Russian Math. Surveys **50**:1 1–68

Uspekhi Mat. Nauk **50**:1 3–68 UDC 514

The geometry of spherical curves and the algebra of quaternions

V. I. Arnol'd

Contents

Introduction

The spreading of a wave over the surface of a two-dimensional sphere may be described by means of various curves—wave fronts, rays, caustics, and so on.

A system of wave fronts is a system of equidistants of a single curve. Even if this curve is smooth the wave fronts in general have singularities. The singularities of all the wave fronts of the system themselves form a curve, called the caustic. The caustic in general has cusps, just as the wave fronts do.

Another description of the spreading of a wave is given by a Lagrangian curve. Such a curve divides the area of the sphere in half but in general has no singularities apart from self-intersections.

The connections between all these curves, their singularities and their topological invariants are clarified by means of three natural complex structures of quaternion space: namely the connections between three projections of the same Legendrian knot along three Hopf bundles $S^3 \to S^2$.

The following are studied below as consequences of these connections:

1) several generalizations of the classical four-vertex theorem for a plane curve;
2) an unusual version of the Gauss–Bonnet formula for an immersed circle in the sphere and for wave fronts;
3) an explicit formula for the Maslov index of a Lagrangian curve on the 2-sphere;
4) unexpected connections between the indices of points with respect to hypersurfaces on even-dimensional spheres;
5) conformal invariants of an immersed circle in the plane that generalize the Bennequin invariant;
6) new—partly proved and partly conjectured—generalizations of the Morse inequalities and theorems of Chekanov and Givental' on Lagrangian intersections and Legendrian links in symplectic and contact topology;
7) theorems on a duality between area and length in spherical geometry.

In a natural way all these facts arise directly from consideration of the influence exerted by the symmetry of the quaternion units on the symplectic geometry of the 2-sphere and the contact geometry of the 3-sphere. Without these connections the facts stated might have remained unnoticed for a long time.

Since, however, the quaternions can be banished from the formulation of the theorems in question (and, if it were desired, from their proof), I begin straight away with their formulation (the motivation of which, for the most part, lies in the symmetry of the quaternions).

CHAPTER I

THE GEOMETRY OF CURVES ON S^2

We present here geometrical constructions which relate caustics and wave fronts on the 2-sphere. The meaning of these constructions would be clearer if considered from the point of view of contact and symplectic geometry. However I begin with very elementary formulations, which do not require any prior knowledge for their understanding.

§ 1. The elementary geometry of smooth curves and wave fronts

With every curve on the 2-sphere of radius 1 we associate two other curves. The first of these is obtained by moving each point of the curve a distance $\pi/2$ along normals to the original curve. This curve consists of the centres of curvature of great circles of the sphere tangent to the original curve. For this reason it is said to be dual to the original curve (Fig. 1).

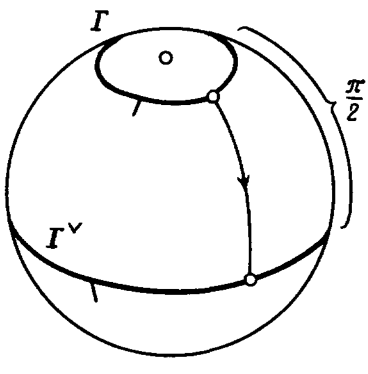

Figure 1

Definition. The curve *dual* to a given co-oriented curve on the sphere is the curve obtained from the original curve by moving a distance $\pi/2$ along the normals on the side determined by the co-orientation.

This definition applies not only to smoothly immersed curves, but also to wave fronts, having cusps (of semicubical type or, in general, of type $x^a = y^{a+1}$).

The dual curve itself is naturally co-oriented and is a wave front equidistant from the original one (lying at a distance $\pi/2$ from it).

The cusps on the original front correspond to points of inflection on the dual front, while points of inflection on the original one correspond to cusps on the dual (a point of inflection of a spherical curve is a point having an order of contact greater than one with the tangent great circle).

The second dual of a front is antipodal to the original one, while the fourth coincides with the original one.

Definition. The *derivative* of a co-oriented curve on the oriented standard sphere S^2 is the curve obtained by moving each point a distance $\pi/2$ along the great circle tangent to the original curve at that point (Fig. 2). The direction of motion along the tangent is chosen so that the orientation of the sphere, given by the direction of the co-orienting normal and the direction of the tangent, is positive.

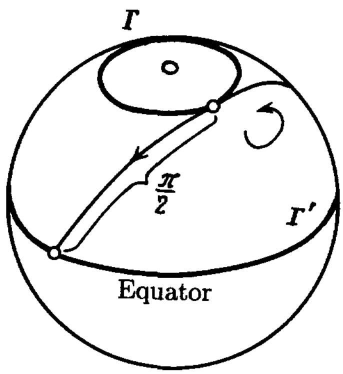

Figure 2

This definiton applies also not only to smoothly immersed curves, but also to wave fronts.[1]

Example. On the unit sphere the parallel of latitude of Euclidean radius $\cos\theta$ is dual to the parallel of lattitude of Euclidean radius $\sin\theta$. The length of the first parallel is equal to the area between the second parallel and the equator of the sphere.

The derivative of any parallel of latitude is the equator parallel to it.

It turns out that the coincidence of length and area and the division by the equator of the sphere into halves is not accidental.

Theorem 1. *The derivative of a wave front coincides with the derivative of any of its equidistants and is a smoothly immersed curve on S^2 even if the original wave front has generic singularities.*

[1]Derivatives of smoothly immersed curves are studied in a recent work by B. Solomon (B. Solomon, Tantrices of spherical curves, Preprint, University of Indiana, 1993) that refers to earlier results of Jacobi and Kagan. The author is grateful to S.L. Tabachnikov for communicating this preprint.

Remark. In the definition of the derivative the movement along tangents by a distance $\pi/2$ may be changed to moving by an arbitrary constant distance s. The curve so obtained is also smoothly immersed in S^2, provided that s is not a multiple of π (as B.A. Khesin has shown to me).

For a plane wave front one can move all the points along tangents by an arbitrary non-zero constant distance: one always obtains a non-singular (smoothly immersed) curve.

The derivative curve of a closed wave front is not an arbitrary curve immersed in the sphere: it satisfies a topological condition of quantization.

If the derivative curve has no points of self-intersection, then this condition consists simply in that, just like the equator, it divides the area of the sphere into two equal parts.

On the other hand if the derivative curve of a closed co-oriented front has points of self-intersection, then the condition of quantization is given by the formula

$$\text{the area bounded by the derivative curve} = -\pi\mu,$$

where μ is the Maslov index of the original wave front. The Maslov index of a wave front, defined in [1], is equal to the algebraic number of cusps of the front. In calculating μ a semi-cubical cusp is considered to be positive if the chords that join points of the branch approaching the cusp to points of the branch leaving it co-orient the curve positively in a neighbourhood of the cusp. The number μ so defined is the Maslov index of the closed curve that corresponds to the front on the two-dimensional Lagrangian manifold of the cotangent bundle of the sphere, that is, on the symplectization of the front (see [1]).

To clarify the quantization condition it remains to define 'the area bounded by an immersed curve on S^2'. Of course, this area is the integral of the standard area form over a 2-chain, whose boundary is the given immersed curve in S^2.

However the 2-chain defined by this condition is not unique, but only modulo the 2-cycle S^2. So 'the area bounded by an immersed curve in S^2' is defined only modulo 4π.

It turns out, however, that among all the 2-chains bounded by a given curve immersed in the 2-sphere there is a distinguished 2-chain (called below the *characteristic chain*).

The characteristic chain appears in an analogue of the Gauss–Bonnet formula for curves immersed in S^2. There is an analogous construction also for hypersurfaces on all even-dimensional spheres.

The geodesic curvature of a co-oriented curve is considered to be positive if the curve turns away from its geodesic tangent on the side of the co-orienting normal.

Theorem 2. *The integral of the geodesic curvature of a closed co-oriented curve immersed in S^2 is equal to the area of its characteristic 2-chain.*

The integral of the area of this characteristic 2-chain for an arbitrary closed co-oriented front is equal to $-\pi\mu$, where μ is the Maslov index of the front.

The co-orientation of the derivative curve is chosen here so that the co-orienting vector together with the orientation form a positively oriented frame of S^2. Such a co-orientation is said to be *correct*.

I now introduce a formula for calculating the characteristic chain of a co-oriented hypersurface on an oriented even-dimensional sphere.

This formula has the form of the product of the volume element of the sphere by a locally-constant function on the complement of the hypersurface. This locally-constant function depends only on the co-orientation of the hypersurface and does not depend either on the orientation of the hypersurface or the orientation of the sphere. On change of sign of the co-orientation of the hypersurface this function changes sign.

We select one of the points of the complement of the hypersurface on the sphere. We call this the 'point at infinity' and denote it by B.

Stereographic projection maps the complement to the point B to an oriented Euclidean space. The hypersurface projects to a compact co-oriented hypersurface lying in this space. This hypersurface of the Euclidean space has a certain index i_B (the degree of the Gaussian map to a sphere). Moreover, for any point A of the complement to this hypersurface there is defined the winding number i_A^B of the hypersurface around A (the degree of the map of the hypersurface to a sphere, sending a point C to the direction of the vector AC).

Theorem 3. *The difference $i_A^B-(i_B/2)$ is conformally invariant (that is, it depends only on the point A and does not depend on B, the chosen point at infinity).*

This difference, considered as a multiplicity function of the point A (integrally or semi-integrally valued), turns the complement to the original hypersurface into a chain whose boundary is the original hypersurface. This is the chain called the *characteristic chain* of the original hypersurface.

The conformal invariant of the winding number of the hypersurface A (corrected by subtracting $i_B/2$) is explained by the following elementary (but apparently not previously observed) identity.

Theorem 4. *There is a formula connecting the indices of points with respect to a hypersurface and the indices of the hypersurface in even-dimensional space, namely*

$$2i_A^B = i_B - i_A.$$

Consider now a closed co-oriented wave front Γ on S^2 with Maslov index μ, its correctly co-oriented derivative curve λ with length element dl, and its characteristic 2-chain c (so that $\partial c = \lambda$).

Theorem 5. *The integral of the area form ω over the characteristic chain, the Maslov index of the original wave front and the geodesic curvature $\varkappa$ of its derivative are connected by the relations*

$$\iint_c \omega = -\pi\mu = \int_\lambda \varkappa\, dl.$$

Remark. The equality of the left and right parts of these equalities is a general formula of Gauss–Bonnet type, satisfied by any closed immersed curve λ (not necessarily occurring as the derivative of a closed front).

A similar formula of Gauss–Bonnet type holds also for hypersurfaces immersed in a sphere of arbitrary even dimension.

The reason for the connection between the derivative curve and the Maslov index is the following, also elementary, fact concerning the geometry of curves on the standard two-dimensional Riemannian sphere.

Let us recall that the derivative curve is obtained from the original wave front on the sphere of radius 1 by moving its points a distance $\pi/2$ along the tangent great circles.

Theorem 6. *The directions of these great circles at points of the derivative curve are parallel in the sense of the standard Riemannian geometry of the sphere S^2.*

The preceding theorems allow one to determine necessary and sufficient conditions on a curve immersed in S^2 for it to be the derivative of a closed co-oriented wave front: the area of its characteristic chain must be a multiple of 2π.

In particular, a curve embedded in S^2 is the derivative of a closed co-oriented front if and only if it divides the sphere into two parts of equal area (like an equator of the sphere).[2]

The cusps of a system of fronts equidistant from one another form the caustic (Fig. 3).

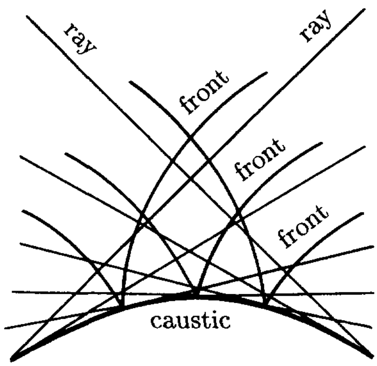

Figure 3

Theorem 7. *The caustic of a system of equidistants from a closed co-oriented wave front on the standard two-dimensional Riemannian sphere is dual to the common derivative curve of any of the fronts equidistant to one another that form the family.*

Remark. In particular, the caustic does not have any points of inflection, since its dual is the derivative curve of fronts smoothly immersed in the sphere and so has no cusps.

[2]The derivative of a smoothly immersed closed curve has full geodesic curvature equal to 0 and does not have an arc of full geodesic curvature π; each closed smoothly immersed curve in the sphere with these properties is the derivative of a smoothly immersed closed curve, as B. Solomon has shown in the work cited earlier.

Theorem 8. *The points of inflection of a family of fronts equidistant from one another lie on two smoothly immersed curves in S^2, each of which bounds an area which is a multiple of 2π, these being the derivative curves of the caustic.*

In fact the points of inflection of a front lie at a distance $\pi/2$ from the cusps of the dual front along the great circle orthogonal to both fronts.

At a cusp of a front this great circle of the sphere is tangent to the caustic. Therefore the points of inflection of all the fronts of the family lie on the pair of derivative curves of the caustic (furnished, for each of the two curves of the pair, with one of the two possible co-orientations).

The topological restriction on the area bounded by the derivative curve is transformed on transfer to the dual curves into a quantization condition on the length of the caustic.

Definition. The *oriented length* of a generic caustic is the alternating sum of the lengths of its segments between successive cusps.

Theorem 9. *The oriented length of the caustic of a system of closed equidistant wave fronts on the standard Riemannian two-dimensional sphere is equal to an integral multiple of 2π (in fact equal to $\pi\mu$, where μ is the Maslov index of any front of the family).*

Which of these segments of the caustic to be counted here as positive will be settled in § 24.

Theorem 9 generalizes a known property of the caustic of a closed plane curve (the oriented length of such a caustic, for example the astroid, arising as the caustic of an ellipse, is equal to zero).

The stated property is explained by the fact that the front is obtained from the caustic 'by unwinding a thread from it' (just as an evolvent from its evolute). The increase in the length of the free part of the thread is equal to the length of the thread unwound from the caustic, so long as the caustic is convex (Fig. 4).

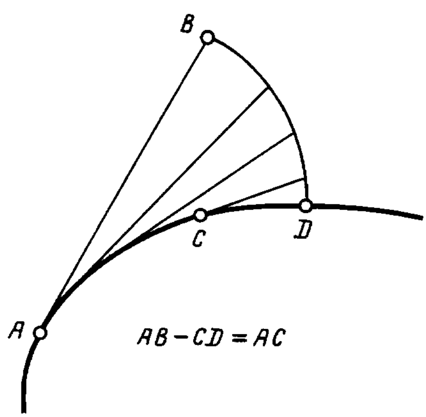

Figure 4

On passing through a cusp of the caustic the direction of convexity changes. It follows that the length of the unwound thread after such a point should be counted with opposite sign.

If the evolvent of a plane curve is closed, then, for a complete circuit of it, the increase in length of the free end of the thread must be equal to zero. Therefore the algebraic sum of the lengths of the segments of the plane caustic between successive cusps (with alternating signs) must be equal to zero.

On the sphere this sum is not zero, but is a multiple of 2π, since the return to the original point of the evolvent no longer requires the return of the length of the free part of the thread to its original value. The increase of this length by 2π corresponds to adding to the free part of the thread a complete circuit of the sphere along the great circle containing the free part of the thread. In that case the increase of the length of the free part of the thread by an integral multiple of 2π does not change the end point of the thread.

The preceding theorem shows that this increase of the length of the free part of the thread after a full circuit of the caustic is equal to $\pi\mu$, where μ is the Maslov index of the front (which for a co-oriented front is always even).

Remark. The equality of the area of the characteristic chain of a closed curve immersed in the sphere and the oriented length of its dual curve is independent of the fact that the immersed curve is a derivative: this is a general 'area-length duality' in the geometry of S^2 (the sign of the length changes on passage through each cusp of a generic front).[3]

To formulate all these formulae with their boring signs with a reasonable degree of generality I must first of all recall some definitions from the general theory of wave fronts and caustics in contact and symplectic geometry. After this, general theorems will be proved (with the aid of quaternions). From these the theorems of this section will follow. Theorem 1 is proved in §§ 6 and 9 of Chapter II. Theorems 2, 5 and 9 are proved in §§ 22, 23 and 24 of Chapter V. Theorems 3 and 4 are proved in § 18 of Chapter IV. Theorems 7 and 8 are proved in § 4. Theorem 6 is proved in § 12 of Chapter III.

