CHARACTERISTIC CLASS ENTERING IN QUANTIZATION CONDITIONS

V. I. Arnol'd

Recently V. P. Maslov gave a mathematically rigorous treatment of multidimensional asymptotic methods of "quasiclassical" type in the large, i.e., for any number of conjugate points [1, 2]. It turned out that there appeared in the asymptotic formulas certain integers, reflecting homological properties of curves on surfaces of the phase space and closely related to the Morse indexes of the corresponding variational problems. In particular, Maslov defined a one-dimensional class of integer-valued cohomologies whose values on the basis cycles enter into the so-called "quantization conditions."

In this note we give new formulas for the calculation of this class of cohomologies. This class is characteristic in the category of real vector bundles, whose complexification is trivial and trivialized, and also in certain wider categories.

§ 1. NOTATION

1.1. Phase Space

Phase space will be 2n-dimensional real arithmetic space

$$\mathbf{R}^{2n} = \{x\}, \quad x = q,\ p; \quad q = q_1, \ldots, q_n; \quad p = p_1, \ldots, p_n.$$

In $\mathbf{R}^{2n}$ we shall consider the following three structures:

1. The Euclidean structure, given by the scalar quadratic

$$(x,\ x) = p^2 + q^2.$$

2. The complex structure, given by the operator

$$I\colon\ \mathbf{R}^{2n} \to \mathbf{R}^{2n},\ I(p,\ q) = (-q,\ p); \quad z = p + iq, \quad \mathbf{C}^n = \{z\}.$$

3. The simplectic structure, given by the skew-scalar product

$$[x,\ y] = (Ix,\ y) = -[y,\ x]. \tag{1}$$

The automorphism groups of $\mathbf{R}^{2n}$ preserving these structures are called the orthogonal group $\mathbf{O}(2n)$, the complex linear group $\mathbf{GL}(n, \mathbf{C})$, and the simplectic group $\mathbf{Sp}(n)$, respectively. From (1) there follows

LEMMA 1.1. An automorphism preserving two of these structures preserves the third also, so that

$$\mathbf{O}(2n) \cap \mathbf{GL}(n,\ \mathbf{C}) = \mathbf{GL}(n,\ \mathbf{C}) \cap \mathbf{Sp}(n) = \mathbf{Sp}(n) \cap \mathbf{O}(2n) = \mathbf{U}(n).$$

The automorphisms preserving two (and thus all three) structures form the unitary group $\mathbf{U}(n)$. The determinant det of a unitary automorphism is a complex number with modulus 1. Thus there arises a mapping of $\mathbf{U}(n)$ onto the circle

$$\mathbf{SU}(n) \to \mathbf{U}(n) \xrightarrow{\det} \mathbf{S}^1, \tag{2}$$

which is obviously a fibering (the fiber is the group $\mathbf{SU}(n)$ of unitary automorphisms with determinant 1).

1.2. The Lagrangian Grassmanian $\Lambda(n)$

We consider an n-dimensional plane $\mathbf{R}^n \subset \mathbf{R}^{2n}$. It is called Lagrangian if the skew-scalar product of any two vectors of $\mathbf{R}^n$ equals zero. For example, the planes $p = 0$ and $q = 0$ are Lagrangian.*

* The name comes from the "Lagrange brackets" in classical mechanics.

Moscow State University. Translated from Funktsional'nyi Analiz i Ego Prilozheniya, Vol. 1, No. 1, pp.1-14, January-March, 1967. Original article submitted September 28, 1966.

The manifold of all (nonoriented) Lagrangian subspaces of $\mathbf{R}^{2n}$ is called the Lagrangian Grassmanian $\Lambda(n)$.

From the complex point of view Lagrangian planes can be called real-similar, since there holds

LEMMA 1.2. The unitary group $\mathbf{U}(n)$ acts on $\Lambda(n)$ transitively with stationary subgroup $\mathbf{O}(n)$.

Proof. Let λ be a Lagrangian plane. By (1) this means that the plane $I\lambda$ is orthogonal to λ. Let $\lambda' \in \Lambda(n)$ and ξ, ξ' be orthogonal frames in λ, λ'. Then the automorphism of $\mathbf{R}^{2n}$ carrying ξ into ξ' and $I\xi$ into $I\xi'$ is unitary.

From this lemma it follows that $\Lambda(n)$ is a manifold, $\Lambda(n) = \mathbf{U}(n)/\mathbf{O}(n)$; thus there is a fibering

$$\mathbf{O}(n) \dashrightarrow \mathbf{U}(n) \to \Lambda(n). \tag{3}$$

1.3 The Mapping Det^2: $\Lambda(n) \to S^1$

The determinant of an orthogonal automorphism $A \in \mathbf{O}(n) \subset \mathbf{U}(n)$ equals ± 1. Therefore the square of the determinant of a unitary automorphism carrying the plane $p = 0$ into the Lagrangian plane λ depends only on λ. In this way a mapping is constructed

$$\mathrm{Det}^2\colon \Lambda(n) \to S^1.$$

Denote by $S\Lambda(n)$ the set of Lagrangian planes $\lambda \in \Lambda(n)$ with $\mathrm{Det}^2 \lambda = 1$. On this set the group $\mathbf{SU}(n)$ of unitary unimodular automorphisms acts transitively, and the stationary subgroup of any point is isomorphic to the rotation group $\mathbf{SO}(n)$. Therefore $S\Lambda(n) = \mathbf{SU}(n)/\mathbf{SO}(n)$ is a manifold.

Thus we obtain a diagram (obviously commutative) of six fiberings:

$$\begin{tikzcd}
\mathbf{SO}(n) \arrow[r] \arrow[d] & \mathbf{O}(n) \arrow[r, "\det"] \arrow[d] & S^0, \arrow[d] \\
\mathbf{SU}(n) \arrow[r] \arrow[d] & \mathbf{U}(n) \arrow[r, "\det"] \arrow[d] & S^1, \arrow[d, "z^2"] \\
S\Lambda(n) \arrow[r] & \Lambda(n) \arrow[r, "\mathrm{Det}^2"] & S^1,
\end{tikzcd}$$

where z^2 is the mapping of the circle ($z = e^{i\varphi} \to e^{2i\varphi} = z^2$).

