DISTRIBUTION OF OVALS OF THE REAL PLANE OF ALGEBRAIC CURVES, OF INVOLUTIONS OF FOUR-DIMENSIONAL SMOOTH MANIFOLDS, AND THE ARITHMETIC OF INTEGER-VALUED QUADRATIC FORMS

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There is an interesting connection between the distribution of the branches of a real algebraic curve on the projective plane, on one hand, and, on the other hand, the topology of certain complex algebraic surfaces. In the present paper this connection is used in order to derive, from simple considerations of four-dimensional topology and the arithmetic of integer-valued quadratic forms, information as to the distribution of ovals of the real planes of algebraic curves.

§1. Formulation of the Result

We consider a nonsingular real algebraic curve of degree n on the real projective plane RP^2. Such a curve is a compact smooth one-dimensional manifold without a boundary. Its connected components are diffeomorphic to circles and are called ovals.

A. Harnack [1] proved that the number of ovals does not exceed g + 1, where $g = (n-1)(n-2)/2$ is the genus of the curve. Curves with g + 1 ovals do exist, and are called M-curves.

The question as to how ovals can be distributed has been considered by many authors (in particular, D. Hilbert [2], K. Rohn [3], I. G. Petrovsky [4], and D. A. Gudkov [5]), but has been answered only for curves of degree 6 and less (see the survey in [5]).

To formulate our result we need to use the partition of ovals, introduced by I. G. Petrovsky, into positive and negative ovals. We assume that the degree of the curve is even: n = 2k (we retain this notation throughout the paper). Then, the ovals lie two-sidedly in RP^2, and each of them has an interior part (diffeomorphic to a circle) and an exterior (diffeomorphic to a Möbius sheet). We shall call an oval positive (or even) if it lies within an even number of others, and negative (or odd) if it lies within an odd number of other ovals. For example, the circle $x^2 + y^2 = 1$ is an even oval.

THEOREM 1. Let p be the number of positive, and m the number of negative, ovals of an M-curve of degree 2k. Then, the following congruence holds:

$$p - m \equiv k^2 \bmod 4. \tag{1}$$

We note that congruence (1) does not exhaust all the constraints on the distribution of the ovals. For example, I. G. Petrovsky [4] proved the inequality

$$|2(p-m)-1| \leqslant 3k^2 - 3k + 1 \tag{2}$$

for any curve of degree 2k (with the necessarily maximal number of ovals), while D. A. Gudkov [5] proved, for M-curves of degree 6, a congruence of the form of (1) but modulo 8. For other constraints, see §9, paragraphs 4, 5, and 6.

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The proof given below of congruence (1) is based on a consideration of the actions of involutions of a certain four-dimensional manifold Y on its two-dimensional homology.

§2. Structure of Four-Dimensional Manifold Y

Manifold Y is a two-sheeted covering of the complex projective plane, ramifying along the set of complex points of the curve in question.

Let $f(x, y) = 0$ be the affine equation of the curve in question. Here, x and y are real variables while f is a nonhomogeneous polynomial of degree 2k with real coefficients. The corresponding homogeneous equation has the form $F(x_0, x_1, x_2) = 0$, where $f(x, y) = F(1, x, y)$.

Here, F is a homogeneous polynomial of degree 2k with real coefficients. Since the degree of F is even, the sign of F is the same at all points (x_0, x_1, x_2) corresponding to one point $(x_0 : x_1 : x_2)$ of the projective plane RP^2 which does not lie on our curve. We can so choose F that, at the points corresponding to the nonorientable component of the complement to the curve in RP^2, the values of F will be negative. We fix such a polynomial F.

We now consider the equation

$$z^2 = F(x_0, x_1, x_2), \tag{3}$$

where x_0, x_1, and x_2 are complex variables not all simultaneously equal to zero. This equation gives a compact complex algebraic surface Y embedded in three-dimensional complex space E of the one-dimensional vector fibration over the complex projective plane

$$P' : E \to (CP^2 = \{(x_0 : x_1 : x_2)\}),$$

whose sections are homogeneous functions of degree k of the variables x_0, x_1, and x_2. From the real point of view, surface Y is a four-dimensional compact smooth orientable connected manifold without boundary.

In the affine map on E corresponding to the affine map (x, y) on CP^2, surface Y is given by the equation $z^2 = f(x, y)$.