§ 2. Contact manifolds, their Legendrian submanifolds and their fronts

A linear hyperplane $\mathbb{R}^{n-1}$ of a tangent space to an n-dimensional smooth manifold M^n is said to be a *contact element* of M^n. A co-orientation of the contact element is the choice of one of the two halves into which it divides the tangent space. A co-orientation of a contact element of a Riemannian manifold is determined by a choice of direction on the line normal to it.

All the (co-oriented) contact elements on M^n form the bundle space E^{2n-1} of a smooth fibration

$$p\colon ST^*M^n \to M^n$$

with fibre S^{n-1} (the cotangent sphere bundle of M^n). This manifold E^{2n-1} is equipped with the natural 'tautological' field of tangent hyperplanes. The hyperplane of the tautological field at a point s of E^{2n-1} is the inverse image by p_* of

[3] The area–length duality was encountered earlier by Santalo in works on integral geometry; it has recently been employed by S.L. Tabachnikov in Russian Math. Surveys 48:6 (1993), 81–109 and by J.C. Alvares (Rutgers University, 1994).

the hyperplane in the space tangent to M^n, which is represented by the point s of the manifold E^{2n-1}.

The tautological field of hyperplanes on E^{2n-1} is called the ***natural contact structure*** of the manifold of contact elements on M^n.

Example. The manifold of co-oriented contact elements of S^2 is the projective space $\mathbb{R}P^3 = S^3/\pm 1$. Its natural contact structure is obtained from the field of planes orthogonal to the fibres of the Hopf bundle $S^3 \to S^2$, under the projection $S^3 \to S^3/\pm 1$.

The identification $ST^*S^2 \approx S^3/\pm 1$ will be described in detail below, in § 6.

Definition 1. A *Legendrian curve* in a three-dimensional contact manifold E^3 is an integral curve of the contact structure, that is, an immersed curve in E^3, whose tangent at each point lies in the plane of the contact field of planes.[4]

Example. Each curve Γ immersed in a surface M^2 determines a Legendrian curve in the contact manifold $E^3 = ST^*M^2$. This curve consists of the contact elements on M^2, tangent to Γ.

A point on M^2 also determines a Legendrian curve in E^3. This curve consists of the contact elements of M^2, applied at the point (that is, it is a fibre of the cotangent sphere bundle).

Definition 2. The *front* of a Legendrian curve $L : S^1 \to E^3$ on the manifold E^3 of contact elements of a surface M^2 is the projection of this Legendrian curve onto the surface $pL(S^1) \subset M^2$.

The front of a generic Legendrian curve on the surface has semicubical cusps and points of transversal self-intersection as its only singularities.

The contact elements forming the original Legendrian curve determine a *co-orientation* of that front, namely a concordant choice of normal to the front (at double points of the front there are two normals).

Such co-oriented fronts Γ of generic Legendrian curves on the 2-sphere will be the starting material of our constructions.

§ 3. Dual curves and derivative curves of fronts

We consider a co-oriented front Γ in an oriented sphere S^2 of radius 1.

The great circles of the sphere, orthogonal to the front, will be called its *rays.* The rays are oriented (by the co-orientation of the front).

On moving each point of the front along the ray by a distance t we obtain a new curve Γ_t, called the *t-equidistant* of the front. For example, $\Gamma_{2\pi} = \Gamma$.

According to the Huygens' principle, an equidistant is orthogonal to the rays emanating from the original front. An equidistant co-oriented by the directions of these rays is the front of a Legendrian curve, diffeomorophic (and contactomorphic) to the original one.

[4] A Legendrian submanifold of the contact manifold E^{2n-1} is an integral submanifold of maximal possible dimension (equal to $n-1$).

Definition 1. The front $\Gamma^\vee$ equidistant by $\pi/2$ from the original one is said to be *dual* to Γ.[5]

Example. The curve dual to a parallel of Euclidean radius $\sin\theta$ on S^2 is a parallel of Euclidean radius $\cos\theta$.

The following lemma (clarifying the connection with projective duality) is obvious.

Lemma 1. *The dual front is formed from the centres of the great circles tangent to the original curve. The second dual of a front is antipodal to the original one*: $\Gamma^{\vee\vee} = -\Gamma$.

In particular, points of inflection of the original front correspond to cusps of the dual (and conversely).

The co-orientation of the original front on the oriented sphere determines at every point of the front (including singularities) a positive tangent direction (such that the frame determined by the co-orientation and orientation directions orients the sphere positively).

Definition 2. The *derivative curve* Γ' of a front Γ is formed from those points of the great circles tangent to Γ which lie at a distance $\pi/2$ from the point of tangency on the positive side.

Example. The derivative of a parallel of latitude of the sphere is its equator.

Remark. The derivative curve may be considered to be the curve of normals in the same way that the dual curve may be considered to be the curve of tangents.

The following lemma (clarifying the term 'derivative') is almost obvious.

Lemma 2. *The derivative of a front coincides with the derivative of an arbitrary front equidistant from it.*

Example. $(\Gamma^\vee)' = \Gamma'$: the derivatives of dual fronts coincide. The derivatives of anitpodal fronts also coincide: $(-\Gamma)' = \Gamma'$. Change of the orientation of a front changes the sign of the derivative.

Theorem 1. *The derivative of the front of a generic Legendrian curve is a smooth curve (immersed in the sphere), bounding the area $-\pi\mu$, a multiple of 2π.*

The topological meaning of the even number μ appearing here (in fact the Maslov index) will be considered in Chapter V.

Theorem 2. *The great circles tangent to a given front form at the points of the derivative curve of this front a framing, parallel in the sense of the Riemannian geometry of the sphere.*

The connection between the integral of the geodesic curvature of the derivative curve and the Maslov index of the front will be discussed below in Chapter V.

---

[5]The differential geometry of a pair of dual hypersurfaces on spheres has been a classical object of study (see, in particular, Chapter 17 of Santalo's book); it is remarkable that here it is still possible to find new results, even for curves on the two-dimensional sphere.

Theorem 3. *The angle between the great circles tangent to two equidistant fronts at the point of their intersection on the derivative curve is equal to the distance between the equidistants (and, in particular, is constant along the derivative curve).*

Theorem 3 is obvious.

The proof of Theorems 1 and 2 will be given below in § 18 and § 12.

§ 4. The caustic and the derivatives of fronts

Consider the family of all the equidistants of a given co-oriented front on the standard sphere. Even if the original front is non-singular, some of its equidistants will have singularities.

A *singular point* of a front is a critical value of the projection of the corresponding Legendrian curve.

The singular points of generic fronts are semicubical cusps. The double points of a front in general are not singular.

Definition 1. The *caustic* of a family of fronts equidistant from one another is the curve formed from their singular points.

The caustic of a generic front has as its only singularities semicubical cusps and points of transversal self-intersection.

Theorem 1. *The derivative of a front is dual to its caustic.*

Proof. The front at a cusp is orthogonal to the caustic. Therefore to move such a point a distance $\pi/2$ along the tangent to the front is the same as moving it a distance $\pi/2$ along the normal to the caustic (Fig. 5).

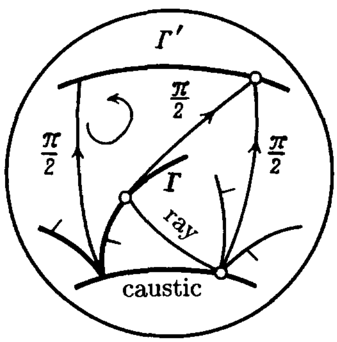

Figure 5

Remark 1. The caustic, in general, is not co-oriented. In this theorem the co-orientation of the front is not taken into account. Therefore the derivative of the front in Theorem 1 is understood to be the union of both (mutually antipodal) components, obtained by moving along the tangent a distance $\pi/2$ in either direction.

Remark 2. The caustic of a co-oriented front on the sphere consists of two different (mutually antipodal) components. This follows easily from the evenness of the number of cusps of a co-oriented curve on the sphere.

Corollary. *The caustic of a family of fronts on the sphere equidistant from one another has no points of inflection.*

In fact, its dual, being the derivative of any of the fronts equidistant from one another, has no cusps (by Theorem 1 of § 3).

Theorem 2. *The points of inflection of all the fronts equidistant from a given one form an immersed curve on the sphere, bounding an area that is a multiple of* 2π *(namely the derivative of the caustic, which is consequently the second derivative of the front).*

Proof. The points of inflection of the front are obtained from the cusps of the dual front (lying on the caustic) by moving a distance $\pi/2$ along the ray orthogonal to the front at the points of inflection and the cusps. At a cusp it is tangent to the caustic, and this proves Theorem 2 (in view of Theorem 1 of § 3).

CHAPTER II

QUATERNIONS AND THE TRIALITY THEOREM

Passing from the wave front to the dual front and to the derivative curve of them both corresponds in algebra to passing from the imaginary unit i to the imaginary units j and k in the space of quaternions (see § 9).

In order to explain this we need some elementary facts from the geometry of quaternions, set out in §§ 5–8. Basic to these facts is the easily proved identity

$$e^{ks}e^{it}e^{-ks} = e^{\varkappa t}, \qquad \varkappa = i\cos 2s + j\sin 2s$$

(where s and t are real). The other computations of this chapter are just as simple, but it is necessary to carry them out in order not to get lost among numerous signs and orientations.

§ 5. Quaternions and the standard contact structures on the sphere S^3

Let $\varkappa$ be an imaginary quaternion of length 1 (for example, i, j or k). Multiplication of all the quaternions by $\varkappa$ on the *right* imparts a complex structure (depending on $\varkappa$) to the space of quaternions $\mathbb{R}^4$.

We associate with each quaternion q the operator of multiplication on the *left* by q in the space of quaternions $\mathbb{R}^4$.

Proposition 1. *The operator of multiplication by the quaternion q on the left is $\varkappa$-complex.*

Proof. In fact, denoting quaternion multiplication by a dot we have $q \cdot (z \cdot \varkappa) = (q \cdot z) \cdot \varkappa$ for every quaternion z.

Proposition 2. *The determinant of the operator of multiplication on the left by a quaternion of length* 1 *is equal to* 1.

Proof. Both length and determinant are multiplicative. Therefore the image of the map det : $S^3 \to \mathbb{C}\backslash 0$ is a compact connected subgroup. The connected compact subgroups of $\mathbb{C}\backslash 0$ are two in number: the point 1 and S^1. But S^3 does not fibre over S^1. Accordingly, $\det(S^3) = 1$.

Corollary. *The determinant of the $\varkappa$-complex operator of multiplication on the left by an arbitrary quaternion is equal to the square of its length.*

Proof.
$$\det(q) = \det(|q|) \cdot \det(q/|q|) = |q|^2 \cdot 1,$$
since the determinant of the operator of multiplication of a vector in the complex plane by a real number is equal to the square of that number.

Example. The vectors 1 and j form an i-complex basis for $\mathbb{R}^4$. This means that the quaternion $x+yi+uj+vk$ may be considered as the complex vector $(x+iy, u-iv)$ (pay attention to the sign!).

With respect to this basis the operator of multiplication on the left by the quaternion $q = a + bi + cj + dk$ has the matrix
$$\begin{pmatrix} a+bi & -c-di \\ c-di & a-bi \end{pmatrix} = \begin{pmatrix} A & -\overline{C} \\ C & \overline{A} \end{pmatrix}, \qquad \det(q) = a^2 + b^2 + c^2 + d^2.$$

Accordingly, the group S^3 of quaternions of length 1 is isomorphic to the group $SU(2)$ of unitary matrices of degree 2 with determinant 1.

If we were to start from another complex structure involving $\varkappa$ instead of i we should obtain another isomorphism $S^3 \to SU(2)$.

For each quaternion $\varkappa$ of length 1 consider now its corresponding Hopf bundle
$$\pi_\varkappa \colon S^3 \to S^2_\varkappa \approx \mathbb{C}\mathrm{P}^1,$$
associating with each non-null point of the $\varkappa$-complex line passing through 0 the direction of that line.

The fibre of this principal S^1-bundle is the circle
$$\{z \cdot e^{\varkappa t} : t \in \mathbb{R}/(2\pi\mathbb{Z})\},$$
where $z \in S^3$ is a point of the fibre. The action by S^1 on the fibre is determined by shifts of the real axis t.

Warning. Our notations are not complex, but quaternionic. If $I_\varkappa$ is the operator of the complex structure on $\mathbb{C}^2$ such that
$$I_\varkappa z = z \cdot \varkappa,$$
then $z \cdot e^{\varkappa t} = e^{I_\varkappa t} z$.

Consider on the sphere S^3 of quaternions of unit length the vector field $\varkappa$, tangent to the fibres of the bundle $\pi_\varkappa$,
$$\varkappa(z) = z \cdot \varkappa = (d/dt)\big|_{(t=0)}(z \cdot e^{\varkappa t}).$$

Proposition 3. *The vector field $\varkappa$ is left-invariant (under the action of the group S^3 of quaternions of length 1 on itself, by multiplication by elements of S^3 on the left).*

Proof.

$$q \cdot (z \cdot e^{\varkappa t}) = (q \cdot z) \cdot e^{\varkappa t}.$$

Consider the field of planes on S^3 orthogonal to the vectors of the field $\varkappa$. This field provides S^3 with a contact structure, which we call the $\varkappa$-structure.

Proposition 4. *The contact $\varkappa$-structure is left-invariant (with respect to the action of the group S^3 of quaternions of length 1 on itself by multiplying elements of S^3 on the left) and gives an S^3-left-invariant connection to the principal S^1-bundle $\pi_\varkappa$.*

Proof. Left shifts preserve the metric on S^3 and the vector field $\varkappa$, and therefore also the field of planes orthogonal to it. The S^1-structure of the fibres is also preserved, since it is given by the parameter t.

We have constructed on S^3, in particular, three S^3-left-invariant vector fields $\mathbf{i}, \mathbf{j}, \mathbf{k}$ and the contact structures that correspond to them and determine S^3-left-invariant connections on the three principal Hopf bundles $\pi_i : S^3 \to S^2_i$, $\pi_j : S^3 \to S^2_j$, and $\pi_k : S^3 \to S^2_k$.

§ 6. Quaternions and contact elements of the sphere S^2_j

Suppose that the i-contact structure is assigned to the space of the j-bundle π_j constructed in § 5.

Theorem 1. *The bundle $\pi_j : S^3 \to S^2_j$ is naturally isomorphic to the double covering over the bundle of co-oriented contact elements of the 2-sphere, $\tilde{\pi}_j : (ST^*S^2 \approx \mathbb{R}P^3) \to S^2_j$. Under this isomorphism the contact structure on S^3 projects to the natural contact structure on the space of contact elements.*

Proof of Theorem 1. We begin with the following obvious remark.

Lemma 1. *The fibres of the bundle π_j are Legendrian (that is, simply integral) curves of the i-structure.*

In fact, the fields $\mathbf{i}$ and $\mathbf{j}$ are orthogonal at every point. Therefore the direction of the fibre of $\mathbf{j}$ is tangent to the orthogonal $\mathbf{i}$-plane of the contact i-structure.

Thus the two-dimensional planes of the contact i-structure project along the fibres of the j-bundle to one-dimensional contact elements of the sphere S^2_j. These elements are co-oriented (by the projection of the direction of the field $\mathbf{i}$).

Definition. The co-oriented contact element of the sphere S^2_j *corresponding to a given plane of the contact i-structure on S^3* is the projection of the plane co-oriented by the vector $\mathbf{i}$ along the fibres of the j-bundle.

We prove that the constructed map $S^3 \to ST^*S^2_j$ is a double covering and has the properties stated in the theorem.

Lemma 2. *On moving a point on S^2 with velocity* 1 *along a fibre of the j-bundle the contact i-plane changes in such a way that its projection on the sphere S_j^2 turns with angular velocity* 2.