1.4. The Cohomology Class $\alpha \in H^1(\Lambda(n), Z)$

LEMMA 1.4.1. The fundamental group $\Lambda(n)$ is free cyclic,

$$\pi_1(\Lambda(n)) = \mathbf{Z},$$

and its generator goes into the generator of S^1 under the mapping induced by Det^2.

The proof is obtained from the exact homotopy sequences of the left column and lower row of the diagram of section 1.3.

COROLLARY 1.4.2. The one-dimensional homology and cohomology groups of $\Lambda(n)$ are free cyclic:

$$H_1(\Lambda(n), \mathbf{Z}) \simeq H^1(\Lambda(n), \mathbf{Z}) \simeq \pi_1(\Lambda(n)) \simeq \mathbf{Z}.$$

For the generator α of the cohomology group $H^1(\Lambda(n), Z)$ we take the number of rotations of Det^2, i.e., the cocycle whose value on a closed curve $\gamma\colon S^1 \to \Lambda(n)$ is equal to the degree of the composition

$$S^1 \xrightarrow{\gamma} \Lambda(n) \xrightarrow{\mathrm{Det}^2} S^1.$$

(Here S^1 is the circle $e^{i\varphi}$, oriented on the side of increasing φ.)

Example 1.4.3. Let λ be a Lagrangian plane: $\lambda \in \Lambda(n)$. Consider the automorphisms $e^{i\varphi}E \in \mathbf{U}(n)$. The Lagrangian planes $e^{i\varphi}\lambda$ ($0 \le \varphi \le \pi$) form a closed curve $\gamma\colon S^1 \to \Lambda(n)$, since $e^{i\pi}E = -E$.

The value of the class α on the curve γ equals n.

Indeed, $\det(e^{i\varphi}E) = e^{in\varphi}$, therefore $\mathrm{Det}^2 e^{i\varphi}\lambda = e^{2in\varphi}\mathrm{Det}^2\lambda$.

1.5. Lagrangian Manifolds

Let M be an n-dimensional submanifold of the phase space R^{2n}. The manifold M is called Lagrangian if its tangent plane at each point is Lagrangian. For example, in the case n = 1 every curve M on the phase plane R^2 is Lagrangian.

Let M be a Lagrangian manifold. We consider the tangential mapping

$$\tau\colon\ M \to \Lambda(n),$$

carrying each point $x \in M$ into the subspace $\tau x \in \Lambda(n)$ parallel to the tangent plane to M at x.

The cohomology class $\alpha \in H^1(\Lambda(n), \mathbf{Z})$ introduced above induces on M a one-dimensional cohomology class

$$\alpha^* = \tau^* \alpha \in H^1(M, \mathbf{Z}).$$

The value of α^* on an oriented closed curve $\gamma\colon\ S^1 \to M$ is defined as the number of rotations of the square of the determinant of the tangent plane, i.e., as the degree of the composition

$$S^1 \xrightarrow{\gamma} M \xrightarrow{\tau} \Lambda(n) \xrightarrow{\mathrm{Det}^2} S^1.$$

The aim of this note is the proof of the following assertion.

THEOREM 1.5. The cohomology class $\alpha^* \in H^1(M, Z)$ coincides with the "index of closed curves on the Lagrangian manifold M" introduced by Maslov in [1].

§ 2. PROOF OF THEOREM 1.5

The Maslov index is defined by him as the index of intersection of a certain two-sided (n-1)-dimensional cycle on M^n – the singular cycle.

2.1. The Singular Cycle

Let M be an n-dimensional Lagrangian manifold. Consider the projection $f\colon M \to \mathbf{R}^n$ of the manifold M onto the plane $p = 0$; $f(p, q) = q$. The set Σ of points of M where the rank of the differential of f is less than n is called the singularity of the mapping f. Regarding the singularity Σ, Maslov formulates the following assertions 1-5 (proofs are given below in § 3 and 4).

THEOREM 2.1. By an arbitrarily small unitary rotation the manifold M can be brought into "general position" relative to the projection f, so that the following assertions are valid:

ASSERTION 1. The singularity Σ consists of an open $(n-1)$-dimensional manifold Σ^1, where df has rank $n-1$, and the boundary $(\Sigma-\Sigma^1)$ dimension strictly less than $n-2$, so that Σ determines an $(n-1)$-dimensional (unoriented) cycle in M.

ASSERTION 2. This cycle is two-sided in M.

The choice of a positive side of Σ can be carried out in the following way.

ASSERTION 3. In a neighborhood of the point $x \in \Sigma^1$ the Lagrangian manifold M is given by n equations of the form

$$q_k = q_k(p_k, q_{\hat{k}}),\ p_{\hat{k}}\,(p_k, q_{\hat{k}}), \tag{4}$$

where $\hat{k} = 1, 2, \ldots, k-1, k+1, \ldots, n$ for some k, $1 \le k \le n$.

Obviously, in a neighborhood of such a point x the singularity Σ^1 is given by an equation $\dfrac{\partial q_k}{\partial p_k} = 0$.

ASSERTION 4. On passing through Σ^1 the quantity $\partial q_k/\partial p_k$ changes sign.

It turns out that for the positive side of Σ^1 we can take the one for which $\partial q_1/\partial p_1 > 0$.

ASSERTION 5. Such a definition of the positive side is correct, i.e., does not depend on which of the coordinate systems $p_k, q_{\hat{k}}$ $(k = 1, \ldots, n)$ we use.

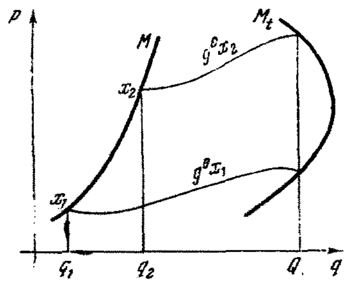

Fig. 1

2.2. The Maslov Index, $\mathrm{ind} \in H^1(M, Z)$

Suppose there is given on the Lagrangian manifold M in "general position" in the sense of Theorem 2.1. a curve γ, transversal to the cycle Σ, with the initial and end points of γ nonsingular:

$$\partial\gamma = x_1 - x_0,\ x_1 \bar{\in} \Sigma,\ x_0 \bar{\in} \Sigma.$$

Maslov calls the index ind γ of the curve γ its index of intersection with Σ, i.e., the number ν_+ of points of passage from the negative side to the positive side minus the number ν_- of points of passage from the positive side to the negative:

$$\mathrm{ind}\,\gamma = \nu_+ - \nu_-.$$

Example. Let n = 1 and let M be a curve in the pq-plane (Fig. 1). For M in general position, Σ consists of separate points a, b, c, ... The indexes of the curves γ_i ($\partial\gamma_i = x_i - x_0$) are equal to 0, 1, 0, 1, 2. respectively.

