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On some problems in singularity theory

By

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Much progress in singularity theory of differentiable maps is based on empirical data. Some of these empirical facts later become theorems. We discuss here some facts, that are not explained today and some conjectures, related to them.

1. Semicontinuity of singularity spectrum

Let $f : (C^n, 0) \to (C, 0)$ be a holomorphic function-germ of finite multiplicity μ. One can associate to such a germ a set of μ rationals (not necessarily all different), which we shall call *singularity spectrum*.

Following Steenbrink [14], we denote spectrum points l_k, $k = 1, \ldots, \mu$. The singularity spectrum has the following properties.

(1) Eigenvalues λ_k of the monodromy are related to the spectrum by exponentiation : $\lambda_k = \exp\,(2\pi i l_k)$.

(2) Let f be quasi-homogeneous, and let $\{x^{m_k}\}$ be a monomial C-basis of the local algebra

$$Q_f = C\,[[x_1, \ldots, x_n]]/(\partial f/\partial x_1, \ldots, \partial f/\partial x_n).$$

Then $\{l_k\}$ is the set of weights of μ differential forms $x^{m_k}\, dx_1 \ldots dx_n$ (here $\deg x_k = \deg dx_k = a_k$, $\deg f = 1$).

(3) Let f be a function in two variables, which is generic among functions with a given Newton diagram Γ (figure 1).

Then the spectrum consists of orders (for the Newton filtration) of monomials, whose exponents are detectable from figure 1 (the order of a monomial x^m in the Newton filtration is the coefficient λ, for which $\lambda m \in \Gamma$).

(4) In the general case l_k mod 1 is defined by 1, and the integer part of l_k — by the Steenbrink conventions :

$[l_k] = q$ if l_k corresponds to an eigenvalue of the monodromy on the space $H^{p,q}$ of the mixed Hodge structure on the vanishing cohomology group.

(5) The spectrum is symmetric, with centre $l = n/2$.

Many examples led to the following conjecture. Let the spectra of singularities be ordered : $l_1 \leqslant l_2 \leqslant \cdots \leqslant l_\mu$.

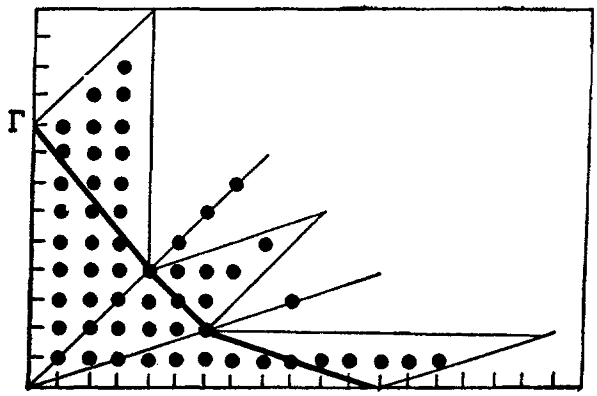

Figure 1. The spectrum of a Newton diagram.

Conjecture. The spectrum is semicontinuous in the following sense: let a singularity S be adjacent to a (simpler) singularity S' (with $\mu' < \mu$), then $l_k \leqslant l'_k$ for $k = 1, \ldots, \mu'$.

Remarks. (1) Even in simple and explicitly calculable cases, like (2) and (3) above, this conjecture is a nontrivial arithmetical assertion on integer points in convex polyhedra.

(2) The conjecture implies the semicontinuity of dimensions of spaces of the Hodge filtration and of the filtration dual to the Hodge filtration in vanishing cohomology, that is the semicontinuity of numbers

$$h^r = \sum_{p \leqslant r} \sum_{q} h^{p,q} \quad \text{and} \quad h_r = \sum_{p > r} \sum_{q} h^{p,q}.$$

(3) In particular, for the case of functions in two variables, these semicontinuities are reduced to the semicontinuities of genus g of the Riemann surface (of the " vanishing cycles manifold ") and of the " cogenus " $\mu - g$. Semicontinuity of both numbers g and $\mu - g$ is clear (the last, as Varchenko has explained to the present author, reduced to the fact that the inclusion of the vanishing cycles manifold of a simpler singularity, to that of a more complicated one is monomorphical at the homology level).

