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The Asymptotic Hopf Invariant and Its Applications*

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The classical Hopf invariant distinguishes among the homotopy classes of continuous mappings from the three-sphere to the two-sphere and is equal to the linking number of the two curves that are the preimages of any two regular points of the two-sphere.

The asymptotic Hopf invariant is an invariant of a divergence-free vector field on a three-dimensional manifold with given volume element. It is invariant under the group of volume-preserving diffeomorphisms, and describes the "helicity" of the field, i.e., the mean asymptotic rotation of the phase curves around each other. The asymptotic Hopf invariant coincides with the classical Hopf invariant for the unitary vector field that is tangent to the Hopf bundle. In the general case the asymptotic Hopf invariant can have any real value (whereas the classical Hopf invariant is always an integer).

The asymptotic Hopf invariant can also be considered as a quadratic form on the Lie algebra of the volume-preserving diffeomorphisms of the three-dimensional manifold that is invariant under the adjoint action of the group on the algebra.

In this paper we present the definition and simplest properties of the asymptotic Hopf invariant, as well as some of its applications to an unusual variational problem that arises in magnetohydrodynamics which was called to the author's attention by Ya. B. Zel'dovich. In connection with this problem there arise a whole series of unsolved mathematical problems, some of which appear to be difficult. The main object of this paper is to discuss these unsolved problems; all the theorems in the paper are obvious.

Attention was first called to the problems considered here by Voltjer [7] in connection with magnetohydrodynamics. Applications to ordinary hydrodynamics were given by Moffatt [4], [5] and Kraichnam [3].

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1. The problem of the minimum magnetic energy of a frozen-in field

Let M be a three-dimensional closed Riemannian manifold, and ξ a divergence-free vector field on M. The *energy* of the field is the integral

$$E = \tfrac{1}{2}\langle \xi, \xi \rangle = \tfrac{1}{2}\int_M (\xi, \xi)\, dv.$$

We are to find the minimum energy for fields obtained from a given field under the action of volume-preserving diffeomorphisms of the manifold M.

Here the action of a volume-preserving diffeomorphism $g: M \to M$ associates with a divergence-free field ξ on M another divergence-free field $g * \xi$ such that the flux of the field ξ across any surface σ is equal to the flux of $g * \xi$ across $g\sigma$. In other words, the field is frozen into a covering of M by an incompressible fluid: the vector field can be thought of as drawn on the elements of the fluid and expanding as these elements expand.

The two-dimensional analog can be formulated as follows:

To find a function that minimizes the Dirichlet integral

$$E = \tfrac{1}{2}\langle \nabla u, \nabla u \rangle = \tfrac{1}{2}\int_M (\nabla u, \nabla u)\, dv$$

among all functions u on the closed two-dimensional Riemannian manifold M obtained from a given function u_0 by the action of an area-preserving diffeomorphism on itself:

$$u(x) = u_0(g^{-1}x).$$

It is clear that similar problems arise for manifolds with boundary, for example, for functions $u(x, y)$ in an ordinary Euclidean disk. The mathematical aspects of investigations of these problems have been highly unsatisfactory.

1.1. The Euler equation

Theorem. *The extremals of the problem stated above are divergence-free fields that commute with their curl.*

Proof. Let η be any divergence-free field. The variation of a field ξ under the infinitesimal diffeomorphism defined by η is given by the Poisson bracket $\delta\xi = \{\eta, \xi\}$ (in terms of the coordinates, $(\eta, \xi) = (\xi\nabla)\eta - (\eta\nabla)\xi$).

Consequently $\delta E = \langle \xi, \delta\xi \rangle = \langle \xi, \{\eta, \xi\}\rangle$. But, by a formula from vector analysis, $\operatorname{curl}[\eta, \xi] = \{\eta, \xi\} - \eta \operatorname{div} \xi - \xi \operatorname{div} \eta$ on any three-dimensional Riemannian manifold. Since our fields ξ and η are divergence-free, $0 = \delta E = \langle \xi, \operatorname{curl}[\eta, \xi]\rangle = \langle \operatorname{curl} \xi, [\eta, \xi]\rangle = \langle \eta, [\xi, \operatorname{curl} \xi]\rangle$. Since η is divergence-free, the vector product $[\xi, \operatorname{curl} \xi]$ is orthogonal to all divergence-free fields. Consequently it is a gradient:

$$[\xi, \operatorname{curl} \xi] = \operatorname{grad} \alpha,$$

whence, taking the curl of both sides, we obtain

$$\{\xi, \operatorname{curl} \xi\} \equiv 0,$$

as was to be proved.

Remark 1. In the two-dimensional case we obtain the equation

$$[\nabla u, \nabla \Delta u] \equiv 0,$$

which says that the gradient of the extremal function is collinear with that of its Laplacian.

Remark 2. A similar calculation leads to the following expression for the second variation:

$$\delta^2 E = \langle \{\eta, \xi\}, \{\eta, \xi\} \rangle + \langle \{\eta, \xi\}, [\operatorname{curl} \xi, \eta] \rangle$$

(where ξ is an extremal whose first and second variations are given by the formula

$$\xi(\epsilon) = \xi + \epsilon\{\eta, \xi\} + \frac{\epsilon^2}{2}\{\eta, \{\eta, \xi\}\} + \cdots, \qquad \varepsilon \to 0,$$

in terms of a divergence-free vector field η).