THEOREM 2.2. The index of a closed oriented curve γ on the Lagrangian manifold in general position M depends only on the homology class of γ and is the value of a one-dimensional class of integer-valued cohomologies of M on the cycle γ: $\mathrm{ind} \in H^1(M, Z)$.

2.3. Index of Curves on the Grassmanian $\Lambda(n)$

Proofs of the formulated theorems (1.5, 2.1, 2.2) are based on the following construction.

In the manifold of Lagrangian planes of $\Lambda(n)$ we single out the sets $\Lambda^k(n)$ of planes having a k-dimensional intersection with a fixed plane $\sigma \in \Lambda(n)$ (namely, the plane q = 0). It turns out that the closure $\overline{\Lambda^1(n)}$ determines a cycle (nonoriented) of codimension 1 (see 3.2.2).

In section 3.5 we prove

THE FUNDAMENTAL LEMMA. $\Lambda^1(n)$ is two-sidedly imbedded in $\Lambda(n)$, i.e., there exists a continuous vector field transversal to $\Lambda^1(n)$ and tangent to $\Lambda(n)$.

Such a vector field is constructed by means of the orbits of action of $S^1 = \{e^{i\theta}\}$ on $\Lambda(n)$. In § 3 we prove

LEMMA 3.5.1. Every circle

$$\theta \to e^{i\theta}\lambda, \quad 0 \leqslant \theta \leqslant \pi, \quad \lambda \in \Lambda(n), \tag{4}$$

is transversal to $\Lambda^1(n)$.

For the positive side of $\Lambda^1(n)$ we choose the one toward which the velocity vectors of the curves (4) are directed.

The two-sidedness of $\Lambda^1(n)$ allows us to define

$$\mathrm{Ind} \in H^1(\Lambda(n), Z)$$

as the index of intersection of oriented closed curves on $\Lambda(n)$ with $\overline{\Lambda^1(n)}$ (Definition 3.6.1).

The index Ind is connected with the Maslov index ind and the cohomology class α of section 1.4. Namely, it turns out that the choice of a positive side of $\Lambda^1(n)$ by means of the curves (4) agrees with the definition of the positive side of Σ^1 from section 1.2. In § 4 we prove

LEMMA 4.3.1. The index Ind generates the Maslov index ind under the tangential mapping τ: $M^n \to \Lambda(n)$; ind $= \tau^*$ Ind, i.e., for every curve γ: $S^1 \to M$ we have ind γ = Ind $\tau\gamma$.

Proof of Theorem 1.5. Calculation of the index of the curves (4) (see Example 3.6.2) gives Ind γ $= n = \alpha(\gamma)$ (Example 1.4.3). But $H^1(\Lambda(n), Z) = Z$ (Corollary 1.4.2). So, Ind = α. By Lemma 4.3.1 ind = τ^* Ind and by Definition 1.5. $\alpha^* = \tau^*\alpha$. Thus ind $= \alpha^*$, which was to be proved.

3. PROOF OF THE FUNDAMENTAL LEMMA

In this section we prove the two-sidedness of the singular cycle $\overline{\Lambda^1(n)}$ and define the index $\mathrm{Ind} \in H^1(\Lambda(n), Z)$.

3.1. Generating Functions

Let M be a manifold in phase space which is given in a simply connected neighborhood of the point $q = q_0$, $p = p_0$ by an equation of the form $p = p(q)$.

LEMMA 3.1.1. The manifold M is Lagrangian if and only if there is a "generating function" s(q) such that

$$p = \frac{\partial s}{\partial q}. \tag{5}$$

Proof. Let $s(q) = \int_{q_0}^{q} p(q)dq$. Independence of this integral of path is equivalent to the differential $d(p\,dq) = dp \wedge dq$ being 0 on M. But the value of $dp \wedge dq$ on the bivector $\xi \wedge \eta$ is exactly equal to the skew-scalar product $[\xi, \eta]$, so that equality of $dp \wedge dq$ to zero on M is equivalent to M being Lagrangian. The function s(q) satisfies (5), proving the lemma.

Remark 3.1.2. The function s(q) is determined up to a constant summand. In the particular case where M is a subspace, this summand can be chosed so that s(q) is a quadratic form. From this there follows

COROLLARY 3.1.3. The set of Lagrangian subspaces of the form $p = p(q)$ (i.e., transversal to the plane $q = 0$) make up in the manifold $\Lambda(n)$ an open cell $\Lambda^0(n)$, diffeomorphic to the linear space D of all real symmetric matrices of order n. The diffeomorphism is given by the mapping

$$\varphi\colon\ D \to \Lambda^0(n), \quad \varphi(S) = \lambda_S \quad (S \in D,\ \lambda_S \in \Lambda^0(n)),$$

where λ_S denotes the plane $p = Sq$.

The proof is obtained from (5) by setting $s(q) = \frac{1}{2}(Sq, q)$.

The space of symmetric matrices D is $R^{n(n+1)/2}$. Thus we have proved

COROLLARY 3.1.4. The manifold $\Lambda(n)$ has dimension

$$\dim \Lambda(n) = n(n+1)/2.$$

3.2. The Singular Cycle $\overline{\Lambda^1(n)}$

Notation 3.2.0. Let σ be the Lagrangian plane $q = 0$. We denote by $\Lambda^k(n)$ the set of all Lagrangian planes $\lambda \in \Lambda(n)$ whose intersection with the plane σ is k-dimensional

LEMMA 3.2.1. The set $\Lambda^k(n)$ is an open manifold of codimension $k(k+1)/2$ in the Lagrangian Grassmanian $\Lambda(n)$.

Proof. We compare with each plane $\lambda \in \Lambda^k(n)$ its intersection with the plane σ. There arises a mapping of $\Lambda^k(n)$ on the Grassman manifold $G_{n,k}$ of all k-dimensional subspaces of the n-dimensional space σ. It is easily verified that this mapping determines a fibering

$$\Lambda^0(n-k) \to \Lambda^k(n) \to G_{n,k}.$$

By Corollary 3.1.4 $\dim \Lambda^0(n-k) = \frac{(n-k)(n-k+1)}{2}$. Since $\dim G_{n,k} = k(n-k)$, we find

$$\dim \Lambda^k(n) = \frac{(n-k)(n-k+1)}{2} + k(n-k) = \frac{n(n+1)}{2} - \frac{k(k+1)}{2} = \dim \Lambda(n) - \frac{k(k+1)}{2},$$

which was to be proved.

