, 3, 5$ in the space of the quadratic (Hermitian, hyperhermitian) forms.

The discriminants of these polynomials with real roots do not change the sign. Probably these polynomials of the coefficients of the forms are sums of squares of several polynomials.

Example. For $N = 2$ the number of squares is 2 in the Euclidean case, 3 – in the Hermitian case and 5 in the hyperhermitian case.

Remark. A simpler but also important example of an almost everywhere positive polynomial is provided by the attempt to complexify and to quaternionnise the determinants of real operators.

A complex linear operator $A : \mathbf{C}^n \to \mathbf{C}^n$ defines a real linear operator ${}^{\mathbf{R}}A : \mathbf{R}^{2n} \to \mathbf{R}^{2n}$, for which $\det({}^{\mathbf{R}}A) = |\det A|^2$. Therefore, the determinant of the real operator ${}^{\mathbf{R}}A$ is the sum of squares of two polynomials of the elements of its matrix (as it should be, the variety of the degenerate complex operators being of real codimension 2).

The variety of the quaternionic-degenerate operators $A : \mathbf{H}^n \to \mathbf{H}^n$ has the real codimension 4 in the space of the quaternionic operators. Hence it is natural to consider the nonnegative polynomial $\det \left({}^{\mathbf{R}}A : \mathbf{R}^{4n} \to \mathbf{R}^{4n}\right)$ of the elements of the quaternions forming the matrix of A and to try to represent it as the sum of (at least four) squares of polynomials.

A quaternion can be represented by a complex 2×2-matrix $\begin{pmatrix} z & -\bar{w} \\ w & \bar{z} \end{pmatrix}$. This operation transforms a quaternionic operator into a complex operator ${}^{\mathbf{C}}A : \mathbf{C}^{2n} \to \mathbf{C}^{2n}$. Since ${}^{\mathbf{R}}A = {}^{\mathbf{RC}}A$, we get the formula

$$\det({}^{\mathbf{R}}A) = |H|^2, \quad \text{where} \quad H = \det({}^{\mathbf{C}}A).$$

The function H, which is defined as a complex-valued polinomial, is in fact a (nonnegative) real polynomial of the elements of the quaternions forming the matrix of A. Therefore $\det({}^{\mathbf{R}}A) = H^2$ is not a sum of two squares but just one square of a nonnegative polynomial H, whose zeroes still form an algebraic variety of real codimension 4 in the space of the quaternionic matricies.

In the case $n = 1$ this polynomial H is the sum of the squares of the four components of the quaternion. One might conjecture that it is still represantable as the sum of at least four squares in the general case.

For $n = 2$ the polynomial H has a simple expression in terms of the minors of the 2×4 complex matrix (z, w):

$$H = |M_{12}|^2 + |M_{14}|^2 + |M_{34}|^2 - 2\mathrm{Re} M_{13}\bar{M}_{24},$$

where $M_{12}M_{34} + M_{14}M_{23} = M_{13}M_{24}$.

This formula is strangely similar (differing only by the presence of the complex conjugation) to the formula of the Klein representation of the space of the lines in $\mathbf{C}P^3$, which is quoted by Atiyah [10] in the description of the Penrose twistors space.

References

[1] V.I.Arnold. *On disposition of ovals of real plane algebraic curves, involutions of four-dimensional manifolds and arithmetics of integer quadratic forms.* Funct. Anal. and Its Appl., 1971, V. **5**, N 3, P. 1-9.

[2] V.I.Arnold. *Ramified covering* $\mathrm{CP}^2 \to S^4$, *hyperbolicity and projective topology.* Siberian Math. Journal 1988, V. **29**, N 5, P.36-47.

[3] V.I.Arnold. *Modes and quasimodes.* Funct. Anal. and Its Appl., 1972, V. **6**, N 2, P. 12-20.

[4] V.Arnold. *Remarks on eigenvalues and eigenvectors of Hermitian matricies, Berry phase, adiabatic connections and quantum Hall effect.* Selecta Math., 1995, V. **1**, N 1, P. 1-19.

[5] V.Arnold. *Symplectization, complexification and mathematical trinities.* CEREMADE(UMR 7534), Université Paris-Dauphine, N^o 9815, 04/03/98, P. 1-20.

[6] V.Arnold. *Topological content of the Maxwell theorem on multipole representation of spherical functions.* Topological methods in nonlinear Analysis. Journal of the Juliusz Schauder Center, Vol. **7,** 1996, p. 205-217.

[7] W.Massey. *The quotient space of the complex projective plane under the conjugation is a* 4*-sphere.* Geometria Dedicata, V.**2**,(1973), p.371-374.

[8] N.H.Kuiper. *The Quotient Space of* $\mathbf{C}\mathrm{P}^2$ *by Complex Conjugation is the* 4*-sphere.* Math. Ann., v. **208** (1974), p. 175-177.

[9] A.Marin. $\mathbf{C}\mathrm{P}^2/\sigma$ *or Kuiper and Massey in the land of conics.* In: A.Marin, L.Guillou, A la recherche de la topologie perdue. Progress in Math., V.**62**, Birkäuser, Boston, 1986, p. 141-152.

[10] M.F.Atiyah. *Geometry of Yang-Mills Fields.* Accademia Nazionale dei Lincei, Scuola Normale Superiore, Lezioni Fermiane, Pisa, 1979 (section III — 1).

Steklov Mathematical Institute and Université Paris-Dauphine