Proof. We compare the directions of the vectors of the field $\mathbf{i}$ at points z and $z \cdot e^{jt}$ of a fibre of the j-bundle. We bring the vector $z \cdot e^{jt} \cdot i$ of this field from the point $z \cdot e^{jt}$ back to the point z by acting on the right with the quaternion e^{-jt} (which does not change the projection onto S_j^2). We obtain at the point z the vector $z \cdot e^{jt} \cdot i \cdot e^{-jt}$.

On varying t this vector turns in the plane of the vectors at z of the fields $\mathbf{i}$ and $\mathbf{k}$, orthogonal to the direction $\mathbf{j}$ of the fibre. The rotation is from the direction $\mathbf{i}$ to the direction $-\mathbf{k}$, and has (constant) angular velocity 2. For example, for small t

$$e^{jt} \cdot i \cdot e^{-jt} \approx i + (ji - ij)t + \dots .$$

From Lemma 2 it follows that after the time $t = 2\pi$ of a complete turn along the fibre of the j-bundle on S^3 the corresponding contact element on S_j^2 has turned through an angle 4π. Consequently, our natural map $S^3 \to ST^*S_j^2$ is fiberwise a double covering of principal S^1-bundles.

The projection of the contact i-structure by this map is the field of planes generated by the projections of the vector fields $\mathbf{j}$ and $\mathbf{k}$. Such a plane contains the velocities of motion of the contact elements for which the velocity of the point of application belongs to the element (since the projection of the vector $\mathbf{k}$ onto the sphere belongs to the element, while the direction $\mathbf{j}$, by Lemma 2, corresponds to the rotation of the element).

The constructed double covering $S^3 \to ST^*S_j^2$ allows one to identify the latter space in a natural way with the factor-space $\mathbb{RP}^3 = S^3/\pm 1$. In fact, $e^{tj} = -1$ for $t = \pi$. Therefore both the antipodal points of S^3 project to one and the same co-oriented contact element on the sphere S_j^2. The theorem is proved.

Note that on the manifold

$$\mathrm{SO}(3) \approx \mathbb{RP}^3 = S^3/\pm 1$$

there are naturally induced from the sphere S^3 a metric, vector fields $\varkappa$ (in particular $\mathbf{i}, \mathbf{j}, \mathbf{k}$), the contact structures orthogonal to them, the principal S^1-bundles

$$\tilde{\pi}_\varkappa \colon S^3/\pm 1 \to S_\varkappa^2$$

and the S^3-invariant connections given by the corresponding contact structures.

Consider an i-Legendrian curve L on S^3.

Project it to the three spheres S_i^2, S_j^2, S_k^2 by means of the three natural projections π_i, π_j, π_k. The projection $\pi_j L$ is a front Γ of the i-Legendrian curve L on the sphere S_j^2. Let us clarify the meaning of the other two projections.

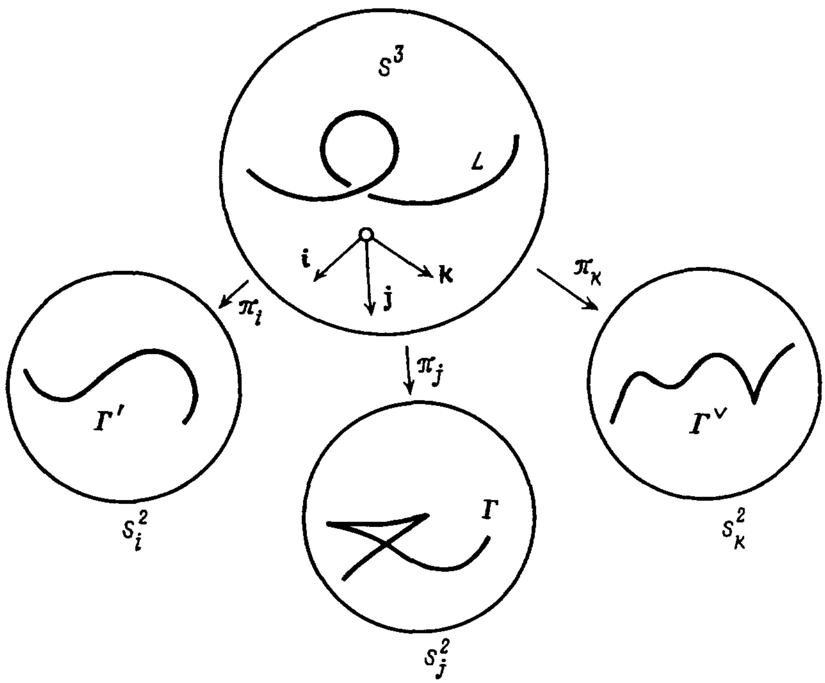

Figure 6

The triality theorem. *The projection $\pi_k L$ is congruent to the front $\Gamma^\vee$ dual to Γ, while the projection $\pi_i L$ is congruent to the derivative Γ' of Γ.*

In this way a front, the dual front, and the derivative of them both, are three personifications of one and the same Legendrian curve (Fig. 6).

The next few sections are dedicated to the proof of the triality theorem, completed below in § 9. The proof establishes also explicit forms of the isometries $S_k^2 \to S_j^2$ and $S_i^2 \to S_j^2$ that realize the congruences stated in the theorem.

To be precise, the first is induced by multiplication of S^3 on the right by $e^{i\pi/4}$, and the second by multiplication by $e^{k\pi/4}$.

From the triality theorem it follows that the derivative curve is smoothly immersed. In fact the i-Legendrian curve L, smoothly immersed by definition, is transversal to the direction of the fibres of the i-bundle. Therefore its projection $\pi_i L$ on the sphere S_i^2 also is smoothly immersed. So Theorems 1 of § 1 and § 3 follow from the triality theorem.

Remark. The proof of the smoothness of the curves obtained from a front by moving along tangents by a distance $2s$ that is not a multiple of π is similar to the proof of the smoothness of the dual curve given below, except that multiplication by $e^{k\pi/4}$ has to be replaced by multiplication by e^{ks}.

§ 7. The action of quaternions on the contact elements of the sphere S_j^2

Let $S^3/\pm 1$ be identified with the space $ST^*S_j^2$ of co-oriented contact elements of the sphere S_j^2 by means of the complex structure j and the contact structure i, as has been described in § 6. Let t be a real parameter.

The action of multiplication on the right by the quaternions e^{it}, e^{jt} and e^{kt} on S^3 induces on $S^3/\pm 1 \approx ST^*S_j^2$ three dynamical systems, described in the following lemmas (Fig. 7).

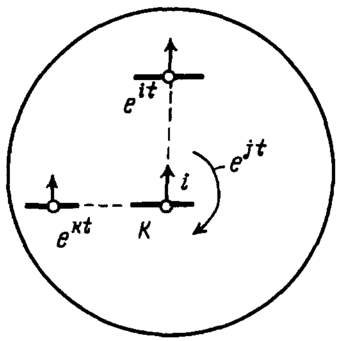

Figure 7

Lemma 1. *Under the action of e^{it} on $ST^*S_j^2$ a contact element moves along a geodesic at all times orthogonal to it in the direction of its co-orientation with speed* 2.

Lemma 2. *Under the action of e^{jt} on $ST^*S_j^2$ a contact element turns around its fixed point of application with angular velocity -2 in the direction given by the j-complex structure on the sphere S_j^2.*

Lemma 3. *Under the action of e^{kt} on $ST^*S_j^2$ a contact element moves with speed* 2 *along a geodesic at all times tangential to it in the direction obtained from the co-orientation direction by turning through an angle $\pi/2$ in the direction given by the j-complex structure on the sphere S_j^2.*

Proof. The base S_j^2 of the bundle π_j is obtained from S^3 by factorizing it by *right* shifts (by elements of the form e^{jt}). Therefore *left* shifts by elements of S^3 act on the factor-space S_j^2.

These left shifts preserve the e^{jt}-right-invariant metric on S^3 and, consequently, preserve the natural metric on the sphere S_j^2. Since the sphere S^3 is connected, the left shifts act on the sphere S_j^2 as motions.

All motions of the sphere S_j^2 are obtainable in this way from left shifts on S^3, since any element of S^3 can be carried by left shifts into any other, and that means that any direction tangent to the sphere S_j^2 may be carried into any other.

Consequently, it is enough to verify Lemmas 1–3 for a specially chosen element of S^3 and it is natural to choose the quaternion 1.

Lemma 4. *Let $\varkappa$ be an imaginary quaternion of length 1, orthogonal to j. Then the projection π_j maps the great circle $\{e^{\varkappa t} : t \in \mathbb{R} \mod 2\pi\}$ of the sphere S^3 onto a great circle of the sphere S^2_j as a double covering.*

Proof of Lemma 4. Consider the circle of quaternions $e^{\varkappa t} = \cos t + \varkappa \sin t$, $t \in \mathbb{R}$, as a curve in the j-complex plane. Choose in this plane the j-complex Hermitian-orthogonal basis $(1, \varkappa)$. A point with the complex coordinates ($z_1 = x_1 + iy_1$, $z_2 = x_2 + iy_2$) in this basis is the quaternion $q = x_1 + y_1 j + x_2 \varkappa + y_2 \varkappa j$.

In the affine coordinate $w = z_2/z_1$ on the projective line $\mathbb{C}\mathrm{P}^1 = S^2_j$ the projection π_j of the circle in $\mathbb{C}^2$ defined above is given by the formula $w = \tan t$, and this proves Lemma 4.

We now prove Lemmas 1 and 3. In the case of Lemma 1 the direction of motion of the point of application on S^2_j is at all times the projection of the vector field $\mathbf{i}$. Consequently, the direction of motion positively co-orients the moving contact element, remaining orthogonal to the direction of the motion. Meanwhile the point of application describes a great circle with speed 2 by Lemma 4.

In the case of Lemma 3 the direction of motion of the point of application on S^2_j is at all times the projection of the vector field $\mathbf{k}$. Consequently, the direction of motion is at all times tangent to the moving contact element. Meanwhile the point of application describes a great circle with speed 2 by Lemma 4.

Lemma 5. *The complex j-structure on the sphere $S^2_j = \mathbb{C}\mathrm{P}^1$ defines on it the orientation in which the frame $(\pi_{j_*}\mathbf{i}, \pi_{j_*}\mathbf{k})$ is positive.*

Proof of Lemma 5. Consider the vectors i and k of the fields $\mathbf{i}$ and $\mathbf{k}$ at the point 1 of the sphere S^3. We compute their projections on the sphere S^2_j in the affine coordinate $w = z_2/z_1 = x + iy$ on the sphere S^2_j, constructed as in the proof of Lemma 4, in which we take $\varkappa$ as i.

The projections of the vectors i and k are $\partial/\partial x$ and $\partial/\partial y$, respectively (at the point $w = 0$). In fact, by the formulae of the proof of Lemma 4 with $z_1 = 1, z_2 = w$ we find that $q = 1 + xi + yk$. On varying the real parameters x and y near 0 we have

$$\partial q/\partial x\big|_{0,0} = i = \mathbf{i}(1), \qquad \partial q/\partial y\big|_{0,0} = k = \mathbf{k}(1),$$

as was asserted. Lemma 5 has been proved.

Lemma 3 is proved at the same time, since the direction $\pi_{j_*}\mathbf{k}$ of the motion of the point of application is obtained from the direction $\pi_{j_*}\mathbf{i}$ that co-orients the element by turning through an angle $\pi/2$ towards the side defined by the given j-complex structure.

Lemma 2 of § 7 follows from Lemma 2 of § 6 and Lemma 5, since the direction of turning 'from $\mathbf{i}$ to $-\mathbf{k}$', obtained in § 6, is opposite to the direction of turning of the given j-complex structure.

§ 8. The action of right shifts on left-invariant fields

It turns out that left-invariant contact structures on the sphere S^3 are mapped into each other by multiplication by appropriate quaternions on the right.

We denote by $R_q : S^3 \to S^3$ multiplication on the right by the quaternion q of length 1.

Theorem. *Multiplication of S^3 on the right by e^{jt} maps the field* $\mathbf{j}$ *to itself while the fields* $\mathbf{i}$ *and* $\mathbf{k}$ *are turned through an angle $2t$ in the plane generated by them (from* $\mathbf{i}$ *towards* $\mathbf{k}$*):*

$$(R_{e^{jt}})_* \mathbf{i} = \mathbf{i} \cos 2t + \mathbf{k} \sin 2t.$$

Proof. The vectors of the fields $\mathbf{i}$ and $\mathbf{k}$ are orthogonal to the direction of the field $\mathbf{j}$. Under the isometries $R_q : S^3 \to S^3, q = e^{jt}$, preserving the direction $\mathbf{j}$ of the fibres of the bundle $\pi_j : S^3 \to S^2_j$, the translated vectors remain orthogonal to the directions of the fibres.

The projection to S^2_j of a vector of the field $\mathbf{i}$ transported by the isometry $R_{e^{jt}}$ does not change under change of the parameter t, while the point of application moves along the fibre (in the direction $\mathbf{j}$) with speed 1.

The projection to S^2_j of a vector of the field $\mathbf{i}$, applied at this moving point, turns with angular velocity -2 from $\pi_{j_*}\mathbf{i}$ towards $\pi_{j_*}\mathbf{k}$, by Lemma 2 of § 7. Consequently, a vector transported by the shifts $R_{e^{jt}}$ turns with angular velocity $+2$ relative to $\mathbf{i}$ in the direction towards $\mathbf{k}$.

Corollary 1. *Multiplication of S^3 on the right by $e^{\pi j/4}$ turns the contact i-structure of S^3 into the k-structure (and the k-structure into the i-structure).*

The vector field $\mathbf{i}$ then becomes the vector field $\mathbf{k}$, while $\mathbf{k}$ becomes $-\mathbf{i}$.

Corollary 2. *Multiplication of S^3 on the right by $e^{\pi j/4}$ turns the bundle $\pi_i : S^3 \to S^2_i$ into the bundle $\pi_k : S^3 \to S^2_k$ and π_k into π_i. The induced maps of the 2-spheres $S^2_i \to S^2_k$ and $S^2_k \to S^2_i$ are isometries. The first of these respects the orientations of these spheres given by the complex i- and k-structures, respectively, while the second changes them. Their compositions are the antipodal involutions of the spheres S^2_i and S^2_k.*

§ 9. The duality of j-fronts and k-fronts of i-Legendrian curves

Let L be an i-Legendrian curve (that is, an integral curve of the i-structure on S^3).

Proposition 1. *Multiplication of S^3 by e^{it} on the right sends an i-Legendrian curve into an i-Legendrian curve.*

Proof. This multiplication is an isometry of S^3. It preserves the field $\mathbf{i}$ and therefore the contact structure orthogonal to it. Therefore integral curves of the $\mathbf{i}$-structure are mapped to integral curves.

Proposition 2. *The fronts of the i-Legendrian curves L and $L \cdot e^{it}$ on the sphere S_j^2 are equidistant (at a distance $2t$ in the direction from $\pi_j L$ towards $\pi_j(L \cdot e^{it})$).*

Proof. This is Lemma 2 of § 7.

Proposition 3. *Cusps of the j-front $\pi_j L$ of the i-Legendrian curve L on S_j^2 correspond to points of inflection to the k-front $\pi_k L$ on S_k^2 and, conversely, the points of inflection on S_j^2 correspond to cusps on S_k^2.*

Proof. At a cusp L has the j-direction, that is, it is tangent to $\{z \cdot e^{jt}\}$. Under the k-projection the circle $\{z \cdot e^{jt}\}$ projects to the great circle on S_k^2 with its normal contact elements. Tangency of the *first* order (between L and $\{z \cdot e^{jt}\}$) in the space $ST^* S_k^2$ is transformed after projection to S_k^2 into tangency of the *second* order of the projections, which implies a point of inflection of the k-front.