1.2. *Study of the extremal fields*

Let ξ be a divergence-free field on a three-dimensional closed orientable Riemannian manifold M for which $[\xi, \operatorname{curl} \xi] = \operatorname{grad} \alpha$. All such fields are extremals for our problem. It turns out that the field lines of ξ have a very special topology.

Theorem. *Every noncritical level set of the function α is diffeomorphic to a torus (or union of tori). In a neighborhood of such a torus we can define coordinates $\varphi_1, \varphi_2 \pmod{2\pi}$, z, such that φ is the angular coordinate on the torus, z indexes the torus, and the field ξ (as well as the field* curl ξ*) has components*

$$\xi = \omega_1(z)\frac{\partial}{\partial\varphi_1} + \omega_2(z)\frac{\partial}{\partial\varphi_2}; \qquad \operatorname{curl} \xi = \omega_1'(z)\frac{\partial}{\partial\varphi_1} + \omega_2'(z)\frac{\partial}{\partial\varphi_2}.$$

Here the coordinate z can be chosen so that the volume element has the form $d\varphi_1 \wedge d\varphi_2 \wedge dz$.

Remark. The coordinates φ_1, φ_2, z are analogs of the action-angle variables of classioal mechanics. The theorem means, in particular, that both the field lines of ξ and of curl ξ lie on the tori $\alpha = \text{const}$. These lines are either closed (if the relative frequency ω is rational) or dense on the torus.

For the proof see [**1**]. It follows from the theorem that, for example, in the analytic case, when $\alpha \neq \text{const}$ the manifold M is divided by the critical level sets of α into a finite number of cells, diffeomorphic to the product of the torus by intervals in each of which the fields ξ and curl ξ are tangent to the torus and generate periodic or conditionally periodic windings of the torus. Consequently, we obtain an explicit description of the topology of the field ξ (or curl ξ).

It remains to consider the case when grad $\alpha \equiv 0$. In this case $[\xi, \text{curl}\, \xi] = 0$, i.e., the fields ξ and curl ξ are collinear at each point. Such fields are called force-free fields in magnetohydrodynamics.

If a force-free field ξ is never zero, then curl $\xi = c\xi$, where c: $M \to R$ is a smooth function. But div curl $\xi \equiv 0$; consequently, (grad c, $\xi) \equiv 0$, i.e., the function c is a first integral of the field ξ (and also of curl ξ). Hence it follows that the connected components of the nonsingular level surfaces of c are tori, and the field lines of ξ are windings on these tori (in the corresponding coordinates φ_1, φ_2, z, the constants along the field lines of ξ will be the frequency ratios, $\dot{\varphi}_1/\dot{\varphi}_2 = \kappa(z)$). Therefore even in the case of a force-free field the field lines lie on two-dimensional tori, provided that the field does not have zeros and c is not constant.

A force-free field with curl $\xi = \lambda\xi$, where λ is a constant, can have a much more complicated topology. An example of such a field on the three-dimensional torus $\{x, y, z, \bmod 2\pi\}$ is given by the components

$$\xi_x = A \sin z + C \cos y, \qquad \xi_y = B \sin x + A \cos z, \qquad \xi_z = C \sin y + B \cos z.$$

The topology of these field lines was investigated experimentally by Henon [**2**], using the computer at the Astrophysical Institute at Paris. As a result he discovered a set of tori filled out by field lines ("magnetic surfaces") together with whole domains of three-dimensional space whose field lines, as far as one can tell from the experimental data, are ergodic, or everywhere dense.

1.3. Discussion

Returning to our extremal problem, we see that a field of minimum energy in a given class of frozen-in fields must either have a very special topology (the field lines fill out tori), or be force-free fields of a special kind. But the topological properties of the field lines are invariant under diffeomorphisms, and therefore if the original field is a general one, then every field obtained from it by a diffeomorphism has the same property. Consequently, a field of minimum energy either does not exist (in the class of smooth fields to which the preceding analysis applies) or is a force-free field of special type.

But force-free fields with curl $\xi = \lambda\xi$ are scarce; they are eigenvectors of the field of the operator curl on the space of divergence-free fields. Hence we must assume that our variational problem apparently does not always admit a smooth solution.

In this connection we consider the following example. Let M be a sphere in three-dimensional Euclidean space, and let the field lines of ξ be horizontal circles with centers on the vertical axis. According to Zel'dovich, the energy of such a field can be made arbitrarily small by means of a suitable diffeomorphism which preserves volumes and is fixed in a neighborhood of the boundary. In fact, let us divide the whole sphere into a number of slender solid tori (doughnuts) formed from the circles of the field, and a remainder of small volume. Then let us deform (preserving its volume) each solid torus (violating the axial symmetry of the field) so that it becomes fat and small, with the hole decreasing almost to zero. Then the field energy in the solid tori is decreased (since the field lines are shortened). It can be seen that the whole construction can be carried out in such a way that the field energy in the remaining small volume is not increased by too much, as a result the total energy remains arbitrarily small.