The restriction of P to mapping P' on manifold Y gives a two-sheeted ramified covering of the complex projective plane. The manifold of the ramifications is the intersection of manifold Y with the zero fiber of fibration P'. We denote this manifold of ramification by A. It is clear from Eq. (3) that PA is precisely the set of complex points of our algebraic curve F = 0. Thus, A is a smooth orientable connected compact submanifold with boundary of the four-dimensional manifold Y. The real dimension (and codimension in Y) of manifold A equals 2.

§3. Involution τ and Form Φ_τ

Multiplication of z by −1 gives a smooth involution τ of manifold Y. The set of fixed points is again our complex curve A.

On all our complex manifolds we choose an orientation in the natural way (by means of the form $\mathrm{Re}z_1 \wedge \mathrm{Im}z_1 \wedge \ldots \wedge \mathrm{Re}z_s \wedge \mathrm{Im}z_s$) such that the indices of intersection of the complex manifolds will be nonnegative. We remark that involution τ retains the orientation of manifold Y.

We denote by $H_2(Y) = H_2(Y, \underline{Z})$ mod Tors the group of two-dimensional integer-valued homologies of space reduced modulo the torsion. The index of intersection (,) of two-dimensional cycles gives, on $H_2(Y)$, a bilinear integer-valued nonsingular (Poincaré duality) form. The involution τ: Y → Y induces the isomorphism $\tau_* : H_2(Y) \to H_2(Y)$. On $H_2(Y)$ we define the bilinear form Φ_τ by the relationship

$$\Phi_\tau(a, b) = (\tau_* a, b), \quad a, b \in H_2(Y). \tag{4}$$

LEMMA 1. Form Φ_τ is symmetric and nonsingular (det $\Phi_\tau = \pm 1$).

Proof. Since involution τ retains orientation on Y, $(\tau_* a, b) = \tau_*(\tau_* a, b) = (a, \tau_* b) = (\tau_* b, a)$. The second assertion follows from the Poincaré duality.

§4. Arithmetic Lemma

Let $\Phi : Z^r \times Z^r \to Z$ be an integer-valued symmetric nonsingular (det $\Phi = \pm 1$) bilinear form. The following lemma is well known (see [6, 7, 8]).

LEMMA 2. There exists an element $w \in Z^r$ such that, for all $x \in Z^r$

$$\Phi(x, x) \equiv \Phi(w, x) \bmod 2. \tag{5}$$

Element w is defined univocally mod $2Z^r$. The magnitude of $\Phi(w, w)$ mod 8 does not depend on the choice of the representative w, and coincides, modulo 8, with the signature of the form.

In what follows we shall need only the fact that a consequence of (5) is

$$\Phi(w', w') \equiv \Phi(w, w) \bmod 8 \quad \text{for} \quad w' = w + 2x. \tag{6}$$

Proof of relationship (6). We have

$$\Phi(w', w') = \Phi(w, w) + 4\Phi(w, x) + 4\Phi(x, x),$$

where, according to (5), the numbers $\Phi(w, x)$ and $\Phi(x, x)$ have the same parity.

Element w (defined modulo 2) will be called the fundamental class of form Φ.

§5. Computing the Fundamental Class of Form Φ_τ

Let $\tau: Y^{4s} \to Y^{4s}$ be an orientation-retaining involution of the oriented compact smooth 4s-dimensional manifold without boundary with the 2s-dimensional oriented manifold of fixed points A^{2s}. We define form Φ_τ on $H_{2s}(Y^{4s})$ by Eq. (4).

LEMMA 3. As the fundamental class of form Φ_τ one can choose the homology class $[A^{2s}]$ represented by cycle A^{2s}.

Proof. To each point of intersection Q, not lying on A^{2s}, of the cycles a and τa there corresponds a second point τQ. At the points of intersection of a and A^{2s} one can achieve pairwise transversality of A^{2s}, a, and τa by small displacements of a. Therefore, the indices of intersection of a with τa and with A^{2s} coincide modulo 2.

Now, let $A \subset Y$ be the manifold of §3.