COROLLARY 3.2.2. $\overline{\Lambda^1(n)}$ determines an (unoriented) cycle of codimension 1 in $\Lambda(n)$.

Proof. The manifold $\Lambda(n)$ can be considered algebraic. The closure $\overline{\Lambda^1(n)} = \bigcup_{k \geq 1} \Lambda^k(n)$ is an algebraic submanifold of codimension $1(k(k+1)/2 \geq 1$ for $k \geq 1)$. Therefore $\overline{\Lambda^1(n)}$ determines a (nonoriented) chain. The singularity of the algebraic manifold $\overline{\Lambda^1(n)}$ is the algebraic submanifold $\overline{\Lambda^2(n)} = \overline{\bigcup_{k\geq 2} \Lambda^k(n)}$ of codimension 3 in $\Lambda(n)$, since $k(k+1)/2 \geq 3$ for $k \geq 2$. Thus the homological boundary of the chain $\overline{\Lambda^1(n)}$ equals 0, which was to be proved.

3.3. Coordinates on $\Lambda(n)$

We consider a Lagrangian plane $\lambda \in \Lambda(n)$. Let $\lambda \in \Lambda^k(n)$, i.e., let the intersection $\lambda \cap \sigma$ be k-dimensional. We introduce coordinates on $\Lambda(n)$ in a neighborhood of λ.

Notation 3.3.0. Let K be a subset of the set 1,2, . . ., n. Denote by σ_K the Lagrangian coordinate plane

$$\sigma_K = \{p, q: p_k = 0, \quad q_l = 0 \quad \forall k \in K, \ \forall l \bar{\in} K\}.$$

LEMMA 3.3.1. The plane $\lambda \in \Lambda^k(n)$ is transversal to one of the C_n^k coordinate planes σ_K, where K has k elements.

Proof. The intersection $\lambda \cap \sigma = \lambda_0$ is k-dimensional. Consequently, the plane λ_0 in σ is transversal to one of the C_n^k $(n-k)$-dimensional coordinate planes $\tau = \sigma_K \cap \sigma$, i.e., for some K we have $\lambda_0 \cap \sigma_K \cap \sigma = 0$. We shall show that the plane σ_K is transversal to λ: $\sigma_K \cap \lambda = 0$.

By the condition, $\lambda_0 + \tau = \sigma$. From the Lagrangian property of λ and σ_K it follows that $[\lambda, \lambda_0] = 0$ (since $\lambda_0 \subset \lambda$) and $[\sigma_K, \tau] = 0$ (since $\tau \subset \sigma_K$). Thus, $[\lambda \cap \sigma_K, \lambda_0 + \tau] = 0$, i.e., $[\lambda \cap \sigma_K, \sigma] = 0$. But the largest number of pairwise skew-orthogonal independent vectors in R^{2n} equals n. Therefore the n-dimensional plane σ is itself a maximal skew-orthogonal plane, thus $(\lambda \cap \sigma_K) \subset \sigma$. So, $(\lambda \cap \sigma_K) \subseteq (\lambda \cap \sigma \cap \sigma_K) = (\lambda_0 \cap \tau) = 0$, which was to be proved.

From the lemma just proved it follows that every plane $\lambda \in \Lambda(n)$ is transversal to one of the 2^n coordinate planes σ_K. This allows us to set up an atlas of $\Lambda(n)$ of 2^n charts.

One of the maps was constructed in section 3.1.1: the region $\Lambda^0(n)$ is diffeomorphic to the space of symmetric matrices $D = R^{n(n+1)/2}$, where the diffeomorphism φ: $D \to \lambda^0(n)$ is defined as

$$\varphi(S) = \lambda_S = \{p, q: p = Sq\} \ \forall S \in D.$$

Notation 3.3.2. We denote by I_K the operator of multiplication by i of the variables $z_\varkappa = p_\varkappa + iq_\varkappa$, $\varkappa \in K$:

$$I_K: \ R^{2n} \to R^{2n},$$

and for $\eta = I_K \xi$

$$q_\varkappa(\eta) = +p_\varkappa(\xi), \quad p_\varkappa(\eta) = -q_\varkappa(\xi), \quad \forall \varkappa \in K,$$

$$q_\nu(\eta) = q_\nu(\xi), \qquad p_\nu(\eta) = p_\nu(\xi), \qquad \forall \nu \bar{\in} K.$$

The operator I_K is unitary, therefore it carries Lagrangian planes into Lagrangian planes. In particular, $I_K\sigma = \sigma_K$. Thus I_K carries the set $\Lambda^0(n)$ of planes transversal to σ into the set of planes $I_K\Lambda^0(n)$ transversal to σ_K. Thus the formula

$$\varphi_K(S) = I_K\lambda_S \in \Lambda(n), \quad S \in D \tag{6}$$

gives a diffeomorphism φ_K: $D \to I_K\Lambda^0(n)$, where $I_K\Lambda^0(n)$ is the set of all Lagrangian planes transversal to σ_K.

By Lemma 3.3.1 the 2^n regions $I_K\Lambda^0(n)$ cover $\Lambda(n)$ entirely, so that formula (6) gives an atlas of $\Lambda(n)$ of 2^n charts.

LEMMA 3.3.3. The set $\Lambda^k(n)$ is covered by C_n^k charts $\varphi_K D$(K consisting of k elements) and in the coordinates $S = \varphi_K^{-1}\lambda$ is given by $k(k+1)/2$ linear equations $S_{\mu\nu} = 0$ $(\forall \mu \in K, \forall \nu \in K)$.

Proof. Let $\dim \lambda \cap \sigma = k$. By Lemma 3.3.1 $\lambda \cap \sigma_K = 0$ for some K of k elements. Therefore the plane $I_K\lambda = I_K^{-1}\lambda$ is transversal to σ and has an equation $p = Sq$. The intersection $(I_K\lambda) \cap \sigma_K = I_K(\lambda \cap \sigma)$ has dimension k. But on σ_K we have $q_l = 0$ $(\forall l \bar{\in} K)$, $p_m = 0$ $(\forall m \in K)$. Therefore on a k-dimensional subspace $q_l = 0$ $(l \bar{\in} K)$ of the plane $p = 0$, k of the functions p_μ $(p = Sq, \ \mu \in K)$ must vanish identically. This is equivalent to the equations $S_{\mu\nu} = 0$, as was to be proved.