It would be of interest to carry out this construction precisely.

In connection with this example, there arises the *question of whether it is possible to reduce the energy of an arbitrary field to an arbitrarily small value* by an appropriate volume-preserving diffeomorphism. We shall see below that this is not the case. An obstacle to the complete annihilation of the energy can be constructed by considering two linked doughnuts of field lines. In this case the shortening of the field lines in one doughnut, shrinking its hole, induces a lengthening of the field lines in the other, so that there is an obstacle to the decrease of the energy. The asymptotic Hopf invariant, which measures the linking of the field lines (not necessarily closed) lets us give a qualitative expression for this situation in the form of a lower bound for the energy.

1.4. *Magnetohydrodynamic discussion*

In magnetohydrodynamics the role of ξ is played by the magnetic field H, frozen into a fluid of finite viscosity, but of infinite conductivity, which fills M. With an appropriate choice of units, the velocity field v and the magnetic field H satisfy the system of equations

$$\frac{\partial v}{\partial t}+(\nabla v, v)=-\operatorname{grad} p-\nu\Delta v+[jH], \qquad \operatorname{div} v=0,$$

$$\frac{\partial H}{\partial t}=\{v, H\}, \qquad \operatorname{div} H=0, \qquad \operatorname{curl} H=j.$$

The magnetic field H and the velocity field v are prescribed at the initial time. In the course of time, the kinetic energy is dissipated because of the viscosity, and the motion ceases "in the end," since each particle approaches some terminal position. The magnetic field, being frozen in, then attains some terminal value. The energy of this terminal field must be a minimum; otherwise the magnetic energy would have been converted into kinetic energy and, on account of the Lorenz force, the fluid would move until it dissipated the excess of magnetic energy above the minimum.

This sort of description of the behavior of solutions of the system presented above is usually given by physicists. Unfortunately, the preceding analysis of the topology of the extremal fields holds out little hope that this description is correct for any general initial conditions: in fact, the initial magnetic field can be taken without having magnetic surfaces, and then the limiting field, if there is one, must be a force-free field of special type; but such fields are too scarce, and one would hardly find a field with the prescribed lines of force among them.

It appears that for a correct description of the actual process it is necessary to take account of the magnetic viscosity, which violates the assumption that the field is frozen in, and was not taken into account in our system of equations.

The question of the extent to which one can use the extremal field to study the behavior of H over an extended period of time during which the ordinary viscosity succeeds in extinguishing the motion of the fluid, but the magnetic viscosity does not extinguish H, is an interesting unsolved problem.

Zel'dovich proposed the problem of the minimum magnetic field in connection with the question of the evolution of the magnetic field of a star. In this case M is a sphere in three-dimensional Euclidean space, and the field is propagated over the whole space with the boundary conditions

$$\operatorname{curl} H = 0 \text{ outside } M, \qquad \operatorname{div} H = 0 \text{ outside } M,$$

$$(H, n) \text{ is continuous on } \partial M,$$

and the condition of decrease at infinity. Consequently, the volume-preserving diffeomorphism of M acts on the field H throughout the whole space. It is necessary to minimize the total energy of the field H (i.e., the integral over all space). The minimizing field must provide a minimum of the magnetic energy inside M with respect to fields obtained from the given diffeomorphism and stationary near the boundary.

We will not discuss the question of how close this simple model is to reality. In what follows we restrict ourselves to a more simple system, in which M is a manifold without boundary.

2. Definition of the invariant

We begin with a dogmatic presentation: we consider an *ad hoc* definition of the invariant, and establish its simplest properties. The interesting meaning of the invariant (and an explanation of how the invariant was found) will be discussed in the following sections.

Let M be a three-dimensional manifold that is closed (compact, without boundary), oriented, and connected, and let v be the volume element (i.e., a 3-form defining the correct orientation) on M. It will be convenient to assume that the total volume of M is 1. Notice that we are given a volume element on M, but *we are not given any particular Riemannian metric.*

2.1. Notation

Every vector field ξ on M generates a differential 2-form ω_ξ according to the formula

$$\omega_\xi(\eta, \zeta) = v(\xi, \eta, \zeta) \qquad \text{for all } \eta, \zeta,$$

and the correspondence $\xi \mapsto \omega_\xi$ is an isomorphism of the linear spaces of fields and 2-forms. The derivative of the form ω_ξ, as for every 3-form, can be written in the form

$$d\omega_\xi = \varphi v,$$

where $\varphi\colon M \to R$ is a smooth function. The function φ is called the *divergence* of the field ξ:

$$\varphi = \operatorname{div} \xi.$$

The velocity field of a flow that preserves the volume element on M is divergence-free; and, conversely, every flow with divergence 0 on M is the velocity field of an incompressible flow (i.e., of a flow that preserves the volume element v on M).

A divergence-free vector field ξ on M is said to be *homologous to zero* if the 2-form ω_ξ corresponding to it is the total differential of a 1-form α on M:

$$\omega_\xi = d\alpha.$$

The 1-form α will be called a *form-potential.* A field is homologous to zero if and only if its flux across every closed surface is zero.