(4) The semicontinuity of l_1, that is of the first spectrum point, is very important for the theory of integral asymptotics. Probably, our conjecture on the semicontinuity of l_1 implies (or is equivalent to) the conjecture on the semicontinuity of oscillating integral asymptotics, which originated from [1], was disproved by Varchenko in [16] and was finally reformulated by Pham [13].

(5) The relation of the spectrum to the set of zeros of Bernstein polynomial (see [3], [12]) seems less proved that one should like, but our conjecture can be reformulated in this setting too (?).

(6) It follows from the spectrum symmetry, that the conjecture implies the two-sided inequalities

$$l_k \leqslant l'_k \leqslant l_{k+(\mu-\mu')}.$$

For instance, if only one point bifurcates from a complicated singularity (S), so that $\mu = \mu' + 1$, then the deformed singularity (S') has a spectrum which divides the spectrum of (S).

The relation between spectra of (S) and (S') is the same as between semiaxes of an ellipsoid in R^{μ} and of its section by $R^{\mu'}$. Does there exist a quadratic form, associated with the singularity, whose eigenvalues (in some Euclidean space) are values of a monotonic function at spectrum points?

(7) We can deform the loop, defining the monodromy in the base space of the versal deformation of a complicated singularity into a product of the loop, defining the monodromy of the simpler one and of some "positive" loops, along which the discriminant argument increases. One can conjecture that the image of the product in the matrix group "rotates more than for the simpler monodromy". This gives some heuristic explanation for the semicontinuity. While these ideas are not at all clear, they are sometimes useful; for instance it was precisely these ideas that have led the present author to the conjecture on the formula for the quasi-homogeneous singularity signature (see [6]). In this case, the "rotation" was defined in terms of two-dimensional invariant planes of the symplectic mapping

$$\begin{pmatrix} 0 & \mathrm{Var} \\ -(\mathrm{Var}^{*})^{-1} & 0 \end{pmatrix}$$

in $H^* + H_*$. The conjecture was related to the positivity of some eigenvalues of this symplectic mapping in the sense of Krein's parametrical resonance theory. However the proof of the conjecture, given later by Steenbrink [15], is quite different.

2. Bifurcation diagrams of complex singularities

Bifurcation diagrams of real functions at critical points of series A ($A_2 = x^3 + y^2$, $A_3 = x^4 + y^2, \ldots$) are very useful for the calculations (and definitions) of generalised Whitehead groups in algebraic K-theory (Cerf [5], Hatcher [9], Wagoner [18], Volodin [17]).

This led to the question, what is the "complex analogue" of these algebraical objects? Such "complex analogues" are perhaps quite different from K-theory. For instance, complex analogue for Morse theory is Picard-Lefschetz theory, but it would be very difficult to reconstruct the second theory, knowing nothing but the first. We also know that the complex analogue for $K(\pi, 0)$ is $K(\pi, 1)$, and for the symmetric group—the braids group. The complex analogue for "boundary" is "two-fold ramified covering". But we have no general methods or axioms for finding such analogues.

As one of the candidates, arising from the complex bifurcation diagrams, we describe a "quasi-resolvent" of the fundamental group of the complement to the singularity bifurcation diagram.

Let Γ_0 be a group, presented as the quotient F_0/R_0, where F_0 is free and R_0 is an invariant subgroup, generated (as an *invariant* subgroup) by elements of the form $(af)f^{-1}$, where $f \in F_0$ and $a \in \mathrm{Aut}\, F_0$, group of automorphisms of F_0.

Let Γ_1 be a group of automorphisms of F_0, leaving invariant every class $f R_0$, and large enough to generate R_0. Let us suppose, that Γ_1 is represented as $\Gamma_1 = F_1/R_1$, and so on : we have a chain of groups $\Gamma_i = F_i/R_i$. We call such a chain *quasi-resolvent* of Γ_0.