LEMMA 4. The index of self-intersection of cycle A in Y equals half the square of the degree of algebraic curve PA:

$$\Phi_\tau([A], [A]) = (A, A) = 2k^2.$$

Proof. Consider projection PA of curve A on CP^2. Let A' and A" be curves, homologous to PA, in CP^2 intersecting transversally, and not on PA. Then, the index of intersection of A' and A" in CP^2 equals the square of the degree of curve PA, i.e., $4k^2$. Consequently, the index of intersection of the cycles $P^{-1}A'$ and $P^{-1}A''$ in Y equals $8k^2$. But in Y we have $P^{-1}A' \sim 2A \sim P^{-1}A''$, so that $(A, A) = 2k^2$.

§6. Real Part Π of Manifold Y

We now investigate the set of real points of manifold Y, i.e., the points to which there correspond real values of coordinates z and $x_0 : x_1 : x_2$.

Consider, on real projective plane RP^2, the set C of points to which there correspond values of $F \geq 0$. In RP^2 set C is a smooth two-dimensional submanifold whose boundaries are the ovals of interest to us. In general, manifold C is not connected; in view of the conventions made in §2 about the sign of F, it is orientable. We note that the Euler characteristic of manifold C equals the difference $p - m$ between the numbers of positive and of negative ovals.

LEMMA 5. Set Π of real points of manifold Y is a smooth compact two-dimensional orientable manifold without boundary, smoothly embedded in Y. Manifold Π is diffeomorphic to the doubled manifold with boundary C.

The proof follows from Eq. (3) and the triviality of fibration P over C; this latter is guaranteed by the choice of the sign of F.

LEMMA 6. The index of self-intersection of the real orientable compact smooth analytic manifold M of dimensionality 2s in its complexification equals $(-1)^s \chi$, where χ is the Euler characteristic of manifold M.

Proof. On M we construct the tangent vector field with simple singular points, and we multiply this by i. The sign of $(-1)^S$ corresponds to the difference between the two orientations of C^{2S}: we can initially take all the coordinate axes real, and then imaginary, and we can select the orientation as specified in §3.

LEMMA 7. The Euler characteristic of surface Π is expressed in terms of the numbers of positive and negative ovals by the formula

$$\chi = 2(p - m).$$

For the proof we note that p−m is the Euler characteristic of the manifold with boundary C, and we then use Lemma 5.

Joining Lemmas 6 and 7, we obtain the following result.

LEMMA 8. The self-intersection index of surface Π in manifold Y equals (Π, Π) = 2(m−p).

§7. Homologies between the Cycles of A and Π

LEMMA 9. The homology classes represented by surfaces A and Π in Y coincide modulo 2; more precisely,

$$[A] = [\Pi] \in H_2(Y) \otimes \mathbf{Z}_2.$$

Proof. We denote by ∞ the "infinitely distant" complex line $x_0 = 0$ in CP^2 with coordinates $x_0 : x_1 : x_2$ from Eq. (3). We shall assume that this line intersects curve PA transversally in n = 2k points. We can consider line ∞ as a cycle (with coefficients from **Z** or from $\mathbf{Z}_2$).

We denote by [∞] the homology class of cycle ∞ in $H_2(CP^2)$. Each cycle c of $H_2(CP^2)$ is homologous to (c, ∞) [∞]. For example, PA ~ 2k[∞].

Further, we denote by ∞_Y the cycle $P^{-1}\infty$ in Y, and by $[\infty_Y]$ its homology class in $H_2(Y)$. Lemma 9 follows from the two relationships

$$[A] = k[\infty_Y] \in H_2(Y), \tag{7}$$

$$[\Pi] = k[\infty_Y] \in H_2(Y, \mathbf{Z}_2). \tag{8}$$

To prove relationship (7) we note that the integer-valued cycle PA−2k∞ is the boundary of some integer-valued three-dimensional chain K_3 in CP^2. We set $K_3' = P^{-1}K_3$. Then, $\partial K_3' = 2A - 2k\infty_Y$, i.e., $[A] - k[\infty_Y]$ is a second-order element in $H_2(Y, Z)$, which also proves Eq. (7) (we recall that $H_2(Y)$ is the torsion-factored homology group).

For the proof of relationship (8) we note that the ovals divide M-curve PA into two parts, the complex conjugate taking one part into the other (this is established in the proof of Harnack's Theorem). Let B be one of these parts. B is a connected compact oriented two-dimensional manifold with boundary. The boundaries of B are just the ovals of curve F = 0 on RP^2.