Remark. If a Riemannian metric is given on M then the 1-form α can be identified with the vector field a for which

$$\alpha(\eta) = (a, \eta) \qquad \text{for every } \eta.$$

In this case $\xi = \operatorname{curl} a$, and the vector field a is called the *vector potential* of ξ. However, it is essential to observe that the forms ω and α (in contrast to the field a) do not depend on the Riemannian metric, but only on the choice of the volume element v.

2.2. Definition

The *(mean) Hopf invariant* of a field ξ that is homologous to zero on the three-dimensional manifold M with volume element v is the integral of the product of the form ω_ξ and its form-potential, i.e., the number

$$I(\xi) = \int_M d \wedge d\alpha, \qquad \text{where} \quad \omega_\xi = d\alpha.$$

Let us show that this definition is consistent, i.e., that *the value of I does not depend on the particular choice of the form-potential* α, *but only on the field* ξ.

In fact, if $\beta = \alpha + \gamma$ is another form-potential, then $d\gamma = 0$, and therefore

$$\int_M \alpha \wedge d\alpha - \beta \wedge d\beta = \int_M \gamma \wedge d\alpha = \int_M d(\gamma \wedge \alpha)$$
$$= \int_{\partial M} \gamma \wedge \alpha = 0.$$

Remark. If a Riemannian metric with volume element v is given on M, then

$$I(\xi) = \int_M (\xi, a)\, dv = \langle \xi, \operatorname{curl}^{-1} \xi \rangle,$$

where a is any vector potential of ξ. Therefore *I is the scalar product of the field with its vector potential.* It is essential to observe, however, that the Riemannian metric *does not enter* into the definition of I.

2.3. Invariance

Corollary. *Every volume-preserving diffeomorphism* $g\colon M \to M$ *carries every field* ξ *that is homologous to zero into a field with the same Hopf invariant.*

In particular, on a Riemannian manifold the scalar product of a divergence-free field and its vector potential is preserved when the field is acted on by a volume-preserving diffeomorphism.

Consequently the invariance of I under diffeomorphisms that preserve the volume element follows from the fact that *I can be defined by using no structures other than the smooth structure of M and the volume element v.*

Remark. The question of whether I is preserved under homeomorphisms that preserve the volume element (transforming the phase flow of ξ into the phase flow of another field ξ') is an interesting unsolved problem, as is the closely related problem of *whether one can define the invariant I directly for one-parameter groups of homeomorphisms that preserve the volume element.*

Remark. In the case when M is a manifold with boundary, the number I is preserved under all volume-preserving diffeomorphisms that are stationary in a neighborhood of the boundary. If, however, ξ is tangent to the boundary, then I is preserved under all volume-preserving diffeomorphisms provided that M is simply connected. The question of whether one can define an invariant analogous to I for general divergence-free fields on a manifold with boundary (including a surface term in I) is an interesting unsolved problem.

2.4. *Examples*

If we take ξ to be a magnetic field, we arrive at the conclusion that *the Hopf invariant of a magnetic field frozen into an incompressible fluid that fills a closed manifold does not change during any flow of the fluid.*

If we interpret the field ξ as the vorticity field of a perfect fluid, we obtain the result that *in the flow of a perfect fluid on a closed three-dimensional manifold, the scalar product of the velocity field and the vorticity field does not change with time.*

If we consider the field ξ as an element of the Lie algebra of the group S Diff M of volume-preserving diffeomorphisms of the three-dimensional manifold M, we obtain the result that *on the Lie algebra of the group S* diff *M there is a symmetric bilinear form that is invariant with respect to the corresponding action of the group on the algebra.* If we give a Riemannian metric on M then

$$I(\xi, \eta) = \langle \xi, \operatorname{curl}^{-1} \eta \rangle,$$

where $\operatorname{curl}^{-1} \eta$ is the vector potential of the field η. In particular, *for every divergence-free field η we have*

$$\langle \{\xi, \eta\}, \operatorname{curl}^{-1} \xi \rangle = 0,$$

which is, of course, easily verified by direct calculation.

For a two-dimensional manifold M we obtain a skew-symmetric form instead of a symmetric form.

3. Asymptotics of the coefficient of linking with a curve

Let M be a closed connected oriented and simply connected three-dimensional manifold with volume element v, let γ be a smoothly embedded closed orientable curve in M, and let ξ be a divergence-free vector field on M. We define an asymptotic coefficient of the linking of the field lines of ξ that issue from the point x with the curve γ. Let $\{g^t: M \to M\}$ be the phase flow of the field ξ. Select a 2-chain σ (of smooth simplexes) for which $\partial\sigma = \gamma$.

3.1. *Asymptotic linking coefficient*

For every pair of points x, y of M we introduce a "short curve" $\Delta(x, y)$ that joins these points and has the following properties:

(1) If x and y do not belong to γ, then Δ does not intersect γ.

(2) The number of intersections of $\Delta(x, y)$ with γ is bounded by a constant independent of x and y.

It is easy to construct such a system of "short curves;" the dependence of Δ on x and y can be made measurable (and even piecewise smooth).