Something of this kind arises from the fundamental group Γ_0 of the complement $C^\mu - \Sigma$ to the bifurcations diagram Σ in the versal deformation base space C^μ of a singularity. Let F_0 be $\pi_1(C^1 \setminus \Sigma)$, where C^1 is a generic line in C^μ. Let us consider in C^μ/C^1 the set Σ_1 of non-generic (with respect to Σ) lines. The group

$$\Gamma_1 = \pi_1\left((C^\mu/C^1) \setminus \Sigma_1\right)$$

acts on $\pi_1(C^1 \setminus \Sigma)$, leaving invariant elements of $\pi_1(C^\mu \setminus \Sigma)$. Choose in C^μ/C^1 a generic line (that is, a generic C^2 containing C^1 in C^μ). We obtain a set of generators for Γ_1, that is we consider the free group

$$F_1 = \pi_1\left((C^2/C^1) \setminus \Sigma_1\right),$$

and so on, finishing at $F_{\mu-1}$.

In the case of A_μ singularities, the group Γ_0 is the Artin braids group with $\mu + 1$ strings, the free group F_0 is generated by the μ standard generators of the braids group, the group Γ_1 can be considered as a "quasi-relations" group for Γ_0 (not to be confused with R_0). In the same way Γ_2 corresponds to the "quasi-relations between quasi-relations" and so on. But even in the A_μ case it is not clear whether Γ_1 coincides with the whole group of automorphisms of F_0, which conserves all elements of Γ_0 (and which belongs to the group of automorphisms, generated by orientation-preserving plane with holes homeomorphisms).

Perhaps for the study of $\{\Gamma_i\}$ the classification of all decompositions of simple singularities A, D, E into simpler ones will be useful. Such a classification for functions (not just for levels, which is much easier) is recently found by Ljashko [10].

Very little is known on the topology of the complements to more complicated singularities bifurcations sets. Looijenga [11] has reported that the complements are $K(\pi, 1)$ for parabolical singularities, but his arguments are not clear.

3. Cohomology of complements to bifurcation diagrams

The imbedding of the versal deformation of a simpler singularity (S') base space $C^{\mu'}$ into the base space for a more complicated singularity (S) defines a cohomology homomorphism :

$$H^*(C^\mu \setminus \Sigma) \leftarrow H^*(C^{\mu'} \setminus \Sigma') \quad (\mu > \mu'),$$

between the cohomologies of complements to bifurcation diagrams.

A question naturally arises whether these homomorphisms are canonical and whether one can define a stable cohomology ring (which is, in a sense, the ring of cohomologies of the complement to the bifurcations diagram for $f \equiv 0$ in the infinite-dimensional versal deformation space). Even if this programme cannot be completely realised, one still can associate stable cohomology classes to (at least) some strata of the natural stratification of the set of function-germs set (or hyperface-germs set).

Since the cohomology classes of complements to bifurcation diagrams for versal deformations define corresponding classes in the base spaces of arbitrary deformation complement to bifurcation sets, one can hope that any information on the "stable ring" can be useful to obtain information on global properties of bifurcation sets for arbitrary families of functions (hypersurfaces, mappings, ...).

4. Modality

The modality (moduli number) of a Lie group action at a point of a manifold is the minimal integer m for which orbits in some neighbourhood of the point can be arranged in a finite number of families with $\leqslant m$ parameters.

Problem. Let $f: (R^n, 0) \to (R, 0)$ be a finite multiplicity real function-germ. Is its modulus number in the real jets space equal to its modulus number in the complex jets space? [The corresponding groups are the groups of jets $(R^n, 0) \to (R^n, 0)$ and $(C^n, 0) \to (C^n, 0)$ acting as "right equivalence" $(gf)(x) = f(g(x))$].

I was told by Professor E. B. Vinberg that there exist real representations of real Lie groups, such that the modality of their complexification is larger than the modality of the initial real action. But it is unknown whether such a case is possible for the right equivalence action of the diffeomorphism group on the functions space.