Consider the surface B + C, where C is the submanifold in RP^2 whose boundaries are the ovals of curve F = 0 (manifold C was defined in §6). The nonsmooth surface B + C is a combinatory, generally nonorientable, connected compact two-dimensional submanifold without boundary in CP^2. Therefore, surface B + C defines, in CP^2, cycles with coefficients in $\mathbf{Z}_2$.

We now show that the index of intersection of cycles B + C and ∞ in CP^2 equals k modulo 2.

Indeed, if the ovals do not intersect the infinitely distant line, then exactly half of the 2k points of intersection of PA and ∞ fall in B (the complex conjugate interchanges the interiors of B and PA−B). If the ovals do intersect the infinitely distant line, then, by continuing this line up to B + C, one can achieve its intersection solely within B, transversally and at precisely k points.

For the proof we choose the real affine coordinates (x, y) so that line ∞ has the equation x = 0, and so that its infinitely distant point lies outside C. The line x = iε, where ε > 0 is sufficiently small, is homologous to the line x = 0, does not intersect C, and transversally intersects B in precisely k points. Indeed, we orient the tangent field ξ of each oval. Then, upon movement in the direction of along an oval in the (x, y) plane, we cross the line x = 0 from left to right as many times as from right to left. To intersections of one type correspond the points of intersection of PA with line x = iε lying in B and, to the other, on PA−B (this follows from the fact that $i\xi$ is a normal to an oval in B). The nonreal points of intersection of PA with $(x = i\varepsilon)$ lie in B and PA−B in equal parts if ε is sufficiently small (because this is the case when $\varepsilon = 0$).

Thus, our earlier assertion is proven. It follows from it that $[B + C] = k[\infty] \in H_2(CP^2, Z_2)$. Therefore, the cycle $B + C - k\infty$ is homologous to 0 as a Z_2 chain: there exists a three-dimensional Z_2 chain in L_3 in CP^2 for which

$$\partial L_3 = B + C - k\infty.$$

Setting $L_3' = P^{-1}L_3$ we find

$$\partial L_3' = 2(P^{-1}B) + \Pi - k\infty_Y, \quad \text{i.e.,} \quad [\Pi] = k[\infty_Y] \in H_2(Y, \mathbf{Z}_2).$$

Lemma 9 is proven.

§8. Proof of Theorem 1

We apply Lemma 2 to form Φ_τ and homology classes $w = [A] \in H_2(Y)$, $w' = [\Pi] \in H_2(Y)$.

According to Lemma 1, the form is symmetric and nonsingular; according to Lemma 3, class w is fundamental. By Lemma 9, class w' differs from w on even elements. Thus, Lemma 2 is applicable, and we find from Eq. (6) that

$$\Phi_\tau([A], [A]) \equiv \Phi_\tau([\Pi], (\Pi) \bmod 8.$$

We note that $\tau_* w = w$, $\tau_* w' = -w'$ (change of sign of z changes the orientation of Π). Thus, $\Phi_\tau([A], [A]) = (A, A)$, $\Phi_\tau([\Pi, \Pi]) = -(\Pi, \Pi)$. Therefore, $(A, A) + (\Pi, \Pi) \equiv 0 \bmod 8$. Substituting the values of the self-intersection indices $(A, A) = 2k^2$ from Lemma 4 and $(\Pi, \Pi) = 2(m-p)$ from Lemma 8, we find

$$2k^2 + 2(m - p) \equiv 0 \bmod 8,$$

q.e.d.

§9. Remarks

1. Assertion (1) (modulo 8 rather than 4) was formulated by D. A. Gudkov in the form of a hypothesis supported by a large number of examples. Although the proof of congruence (1) does not use the results of D. A. Gudkov, the present paper could not have been produced had not D. A. Gudkov communicated his hypothesis to the author.

2. Manifold Y was studied by V. A. Rokhlin in a recent work [9] by means of the Hirzebruch-Atiyah-Zinger signature formulas ([10], § 6). By joining these computations with ours and with the Lefschetz-Dold-Atiyah-Bott formulas given by Hirzebruch in [11], we can obtain additional information on manifold Y and its involutions.

We note that the real differentiable type of manifold Y and its involution τ depends only on the degree of curve A, which may not even be real. However, the complex conjugate σ: $Y \to Y$ depends on the distribution of the real ovals of the curve.