We fix a system of curves Δ and consider the segment of the orbit $g^t x$ of x corresponding to $0 \leq t \leq T$. We join the last point $g^T x = y$ with the first by $\Delta(y, x)$; then we have a closed curve $\Gamma_T(x)$. We assume that this curve does not intersect γ.

Let $N_T(x)$ denote the linking coefficient of $\Gamma_T(x)$ with γ (i.e., the index of the intersection of $\Gamma_T(x)$ with σ).

Theorem. *For almost all x in M the limit*

$$\lim_{T\to\infty} \frac{1}{T} N_T(x) = \lambda(x)$$

exists (where T runs through the values for which $\Gamma_T(x)$ does not intersect γ). This limit belongs to $L_1(M, v)$ and, as an element of L_1, is independent of the system of curves Δ.

The limit $\lambda(x)$ is called *the asymptotic linking number of the orbit $g^t x$ with the curve γ.*

To prove the theorem it is convenient to give a different definition of the asymptotic linking number, and then prove that it is equivalent to the definition given above.

3.2. *Second definition of the asymptotic linking number*

On the manifold $M - \gamma$ we can construct a closed differential 1-form α with the following properties:

(1) The linking number with γ of every closed curve δ in $M - \gamma$ is equal to the integral of α over δ.

(2) There is a diffeomorphic embedding $u: S^1 \times D^2 \to M$ of the direct product of a circumference and a disk into M such that the circumference $S^1 \times 0$ maps to γ and the form α induces, on the complement of this circumference, the standard form $u * \alpha = (1/2\pi) \arctan(y/x)$ (where x, y are the coordinates in D^2).

We select a form α with these properties, and consider the limit

$$\hat{\lambda}(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \alpha\left(\frac{d}{dt} g^t x\right) dt.$$

Theorem. *The limit exists for almost all x and is independent of the choice of the 1-form α satisfying hypotheses (1) and (2).*

Proof. Consider the function

$$f(x) = \alpha(\xi(x)), \qquad \text{where} \quad \xi(x) = \left.\frac{d}{dt}\right|_{t=0} g^t x.$$

This function belongs to the space $L_1(M, v)$ by condition (2) on α. By Birkhoff's ergodic theorem, the time average of f exists almost everywhere. This establishes the first part of the theorem, since $\hat{\lambda}(x)$ is the time average of f.

To prove the second part, we observe that α is defined on $M - \gamma$ up to a differential of a single-valued function. If φ is a smooth function on $M - \gamma$, then

$$\int_0^T d\varphi\left(\frac{d}{dt} g^t x\right) dt = \varphi(g^T x) - \varphi(x).$$

Now we observe that if $g^t x$ does not approach γ asymptotically as $t \to \infty$, we can choose a sequence $T_i \to \infty$ such that the distance of the points $g^{T_i} x$ from γ remains bounded below. But φ is bounded above by a constant $C(\epsilon)$ outside an ϵ-neighborhood of γ. Consequently, for all points x that are not asymptotic to γ there is a sequence $T_i \to +\infty$ along which $\varphi(g^{T_i} x)$ is bounded above. Therefore the limit $\hat{\lambda}(x)$ is the same for any two 1-forms α for all points x except those asymptotic to γ (and those points for which one of the limits does not exist). But the points asymptotic to γ form a set of measure 0 (since the field ξ is divergence-free), and we have established that $\hat{\lambda}$ is independent of α for almost all x.

3.3. Equivalence of the definitions

The theorem of Section 3.1 is a consequence of the following theorem.

Theorem. *For almost all x, the limit $\lambda(x)$ exists and is equal to $\hat{\lambda}(x)$.*

Proof. By property (1) of the form α, it is enough to prove that for almost all x

$$\lim_{T \to \infty} \frac{1}{T} \int_{\Delta(g^T x, x)} \alpha(\xi) \, dt = 0.$$

But since the number of intersections of Δ and σ is bounded (see property (2) of the curves Δ), it follows that the integrals of α along Δ are uniformly bounded; consequently, the limit (over a sequence of values of T for which Δ does not intersect γ) is zero; this establishes the theorem.

Remark. We have simultaneously proved that $\lambda(x)$ is independent of the family of short curves Δ.

Remark. It is obvious from the theorem that the asymptotic linking number is invariant under volume-preserving diffeomorphisms, in the sense that if a diffeomorphism h carries the system (M, v, γ, ξ, x) to $(M', v', \gamma', \xi', x')$ then

$$\lambda_{\xi,\gamma}(x) = \lambda_{\xi',\gamma'}(x').$$

The question of *whether the asymptotic linking number is invariant under volume-preserving homeomorphisms is an unsolved problem, as is the related question of the possibility of defining an asymptotic linking number with a topological curve γ for a one-parameter group of volume-preserving homeomorphisms.*

3.4. The mean linking number with a curve

Let $\{g^t\}$ be the phase flow of a divergence-free field ξ on a simply connected three-dimensional manifold M with volume element v. Let $\gamma = \partial\sigma$ be an oriented smooth curve in M, and let σ be a piecewise smooth 2-chain. *The mean linking number of $\{g^t\}$ with γ* is the average of the asymptotic linking number with respect to M:

$$\lambda = \int_M \lambda(x) v.$$

Theorem. *The mean linking number λ is equal to the flux of the field ξ through the surface σ.*

Proof. The number $\lambda(x)$ is the time average of $f(x) = \alpha(\xi(x))$. Consequently, the space averages of f and λ are the same, i.e.,

$$\lambda = \int_M \alpha(\xi) v = \int_M \alpha \wedge \omega_\xi.$$

Now the theorem follows from the homology of the 2-chain σ and the 1-form α as de Rham flows in $M - \gamma$ (strictly speaking, we should consider not $M - \gamma$, but the complement in M of an ϵ-neighborhood of γ, and then let $\epsilon \to 0$).