Proposition 4. *The front $\pi_j(L \cdot e^{it})$ of the shifted i-Legendrian curve $L \cdot e^{it}$ is congruent to the front $\pi_\varkappa L$ of the original i-Legendrian curve, where $\varkappa = j \cos 2t - k \sin 2t$.*

Proof. Multiplication on the right by e^{it} rotates the fields $\mathbf{j}$ and $\mathbf{k}$ in their plane with angular velocity 2 from $\mathbf{k}$ towards $\mathbf{j}$ (by the Theorem of § 8) with cyclic permutation of the units. The field $\mathbf{j}$ is sent by such a rotation into the field $\boldsymbol{\varkappa}$.

Theorem. *The k-front of an i-Legendrian curve on S_k^2 is congruent to the $\pi/2$-equidistant of its j-front on S_j^2.*

Proof. Setting $t = \pi/4$ in Proposition 4 we obtain the congruence of $\pi_j(L \cdot e^{i\pi/4})$ and $\pi_k L$. By Proposition 2, $\pi_j(L \cdot e^{i\pi/4})$ is the $\pi/2$-equidistant of the front $\pi_j L$.

Proof of the triality theorem from § 6. The assertion concerning the dual front has just been proved. The assertion concerning the derivative is obtained from that proof by cyclic permutation of the imaginary units i, j and k.

In fact, multiplication of S^3 on the right by e^{kt} for $t = \pi/4$ takes the field $\mathbf{j}$ to the field $\mathbf{i}$, and the field $\mathbf{i}$ to the field $-\mathbf{j}$. Consequently it takes the projection π_i to the projection π_j. The induced isometry $S_i^2 \to S_j^2$ takes the curve $\pi_i L$ to the curve $\pi_j(L \cdot e^{k\pi/4})$. The latter curve on the sphere S_j^2 is the derivative of the front $\Gamma = \pi_j L$ (by Lemma 3 of § 7). The theorem is proved.

Remark. It is not unprofitable to remark that the isometry $S_k^2 \to S_j^2$ constructed in the proof (induced by multiplication on the right by $e^{i\pi/4}$) respects the complex orientations of the spheres S_k^2 and S_j^2, while the isometry $S_i^2 \to S_j^2$ (induced by multiplication on the right by $e^{k\pi/4}$) does not.

CHAPTER III

QUATERNIONS AND CURVATURES

Here the algebra of quaternions is used to obtain additional information about spherical curves. First of all we consider the calculation of the geodesic curvatures of curves on the standard two-dimensional sphere. These results are used below in the proof of a generalized Gauss–Bonnet theorem and formulae for the Maslov index.

§ 10. The spherical radii of curvature of fronts

Consider the front $\Gamma = \pi_j L$ on the sphere S_j^2 of an i-Legendrian curve $L \subset S^3$. The *centre of spherical curvature* of the front at one of its points is the point of intersection of infinitesimally close great circles orthogonal to the front at the point. There are two such centres and they are antipodal.

Definition. The *spherical radius of curvature* of the front at a point is the distance from this point to the centre of spherical curvature along the normal that co-orients the front.

This radius is defined modulo π.

Theorem. *If an i-Legendrian curve L in S^3 has at a given point the direction*

$$\varkappa = \mathbf{j}\cos\theta + \mathbf{k}\sin\theta,$$

then its j-front $\pi_j L$ has at the corresponding point of the sphere S_j^2 the spherical radius of curvature $\theta \mod \pi$.

Proof The centre of curvature of the front at the given point lies on the caustic of the family of its equidistants: it is a singular point of the corresponding equidistant, lying on the normal to the front at that point.

On multiplication of the i-Legendrian curve L on the right by e^{it} its j-front is turned into its $2t$-equidistant (Proposition 2 of § 9). This equidistant is singular when the i-Legendrian curve $L \cdot e^{it}$ is tangent to the directions $\pm\mathbf{j}$ of the fibres of the bundle π_j.

A vector $\varkappa$, orthogonal to $\mathbf{i}$, is turned by multiplication on the right by e^{it} in the moving plane of the fields $\mathbf{j}$ and $\mathbf{k}$ through an angle $2t$ in the direction from $\mathbf{k}$ towards $\mathbf{j}$ (by the Theorem of § 8). Therefore the direction of the curve $L \cdot e^{it}$ (at the given point of L) becomes tangent to the fibre ($\theta = 0 \mod \pi$) after turning the vector $\varkappa$ through an angle $\theta \mod \pi$.

For such a choice of $t (= \theta/2)$ the front of the curve $L \cdot e^{it}$ is $\theta \mod \pi$-equidistant from the original front $\pi_j L$. Consequently, the distance to the centre of curvature of the front along its co-orienting normal is equal to $\theta \mod \pi$.

The theorem just proved allows us to consider the radii of curvature of the j-fronts of i-Legendrian curves, without worrying about their smoothness.

Definition. The *reduced radius of spherical curvature* of a front at one of its points is the minimum of the absolute values of its radii of spherical curvature, $\min |n\pi + \theta|$. This radius does not exceed $\pi/2$.

Example. The reduced radius of spherical curvature of a parallel of latitude is equal to the distance to the nearest pole along a meridian.

From the theorems proved there follow:

Corollary 1. *The sum of the reduced radii of spherical curvature at corresponding points of a front and its dual front is equal to $\pi/2$.*

Corollary 2. *The points of spherical inflection of fronts equidistant from the given one lie on the derivative curve of the caustic.*

In fact, at a point of the caustic the radius of curvature of the corresponding front is equal to 0, while at a point of inflection of the front it is equal to $\pi/2$. Moreover the caustic touches the ray (orthogonal to the front) at a cusp of the front, lying on the caustic.

The theorems proved above enable one to relax the regularity of fronts and caustics in these elementary assertions, requiring only that the i-Legendrian curve is immersed in S^3.

§ 11. Quaternions and caustics

Consider an i-Legendrian curve L on S^3. To it there corresponds the j-front $\pi_j L$ on the sphere S_j^2. The equidistants of this front are the projections of the i-Legendrian curves $L \cdot e^{it}$.

Consider the caustic of this family of equidistants. The family of i-Legendrian curves $L_t = L \cdot e^{it}$ defines a torus immersed in S^3:

$$T^2 = \{l \cdot e^{it} : l \in L, \quad t \in \mathbb{R} \mod 2\pi\}.$$

Definition. The *caustic of the j-projection of an i-Legendrian curve* is the set of critical values of the restriction $p : T^2 \to S_j^2$ of the projection $\pi_j : S^3 \to S_j^2$ to the torus T^2.

Theorem. *The caustic of the j-projection of an i-Legendrian curve is the j-projection of a k-Legendrian curve.*

Proof. Fibre the torus T^2 by i-Legendrian curves L_t with fixed t. Such an i-Legendrian curve has at the point $l \cdot e^{it}$ the j-direction for some $t(l)$, to be precise for $t(l) = \theta(l)/2 \mod \pi$, by the Theorem of § 10.

Definition. The curve $\{l \cdot e^{it(l)} : l \in L\}$ is the *j-critical curve of the i-Legendrian curve L*.

Lemma 1. *The j-critical curve of an i-Legendrian curve L is the set of critical points of the projection $p : T^2 \to S_j^2$.*

Proof of Lemma 1. The tangent plane of the torus at each point is generated by a vector of the field $\mathbf{i}$ and a vector which is orthogonal to it and tangent to the curve L_t. This plane contains the $\mathbf{j}$-direction if and only if the curve L_t is tangent to it.

The surprising resemblance of one-dimensional caustics to one-dimensional wave fronts is clarified by the following lemma.

Lemma 2 (basic). *The j-critical curve of an i-Legendrian curve is k-Legendrian.*

Proof of Lemma 2. Consider an i-Legendrian curve, having the $\mathbf{j}$-direction at a given point. We introduce a local parameter s on this curve, reducing to 0 at this point. A point of the curve corresponding to a small value of s has the form

$$l(s) = z + asz \cdot j + o(s), \qquad a \neq 0.$$

The angle of the tangent vector of this i-Legendrian curve with the $\mathbf{j}$-direction is given by the formula

$$\theta(s) = bs + o(s),$$

since for $s = 0$ this angle is zero.

Consequently, the corresponding point of the j-critical curve has the form

$$\begin{aligned} l(s) \cdot e^{i\theta(s)/2} &= (z + asz \cdot j + o(s)) \cdot (1 + ibs/2 + o(s)) \\ &= z + s(az \cdot j + (b/2)z \cdot i) + o(s). \end{aligned}$$

The vector in the brackets that is tangent to the j-critical curve at the point z is a real linear combination of the vectors $z \cdot j$ and $z \cdot i$ of the fields $\mathbf{j}$ and $\mathbf{i}$ at the point z. It is orthogonal to the vector of the field $\mathbf{k}$, as the lemma asserts.

The theorem is proved at the same time.

Thus a caustic and front on the sphere S^2_j are the j-projections of k-Legendrian and i-Legendrian curves. Therefore the symmetry between the imaginary units allows us to extend to caustics what we already know about fronts.

§ 12. The geodesic curvature of the derivative curve

Here we relate the geodesic curvature of the derivative curve to the derivative of the geodesic curvature of the original curve.

Theorem 1. *The geodesic curvature $\varkappa$ of the original curve and the angle α of its derivative curve with the great circle tangent to the original curve are connected by the relation*

$$\varkappa = \tan \alpha.$$

Proof. Consider an infinitely thin spherical triangle, formed by two infinitely close arcs of great circles that are tangent to the original curve (at points ds apart) and of length $\pi/2$, and the derivative curve. The angles of this triangle adjacent to one of the circular arcs are equal to $\varkappa ds$ and α (Fig. 8). The altitude dropped to that side is equal to $(\tan \alpha) ds$, since the distance of its base from the vertex of the angle α is equal to ds. Since this same altitude is equal to $\varkappa ds$ (being the side of a spherical triangle with sides $\pi/2$ lying opposite the angle $\varkappa ds$) one obtains the relation stated in the theorem.

Example. If the original curve is a parallel of latitude with Euclidean radius $\sin \theta$, then $\varkappa = \cot \theta$ and $\alpha = (\pi/2) - \theta$.

Theorem 1 follows from this example, since the *existence* of a universal relation between $\varkappa$ and α is obvious.

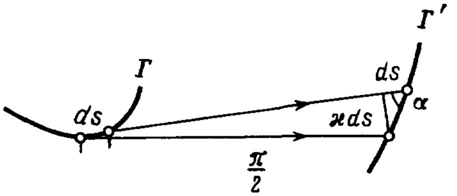

Figure 8

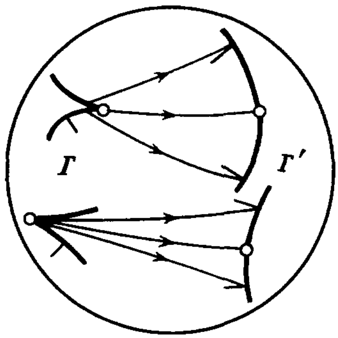

Figure 9

Corollary 1. *The derivative curve of a front is orthogonal to the great circle tangent to the front if and only if this circle is tangent to the front at a cusp* (Fig. 9).

Corollary 2. *The derivative curve of a front is tangent to the great circle tangent to the front if and only if this circle is tangent at a point of inflection* (Fig. 10).

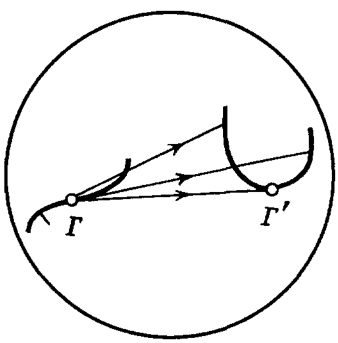

Figure 10

Both these corollaries allow one to estimate from below the number of cusps and points of inflection of the front (in terms of the increase of the angle α along the derivative curve).

Theorem 2. *The great circles touching the original front form a framing on its derivative curve that is parallel with respect to the standard Riemannian metric on the sphere.*

Proof. Consider two points of the original front at a small distance s from each other. We transport in parallel the tangent to the great circle touching the original front at the first point, from a point I of this circle on the derivative curve of the front along the derivative curve to a point II of the circle touching the original curve at the second point (Fig. 11).

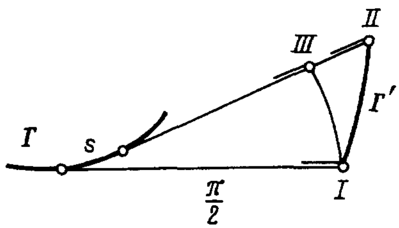

Figure 11

We replace the transfer along the segment (I, II) of the derivative curve by transfer along a path (I, III, II) along the trajectory orthogonal to the family of great circles touching the original front up to a point III on the second tangent circle, and then along this circle touching the front to the point II on the derivative curve.

The area of the triangle (I, II, III) has order s^2 and so the result of transfer along the broken path (I, III, II) differs from the result of transfer along the original curve by $O(s^2)$.

The path (I, III) has geodesic curvature 0 at the point I, since the distance from the point I to the centre of curvature of the path is equal to $\pi/2$. At a point of this path at a distance x from I the distance to the centre of curvature is equal to $\pi/2+O(x)$. Therefore the geodesic curvature at this point is $O(x)$. The distance (I, III) is $O(s)$. Therefore the total angle of turn of a vector carried along the path (I, III) with respect to the directions of the normals to the path (I, III) is $O(s^2)$. The path (II, III) is geodesic, and under parallel transport of a vector along this path the angle between the vector and the direction of the path does not change.

Finally, the direction of the great circle touching the front carried from the point I to the point II along the path (I, III, II) is transformed to the direction of the circle touching the front at the point II, turned through an angle $O(s^2)$. Under transfer along the derivative curve one obtains a result differing from this by another $O(s^2)$. Therefore the derivative of the angle between the transported vector and the directions of the great circles is equal to zero, and that means that the framing is parallel.

Corollary. *The framing of the curve on the sphere S_j^2, obtained by $\tilde{\pi}_j$-projection from the k-vectors along a j-Legendrian curve, is parallel with respect to the Riemannian metric of the sphere S_j^2.*

In fact, we apply Theorem 2 to the front Γ on S_j^2 of an i-Legendrian curve L on $S^3/\pm 1$. The derivative curve of the front is the $\tilde{\pi}_j$-projection of the image L' of the curve L under the action of multiplication on the right by $e^{k\pi/4}$. The framing by the directions of the great circles is the framing by the projections $\tilde{\pi}_{j*}$ of vectors of the field $\mathbf{k}$ along the curve L'. The curve L' is j-Legendrian if L is i-Legendrian, since multiplication on the right by $e^{k\pi/4}$ maps the field $\mathbf{i}$ to the field $\mathbf{j}$.

Thus by Theorem 2 for such a j-Legendrian curve L' its framing by the k-vectors projects onto S_j^2 as a parallel framing.

Any j-Legendrian curve may be obtained from an appropriate i-Legendrian curve by multiplication on the right by $e^{k\pi/4}$. On applying Theorem 2 to the front Γ of this i-Legendrian curve we have the corollary.

Remark. The quaternion units may be permuted. For example, along the projection of an i-Legendrian curve in S_i^2 the projections of the j-vectors (and also of the k-vectors) are parallel.

Theorem 2 enables one to compute explicitly the geodesic curvature of the derivative curve. In fact, this curvature, by Theorem 2, is equal to the derivative of the angle α between the derivative curve and the direction of the framing great circle with respect to the natural parameter l (arc length) of the derivative curve.

Theorem 3. *The geodesic curvature $K = d\alpha/dl$ of the derivative curve is expressed in terms of the geodesic curvature $\varkappa(s)$ of the original curve by the formula*

$$K = \frac{d\varkappa/ds}{(\sqrt{1+\varkappa^2})^3},$$

which can also be written in the form

$$K = \cos^3\alpha\, d\varkappa/ds = d\sin\alpha/ds.$$

Corollary. *The points of inflection ($K = 0$) of the derivative curve correspond to the vertices ($d\varkappa/ds = 0$) of the original curve.*

Proof of Theorem 3. Consider once again the infinitely thin spherical triangle (with sides $\pi/2$, $(\pi/2) - ds$ and dl and the angle α). From this triangle we find that the length dl of the segment of the derivative curve is $ds/\cos\alpha$. But from Theorem 1 we get

$$d\alpha = \frac{d\varkappa}{1+\varkappa^2} = \cos^2\alpha\, d\varkappa.$$

These relations give for $d\alpha/dl$ the expression stated in the theorem.