Involutions σ and τ commute, so that on Y there acts the group $H = \mathbf{Z}_2 + \mathbf{Z}_2$ of the four elements $1, \sigma, \tau, \sigma\tau$. We denote by Φ_h (where $h \in H$) the form $\Phi_h(x, y) = (h_* x, y)$ on $H_2(Y)$. We denote by Π' the submanifold consisting of those points in Y for which the point $x_0 : x_1 : x_2$ is real while z is pure imaginary. Then, for any real curve $F = 0$ (not necessarily with maximal number of ovals), the relationships shown in Table 1 hold:

h	1	σ	τ	$\sigma\tau$
Fixed points of involution h;	Y	Π	A	Π'
Fundamental class of form Φ_h	$(k+1)[\infty_Y]$	$k[\infty_Y]$	$k[\infty_Y]$	$(k+1)[\infty_Y]$
Trace of involution h_*	$2+2g$	$2(p-m)-2$	$-2g$	$2(m-p)$
Signature of form Φ_h	$2-2k^2$	$2(m-p)$	$2k^2$	$2(p-m)-2$

Not all these results were used for our proof of congruence (1), but without the computations of V. A. Rokhlin this proof would hardly have been found. To be more specific, the formulation of Lemma 9 resulted from a comparison of the data of Table 1 with the hypothesis of D. A. Gudkov.

3. The connection we have observed between the distribution of ovals and the involutions of four-dimensional manifold Y can also be used in the contrary direction, obtaining information on the involutions of manifold Y on the basis of data on real curves. For example, we find from relationships (1) and (2) that

$$2 + \operatorname{Tr}\sigma_* \equiv 2k^2 \bmod 8 \text{ (for M curves)}; \quad |1 + \operatorname{Tr}\sigma_*| \leqslant 3k^2 - 3k + 1.$$

It also follows from Table 1 that forms Φ_σ and Φ_T have the same parity as the number k, while forms Φ_1 and $\Phi_{\sigma T}$ have that of k + 1. Indeed, for even k, the signature of form Φ_1, and for odd k the signature of form Φ_T, is not divisible by 8. Consequently, for all k there exists a cycle whose index of intersection with ∞_Y is odd. By combining this information with the form of the fundamental classes, we obtain our assertion.

From Table 1, congruence (1), and inequality (2), it is clear that forms Φ_1 and $\Phi_{\sigma T}$ are not sign-definite, while forms Φ_σ, Φ_T are sign definite only when k = 1 and k = 2. Theorems on the structure of quadratic forms ([8], Chapter 5) therefore permit a complete reconstruction of the canonical form of forms Φ_1, Φ_σ, Φ_T, $\Phi_{\sigma T}$ for all k. For example, when k = 2 (i.e., for curves of degree 4), the form $-\Phi_\sigma = \Phi_T$ is, in the notation of [8], the form Γ_8 corresponding to Lie algebra E_8.

4. It is of interest to note that our involution σ allows us to provide a simple proof of the inequality of I. G. Petrovsk given in our (2). To this end we consider the linear space $E = H_2(Y, R)$ with scalar product given by the index of intersection. Involution σ_* acts in E, retaining scalar products, so that E decomposes into the direct sum of two orthogonal proper subspaces E_1 and E_{-1}, corresponding to eigenvalues 1 and -1 of operator σ_*.

We mention that the scalar product is not degenerate. Therefore, each of the spaces E_1 and E_{-1} can be presented in the form of the direct sum of orthogonal subspaces on which the scalar square is positive (negative) definite:

$$E_1 = E_1^+ + E_1^-, \quad E_{-1} = E_{-1}^+ + E_{-1}^-.$$

We have used here the following notation: $\dim E_1^+ = a$, $\dim E_1^- = b$, $\dim E_{-1}^+ = c$, $\dim E_{-1}^- = d$. In this notation,

$$\operatorname{Tr}\sigma_* = a + b - c - d, \qquad \operatorname{Tr} 1_* = a + b + c + d,$$
$$\operatorname{Sgn}\Phi_\sigma = a - b - c + d, \qquad \operatorname{Sgn}\Phi_1 = a - b + c - d.$$

We substitute into the left sides of these equations the values of traces and signatures from Table 1. Adding and subtracting the equations thus obtained, we find

$$a = (k-1)(k-2)/2, \; c = a + 1, \; b + d = 3k^2 - 3k + 1, \quad b - d = 2(p - m) - 1.$$

Since b and d are non-negative, $|b-d| \leq b + d$, which also proves inequality (2).