For quasi-homogeneous singularities there exists a notion of "inner modality" which can be calculated as the number of monomials of positive degrees in the monomial basis of the corresponding versal deformations module

$$T_f = (C[x_1, \ldots, x_n])^p \Big/ \Big(\sum_{j=1}^{p} \frac{\partial f_j}{\partial x_i} \frac{\partial}{\partial y_j},\ f_r \frac{\partial}{\partial y_q} \Big),\quad i = 1, \ldots, n;$$
$$r, q = 1, \ldots, p.$$

Here $f_j \in C[x_1, \ldots, x_n]$ are quasi-homogeneous polynomials of degrees D_j, where $\deg x_i = A_i$; the module $(C[x])^p$ is generated by p free generators $\partial/\partial y_q$, whose degrees are $-D_q$; one supposes that $p \leqslant n$ and that $\dim_C T < \infty$.

One conjecture that the inner modality for $p < n$ is equal to the modality (of the contact group action), but it is not proved (as it is a known problem for the case of right-equivalence of functions). The above conjecture is partially confirmed by the theorem (due to I. G. Scherbak) that inner modality 0 complete intersections (quasi-homogeneous curves) in C^3 coincide with contact modality 0 complete intersections (which are all quasi-homogeneous). The classification of these as a standard exercise on Newton diagrams, was at Moscow a known examination problem (1973). By the way, this classification disproves some of the classification results in the sixth part of J. Mather's celebrated paper on singularities.

The Scherbak theorem (containing also quasi-homogeneous unimodular singularities list) was proved with the help of the following formula for the Poincaré polynomial of the graded module T_f for $p = n - 1$ (that is, for singularities of generic quasi-homogeneous curves) :

$$p(t) = \frac{\Pi(1 - t^{D_j})}{\Pi(1 - t^{A_i})}(\Sigma\, t^{-D_j} - \Sigma\, t^{-A_i} + 1) + t^{\Sigma D_j - \Sigma A_i}.$$

One can rewrite this as

$$(*)\ \ p(t) = t^{\Sigma D_j - \Sigma A_i} h(1/t)$$

denoting by h the Poincaré polynomial of the relative differential forms graded module H-, calculated by Hamm [7]. The formula (*) was found empirically for $n = 3$, $p = 2$. It seems that Ljashko can prove it, but his proof does not explain the duality between modules T and H for quasi-homogeneous curves singularities.

In the more general case of generic complete intersections of positive dimension the Ljashko's formula for the Poincaré polynomial of T_f is

$$p(t) = (-1)^{n-p}$$
$$t^{\Sigma D_j - \Sigma A_i}\left[-1 + \operatorname*{res}_{s=0} \prod \frac{1 - st^{A_i}}{s - st^{A_i}} \prod \frac{s - st^{D_j}}{1 - st^{D_j}} \frac{ds}{1 - s}\right]$$
$$+ \frac{\Pi(1 - t^{D_j})}{\Pi(1 - t^{A_i})}[\Sigma\, t^{-D_j} - \Sigma\, t^{-A_i} + 1].$$

Recently Hamm and Gruel have proved $p(1) = h(1)$ for $n - p > 0$, but they give no formula for $p(t)$.

5. Real singularities topology

Mixed Hodge structure defines for every singularity of a function a large set of integers $h_\lambda^{p,q}$ (where λ are monodromy eigenvalues). It seems that these numbers are closely related to the real geometry of the function, its level sets and its morsifications and bifurcations diagrams topology.

A simple example of this is the generalised Petrovski inequality

$$|\operatorname{ind}| \leqslant h_1^{n/2, n/2},$$

for the local degree of the gradient mapping of a real smooth function-germ $f : (R^n, 0) \to (R, 0)$, n even : see [1] for more details.

Other geometrical invariants of the real singularity, which would be interesting to compare with Hodge numbers, are, for instance :

(i) Betti numbers of real non-singular neighbour level set (or their partial sums, may be alternate);

(ii) Numbers of critical points of different indexes, arising from different real morsifications (or their partial sums, may be alternate);

(iii) Numbers describing the possible complication of one real level set of a morsification, for instance, the maximal number of singular points on the same level set.