Example 1. For a parallel of latitude $\varkappa = \text{const}$ and its derivative curve is the equator ($K \equiv 0$).

Example 2. Near an ordinary (semicubical) cusp the square of the radius of curvature of the front is proportional to the distance to the cusp.

In this case $\varkappa(s) \sim 1/\sqrt{s}$, $d\varkappa/ds \sim 1/(\sqrt{s})^3$. The value of K remains bounded as $s \to 0$, as should be the case for an immersed curve. Therefore the K is positive for positive cusps and negative for negative cusps.

Example 3. Near a degenerate cusp of a front ($x \sim p^a, y \sim p^{a+1}$) the curvature $\varkappa$ has order $s^{(1-a)/a}$. In this case $\varkappa(s)$ has order $s^{(a-2)/a} \sim p^{a-2}$ (p is a regular parameter along the derivative curve).

Corollary. *The geodesic curvature of the derivative curve of a front has a zero of order $\mu - 1$ at a point corresponding to a cusp of the front of multiplicity μ.*

Here $\mu = \alpha - 1$: at an ordinary cusp (of multiplicity 1) $a = 2$, $\mu = 1$ and the geodesic curvature of the derivative curve at the origin is not zero.

We conclude that, at a cusp of the original front of multiplicity $\mu, \mu - 1$ vertices of the front vanish. For $\mu = 2$ after a small perturbation the front has a pair of cusps, and the vanishing vertex lies halfway between them.

The formulae obtained lead to new generalizations of the four-vertex theorem (for spherical curves), but I do not elaborate on this here.

§ 13. The derivative of a small curve and the derivative of curvature of the curve

The formulae of the preceding section can be used for the study of plane curves also.

If the original curve is concentrated around the pole on the sphere, then its derivative is concentrated near the equator. The geodesic curvature of a small curve on the sphere placed near the pole is asymptotically equal to the curvature of the plane curve obtained by projecting it to the tangent plane at the pole. The curvature of the derivative curve which is close to the equator also admits a simple asymptotic expression. By combining these two formulae we obtain a rather remarkable expression for the derivative of the radius of curvature of a *plane* curve, given by the following known theorem (see for example the text-book by Santalo).

Consider a curve Γ on the Euclidean plane with Cartesian coordinates (x, y), the parameter on which is the azimuth φ of the normal equator (so that $(\cos\varphi)dx + (\sin\varphi)dy \equiv 0$ along the curve). We denote the radius of curvature of the curve by R and we let $A = x\sin\varphi - y\cos\varphi$ (this measures the distance from the origin to the normal to the curve). We shall denote the derivatives with respect to the parameter φ by dashes.

Theorem.

$$R' = A + A''.$$

Remark. From this formula there follows at once the four-vertex theorem for a convex plane curve, namely: *R' has no fewer that four zeros.*

In fact $-A$ is the derivative by φ of the supporting function $B = x\cos\varphi + y\sin\varphi$ of the curve, so that

$$R' = -(B' + B''').$$

A function of the form $B' + B'''$ has no fewer than four zeros on the circle by the theorems of Sturm (Kellogg and Tabachnikov...), asserting that the Fourier series

$$F(\varphi) = \sum_{n \geqslant m} a_n \cos(n\varphi) + b_n \sin(n\varphi)$$

has no fewer zeros than the number $2m$ of zeros of the first of its harmonics (see [2], [3], [4], [5] for details).

Proof of the theorem. The formula to be proved is a limiting case of Theorem 3 of § 12. Let ε be a small parameter. Consider the small curve $\varepsilon\Gamma$ on the Euclidean plane, tangent to the unit sphere at the North pole. Its curvature is $\varkappa/\varepsilon$, where $\varkappa = 1/R$ is the curvature of Γ. The natural parameter on $\varepsilon\Gamma$ is εs, where s is the natural parameter on Γ.

Project $\varepsilon\Gamma$ to the sphere (for example, along the axis through the pole, or from the centre, or from the South pole). On the sphere we obtain a small curve Γ_ε concentrated near the North pole. The geodesic curvature $\varkappa_\varepsilon$ of this curve is asymptotically equivalent to $\varkappa/\varepsilon$ as $\varepsilon \to 0$, while the natural parameter s_ε is asymptotically equivalent to εs. Consequently the expression of Theorem 2 of § 12 for the geodesic curvature of the derivative of Γ_ε takes the form

$$\frac{d\varkappa_\varepsilon/ds_\varepsilon}{(\sqrt{1+\varkappa_\varepsilon^2})^3} \sim \frac{\varepsilon^{-2}\, d\varkappa/ds}{\varepsilon^{-3}\varkappa^3} = -\varepsilon\frac{dR/ds}{\varkappa} = -\varepsilon\frac{dR/ds}{d\varphi/ds} = -\varepsilon\frac{dR}{d\varphi}. \tag{1}$$

Consider now the derivative of the small curve Γ_ε (Fig. 12). As $\varepsilon \to 0$ this derivative curve tends to the equator. It smoothly depends on ε. Let us introduce on the sphere the coordinates of longitude ψ ($x = \cos\psi, y = \sin\psi$) and latitude λ ($z = \sin\lambda$).

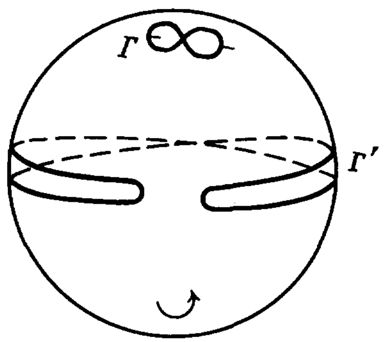

Figure 12

Lemma 1. *The derivative of Γ_ε for small ε is given locally by the equation*

$$\lambda = \Lambda(\psi, \varepsilon), \qquad \Lambda(\psi, 0) = 0.$$

In fact, consider the great circle tangent to Γ_ε at the point corrresponding on Γ to the point whose normal has the azimuth φ. This circle intersects the equator at the point with longitude $\psi = \varphi + (\pi/2) + O(\varepsilon)$. The distance along the circle from the point of tangency to the equator is equal to $(\pi/2) + \varepsilon A(\varphi) + O(\varepsilon^2)$. The expression for λ now follows from the implicit function theorem, since the circle constructed differs from the meridian of longtitude $\varphi + (\pi/2)$ by a quantity of order ε.

Lemma 2. *The geodesic curvature of a smooth curve close to the equator*

$$\lambda = \Lambda(\psi, \varepsilon) = \varepsilon M(\psi) + o(\varepsilon)$$

is given by the formula

$$K = \varepsilon(M + M_{\psi\psi}) + o(\varepsilon).$$

Proof. It is sufficient, for example, to apply the Gauss–Bonnet formula to a small curvilinear quadrilateral bounded by the equator, two nearby meridians and our curve close to the equator.

The area of the quadrilateral is proportional to M, while the difference of the angles between our curve and the meridians is proportional to $M_{\psi\psi}$ (in a first approximation with respect to ε). From this one obtains the required expression for the geodesic curvature of the only side of the quadrilateral that is not a geodesic.

Comparison of Lemmas 1 and 2 leads at once to an asymptotic expression for the geodesic curvature of the derivative of the curve Γ_ε at the point corresponding to the point on Γ where the normal has the azimuth φ:

$$K(\varepsilon) = \varepsilon\big[A(\varphi) + A_{\varphi\varphi}(\varphi)\big] + o(\varepsilon).$$

Comparing this formula with formula (1) we obtain (for the appropriate choice of sign of the geodesic curvature)

$$dR/d\varphi = A + A_{\varphi\varphi},$$

as had to be proved.

Remark. The resemblance of the passage from the small curve to its derivative used here with the contactomorphism

$$ST^*\mathbb{R}^n \to J^1(S^{n-1}, \mathbb{R})$$

of the space of co-oriented contact elements of Euclidean space to the space of the 1-jets of functions on the sphere used in [4] deserves special study.

CHAPTER IV

THE CHARACTERISTIC CHAIN AND SPHERICAL INDICES OF A HYPERSURFACE

The index of a point with respect to a closed plane curve is not conformally invariant. It changes if one takes for the point at infinity some other point of the Riemann sphere.

It turns out that one can adjust the indices of all points (deducting from them a suitable number) so that the adjusted indices are conformally invariant.

In this way, with each component of the complement to a curve on the sphere one can associate its spherical index. The components of the complement, furnished with this index as a multiplicity, form a 2-chain, whose boundary is the original curve.

This remarkable 2-chain is called the characteristic chain of the curve. In an analogous manner one can define the characteristic n-chain of an oriented $(n-1)$-dimensional hypersurface on an even-dimensional sphere and, more generally, of a wave front on an even-dimensional sphere.

Integrals over the characteristic chain appear in the Gauss–Bonnet formula for immersed curves, in the formula for the Maslov index, and so on. Unfortunately the simplicity and naturalness of the constructions of this and the following chapters is somewhat dimmed by the need to pay due heed and attention to endless signs and orientations.

§ 14. The characteristic 2-chain

Consider a smooth immersion of an oriented circle in general position on the oriented 2-sphere. The characteristic 2-chain of such an immersion is the chain consisting of the closures of the regions of the complement to the image with special (integral or semi-integral) coefficients. The curve itself is the boundary of its characteristic chain. The coefficients are chosen in the following way.

Select an auxiliary point B on the sphere outside the immersed curve. We regard it as the 'point at infinity' and identify the complement of this point with the oriented plane. The immersed curve on the sphere transforms to an immersed curve in the plane. This curve has a well-defined index (the number of complete turns of the normal as one travels right round the curve). We denote this by i_B (it depends on the choice of B).

For each point A of the plane outside the curve so obtained there is defined the winding number of the curve around A. Since this number depends not only on A but also on the choice of B, we denote it by i_A^B. The number i_A^B depends on the components of the complement to the curve in which A and B lie (while i_B depends on the component of B).

Theorem 1. *The difference*

$$\tilde{i}(A) = i_A^B - (i_B/2)$$

is a conformally invariant function on the complement to the curve on the sphere: it does not depend on the choice of the point B at infinity.

Example 1. Consider the equator on the oriented sphere. We orient it as the boundary of the Northern hemisphere.

This means that the frame (the outward normal of the Northern hemisphere, the vector orienting the equator) positively orients the sphere.

The choice of orientation and co-orientation of a hypersurface in an oriented manifold is said to be *correct* if on placing after the co-orienting normal an orienting frame one orients the manifold positively.

In the Northern hemisphere $\tilde{i}(A) = 1/2$, while in the Southern hemisphere $\tilde{i}(A) = -1/2$ (Fig. 13, see p. 32). A correctly co-orienting vector is directed to the side of smaller values of $\tilde{i}$.

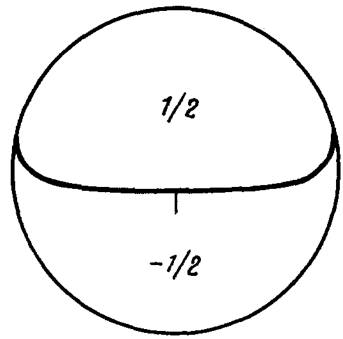

Figure 13

Example 2. Consider a figure of eight on the sphere (with correct orientation and co-orientation). The figure of eight divides the sphere into three regions (Fig. 14). One of these (R) is bounded by both loops, the second region (P) is bounded by the loop with co-orientation directed into the region, while the third region (Q) is bounded by the loop with co-orientation directed outside the loop. Calculations show that $\tilde{\imath}(P) = -1$, $\tilde{\imath}(Q) = 1$, $\tilde{\imath}(R) = 0$.

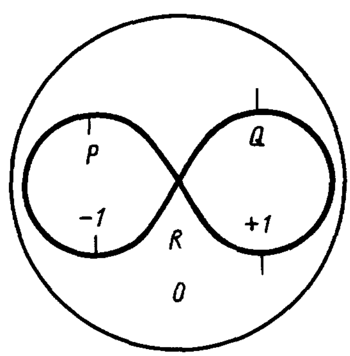

Figure 14

Definition. The *characteristic chain* of an immersed curve in general position on the sphere is the 2-chain consisting of the closures of the regions of the complement with the coefficients $\tilde{\imath}(A)$.

Suppose that we choose the point B to lie in the same component of the complement of the immersed curve in which A lies. Then $i_A^B = 0$. This proves the following corollary.

Corollary. *The following formula holds for the characteristic chain:*

$$\tilde{\imath}(A) = -i_A/2.$$

Although this formula appears to be simpler, in practice it is easier to do calculations using the formula of Theorem 1, since in it the point at infinity is fixed.

Let us consider now the lift of the immersion $S^1 \to S^2$ to the Legendrian curve of the normals formed from the correctly co-oriented contact elements of the sphere S^2 (tangent to the immersed curve):

$$\gamma : S^1 \to ST^*S^2 \approx \mathbb{R}\mathrm{P}^3 \approx S^3/\pm 1.$$

I remark that the bundle $p : ST^*S^2 \to S^2$ doubly covers the Hopf bundle $S^3 \to S^2$.

'The curve of normals' $\gamma(S^1)$ lies in $\mathbb{R}\mathrm{P}^3$ and is doubly covered in S^3 by a curve $\tilde{\gamma}$. The covering curve consists of one or two components, depending on the parity of the index of the corresponding curve on the plane. If the index i_B is even, as for the figure of eight, then there are two components, while if it is odd, then there is one (the parity of the index does not depend on the choice of B, it is opposite to the parity of the number of double points on the immersed curve).

Theorem 2. *The linking coefficient of the Legendrian curve $\tilde{\gamma}$ in S^3 covering the curve of normals, with the fibre of the Hopf bundle lying over the point $A \in S^2 \backslash f(S^1)$, is equal to*

$$2\tilde{i}(A) = 2i_A^B - i_B,$$

independently of the parity of the index ($i_B \mod 2$).

We denote by F the fibre of the bundle $ST^*S^2 \to S^2$ (with its natural orientation).

Theorem 3. *Consider the 2-chain σ in ST^*S^2 whose boundary is the curve of normals γ if the index of the original curve is even, and the curve $\gamma - F$ if it is odd.*

The projection $p_*\sigma$ of the chain σ on S^2 is the characteristic 2-chain $\tilde{i}$ of Theorem 1 (independently of the chioce of σ) for even index ($i_B \mod 2$), while $p_*\sigma = \tilde{i} + (1/2)$ for odd index.

The proofs of Theorems 1–3 are given in § 18 in the more general situation of wave fronts on even-dimensional spheres. But since some facts that we require remain true also for spheres of any dimension, I begin with these more general facts.

§ 15. The indices of hypersurfaces on a sphere

Consider an oriented smoothly immersed hypersurface M^{n-1} on an oriented sphere S^n.

Definition. The *correct co-orientation* of the hypersurface M is the field of unit normals ν such that the frame (normal, tangent frame positively orienting M) positively orients the sphere.

Remark. If M is the boundary of a region (with its usual orientation), then the normal ν is the outward normal of this region.

Consider two points A and B on the sphere outside M. On taking B to be the point at infinity and A to be the origin we may identify $S^N \backslash B$ with $\mathbb{R}^n$ and A with $0 \in \mathbb{R}^n$. The image of M in $\mathbb{R}^n$ will be denoted by M_A^B. The space $\mathbb{R}^n$ and the hypersurface M_A^B immersed in it are oriented.

Definition. The *index* $i_A^B(M)$ of the hypersurface M on the sphere relative to the pair of points (A, B) is the degree of the map of the hypersurface M_A^B to the sphere S^{n-1} by the rays from $A \approx 0$:

$$i_A^B(M) = \deg(s_A^B : M_A^B \to S^{n-1}), \qquad s_A^B(x) = x/|x|$$

(the sphere S^{n-1} is oriented here as the boundary of the ball $|x| \leqslant 1$ in $\mathbb{R}^n$).

Theorem 4. *The index of a smoothly immersed hypersurface on the sphere with respect to a pair of points of the complement depends only on the components of the complement to which these points belong. The function i_A^B on the sphere of a pair of regions of the complement to the hypersurface is skew-symmetric, and may be represented in the form*

$$i_A^B = \varphi(B) - \varphi(A),$$

where φ is a function of the region of the complement.