5. Our constructions also lead to new constraints on the distribution of ovals. In order to formulate these constraints, we partition all ovals into three classes as a function of the sign of the Euler characteristic of that component of the complement to the curve for which the oval is an exterior boundary. We denote the numbers of positive ovals bounding domains with positive, zero, and negative Euler characteristic by p_+, p_0, and p_- respectively, and the numbers of negative ovals by m_+, m_0, and m_-, so that

$$p = p_+ + p_0 + p_-, \quad m = m_+ + m_0 + m_-.$$

For example, p_+ is the number of positive ovals containing no other ovals inside themselves.

<u>THEOREM 2.</u> For any curve of degree 2k,

$$p_- \leqslant \frac{(k-1)(k-2)}{2}, \quad m_- \leqslant \frac{(k-1)(k-2)}{2}, \quad p_+ \leqslant b, \quad m_+ \leqslant d,$$

where numbers b and d are defined in paragraph 4.

Proof. Consider connected component Π_i of surface Π, projected in the region of RP^2 bounded externally by the given oval γ_i. Of the p surfaces Π_i, the p_- have a negative Euler characteristic.

The homology classes represented by these p_- surfaces (of whatever orientation) $[\Pi_i] \in H_2(Y)$ are pairwise Φ_1-orthogonal, and the quadratic form Φ_1 on the p_- classes $[\Pi_i]$ assumes positive values. Moreover, $\sigma_* [\Pi_i] = [\Pi_i]$.

It follows from this that the p_- classes $[\Pi_i]$ are linearly independent, and that on the plane L spanning them, form Φ_1 is positive definite, but $\sigma_* = 1$. Therefore,

$$p_- = \dim L \leqslant \dim E_1^+ = \frac{(k-1)(k-2)}{2},$$

which also proves our first inequality.

The other three inequalities are proven analogously (in considering m_- it is necessary to take into account the nonorientable component of the complement).

6. From the linear independence of all the Π_i would follow the stronger inequalities

$$p_- + p_0 \leqslant \frac{(k-1)(k-2)}{2}, \quad m_- + m_0 \leqslant \frac{(k-1)(k-2)}{2}, \quad p_+ + p_0 \leqslant b, \quad m_+ + m_0 \leqslant d$$

and, for M-curves of degree 2k, the following lower bound on the number of empty ovals: $p_+ + m_+ \geq k^2$.

According to D. A. Gudkov, in all the interesting examples of M-curves, $p_+ + m_+ \geq k^2 + (k-1)(k-2)/2$, where, for any odd k, there exists an M-curve of degree 2k with $p_+ = k^2$, $p_0 = m_+ = (k-1)(k-2)/2$.

7. We also note that the factor space $X = Y/\tau\sigma$ from the naturally arising commutative diagram of the two-sheeted ramified coverings

$$\begin{array}{ccc} Y & \to & X \\ \downarrow & & \downarrow \\ CP^2 & \to & S^4 \end{array}$$

is a closed connected singly-connected oriented four-dimensional smooth manifold for which

$$\operatorname{Sgn} X = \frac{1}{2}(\operatorname{Sgn} Y + \operatorname{Sgn} \Phi_{\tau\sigma}) = p - m - k^2 = a - d,$$

where, for all known M-curves,

$$a \leqslant d, \quad c \leqslant b, \quad a - d \equiv 0 \pmod 8).$$

From the validity of these relationships for all M-curves would follow both the validity of the Gudkov hypothesis modulo 8 and the inequality

$$|p - m - 1| \leqslant k^2 - 1,$$

which is a strengthening of the Petrovsky inequality for M-curves.

8. For the first proof of Lemma 9 the author is indebted to A. N. Varchenko, to whom the author communicated this lemma in the form of a hypothesis. Although there were flaws in the proof of A. N. Varchenko, it convinced the author of the validity of the Lemma, without which the present paper could not have been produced. Our proof of Lemma 9 uses some ideas from the reasoning of A. N. Varchenko. Thus, the proof of congruence (1) is the result of the joint efforts of the author and A. N. Varchenko. Unfortunately, A. N. Varchenko would not agree to consider himself the co-author of this paper.

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