3.4. Unitary Paramaterization

By means of the coordinates S introduced above it is possible to express the unitary transformations carrying the "purely imaginary" plane $p = 0$ into the plane $\lambda_S \in \Lambda^0(n)$.

It is obvious that we have

LEMMA 3.4.1. Let S, U be square $n \times n$ matrices with complex elements. Then

$$\left(U = \frac{E - iS}{E + iS} \right) \Leftrightarrow \left(S = -i \frac{E - U}{E + U} \right), \tag{7}$$

and for S, U, related by formulas (7),

S is self-adjoint if and only if U is unitary,

S is symmetric if and only if U is symmetric.

Thus, formulas (7) set up a diffeomorphism between the space D of real symmetric matrices S and the manifold of unitary symmetric nonsingular matrices U. (The unitary matrix U is nonsingular if -1 is not a proper value; for a real symmetric S we always have $\det(E + iS) \neq 0$).

It is always possible to take the square root of a nonsingular unitary matrix, defining it by continuity, beginning from $\sqrt{E} = E$.

LEMMA 3.4.2. Let $\lambda_S \in \Lambda^0(n)$ be a plane $p = Sq$. Then the matrix

$$\sqrt{U} = \frac{E - iS}{\sqrt{E + S^2}}$$

gives a unitary transformation carrying the plane $p = 0$ into the plane λ_S.

Proof. Since S is symmetric and real, $\sqrt{E + S^2}$ is real, and $\sqrt{U}$ carries the plane $p = 0$ into the same image as $E - iS$. The latter transformation carries the point

$$(0,\ q) \in \mathbf{R}^{2n}, \quad \text{i.e.}, \quad iq \in i\mathbf{R}^n \subset \mathbf{C}^n$$

into the point

$$iq + Sq \in \mathbf{C}^n, \quad \text{i.e.}, \quad (Sq,\ q) \in \lambda_S \subset \mathbf{R}^{2n},$$

which was to be proved.

COROLLARY 3.4.3. The mapping $\mathrm{Det}^2\colon \Lambda(n) \to S^1$ of section 1.3 is given by the formula

$$\mathrm{Det}^2 \lambda_S = \det \frac{E - iS}{E + iS}.$$

COROLLARY 3.4.4. The symmetric nonsingular unitary matrix U for which $\sqrt{U}$ carries the plane $p = 0$ into the plane λ is uniquely determined by this plane $\lambda \in \Lambda^0(n)$.

In fact, by 3.4.1 U is uniquely determined by S, and by 3.4.2 S is uniquely determined by λ.

3.5 Two-Sidedness of the Singular Cycle

Let λ be a Lagrangian plane. Then each of the planes $e^{i\theta}\lambda$ is Lagrangian.

LEMMA 3.5.1. If $\lambda \in \Lambda^1(n)$, then the curve $\gamma\colon S^1 \to \Lambda(n), e^{i\theta} \to e^{i\theta}\lambda$ is transversal to the cycle $\Lambda^1(n)$ at the point $\theta = 0$.

Thus, the velocity vectors $v(\lambda) = \left.\frac{d}{d\theta}\right|_{\theta=0} (e^{i\theta}\lambda)$ form a transversal structure to $\Lambda^1(n)$, from which there follows the

FUNDAMENTAL LEMMA. The singular cycle $\overline{\Lambda^1(n)}$ is two-sidedly imbedded in $\Lambda(n)$.

We will carry out the proof of Lemma 3.5.1 in three stages.

A. First let $\lambda \in \Lambda^0(n)$, $\lambda = \lambda_S$, where $S \in D$ is a real symmetric matrix. We shall compute the coordinates of the velocity vector of the curve $e^{i\theta}\lambda$ in this coordinate system.

LEMMA 3.5.2. For any matrix $S \in D$

$$\left.\frac{d}{d\theta}\right|_{\theta=0} \varphi^{-1} e^{i\theta} \lambda_S = -(E + S^2).$$

Proof of Lemma 3.5.2. According to section 3.4 the plane λ is in a unique correspondence with a nonsingular unitary symmetric matrix U so that

$$\lambda = \lambda_{S(U)}, \qquad S(U) = -i\frac{E-U}{E+U},$$

and $\sqrt{U}$ carries the plane $p = 0$ into λ.

Let $U(\theta) = e^{2i\theta}U$. The matrix $U(\theta)$ is unitary, symmetric, and for small $|\theta|$ nonsingular, so that $\sqrt{U(\theta)} = e^{i\theta}\sqrt{U}$. Therefore $\sqrt{U(\theta)}$ carries the plane $p = 0$ into $e^{i\theta}\lambda$, so that

$$\lambda_{S(U(\theta))} = e^{i\theta}\lambda_{S(U)}, \text{ or } \varphi^{-1}e^{i\theta}\lambda_{S(U)} = S(U(\theta)). \tag{8}$$

The vector $\left.\frac{d}{d\theta}\right|_{\theta=0} \varphi^{-1}e^{i\theta}\lambda$ lies in the tangent space to the linear space of symmetric matrices D; this tangent space is naturally identified with D itself. With this identification, by the formula of 3.4.1,

$$-\left.\frac{d}{d\theta}\right|_{\theta=0} S(U(\theta)) = \left.\frac{d}{d\theta}\right|_{\theta=0} i\frac{E-e^{2i\theta}U}{E+e^{2i\theta}U} = \frac{4U}{(E+U)^2} = E+S^2,$$

which together with the formulas (8) proves Lemma 3.5.2.

B. Now let $\lambda \in \Lambda^1(n)$. By Lemma 3.3.3 the point $\lambda \in \Lambda^1(n)$ belongs to one of the n charts $\varphi_K D$, where K consists of one element $\varkappa$, $1 \le \varkappa \le n$. In other words, in the notation of 3.3.2

$$\lambda = I_K\lambda_S, \text{where } S \in D, \quad \lambda_S \in \Lambda^0(n).$$

It is easy to calculate the velocity vector of the curve γ: $S^1 \to \Lambda(n)$, $e^{i\theta} \to e^{i\theta}\lambda$ for $\theta = 0$ in the coordinate system $\varphi_K^{-1}\lambda = S$.