Consider an n-chain c on S^n, consisting of the closures of regions of the complement of a hypersurface M with coefficients

$$c(A) = i_A^B,$$

where B is an arbitrary fixed point of the complement. The sphere and the components here are oriented.

Theorem 5. *The boundary of the n-dimensional chain c is the original oriented hypersurface*:

$$\partial c = M^{n-1}.$$

Consider a point B outside the image of the hypersurface M on the sphere. Taking B to infinity we identify $S^n \backslash B$ with $\mathbb{R}^n$, as before.

Definition. The *index* $i_B(M)$ *of the immersed hypersurface M with respect to the point B* is the degree of the Gauss map

$$i_B(M) = \deg(s_B : M^{n-1} \to S^{n-1}),$$

where $s_B(x)$ is the normal vector of length 1 at x which co-orients the hypersurface M^{n-1} correctly.

Theorem 6. *On an even-dimensional sphere the indices of a hypersurface relative to points and pairs are connected by the relation*

$$2i_A^B = i_B - i_A.$$

In particular, the parity of all the indices i_A is the same.

Theorem 7. [6] *On an odd-dimensional sphere the indices of a hypersurface relative to all points of its complement are the same and equal to half the Euler characteristic of the immersed hypersurface*:

$$i_B(M) = \chi(M)/2.$$

The proofs of Theorems 1–7 will be given below, in § 18.

§ 16. Indices as linking coefficients

Consider the manifold $E^{2n-1} = ST^*S^n$ of co-oriented contact elements of the sphere. As with any contact manifold defined by a global contact form α, it has a canonical co-orientation $\alpha \wedge (d\alpha)^{n-1}$.

[An orienting frame can be obtained, for example, by adjoining to a frame orienting the base S^n a frame orienting the fibre S^{n-1}. The orientation of the fibre may be obtained from the orientation of the base as the orientation of the boundary of a small disk of radius r. The outward normals of the disk form the fibre, as $r \to 0$. Here, as usual, the cotangent vectors are identified with the tangent vectors by means of a metric.]

Thus in E^{2n-1} one can consider the indices of intersection of transversal chains of complementary dimension (and also of cycles of complementary dimension).

In particular, we can intersect Legendrian $(n-1)$-dimensional chains with n-dimensional ones.

Definition. The *Legendrian manifold of normals* L^{n-1} of an immersed oriented hypersurface M^{n-1} in the oriented sphere S^n is the manifold of correctly co-oriented contact elements of the sphere tangent to the immersed hyperpsurface M^{n-1}.

If the original immersed manifold M^{n-1} is in general position in S^n, then the Legendrian submanifold L^{n-1} of normals is embedded in E^{2n-1}.

The submanifold L^{n-1} of normals is homologous to the fibre F_b^{n-1} over an arbitrary point $b \in S^n \backslash M^{n-1}$ with some coefficient (since the cyclic group $H_{n-1}(E)$ is generated by $[F]$): there is an n-chain σ such that

$$\partial\sigma = L - kF_b, \qquad k \in \mathbb{Z}.$$

[Here we use the homology groups of E^{2n-1}:
if n is even, then the group is $\mathbb{Z}_2$ in dimension $n-1$ and 0 in dimension n;
if n is odd, then it is $\mathbb{Z}$ in dimensions $n-1$ and n (and, of course, always $\mathbb{Z}$ in dimensions 0 and $2n-1$);
the $\mathbb{Z}_2$-homology groups are $\mathbb{Z}_2$ in dimensions 0, $n-1$, n and $2n-1$.]

Theorem 8. *The index of an immersed hypersurface M relative to a pair of points is equal to the difference between the indices of intersection of the chain σ with the fibres over these points, to be precise*

$$i_A^B = \sigma \cap F_A - \sigma \cap F_B.$$

The non-uniqueness of the choice of σ has no effect on the difference.

Now suppose that the dimension of the ambient sphere S^n is even.

[6] On this matter see also the work of Tabachnikov [15] and the results of White discussed there on Legendrian links in a contact manifold of dimension $4n+1$.

Theorem 9. *On an even-dimensional sphere the index of intersection of the chain σ with the fibre F_A is equal to*

$$\sigma \cap F_A = -i_A/2 + \varepsilon/2,$$

where $\varepsilon = 0$ if the index i_A (and therefore also any index i_B) is even, while $\varepsilon = 1$ if the index i_A is odd.

In particular, the projection of the n-chain σ on the sphere S^n does not depend for even n on the choice of σ, and the boundary of this projection is the original oriented hypersurface M^{n-1}.

Remark. Theorem 9 makes clear why the indices

$$\tilde{i}(A) = i_A^B - (i_B/2) = -i_A/2$$

can be interpreted as linking coefficients. In fact, for a hypersurface of even index on an even-dimensional sphere under arbitrary choice of the point B

$$\tilde{i}_A = \sigma \cap F_A.$$

Since in this case $\partial\sigma = L$, Theorem 2 means that $\tilde{i}_A$ *is the linking coefficient of the Legendrian manifold L of normals of M with the fibre F_A over the point A* (this linking coefficient is defined as the index of intersection of the chain σ filling out L with this fibre).

The proofs of Theorems 8 and 9 are carried out in § 18 with the help of the following constructions.

§ 17. The indices of hypersurfaces on a sphere as intersection indices

Consider two distinct points A and B of the sphere S^n. For this pair of points we construct the n-chain in the manifold E^{2n-1} of co-oriented contact elements of the sphere (to intersect the Legendrian submanifolds with it).

Definition 1. The *cylinder V_A^B over the segment* (A, B) is the n-dimensional submanifold (with boundary) in E^{2n-1} of all contact elements of the sphere applied at the points of the segment (A, B) smoothly embedded in the sphere.

By the segment (A, B) we understand here any smooth embedding of a segment in the sphere joining A to B (for example, a geodesic segment).

The cylinder V_A^B carries a natural orientation.

Definition 2. The orientation of the cylinder V_A^B is counted as *positive* if its intersection index with the Legendrian sphere formed from the outward normals of a small disk on S^n with centre at A is equal to $+1$.

Theorem 10. *The index of an immersed oriented hypersurface M^{n-1} with respect to a pair of points A and B of the oriented sphere S^n is equal to the intersection index of the Legendrian manifold L of normals of the immersed hypersurface M with the cylinder over a segment (A, B)*:

$$i_A^B(M) = L \cap V_A^B.$$

Remark. Here it does not matter whether the hypersurface M is correctly oriented, provided only that it is the projection on S^n of the Legendrian manifold L. In fact, changing the orientation of M changes also the orientation of L and therefore also the signs on both the left- and the right-hand sides of the equality.

Theorem 11. *The boundary of the cylinder V_A^B is equal to the difference of the coherently oriented fibres over the points A and B, to be precise*

$$\partial V_A^B = (-1)^n (F_A - F_B).$$

Consider a point B on S^n. For this point we construct the n-chain in the manifold E^{2n-1} of co-oriented elements of the sphere (to intersect Legendrian manifolds with it).

Definition 3. A *B-section* V_B is the n-chain in E obtained from any constant vector field on $\mathbb{R}^n$ under stereographic projection with centre B, $\mathbb{R}^n \to S^n \backslash B$.

Remark. The stereographic projection sends a constant field on $\mathbb{R}^n$ to a manifold diffeomorphic to an open ball of dimension n. Choosing an appropriate diffeomorphism we get a smooth map which extends to the *closed* ball. On the boundary we then obtain a map of the sphere $S^{n-1} \subset \mathbb{R}^n$ to the sphere F_B (the fibre over B), given by the 'folding' map

$$\omega \to e - 2(\omega, e)\omega, \qquad (*)$$

where e is a constant vector of length 1.

The folding map $(*)$ sends any vector ω of the unit sphere S^{n-1} to the reflection of the constant basis vector e in the mirror orthogonal to ω.

One can easily prove the following lemma.

Lemma. *The folding map $(*)$ is realizable geometrically as the stereographic projection of a sphere from its centre onto another sphere of the same radius passing through this centre.*

Example. For $n = 2$ the folding map $(*)$ of a circle on a circle is the ordinary double covering '$\varphi \to 2\varphi$' (Fig. 15).

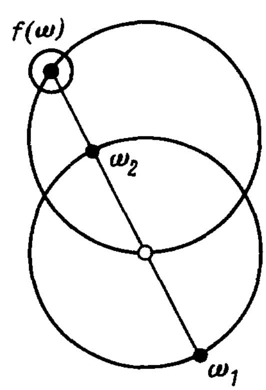

Figure 15

The map of a *closed* ball to E^{2n-1} described above defines an n-chain V_B. Its orientation is induced from the orientation of the base $\mathbb{R}^n \approx S^n \setminus B$.

Theorem 12. *The index of a hypersurface M^{n-1} with respect to B on S^n is equal to the index of intersection of the Legendrian manifold L^{n-1} of normals of the immersed hypersurface M^{n-1} with the B-section V_B:*

$$i_B(M) = L \cap V_B.$$

Remark. Here it does not matter whether the mainfold M is correctly oriented, provided only that it is the projection of a Legendrian manifold L to the sphere S^n. For a change in the orientation of M implies a change in the orientation of L and change in the signs on both the left- and the right-hand sides of the equality.

Theorem 13. *The boundary of the B-section V_B is the image of the sphere S^{n-1} by the folding map $(*)$, to be precise (taking account of the orientations)*

$$\partial V_B = -(1+(-1)^n)F_B.$$

In other words, on even-dimensional spheres $\partial V_B = -2F_B$, while on odd-dimensional spheres $\partial V_B = 0$.

The geometrical Theorems 10–13 are verifiable directly, from the definitions. For example in Theorems 10 and 12 the contributions of each point of intersection in the left- and in the right-hand side of the equality is the same. It is only necessary to follow accurately through all the orientations. Theorems 1–9 are derived below from these four geometrical theorems.

§ 18. Proofs of the index theorems

Proof of Theorem 6. Let n be even. From Theorems 11 and 13 we have

$$\partial(2V_A^B - V_B + V_A) = 0.$$

Therefore from Theorems 10 and 12 it follows that

$$2i_A^B - i_B + i_A = 0,$$

that is, Theorem 6.

Proof of Theorem 4. From Theorem 11 it follows (for arbitrary n) that

$$\partial(V_A^B + V_B^A) = 0, \qquad \partial(V_A^B + V_B^C + V_C^A) = 0.$$

Applying Theorem 10 we conclude that

$$i_A^B = i_C^B - i_C^A.$$

By this Theorem 4 is proved.

Proof of Theorem 5. For a fixed point B the index $i_A^B(M)$ can be computed as the flow of the radial field in $\mathbb{R}^n \approx S^n \setminus B$ with centre at the point A. The change of this flow corresponding to transversal positive intersection of the hypersurface M by the point A is equal to -1. This proves Theorem 5.

Suppose now that the dimension of the sphere S^n is even.

Proof of Theorem 7. Consider M as an immersed hypersurface in general position in $\mathbb{R}^n \approx S^n \setminus B$. Construct two B-sections, corresponding to the constant fields e_1 and $-e_1$ in $\mathbb{R}^n$. On calculating the degree of the Gaussian map over the points $\pm e_1 \in S^{n-1}$ we obtain

$$i_B = m_0^+ - m_1^+ + \cdots + m_{n-1}^+,$$
$$i_B = m_0^- - m_1^- + \cdots + m_{n-1}^-,$$

where $m_k^\pm$ is the number of points where the co-oriented normal has the direction $\pm e_1$ and for which the restriction of the coordinate function $\pm x_1$ has a Morse critical point of index k.

The restriction of the function x_1 to M has therefore in all $m_k = m_k^+ + m_k^-$ critical points of index k. Thus, $\chi = m_0 - m_1 + \ldots + m_{n-1} = 2i_B$, and Theorem 7 is proved.

Proof of Theorem 8. For n-chains σ and τ in E^{2n-1}

$$(\partial\sigma) \cap \tau = (-1)^n \sigma \cap (\partial\tau).$$

Let $\partial\sigma = L - kF_C$, $\tau = V_A^B$. Then $(\partial\sigma) \cap \tau = i_A^B$ by Theorem 10, since $F_C \cap V_A^B = 0$ (the segment (A, B) may be chosen so as not to pass through C). On the other hand,

$$\sigma \cap (\partial\tau) = (-1)^n(\sigma \cap F_A - \sigma \cap F_B),$$

according to Theorem 11. Finally,

$$i_A^B = (\partial\sigma) \cap \tau = (-1)^n \sigma \cap (\partial\tau) = \sigma \cap F_A - \sigma \cap F_B,$$

which proves Theorem 8.

Proof of Theorem 9. Let n be even. Put $\tau = -V_A$. Then $2\tau = F_A$ by Theorem 13. The cycle $L - \varepsilon F_B$ is homologous to zero, so there is an n-chain σ with boundary $\partial\sigma = L - \varepsilon F_B$. Now $(\partial\sigma) \cap \tau = \sigma \cap (\partial\tau)$. But we have

$$(\partial\sigma) \cap \tau = (L - \varepsilon F_B) \cap V_A = \varepsilon F_B \cap V_A - L \cap V_A,$$

while also $\sigma \cap (2\tau) = 2\sigma \cap F_A$. According to Theorem 12, $L \cap V_A = i_A(M)$. Besides this, $F_B \cap V_A = 1$ due to the choice of the orientation of the fibre F_B and the A-section V_A. Thus, $\varepsilon - i_A = 2\sigma \cap F_A$, which proves Theorem 9.

Proof of Theorem 3. This is a particular case of Theorem 9 (the independence of the chain $p_*\sigma$ from the choice of σ follows also from the fact that $H_n(ST^*S^n) = 0$ for even n).

Proof of Theorem 1. This is a particular case of Theorem 6.

Proof of Theorem 2. We use Theorem 9. Denote by $\hat{\sigma}$ the inverse image of the chain σ under the double covering $S^3 \to ST^*S^2$. The fibre of the Hopf bundle over A is the double covering $\hat{F}_A$ of the fibre F_A of the bundle $ST^*S^2 \to S^2$ over A. We have $\partial\hat{\sigma} = \hat{\gamma} - \varepsilon\hat{F}_C$. Denote by D_C a 2-chain in S^3 with boundary $\partial D = \hat{F}_C$. Then the 2-chain $\hat{\sigma} - \varepsilon D_C$ has boundary $\widetilde{\gamma}$. Therefore the linking coefficient sought for is

$$l(\hat{\gamma}, \widetilde{F}_A) = (\hat{\sigma} - \varepsilon D_C) \cap \hat{F}_A = \hat{\sigma} \cap \hat{F}_A - \varepsilon D_C \cap \hat{F}_A.$$

According to Theorem 9, $\hat{\sigma} \cap \hat{F}_A = 2\sigma \cap F_A = \varepsilon - i_A$. The linking coefficient of the fibres $\hat{F}_C$ and $\hat{F}_A$ of the Hopf bundle is equal to 1, and therefore $D_C \cap \hat{F}_A = 1$. Thus $l(\hat{\gamma}, \hat{F}_A) = -i_A$. But, by Theorem 6, $-i_A = 2i_A^B - i_B$. And so Theorem 2 is proved.

§ 19. The indices of fronts of Legendrian submanifolds on an even-dimensional sphere

Let L^{n-1} be an oriented Legendrian immersed submanifold of the contact manifold ST^*S^n of co-oriented elements of the oriented sphere S^n.

Consider the front of L (its projection to the sphere) as an oriented and co-oriented (singular) hypersurface of the sphere.

Example. Let L be the manifold of normals of an oriented hypersurface immersed in the sphere. Then for the front we get this immersed hypersurface with its regular co-orientation.

Remark. If one changes the *orientation* of L, then for the oriented and co-oriented front one obtains the same co-oriented hypersurface but with changed orientation. In such a case the co-orientation of the oriented immersed hypersurface forming the front is not correct.

In the general case the front has singularities. In such a case there is no relation between the orientation and co-orientation of the front and the orientation of the ambient sphere. Therefore in transferring the preceding results to fronts one needs to omit the correctness of the co-orientation everywhere (replacing it by the agreement of the orientations of the Legendrian manifold L and its front). With this change all the results of the previous sections hold.