Remark. One can obtain similar results for the case when γ is not smoothly embedded, but is a piecewise smooth curve. In addition, one can assume that M is n-dimensional and that the chain γ is $(n-2)$-dimensional.

4. Asymptotic linking number of a pair of trajectories

Let M be a three-dimensional closed simply connected manifold with volume element v, let ξ be a divergence-free field on M, and let $\{g^t\}$ be its phase flow.

4.1. *Definition of the asymptotic linking number of a pair of trajectories*

We consider a pair x_1, x_2 of points of M. We are going to associate with this pair of points a number that characterizes the "asymptotic linking" of the trajectories of $\{g^t\}$ that issue from them. For this purpose we first join any two points of M by a "short path" connecting the points (the conditions imposed on a short path were described above and are satisfied for "almost any" choice of the short path).

We select two large numbers T_1 and T_2, and close the segment $g^{t_k}x_k$ $(0 \le t_k \le T_k)$ of the trajectories issuing from x_1 and x_2 by short paths $\Delta(g^{T_k}x_k, x_k)$ $(k = 1, 2)$ so that we obtain two closed curves $\Gamma_k = \Gamma_{T_k}(x_k)$. We assume that these curves do not intersect (which is true for almost all pairs x_1, x_2 for almost all T_1, T_2). Then the linking number $N_{T_1,T_2}(x_1, x_2)$ of Γ_1 and Γ_2 is defined as follows.

Definition. The asymptotic linking number of the pair of trajectories $g^t_{x_1}, g^t_{x_2}$ is defined as the limit

$$\lambda(x, y) = \lim_{T_1, T_2 \to \infty} \frac{N_{T_1,T_2}(x_1, x_2)}{T_1 T_2}$$

(T_1 and T_2 are to vary so that Γ_1 and Γ_2 do not intersect).

We are going to prove that this limit exists almost everywhere and is independent of the system of short paths (as an element of $L_1(M \times M)$).

4.2. *Digression on Gauss's formula*

It will be useful to have the formula given by Gauss for the linking number of two closed curves in three-dimensional Euclidean space. There is also a similar formula for a simply connected manifold: see de Rham's book *Variétés différentiables.*

In order to state Gauss's formula, we introduce the following notation.

Let x_1: $S^1_1 \to \mathbf{R}^3$ and x_2: $S^1_2 \to \mathbf{R}^3$ be smooth mappings of a circumference in three-dimensional Euclidean space, with disjoint images. Let t_1 (mod T_1) and t_2 (mod T_2) be coordinates on the first and second circumference; then we denote by $\dot{x}_1 = \dot{x}_1(t_1)$ the velocity vector of the flow on the first, and by $\dot{x}_2 = \dot{x}(t_2)$ that on the second.

We assume that the circumferences are oriented by the choice of the parameters t_1 and t_2, and we fix an orientation for $\mathbf{R}^3$. Then we can define vector products and triple scalar products in $\mathbf{R}^3$.

Gauss's Theorem. *The linking number of the closed curves* $x_1(S^1)$ *and* $x_2(S^2)$

is equal to

$$N_{1,2}=+\frac{1}{4\pi}\int_0^{T_2}\int_0^{T_1}\frac{(\dot{x}_1,\dot{x}_2,x_1-x_2)}{|x_1-x_2|^3}\,dt_1\,dt_2.$$

Proof. Consider the mapping

$$f\colon T_2\to S^2$$

of the torus on the sphere, making a pair of points on our circumferences correspond to the vector of unit length directed from $x_2(t_2)$ to $x_1(t_1)$: $f=F/\|F\|$, where $F(t_1,t_2)=x_1(t_1)-x_2(t_2)$.

We orient the sphere by the inner normal and the torus by the coordinates t_1, t_2. *The degree of the mapping is equal to the linking number* $N_{1,2}$. In fact, this is true for widely separated small circumferences: both the linking number and the degree of the mapping f are 0. Furthermore, it is easy to verify that under a deformation of a curve by any passage of one curve through another both the linking number and the degree of the mapping change by 1, in the same direction. Now the equation $N_{1,2}=\deg f$ is established, in view of the connectedness of the set of smooth mappings $S^1\to\mathbf{R}^3$.