LEMMA 3.5.3. For any matrix $S \in D$, $\lambda = I_K\lambda_S$,

$$\left.\frac{d}{d\theta}\right|_{\theta=0} \varphi_K^{-1}e^{i\theta}\lambda = -(E+S^2).$$

Indeed, by Definition 3.3.2, $\varphi_K = I_K\varphi$, and I_K commutes with $e^{i\theta}$. Therefore

$$\varphi_K^{-1}e^{i\theta}\lambda = \varphi^{-1}I_K^{-1}e^{i\theta}I_K\lambda_S = \varphi^{-1}e^{i\theta}\lambda_S,$$

and Lemma 3.5.3 follows from Lemma 3.5.2.

C. The singular cycle $\Lambda^1(n)$ in the coordinates $S = \varphi_K^{-1}\lambda$ has an equation $S_{\varkappa\varkappa} = 0$ (Lemma 3.3.3). The velocity vector $v = \left.\frac{d}{d\theta}\right|_{\theta=0} \varphi_K^{-1}e^{i\theta}\lambda$ by Lemma 3.5.3 is a negative definite matrix, $-v_{\varkappa\varkappa} \ge 1$. Thus, v and $\Lambda^1(n)$ are transversal, which proves Lemma 3.5.1.

Remark 3.5.4. At the same time we have shown that the vector v is directed to the side of $\Lambda^1(n)$ where $S_{\varkappa\varkappa} > 0$.

3.6. The Index Ind of Curves on $\Lambda(n)$

Let γ be an oriented curve in $\Lambda(n)$, transversal to $\Lambda^1(n)$, and let $v(\lambda)$ be the velocity field of Lemma 3.5.1.

DEFINITION 3.6.1. By Ind γ we shall denote the index of intersection of the curve γ with the cycle $\Lambda^1(n)$, equipped with the field $v(\lambda)$.

In other words, Ind $\gamma = \nu_+ - \nu_-$, where ν_+ is the number of points of intersection of γ with $\Lambda^1(n)$ in which the vectors $\dot\gamma$ and v lie on the same side of $\Lambda^1(n)$, and ν_- is the number of those on which they lie on opposite sides.

The index of the closed curve γ, like every index of intersection, is determined by the homology class of γ and can be considered as a one-dimensional cohomology class

$$\mathrm{Ind} \in H^1(\Lambda(n), Z).$$

EXAMPLE 3.6.2. The index of the closed curve γ: $S^1 \to \Lambda(n)$, formed by $e^{i\theta}\lambda$, $0 \le \theta \le \pi$, is n:

$$\mathrm{Ind}\,\gamma = n.$$

Proof. We have that $\dim \Lambda^2(n) = \dim \Lambda^1(n)$ is 2 (Lemma 3.2.1). Therefore for almost all λ the curve $e^{i\theta}\lambda$ does not intersect $\overline{\Lambda^2(n)}$. Such a curve is transversal to $\Lambda^1(n)$ at every point of intersection (Lemma 3.5.1). In this case Ind γ is simply the number of these points of intersection (Definition 3.6.1).

Let $\lambda \in \Lambda^0(n)$. By Lemma 3.4.4 we have $\lambda = \lambda_{S(U)}$, where U is a unitary symmetric nonsingular matrix. We may consider the plane λ to be such that all proper vales of the matrix U

$$e^{i\alpha_k}, \ 1 \leqslant k \leqslant n, \quad |\alpha_k| < \pi,$$

are distinct.

But

$$e^{i\theta}\lambda = \lambda_{S(e^{2i\theta}U)}$$

by formula (8) of 3.5.2, and

$$(\lambda_{S(e^{2i\theta}U)} \overline{\in} \Lambda^0(n)) \Leftrightarrow (\det(E + e^{2i\theta}U) = 0)$$

by Lemma 3.4.2. In other words, in the points of intersection of γ with $\Lambda^1(n)$

$$\theta \equiv \frac{\pi - \alpha_k}{2} \pmod{\pi}.$$

There are precisely n such θ on the interval $0 \le \theta \le \pi$. Thus, Ind $\gamma = n$, as was to be proved.

§ 4. PROOFS OF THE THEOREMS ON GENERAL POSITION

Here we prove Theorems 2.1 and 2.2 of § 2.

4.1. Transversality

Notation 4.1.1. Let A be a smooth manifold, and let $a \in A$. By TA_a we denote the tangent space to A at the point a. If f: $A \to B$ is a smooth mapping, then by f_*: $TA_a \to TB_{f(a)}$ we denote the differential of f at a.

Let f: $A \to B$, h: $C \to B$ be two smooth mappings. The mappings f, h are called transversal at the point $b \in B$ if

$$f_* TA_a + h_* TC_c = TB_b$$

for every pair of points $a \in A$, $c \in C$, for which $f(a) = h(c) = b$. The mappings f and h are transversal if they are transversal at every point $b \in B$.

In the particular case where f or h is an imbedding, we may speak of the transversality of a mapping to a submanifold or of the transversality of two submanifolds.

The concept of transversality extends also to the case where A is the union of several manifolds, $A = \cup A_k$ [for example, $\overline{A^1(n)} = \bigcup_{k \geqslant 1} \Lambda^k(n)$ in § 3] — in this case the restriction of f to each A_k must be transversal to h.

It is easy to prove (see, for example, [3]) the lemma of M. Morse and A. Sard

LEMMA 4.1.2. Let f: $A \to B$ be a smooth mapping. Then the measure of the set of points $b \in B$ not transversal to f equals 0 (the point $b \subset B$ is a zero-dimensional submanifold of B).

From Lemma 4.1.2 there follows (see, for example, [4])

LEMMA 4.1.3. Let B be a homogeneous space on which a Lie group G acts transitively ($\forall g \in G$, g: $B \to B$ is a diffeomorphism). Let $C \subset B$ be a smooth submanifold of B and let f: $A \to B$ be a smooth mapping. Then the measure of the set of points $g \in G$ for which the mapping

$$f_g\colon A \to B, \quad f_g(a) = gf(a)$$

is not transversal to C is zero.

For completeness we carry out the proof of Lemma 4.1.3.