Let A and B be points of the complement of a front in the sphere. Identifying B with the point at infinity, we get in $\mathbb{R}^n$ an oriented and co-oriented front M_B^{n-1}.

Definition. The *index* i_B is defined as the degree of the Gauss map $L^{n-1} \to S^{n-1}$ (associating with a point of the Legendrian manifold the direction of the normal which co-orients the front in $\mathbb{R}^n$).

The *index* i_A^B is defined as the degree of the radial map $L^{n-1} \to S^{n-1}$ (associating with a point of the Legendrian manifold the direction of the vector from A to the point of the front).

The quantity

$$\widetilde{i}(A) = i_A^B - (i_B/2)$$

in this new situation is as before a locally constant function on the complement of the front that does not depend on B (it is equal to $-i_A/2$, as in the case of an immersion).

Example. The values of the function $\tilde{\imath}(A)$ for some fronts with two cusps are shown in Fig. 16.

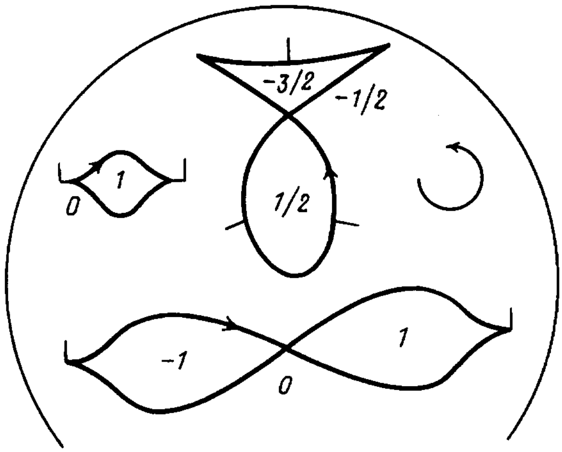

Figure 16

Definition. The *characteristic n-chain* of an oriented and co-oriented front in an even-dimensional oriented sphere is the n-chain consisting of the oriented closures of the components of the complement of the front with coefficients $\tilde{\imath}(A)$.

Remark. On changing the orientation of the ambient sphere the characteristic n-chain *does not change.* One can say that an oriented and co-oriented front determines the orientations as well as the multiplicities of the regions of the complement.

On change of the co-orientation of a front the characteristic n-chain also does not change.

On change of the orientation of a front (that is, of the Legendrian manifold corresponding to it) the characteristic chain changes sign: its boundary is always the oriented front.

From these properties of the characteristic chain it is evident that it is possible to define it also for (oriented and co-oriented) fronts on a projective space of even dimension.

To be precise, for the two-sheeted cover of a projective space by the sphere the full inverse image of the characteristic chain of a front on the projective space is the characteristic chain of the full inverse image of the front on the sphere.

Example. The characteristic 2-chain of an oriented circle on the projective plane is the disk bounded by this circle (with its orientation chosen so that its boundary is the given oriented circle). On the Möbius strip complementary to the disk $\tilde{\imath} = 0$. All this does not depend on the co-orientation of the front.

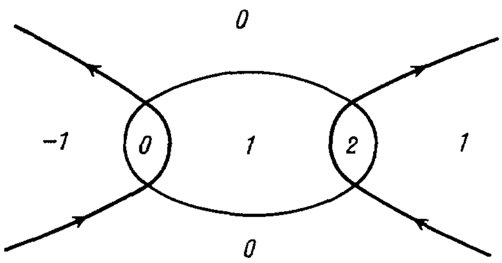

Figure 17

The characteristic 2-chain of the front consisting of a confocal ellipse and hyperbola is shown in Fig. 17.

Theorem 14. *Theorems* 1–6 *and* 8–13 *of this chapter remain true for fronts.*

Even in the proofs practically nothing has to be changed (if we define the indices as mentioned above and define the orientation of the front M independently from its co-orientation as the orientation of its parametrizing Legendrian manifold L).

The proofs of Theorems 10, 12 and 5 are the most crucial. Consider, for example, the assertion of Theorem 12:

$$i_B(M) = L \cap V_B.$$

The proof consists in calculating the contributions of points of transversal intersection of L with V_B both to the left-hand side and to the right. It is asserted that both these contributions are equal for each point separately.

In terms of the Euclidean space $\mathbb{R}^n \approx S^n \setminus B$ the manifold $V_B \subset ST^*\mathbb{R}^n$ is describable as a constant generic (co)vector field v. The contribution to the right-hand side is given only by those points of L where the co-orienting normal to M has directon v.

By choosing the field v to be generic we can arrange that the corresponding points of the front M are non-singular. At these and only at these points the vector co-orienting the front has the direction v.

In computing the number $i_B(M)$ as the degree of the Gauss map $L^{n-1} \to S^{n-1}$ it is enough to count the signs of all these inverse images of the direction v.

At these points the front is a smooth oriented and co-oriented hypersurface. If its co-orientation is correct for the given orientation, then such a point gives the same contribution to both sides of the equality in Theorem 12 (its proof therefore consists in this verification of the coincidence of the contributions).

If the co-orientation of M is not correct for the given orientation, then we change (locally) the *orientations* of L and M. Then the co-orientation becomes correct. The equality of the contributions to the left- and right-hand sides will then hold at the point under consideration. But the contributions both on the left and on the right have changed sign under the change of the orientations. This means that they were equal to start with. By this the assertion of Theorem 12 is proved for the case of fronts.

The assertions of Theorems 10 and 5 are proved in exactly the same way.

The assertion of Theorem 7 for fronts in S^3 or in $\mathbb{R}^3$ is clearly not true: the index i_B depends on the choice of point B.

In the case of S^2, where the front is one-dimensional, all the indices and characteristic chains of generic fronts may be computed also by means of the following construction of averaging the indices of smoothings of the front.

Apart from self-intersections, the only singular points of a generic front are semicubical cusps. There are two ways of smoothing a cusp (Fig. 18): in the one case near the cusp there appears a point of self-intersection, while in the other case there does not. In either case a co-orientable front becomes non-co-orientable, while a non-co-orientable front becomes co-orientable.

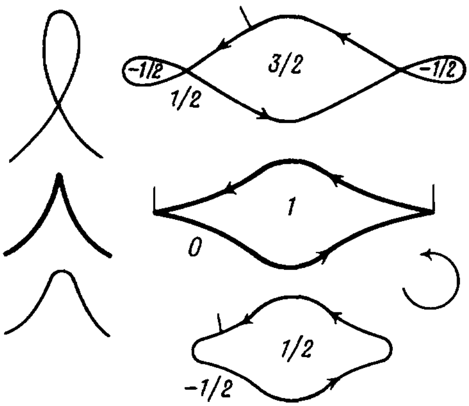

Figure 18

On smoothing in one way or the other all the cusps of a co-orientable front we obtain two new co-orientable fronts (since the number of cusps of a co-orientable front is even). Note that both the fronts so obtained are oriented (like the original front away from its cusps).

Theorem 15. *For the computation of the indices i_B, i_A^B and $\tilde{i}_A^B$ is enough to smooth the original front near all its cusps both ways and to take the arithmetic mean of the values of the corresponding indices for the two immersed curves so obtained.*

Example. Fig. 18 shows the characteristic chains of both smoothings of the simplest curve with two cusps.

Remark. One can form the mean of all the 2^N possible smoothings resulting from smoothing N cusps—the result will be just the same.

Proof. It is sufficient to prove the assertion for plane curves. We denote by $\varphi(t)$ the azimuth of the co-orienting contact element of the original front at the point corresponding to t on the Legendrian curve. For the smoothed curves we denote the corresponding functions by φ_+ and φ_-. Choose the values of the azimuth so that at one non-singular point of the front all three functions have the same value.

On passing through the nearest cusp the differences $\varphi_+ - \varphi$ and $\varphi_- - \varphi$ each change by π but in opposite directions. The same happens on passing through the next cusp. Finally on returning to the original point the difference $\varphi_+ - \varphi$ has changed by an (even) multiple of π in the one direction, and the difference $\varphi_- - \varphi$ in the other. This means that

$$2 \cdot (\text{the increase in } \varphi) = (\text{the increase in } \varphi_+) + (\text{the increase in } \varphi_-),$$

that is, for the index i_B of the original front we have the expression

$$2i_B = i_B(+) + i_B(-)$$

in terms of the indices of both the smoothings.

The number of circuits of the front around different points of its complement in the plane does not change at all for a sufficiently small smoothing:

$$i_A^B = i_A^B(+) = i_A^B(-).$$

Therefore the characteristic chain also coincides with the arithmetic mean of the smoothed chains outside a small neighbourhood of the front:

$$2\widetilde{i}_A = \widetilde{i}_A(+) + \widetilde{i}_A(-).$$

In particular, from this we conclude that even for a front with cusps

$$\widetilde{i}_A = -i_A/2,$$

so that for the indices of fronts with cusps one has conformal invariance, the formula of Theorem 6, and so forth.

CHAPTER V

EXACT LAGRANGIAN CURVES ON A SPHERE AND THEIR MASLOV INDICES

The simplest exact Lagrangian curve on the symplectic sphere is an embedded curve, dividing the area of the sphere in half. Here we calculate the Maslov index of an exact immersed Lagrangian curve: it is expressed in terms of the area of the characteristic chain of the immersed curve, introduced in Chapter IV.

Our construction leads to an unusual version of the Gauss–Bonnett formula and to proofs of the theorems on the dual and derivative curves, formulated in § 1 (Theorems 5 and 9 of § 1 and Theorem 2 of § 3), and to the theorems of duality between area and length.

§ 20. Exact Lagrangian curves and their Legendrian lifts

Consider a smoothly immersed curve on the standard sphere S^2 of radius 1.

Definition. A curve is said to be *exact* if the area that it bounds is a multiple of 2π (half the area of the sphere).

Remark. The area bounded by a curve is the integral of the area form over the 2-chain whose boundary is the curve.

Such a 2-chain is not unique: it is defined up to the addition of the entire sphere. Therefore the area mentioned is defined modulo 4π, which means that the fact of it being a multiple of 2π is well defined. Moreover the distinction between exact curves, the one bounding an area 2π (mod 4π) and the other an area 0 (mod 4π), is also well defined.

Example. An embedded curve is exact if and only if it divides the area of the sphere in half.

A figure of eight is exact if and only if the absolute values of the areas of both the regions bounded by a loop are the same or differ by 2π.

A parallel of latitude, described p times, is exact if and only if it bounds a disk of area $2\pi k/p$, with k an integer.

Consider now the S^1-bundle $\widetilde{\pi}_i : S^3/\pm 1 \to S_i^2$ from § 6.

Theorem. *Exact Lagrangian curves on the sphere S_i^2 are precisely the projections on that sphere of i-Legendrian curves lying in the manifold $S^3/\pm 1$ with its standard contact i-structure.*

Proof. The contact i-structure provides the principal bundle $\widetilde{\pi}_i$ with an S^1-connection. A curve in the space of the bundle is i-Legendrian if and only if it is the lift of its projection on the base as an integral curve of the connection. This S^1-connection is invariant with respect to rotations of the sphere. Let us compute its curvature.

For future calculations it is necessary to introduce on $S^3/\pm 1$ a basis for the differential 1-forms.

Definition. The *form* α_i is defined by its values on the vector fields $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$:

$$\alpha_i(\mathbf{i}) = 2, \qquad \alpha_i(\mathbf{j}) = 0, \qquad \alpha_i(\mathbf{k}) = 0$$

(the coefficient 2 is introduced for convenience, bearing in mind its appearance in Lemmas 1, 2 and 3 of § 7).

The forms α_j and α_k are defined by cyclic permutation of the quaternionic units, so that

$$\alpha_j(\mathbf{j}) = 2, \qquad \alpha_k(\mathbf{k}) = 2.$$

Since the fields $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are left-invariant, all three constructed forms are also left-invariant.

The basic fact in what follows is the following lemma.

Lemma 1. *The differential of the form α_i is the lift of the standard area form ω_i from the complex sphere S_i^2:*

$$d\alpha_i = \widetilde{\pi}_i^* \omega_i.$$

Proof. The differential of α_i in the direction of the field $\mathbf{i}$ is equal to zero. By the homotopy formula

$$0 = L_{\mathbf{i}}\alpha_i = i_{\mathbf{i}}(d\alpha_i) + d(i_{\mathbf{i}}\alpha_i) = i_{\mathbf{i}}(d\alpha_i),$$

so $i_{\mathbf{i}}\alpha_i = 2$ is constant. Conseqently, the form $d\alpha_i$ vanishes when evaluated on the vector $\mathbf{i}$ tangent to a fibre.

This means that the 2-form is the lift of some 2-form from the factor-space S_i^2.

The left invariance of α_i implies that this 2-form on S_i^2 is invariant under rotations of the sphere S_i^2. This means that it is proportional to the standard area form ω_i, that is,

$$d\alpha_i = c\widetilde{\pi}_i^* \omega_i.$$

It remains only to compute the constant c.

For the computation of c we use the following fact.

Lemma 2. *The integral of the form α_i along a fibre of the bundle $\widetilde{\pi}_i : S^3/\pm 1 \to S_i^2$, oriented by the field* $\mathbf{i}$, *is equal to 2π.*

Proof. In the notations of Lemma 2 of § 7 (where the matter concerns the sphere S_j^2, so the quaternion units must be permuted cyclically) a full revolution is accomplished in time 2π (and in the direction opposite to the direction of the complex structure, if the bundle is identified with $ST^*S_i^2$ in the usual way). The time stated is the integral of the form along the fibre oriented by the field $\mathbf{i}$.

Lemma 3. *Consider a vector field with one singular point B on the complex sphere S^2 as a 2-chain σ, oriented by the complex orientation of the sphere. Then $\partial\sigma = -2F_B$, where F_B is the fibre over B (with the orientation given by the direction of the complex-positive rotation of the sphere).*

Proof. This is a particular case of Theorem 13 of § 17 (actually that theorem is obvious). The index of the only singular point of the field is equal to $+2$. Consider the field on the boundary circle of the disk covering this sphere except for a small neighbourhood of the point B. This boundary of the big disk goes around B along a small circle oriented negatively (with respect to the direction of the complex rotation around B). This means that as one goes round this circle a vector of the field turns so that it completes two full revolutions in the negative direction, from which Lemma 3 follows.

Conclusion of the proof of Lemma 1. We compute the integral of the form $d\alpha_i$ over the chain σ. By Stokes' formula

$$\iint_\sigma d\alpha_i = \int_{\partial\sigma} \alpha_i = \int_{-2F} \alpha_i = 4\pi.$$

In fact, the integral of the form α_i over the fibre oriented by the direction of the field $\mathbf{i}$ is equal to 2π. Moreover the direction of the field $\mathbf{i}$ is opposite to that of the complex rotation of the sphere S_i^2 (by Lemma 2 of § 7). Thus $-F$ is the fibre oriented by the direction of the field $\mathbf{i}$. We see that

$$\iint_{S_i^2} c\omega_i = \iint_\sigma d\alpha_i = 4\pi$$

and consequently $c = 1$.

Remark. It is useful to remark that the left-invariant forms α_i, and so on, satisfy the equations

$$d\alpha_i = -\alpha_j \wedge \alpha_k$$

and so on.

In fact, the forms on either side are left-invariant and reduce to zero on the vectors of the $\mathbf{i}$-direction. Therefore it is enough to verify the coincidence only on the pair $(\mathbf{j}, \mathbf{k})$.

On this pair the left-hand side has the value -4, since the complex orientation of the sphere S_i^2 is the orientation from $\mathbf{k}$ to $\mathbf{j}$ (by Lemma 5 of § 7). The right-hand side also has the value -4.

Of course, one can do all this also by direct coordinate calculations, simply multiplying quaternions and explicitly calculating the components of the forms at the point 1, neglecting quantities of the second order of smallness.

Actually all these computations are simply the computations of the structural constants of the group $SO(3)$. But I have given them in full, so as to be sure about the correctness of those cursed signs.