For empirical works in this direction it is possible to use the Steenbrink conjecture [14] (while this conjecture is wrong as it is stated). A comparison of Hodge numbers, calculated from the Newton diagram by formulas of Steenbrink, Danilov and Kirillov, with the values of geometrical invariants, like those above will, perhaps, generate new (and best possible) inequalities for the real geometry invariants.

However even the Petrovski inequality is known to be best possible only in the simplest cases. For example, it is not known how large the Poincaré index (= the local degree of the gradient map) can be for a real homogeneous function f of degree m in n variables, $f: R^n \to R$.

We only know that the Petrovski inequality gives the exact maximum for $n = 3$ (that is, for curves in RP^2) or for $m = 3$ [the extremal function $f = (x_1 + \ldots + x_n)^3 - x_1^3 - \ldots - x_n^3$ was constructed by D B Fuks]. For $m = n = 4$ (surfaces of degree 4 in RP^3) the inequality is the best possible too.

The classification of real projective surface of degree 4 in RP^3 was one of the questions in the 16th Hilbert problem, at present this classification is known completely, after the works of V M Harlamov (see [8]) and V V Nikulin; they find not only the topological types, but also all possible isotopical types in RP^3 and even classify components of the complement to the degenerate surfaces set in the space of all real surfaces of degree 4 in RP^3.

The Petrovski inequality is still true for nongradient vector fields (see [1]).

It gives the best possible bound for the Poincaré index of a vector field in R^n, whose components are homogeneous polynomials of degree $m - 1$ (A G Hovanski).

6. Maxima singularities

Let

$$F(y) = \max f(x, y)$$

be the maxima function $F: B \to R$ of a family $f: M \times B \to R$ of real functions on a compact closed manifold M, depending on a parameter y, belonging to a " base space " B, which is an (open) manifold of dimension n.

The maxima function is continuous, but generically, is not smooth. Empirical data lead to a conjecture : the maxima function for a generic family is topologically equivalent (in some neighbourhood of every base space point y) to a Morse function (that is, either to a non-zero linear function or to a sum of a constant and of a non-degenerate quadratic form at point 0).

For $n \leqslant 6$ this is proved by Brisgalova (see [4]).

For the general case, the arguments are :

(i) Suppose for a given $y \in B$ there is only one maximum point x (may be degenerate). In this case the graph of F has at y a tangent plane, and the maxima function is generically topologically linear.

(ii) Suppose for a given $y \in B$ there are $n+1$ maxima points. Then the graph of F is a pyramid with $n+1$ faces, each face having at y a tangent plane. The maxima function is generically either topologically linear or topologically equivalent to a Morse function at a minimum point.

(iii) In a generic family the maximum is obtained at s points for y on a manifold of codimension $s-1$. Along this submanifold S the graph of $F|_S$ has tangent planes and in the transversal direction we can use argument (ii).

(iv) In the particular case of a germ f, which is R^+-equivalence stable (see [4]) one can prove that the set $\{y, z : z \geqslant F(y)\}$ is locally diffeomorphic to a convex body. For instance, stability is generic for families with $n \leqslant 6$ parameters. It is natural to ask whether the set described above is still locally diffeomorphic to a convex body for generical families maxima function singularities, if $n > 6$.

If it is true, this will be one more confirmation for a general principle of fragility of all good things.

To explain this principle let us consider, for example, the set of all polynomials $x^n + a_1x^{n-1} + \cdots + a_n$ (a real), having only real zeros (or only non-real, or only zeros with negative real parts). This set, at singularities of its boundary, fills *less than one half* of the neighbourhood space. Thus under deformation the corresponding property of being good (elliptic, hyperbolic, stable and so on) will rather disappear than persist. The theorems, generated by this principle, (and describing the cones of velocities of curves, moving from the boundary point inside the good set) were recently proved by L. V. Levantovski.

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