Remark 4.1.4. Since the union of a countable number of sets of measure zero has measure zero, it is sufficient to prove Lemma 4.1.3 for a neighborhood A_0 of a point $a_0 \in A$, a neighborhood C_0 of a point $c_0 \in C$ and a neighborhood of the identity e in the group G.

From the transitivity of the action of G there follows easily

ASSERTION 4.1.5. There exists a diffeomorphism of the product of spheres

$$u : D_1 \times D_2 \to G,$$

$$D_i = \{x \in R^{\nu_i}, |x| < 1\}, \quad \nu_1 = \dim B - \dim C, \quad \nu_2 = \dim G - \nu_1,$$

such that $u(0, 0) = e$, and the mapping

$$\beta: D_1 \times D_2 \times C_0 \to B \times D_2,$$

given by the formula

$$\beta(x, y, c) = (u(x, y)c, y), \forall x \in D_1, y \in D_2, c \in C_0,$$

is a diffeomorphism of $D_1 \times D_2 \times C_0$ onto some neighborhood E of the point $(c_0, 0)$ in $B \times D_2$.

Now define the projection of $E \subset B \times D_2$ on $D_1 \times D_2$

$$\Phi: E \to D_1 \times D_2 \text{ by the formula } \Phi(\beta(x, y, c)) = (x, y).$$

Further, define the mapping

$$\hat{f}: A \times D_2 \to B \times D_2 \text{ by the formula } \hat{f}(a, y) = (f(a), y).$$

We apply Lemma 4.1.2 to the composite mapping

$$\Theta = \Phi \circ \hat{f}: A_0 \times D_2 \to D_1 \times D_2.$$

ASSERTION 4.1.6. Let $x,y \in D_1 \times D_2$ be a point transversal to the mapping Θ. Then the mapping

$$f_g: A_0 \to B, \; g = (u(x, y))^{-1},$$

is transversal to the imbedding $C_0 \subset B$.

Proof of Assertion 4.1.6. Consider $\Phi^{-1}(x, y) (x \in D_1, y \in D_2)$. Obviously, $\Phi^{-1}(x, y) = (C_{x,y}, y)$, where $C_{x,y} = u(x, y)C_0 \subset B$. The kernel of the differential $\Phi_*: T(B \times D_2)_{b,y} \to T(D_1 \times D_2)_{x,y}$ is exactly the tangent space to $(C_{x,y}, y)$:

$$\ker \Phi_* = T(C_{x,y}, y)_{b,y}.$$

Therefore the transversality of the mapping $\Theta = \Phi \circ \hat{f}$ to the point x,y implies

$$\hat{f}_* T(A_0 \times D_2)_{a,y} + T(C_{x,y}, y)_{f(a),y} = T(B \times D_2)_{b,y}$$

for all $a \in A_0$, for which $f(a) = b \in C_{x,y}$. Thus, the mapping $f: A_0 \to B$ is transversal to the imbedding $C_{x,y} \subset B$. Applying the diffeomorphism $g = (u(x, y))^{-1} \in G$, we see that $gf: A_0 \to B$ is transversal to $gC_{x,y} = C_0$, as required.

Proof of Lemma 4.1.3. We apply Lemma 4.1.2 to Θ. The set of points $x, y \in D_1 \times D_2$ not transversal to Θ has measure zero. The corresponding set of points $g = (u(x, y))^{-1} \in G$ has measure zero in G. For the remaining g close to e the mapping f_g is transversal to C_0 by 4.1.6. This proves Lemma 4.1.3 according to remark 4.1.4.

4.2. Proof of Theorem 2.1.

We apply Lemma 4.1.3 to the case where A is a Lagrangian manifold M^n, B is the Lagrangian Grassmanian $\Lambda(n)$, f is the tangential mapping $\tau: M^n \to \Lambda(n)$, C is the submanifold $\Lambda^k(n) \subset \Lambda(n)$, and G is the unitary group $\mathbf{U}(n)$.

From Lemma 4.1.3 it follows that for almost all $u \in \mathbf{U}(n)$ the manifold uM^n is such that its tangential mapping τ is transversal to every $\Lambda^k(n) \subset \Lambda(n)$, $k = 1, 2, \ldots$. Let us show that such a manifold uM^n is in "general position" in the sense of Theorem 2.1.

Assertion 1 of Theorem 2.1 follows from the implicit functions theorem and Lemmas 3.2.1 and 3.2.2. Assertion 2 follows from the fundamental lemma of section 3.5. Assertion 3 is deduced from Lemma 3.3.1. Assertion 4 is obtained from Lemma 3.3.3 for k = 1. Finally, Assertion 5 follows from Lemma 3.5.1 and remark 3.5.4. Theorem 2.1 is thus proved.

4.3. Proof of Theorem 2.2

Let M^n be a Lagrangian manifold in general position, and let $\gamma: S^1 \to M^n$ be an oriented closed curve, transversal to the singular cycle Σ.

LEMMA 4.3.1. Let $\tau\circ\gamma: S^1 \to \Lambda(n)$ be the tangential mapping of M^n onto the curve γ. Then

$$\operatorname{ind}\gamma = \operatorname{Ind}\tau\circ\gamma.$$

In fact, $\Sigma^1 = \tau^{-1}\Lambda^1$ (n) (definitions of sections 2.1 and 3.2). Further, the positive (in the sense of section 2.1) side of Σ^1 is carried under the mapping τ into the positive (in the sense of Definition 3.6.1) side of $\Lambda^1(n)$ —this follows from remark 3.5.4. Thus, each point of intersection of γ with Σ^1 gives to ind the same contribution as the corresponding point of intersection of $\tau\circ\gamma$ with $\Lambda^1(n)$ to Ind, which proves Lemma 4.3.1.

At the same time we have proved Theorem 2.2. also, since Ind $\tau\circ\gamma$ does not change under replacement of γ by a homologous curve γ' [this follows from the fact that $\dim \overline{\Lambda^2(n)} = \dim \Lambda^1(n) - 2$].

§ 5. A QUASICLASSICAL ASYMPTOTIC EXPRESSION

Here we give without proof the asymptotic formulas of Maslov, in which the index plays a role, for the simplest example.