Let us show that *the degree of the mapping f is given by the integral formula of Gauss.* In fact, by the definition of the degree,

$$\deg f=\frac{1}{4\pi}\iint_T f*\omega^2,$$

where ω^2 is the area element on the unit sphere. By the definition of f, the value of the form $f*\omega^2$ on the pair of vectors ξ_1,ξ_2, tangent to the torus, is equal, at t, to its triple scalar product with the vector $-f=-f(t)$ (we oriented the sphere by the *inner* normal),

$$\omega^1(f*\xi_1, f*\xi_2)=(f*\xi_1, f*\xi_2, -f).$$

Differentiating f, we obtain $f*\xi=F*\xi/\|\xi\|+c(\xi)f$, and therefore

$$\omega^2(f*\xi_1, f*\xi_2)=(F*\xi_1, F*\xi_2, -F)/\|F\|^3.$$

Since $F=x_1-x_2$, we obtain, for an element of the spherical image of the torus, the expression

$$f*\omega^2=+(\dot{x}_1,\dot{x}_2,x_1-x_2)\|x_1-x_2\|^{-3}\,dt_1\wedge dt_2,$$

as was to be shown.

4.3. A second definition of the asymptotic linking number

Let $\{g^t\}$ be a phase flow, defined by a divergence-free field ξ in a three-dimensional compact Euclidean domain M. The field is assumed to be tangent to M on the boundary of M. We set

$$\hat{\lambda}(x_1, x_2) = \lim_{\substack{T_1 \to \infty \\ T_2 \to \infty}} \frac{1}{T_1 T_2} \frac{1}{4\pi} \int_0^{T_2} \int_0^{T_1} \frac{(\dot{g}^{t_1} x_1, \dot{g}^{t_2} x_2, x_1 - x_2)}{\|x_1(t_1) - x_2(t_2)\|^3} \, dt_1 \, dt_2.$$

We shall show that:

(1) the limit $\hat{\lambda}(x_1, x_2)$ exists almost everywhere on $M \times M$;

(2) the number $\hat{\lambda}(x_1, x_2)$ coincides, for almost all x_1, x_2, with the number $\lambda(x, y)$ defined above.

To prove the first statement it is enough to verify that $\hat{\lambda}$ is the "time average" of a summable function on the manifold $M \times M$, on which the commutative group $\{g^{t_1}\} \times \{g^{t_2}\}$ acts.

The integrand

$$\Phi(x_1, x_2) = (\xi_1, \xi_2, x_1 - x_2)/\|x_1 - x_2\|^3 \qquad \left(\xi_k = \left.\frac{d}{dt_k}\right|_{t_k = 0} g^{t_k} x_k\right)$$

has a singularity on the diagonal of $M \times M$ of order no higher than r^{-2} (where r is the distance to the diagonal): since the codimension of the diagonal is 3, the function Φ belongs to the space $L_1(M \times M)$, as was to be proved.

To compare $\hat{\lambda}$ with λ we represent the linking coefficient of the curves $\Gamma_{T_1} x_1$ and $\Gamma_{T_2} x_2$ by Gauss's integral with $0 \le t_1 \le T_1 + 1$, $0 \le t_2 \le T_2 + 1$, and using the value of the parameter t_k from T_k to T_{k+1} for parametrizing the "short path" that joins $g^{T_k} x_k$ to x_k.

Definition. A system of short paths joining the points $x, y \in M$ is a system of paths, depending in a measurable way on x and y, such that the integrals of Gauss type for every pair of nonintersecting paths of the system, and also for any nonintersecting pairs (paths of the system, segments of phase curves $g^t x$, $0 \le t \le \tau < 1$), are bounded independently of the paths by a constant c.

It is easy to verify that systems of short paths exist (it is useful to keep in mind that an integral of Gauss type for a pair of straight-line segments remains bounded when the segments approach each other).

Now the difference

$$\int_0^{T_2 \cdot 1} \int_0^{T_2 \cdot 1} - \int_0^{T_2} \int_0^{T_1}$$

of integrals of Gauss type can be estimated by the sum of at most $[T_1] + [T_2] + 3$ terms, none of which exceeds c. Consequently,

$$\lambda(x, y) - \hat{\lambda}(x, y) = \lim_{T_{1,2} \to \infty} \frac{1}{T_1 T_2} \left(\int_0^{T_2 \cdot 1} \int_0^{T_1 \cdot 1} - \int_0^{T_2} \int_0^{T_1} \right)$$

(where T_1 and T_2 tend to infinity on any sequence for which the curves $\Gamma_{T_1}x_1$ and $\Gamma_{T_2}x_2$ do not intersect).

4.4. *Average linking number*

Theorem. *The mean value of the asymptotic linking number of a pair of trajectories,*

$$\iint_{M\times M} \lambda(x_1, x_2)\, dv_1\, dv_2 \Big/ \iint_{M\times M} dv_1\, dv_2,$$

is equal to the asymptotic Hopf invariant of the phase velocity field,

$$I(\xi) = \langle \operatorname{curl}^{-1} \xi, \xi\rangle.$$

Proof. Consider the Biot-Savart integral

$$\eta(x_2) = -\frac{1}{4\pi}\int_M \frac{[\xi(x_1), x_1 - x_2]}{\|x_1 - x_2\|^3}\, dv(x_2),$$

where [,] denotes the vector product. Then curl $\eta = \xi$ and therefore

$$\langle \eta, \xi\rangle = \langle \operatorname{curl}^{-1} \xi, \xi\rangle = +\frac{1}{4\pi}\iint_{M\times M} \frac{[\xi(x_1), \xi(x_2), (x_1 - x_2)]}{\|x_1 - x_2\|^3}\, dv(x_1)\, dv(x_2),$$

as was to be proved.

Remark. There is a similar result for any compact simply connected three-dimensional Riemannian manifold M, but the Gauss integral has to be replaced by the integral of de Rham's "double form;" this form cannot be written as explicitly, but has similar properties.

Remark. The question of *whether the asymptotic and mean linking numbers are invariant for a pair of trajectories under homeomorphisms that preserve the volume element* remains open, as does the closely related question of *whether one can define asymptotic and mean linking numbers for trajectories of one-parameter groups of volume-preserving homeomorphisms.*

5. Applications to the variational problem

From the existence of the Hopf invariant there follow some lower bounds for the energy of any field obtained from a given field by a volume-preserving diffeomorphism. In particular, on any three-dimensional Riemannian manifold one can find a field that is minimal in its class. In particular, certain special force-free fields have this property.

5.1. *Minimal force-free fields*

Let M be a three-dimensional closed Riemannian manifold. We consider the operator curl on divergence-free fields that are homologous to zero (i.e., have a single-valued divergence-free potential). By Weyl's lemma on orthogonal projctions, we can define a single-valued inverse of the operator curl on our space, so that there is an inverse (integral) operator curl^{-1} from the space of divergence-free fields that are homologous to zero, onto itself. This operator is symmetric, and its spectrum accumulates at zero on both sides.

Theorem. *The eigenfield of* curl^{-1} *corresponding to the eigenvalue* v *of largest modulus has minimum energy in the class of divergence-free fields obtained from the eigenfield under the action of volume-preserving diffeomorphisms.*

Proof. Let v_- and v_+ be the smallest and largest eigenvalues of the operator curl^{-1}. Then for every field ξ that is homologous to zero we have

$$v_-\langle \xi, \xi\rangle \leq \langle \text{curl}^{-1}\, \xi, \xi\rangle \leq v_+\langle \xi, \xi\rangle, \qquad v_- < 0 < v_+.$$

Consequently, we have the following bound for the energy in terms of the Hopf invariant:

$$\langle \xi, \xi\rangle \geq \langle \text{curl}^{-1}\, \xi, \xi\rangle / v,$$

where v denotes the value v_+ or v_- of larger modulus.

The inequality becomes an equality for the eigenfield with the eigenvalue v. The right-hand side of the inequality is invariant under volume-preserving diffeomorphisms (see Section 2). Consequently, under the action of such a diffeomorphism on the eigenfield with eigenvalue v, the field energy can only increase. This completes the proof of the theorem.

5.2. *Examples*

Let us take M to be the three-sphere with the usual Riemannian metric. The eigenfield of the operator curl^{-1} can be calculated explicitly. The eigenfields with largest and smallest eigenvalues are the Hopf field and its symmetric field (corresponding to Hopf invariant -1). The moduli of these eigenvalues are equal.

Corollary. *The Hopf field on the three-sphere has minimum energy among all fields obtained from it by the action of a volume-preserving diffeomorphism.*

(The field lines of the Hopf field are circles, and the linking coefficient of any two of them is 1.)

As another example, we consider the three-dimensional torus with the usual Riemannian metric. The eigenfields of the operator curl^{-1} with largest and smallest eigenvalues can be written explicitly in terms of sines and cosines. We obtain the following corollary:

Corollary. *Each of the fields*

$$\xi_x = A \sin z + C \cos y, \qquad \xi_y = B \sin x + A \cos z, \qquad \xi_z = C \sin y + B \cos z$$

on the three-dimensional torus has minimum energy among all fields obtained from it under volume-preserving diffeomorphisms.

Consequently, a minimal force-free field can have a complicated topology for its field lines, as is the case for generic nonintegrable systems (some field lines cover two-dimensional tori densely, others do not lie on any two-dimensional surfaces: see the experiment of Henon mentioned in Section 1.2).

In conclusion, we remark that we can extract from the asymptotic linking number $\lambda(x, y)$ more invariants than the mean linking number λ; for example, the measure $m(\lambda_0)$ of the set $\{x, y \in M \times M: \lambda(x, y) < \lambda_0\}$, or the value of the Hopf invariant for various regions that are invariant under the flow of a given field ξ. By using such invariants one can sometimes give lower bounds for the energy of a field obtained from a given field by the action of diffeomorphisms, more precisely than those found by using only the Hopf invariant.

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Note added June 6, 1985. A survey of modern generalizations of the asymptotic Hopf invariant is given in [6]. In the simplest generalization one begins with two closed 2-forms a, b on S^4 such that $as^2 = a \wedge b = b^2 = 0$ and considers $I(a, b) = p \int a \wedge d^{-1}a \wedge d^{-1}b + q \int b \wedge d^{-1}a \wedge d^{-1}b$. Such forms define two foliations of S^4 into surfaces intersecting along lines and the functional I probably has an asymptotic ergodic description similar to that given here for the Hopf invariant.

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