5.1. Asymptotic Expression as $h \to 0$ of the Solution of the Schrödinger Equation

$$ih\frac{\partial\psi}{\partial t} = -\frac{h^2}{2}\Delta\psi + U(q)\psi,\ \psi = \psi(q, t),\ q\in \mathbf{R}^n \tag{9}$$

with the initial condition

$$\psi|_{t=0} = \varphi(q)e^{\frac{i}{h}f(q)}, \text{where } \varphi(q) \text{ is a finite function.} \tag{10}$$

To the Schrödinger equation there corresponds the classical dynamical system given in the 2n-dimensional phase space $\mathbf{R}^{2n}$ by the Hamilton equations

$$\dot q = \frac{\partial H}{\partial p},\ \dot p = -\frac{\partial H}{\partial q},\quad H = \frac{1}{2}p^2 + U(q).$$

The solutions of the Hamilton equation define a one-parameter group of canonical* diffeomorphisms of the phase space—the phase flow $g^t: \mathbf{R}^{2n} \to \mathbf{R}^{2n}$.

To the initial condition (10) there corresponds a function $\varphi(q)$ on a surface M^n, given in the phase space $\mathbf{R}^{2n}$ by the equations

$$M = \left\{p, q\colon p(q) = \frac{\partial f}{\partial q}\right\}.$$

The surface M is projected uniquely onto the q-plane. It is Lagrangian by Lemma 3.1.1. The phase flow g^t carries M into another Lagrangian surface $g^t M = M_t$. The surface M_t is no longer necessarily projected uniquely onto the q-plane. There arises a mapping $Q(q) = q(g^t(p(q), q))$ (Fig. 2).

Let $x_j = (p_j, q_j)$ be points of M such that $g^t x_j = (P_j, Q) \in M_t$. Assume that the Jacobian $\left.\frac{DQ}{Dq}\right|_{q=q_j} \neq 0$.

Maslov proved the following asymptotic formula for the solution of equation (9) with the condition (10).

THEOREM 5.1. As $h \to 0$

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*A diffeomorphism $g: \mathbf{R}^{2n} \to \mathbf{R}^{2n}$ is called canonical, if for every closed curve $\gamma: S^1 \to \mathbf{R}^{2n}$ we have $\oint_\gamma pdq = \oint_{g\gamma} pdq$. The differential g_* has then at each point a simplectic matrix: under a canonical transformation the 2-form $dp \wedge dq$ goes into itself. Therefore a Lagrangian manifold under a canonical diffeomorphism goes into a Lagrangian.

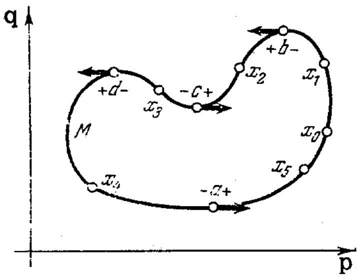

Fig. 2

$$\psi(Q, t) = \sum \varphi(q_j) \left| \frac{DQ}{Dq_j} \right|^{-1/2} e^{\frac{i}{h} S_j(Q,t) - \frac{i\pi}{2} \mu_j} + O(h),$$

where $S_j(Q, t)$ is the action along the trajectory $g^\theta x_j$:

$$S_j(Q, t) = f(q_j) + \int_0^t L \, d\theta, \quad L = \frac{q^2}{2} - U(q); \quad (p(\theta), q(\theta)) = g^\theta x_j,$$

and μ_j is the Morse index of the trajectory $g^\theta x_j$, i.e., the number of points $q(g^\theta x_j)$, $0 < \theta < t$, focal to M.

5.2. Connection between the Maslov and Morse Indexes

In Theorem 5.1 there appeared the Morse index μ. The Morse index is a special case of the Maslov index; there holds

THEOREM 5.2. Consider in the $(2n + 2)$-dimensional phase space $\mathbf{R}^{2n+2} = \{\hat{p}, \hat{q}\}$, $\hat{p} = p_0, p$; $\hat{q} = q_0, q$; $(p, q) \in \mathbf{R}^{2n}$, the $(n + 1)$-dimensional manifold $\hat{M}$: $(p, q) \in M_t$, $q_0 = t$, $p_0 = -H(p, q)$. Then the manifold $\hat{M}$ is Lagrangian, and the Morse index of the trajectory $g^\theta x$, $0 < \theta < t$, is the Maslov index of the curve $(\theta, -H, g^\theta x)$ on the manifold $\hat{M}$.

The proof follows easily from the definitions of the indexes μ and ind: since $\left(\frac{\partial^2 H}{\partial p^2} \xi, \xi\right) > 0$, a simple focal point gives a contribution of + 1 to ind.

COROLLARY. For any curve γ on M

$$\operatorname{ind} g^t \gamma - \operatorname{ind} \gamma = \mu(g^\theta \gamma^+) - \mu(g^\theta \gamma^-),$$

where $g^\theta \gamma^+$, $g^\theta \gamma^-$ $(0 \le \theta \le t)$ are trajectories with endpoints γ, $\partial\gamma = \gamma^+ - \gamma^-$.

For, the quadrilateral γ, $g^\theta \gamma^+$, $(g^t \gamma)^{-1}$, $(g^\theta \gamma^{-1})^{-1}$ on $\hat{M}$ is, obviously, homologous to zero; therefore its Maslov index equals zero (Theorem 2.2), which in view of Theorem 5.2 proves the desired relationship.

5.3. Quantization Conditions

In Theorem 5.1 there appear indexes of curves which are not closed. The indexes of closed curves enter into asymptotic formulas for stationary problems.

Let M be an invariant Lagrangian manifold of the phase flow g^t, lying on the hypersurface $H = E$ (such invariant manifolds exist not only for integrable systems: see [5]).

Maslov proved

THEOREM 5.3. The equation

$$\frac{1}{2} \Delta\psi = \lambda^2 (U(q) - E) \psi$$

has a series of proper numbers $\lambda_N \to \infty$ with asymptotic expressions $\lambda_N = \mu_N + O(\mu_N^{-1})$ if for every $\gamma \in H_1(M, Z)$

$$\frac{2\mu_N}{\pi} \oint_\gamma p \, dq \equiv \operatorname{ind} \gamma \pmod 4. \tag{11}$$

In this case the characteristic functions ψ_N are also related to the manifold M (in a sense defined precisely in [1] and under assumptions of the type of a simple spectrum).

In the particular case $n = 1$ the index of the circle equals 2, and formula (11) becomes the classical "quantization condition"

$$\mu_N \oint_\gamma p \, dq = 2\pi (N + 1/2).$$

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