Proof of the theorem. Let L be an i-Legendrian curve on $S^3/\pm 1$ (so that α_i reduces to zero on L). Consider a 2-chain σ with boundary $L - \varepsilon F$ (where $\varepsilon = 0$ if the curve L is homologous to 0 and equal to 1 if not). Here F is a fibre of the bundle $\widetilde{\pi}_i : S^3/\pm 1 \to S_i^2$ with the orientation given by the complex structure of the sphere (so that $\int_F \alpha_i = -2\pi$).

By Stokes' formula

$$\iint_\sigma d\alpha_i = \int_L \alpha_i - \int_{\varepsilon F} \alpha_i = \varepsilon 2\pi.$$

By Lemma 1, $d\alpha_i = \widetilde{\pi}_i^* \omega_i$ and consequently

$$\iint_\sigma d\alpha_i = \iint_\sigma \widetilde{\pi}_i^* \omega_i = \iint_{\widetilde{\pi}_{i*}\sigma} \omega_i$$

is the area bounded by the projection of L on the sphere S_i^2.

Thus this area is an integral multiple of 2π.

Conversely, start with an arbitrary closed Lagrangian curve on S_i^2. It is covered by a segment of a Legendrian curve L', ending above its initial point. Join the end point to the initial point by a segment Δ of the fibre F_B. We get a closed curve $L \cup \Delta$ in $S^3/\pm 1$. We construct a 2-chain σ with boundary $L - \varepsilon F$ just as before.

Now for the area bounded by the original Lagrangian curve (mod 4π) we have the expression

$$\iint_{\widetilde{\pi}_{i*}\sigma} \omega_i = \iint_\sigma d\alpha_i = \int_L \alpha_i - \int_{\varepsilon F} \alpha_i = \int_\Delta \alpha_i + 2\pi\varepsilon.$$

If the original Lagrangian curve is exact, then the area on the left, and that means also the integral of the form α_i over the segment Δ, is a multiple of 2π. This means that the beginning and end points of the segment coincide, and the theorem is proved.

§ 21. The integral of a horizontal form as the area of the characteristic chain

Consider a closed Legendrian curve L in the space of co-oriented contact elements of a standard sphere S^2 of radius 1. We set about the computation of the integral of a horizontal contact form along this curve. First I recall some notations from § 6.

Consider two bundles over the standard sphere S^2_j,

$$\widetilde{\pi}_j : S^3/\pm 1 \to S^2_j; \qquad p\colon ST^*S^2_j \to S^2_j.$$

The fibres of the first bundle are the orbits of multiplication by $\{e^{jt}\}$ on the right. The fibres of the second bundle are the circles consisting of the co-oriented contact elements applied at a point of the sphere.

Definition. The *natural i-identification* of both these bundles is the map associating with a point z of $S^3/\pm 1$ the contact element on S^2_j orthogonal to the projection of the vector of the field $\mathbf{i}$ at z (co-oriented by this projection).

By means of the identification we carry over to the manifold of the co-oriented contact elements the differential 1-form $\alpha_j = \alpha$ (defined in § 20 in terms of $S^3/\pm 1$).

This 1-form is horizontal: planes $\alpha = 0$ are orthogonal to fibres of the bundle. It is left-invariant and invariant with respect to multiplication by e^{jt} on the right. Its integral along the fibre F of our bundle (oriented by the complex structure of the sphere) is equal to -2π (see § 20).

Main theorem. *The integral of the horizontal form α along a Legendrian curve L is equal to the area of the characteristic 2-chain of the front of this curve*:

$$\int_L \alpha = \iint_{c(p_* L)} \omega,$$

where ω is the standard area form on the complex sphere S^2_j.

Proof. We construct a 2-chain σ with boundary $\partial\sigma = L - \varepsilon F$ (where $\varepsilon = 1$ or 0). By Stokes' formula

$$\iint_\sigma d\alpha = \int_L \alpha - \varepsilon \int_F \alpha = \int_L \alpha + 2\pi\varepsilon.$$

According to Theorem 3 of § 14 the projection of the chain σ to the sphere is $p_*\sigma = c + \varepsilon S_j^2/2$. By the theorem of § 20 the differential of the form α is the area of the projection: $d\alpha = p^*\omega$. Thus we find that

$$\iint_\sigma d\alpha = \iint_\sigma p^*\omega = \iint_{p_*\sigma} \omega = \iint_c \omega + \varepsilon 2\pi.$$

Consequently, and finally,

$$\int_L \alpha + 2\pi\varepsilon = \iint_c \omega + 2\pi\varepsilon,$$

as was asserted.

Remark. The standard i-identification may be changed here to an arbitrary τ-identification, where the quaternion τ is orthogonal to j (for example, a k-identification).

§ 22. A horizontal contact form as a Levi-Civita connection and a generalized Gauss–Bonnet formula

The contact structure $\alpha_j = 0$ in $S^3/\pm 1$ determines a connection in the S^1-bundle $\widetilde{\pi}_j : S^3/\pm 1 \to S_j^2$.

Theorem 1. *The contact connection $\alpha_j = 0$ is transformed into a Levi-Civita connection of the bundle of (co)tangent vectors of unit length on the sphere S_j^2 under the natural i-identification of the bundles*

$$\widetilde{\pi}_j : S^3/\pm 1 \to S_j^2, \qquad p\colon ST^*S_j^2 \to S_j^2$$

(cotangent vectors are identified with tangent vectors by means of a metric on the sphere).

Here, as in § 21, the i-identification may be changed into an arbitrary τ-identificaton, where the quaternion τ is orthogonal to j.

Proof. The vectors of the fields $\mathbf{i}$ and $\mathbf{k}$ at each point of $S^3/\pm 1$ generate the plane $\alpha_j = 0$. An integral curve of the field $\mathbf{k}$ is i-identified in $ST^*S_j^2$ with the motion of the co-orienting vector of the normal along the geodesic everywhere orthogonal to it, while the field $\mathbf{i}$ is everywhere tangent to it (by Lemmas 3 and 1 of § 7). Both these motions are parallel transports in the sense of the Levi-Civita connection (according to its definiton). The theorem is proved.

Definition. The geodesic curvature of a co-oriented curve is considered to be *positive* if the curve bends away from the geodesic tangent to it towards its co-orienting vector.

Example. The parallel of latitude, at a distance θ from the pole of the standard sphere of radius 1 and co-oriented by the direction to the pole, has geodesic curvature $\varkappa = \cot\theta$.

From this example we have the following result.

Lemma 1. *The geodesic curvature $\varkappa$ of a co-oriented curve on the sphere of radius* 1 *is connected with the spherical radius of curvature of the curve, θ, by the relation $\varkappa = \cot\theta$.*

Consider now a Legendrian curve L in the space $ST^*S^2_j$ of co-oriented contact elements of the sphere S^2_j. Recall that we defined above the horizontal 1-form α_j in the space of the fibration $p : ST^*S^2_j \to S^2_j$. The sphere S^2_j is oriented by its complex structure.

Lemma 2. *The value of the horizontal form α_j on any tangent vector of a Legendrian curve L is equal to $\varkappa ds$, where $\varkappa$ is the geodesic curvature of the co-oriented front pL of this curve, while ds is the length of the projection of the vector on the sphere (with plus sign if the projection orients the front correctly, and minus if it does not).*

I remind you that the orientation of a co-oriented curve on an oriented surface is *correct* if the frame (co-orienting vector, orienting vector) imparts a positive orientation to the surface.

Proof of Lemma 2. The tangent vector of an i-Legendrian curve is orthogonal to the $\mathbf{i}$-direction. Therefore one can represent it as a combination of the vectors of the fields $\mathbf{j}$ and $\mathbf{k}$:

$$\boldsymbol{\xi} = \mathbf{j}A\cos\theta + \mathbf{k}A\sin\theta, \qquad A > 0.$$

According to the Theorem of § 10, the spherical radius of curvature of the front is θ. According to Lemma 1 the geodesic curvature of the front is $\varkappa = \cot\theta$.

The value of the form α_j on a vector of the field $\boldsymbol{\xi}$ is equal to $2A\cos\theta$ (from the definition of this form). We prove that *the oriented length of a vector of the field $\boldsymbol{\xi}$ is equal to* $2A\sin\theta$.

In fact, under the projection p along the j-directon we have

$$p_*\boldsymbol{\xi} = (A\sin\theta)p_*\mathbf{k}.$$

But the oriented length of the projection $p_*\mathbf{k}$ of the vector of the field $\mathbf{k}$ on the sphere S^2_j is equal to 2 by Lemma 3 of § 7. Thus, $ds(\xi) = 2A\sin\theta$.

Finally we find that

$$\alpha_j(\boldsymbol{\xi}) = 2A\cos\theta = (\cot\theta)2A\sin\theta = \varkappa\, ds(\boldsymbol{\xi}),$$

which is what had to be proved.

Combining Lemma 2 with Theorem 1 and the main theorem of § 21, we obtain the following generalized Gauss–Bonnet formula.

Consider a co-oriented curve K smoothly immersed in the sphere with geodesic curvature $\varkappa$.

Theorem 2. *There is a Gauss–Bonnet formula for immersions*

$$\int_K \varkappa\, ds = \iint_c \omega,$$

where c is the characteristic chain of a correctly oriented curve K and ω is the Gaussian curvature form (that is, for the standard sphere of area 1 simply the area form).

I remind you that a correct orientation of a curve is defined by the condition that the frame (co-orienting vector, orienting vector) positively orients the sphere.

Proof. At first let the metric be standard. By Lemma 2 the left-hand side is then equal to the integral of the horizontal form α along the curve of normals. By the main theorem of § 21 this integral is equal to the double integral on the right-hand side of the formula. So this formula is proved for the standard metric.

Any metric can be obtained from a standard one by a sequence of local deformations. Under a local deformation the equality of the left- and right-hand sides cannot be broken. This follows from the fact that the Gauss–Bonnet formula is true for embedded curves.

Remark. A direct proof of Theorem 2 for an arbitrary metric repeats the proof of the main theorem. The topological equality $p_*\sigma = c + \varepsilon S^2/2$ does not depend on the metric. On the other hand, the relations

$$\alpha(\boldsymbol{\xi}) = \varkappa\, ds(\boldsymbol{\xi}), \qquad d\alpha = p_*\omega, \qquad \int_F \alpha = -2\pi, \qquad \iint_{S^2} \omega = 4\pi$$

are satisfied for the form α of the Levi-Civita connection and its curvature form ω for any metric of the sphere.

Consider now an oriented and co-oriented front on an oriented sphere.

Definition. A point of a front is said to be *positive* if the pair (co-orienting vector, orienting vector) positively orients the sphere.

Example. A semicubical cusp of a front separates its positive branch from its negative one.

Theorem 3. *The following Gauss–Bonnet formula holds for fronts:*

$$\int_K \varkappa\, ds = \iint_c \omega,$$

where c is the characteristic chain of the front K and ds is the element of the oriented length of the front (positive at the positive points of the front and negative at the negative points).

The proof is the same as that of Theorem 2.

Already from Theorem 2 we have an important corollary.

Corollary. *Every exact Lagrangian curve is the derivative of a closed front.*

Proof. Select at a point of the Lagrangian curve the great circle passing through this point. Transport the tangent vector of this circle at this point parallel to itself (in the sense of the standard Riemannian metric on the sphere) along our closed

Lagrangian curve. On returning to the original point we get the initial vector. In fact, the rate of change of the angle of the vector carried parallel to itself along the curve with respect to the tangent vector of the curve is equal to the geodesic curvature of the curve. Therefore for a complete circuit the transported vector turns with respect to the tangent vector through the integral of the geodesic curvature. By Theorem 2 this angle is equal to the area of the characteristic chain. If the original Lagrangian curve is exact, then the angle is a multiple of 2π and the vector returns to its place (and if not, not).

Consider now the great circles intersecting the Lagrangian curve in the directions of the transported vectors. This family of circles has an envelope.

Lemma. *The distance along each of these circles from the original Lagrangian curve to the envelope is equal to* $\pi/2$.

Proof of the lemma. Replace a small segment of our curve by a small great circle arc. The parallelism of the circles constructed above along our curve means that the difference between the angles of this arc with the circles of our family at its ends is of an order of smallness no higher than the first with respect to the length of the arc. But infinitely close great circles intersecting the equator at the same angle intersect at a distance $\pi/2$ from the equator along each of them. This means that infinitely near great circles of our family also intersect at a distance $\pi/2$ along them from the Lagrangian curve.

The derivative of the envelope constructed in this way is our original exact Lagrangian curve. The corollary has been proved.

§ 23. Proof of the formula for the Maslov index

I recall that the Maslov index μ of an oriented and co-oriented front in general position is equal to the difference between the numbers of positive and negative cusps (Fig. 19).

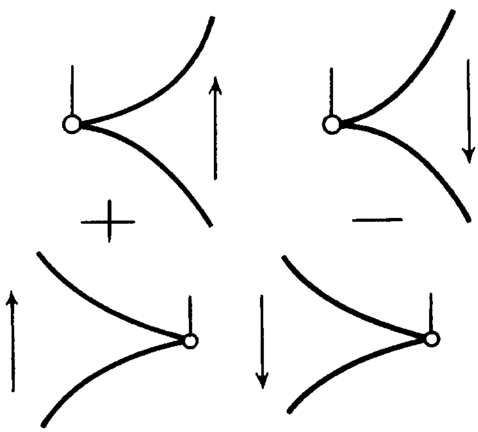

Figure 19

Theorem. *The Maslov index μ of an oriented and co-oriented front on an oriented sphere is connected to the characteristic chain c (that means with the geodesic curvature $\varkappa$ also) of its correctly co-oriented derivative K by the relations*

$$\iint_c \omega = -\pi\mu = \int_K \varkappa\, ds,$$

where ω is the area element of the sphere and ds is the element of length.

Proof. A cusp of the front corresponds to a point where the derivative curve is orthogonal to the family of the framing great circles, with the help of which the derivative has been constructed (see § 12).

So at a generic cusp the angle α between the derivative curve and the framing direction passes through the value $\pm\pi/2$ with non-zero velocity. The sign of the velocity is determined by the sign of the cusp (according to Theorem 3 of § 12 and Example 2 following it).

We conclude that the number μ of cusps, taking account of their signs, is equal to twice the number of turns of the framing vector with respect to the normal vector of the derivative curve.

But the framing vectors form a parallel framing (Theorem 2 of § 12). Consequently the angle of turn of the normal vector with respect to the framing vector after one circuit of the derivative curve is equal to the integral of its geodesic curvature.

The angle of turn of the framing vector with respect to the normal is the opposite, therefore it is equal to $-\int_K \varkappa ds$. Consequently the number of turns is equal to $-\int_K \varkappa ds/(2\pi)$. Therefore twice this number of turns, namely $-\int_K \varkappa ds/\pi$, is equal to the Maslov index μ. Hence the second equality of the theorem is proved.

The first follows from the second by Theorem 2 of § 22.

Remark. The metric of the sphere in the proved theorem is the standard one. Generalizations of this theorem to the case of other metrics are not known to me (the definition of the derivative uses the standard metric).

To verify the sign in the preceding theorem it is necessary to apply it to a front with a non-zero Maslov index.

Example. Consider a system of fronts whose caustic is a parallel of latitude of the standard sphere of radius 1 traversed p times (at a distance $\theta < \pi/2$ from the North pole along meridians). Suppose that the length of the parallel ($2\pi \sin\theta$) is equal to $2\pi/p$. One of the fronts, Γ (for the case $p = 3$), is shown in Fig. 20. The orientation of the sphere is also shown in Fig. 20.

Choose the co-orientation and orientation of the front as shown in Fig. 20. Both the cusps are positive, so that $\mu = 2$. The derivative curve of the front is the ($p = 3$ times traversed) parallel of latitude, at a distance $(\pi/2) - \theta$ from the North pole along meridians. The orientation of the front gives the orientation of the derivative curve Γ', shown in Fig. 20 (see p. 54). The correct co-orientation of this derivative curve is to the side of increasing distance from the North pole.

The geodesic curvature $\varkappa$ of the correctly co-oriented derivative curve is therefore equal to $-\tan\theta$. The integral of this curvature along the derivative curve

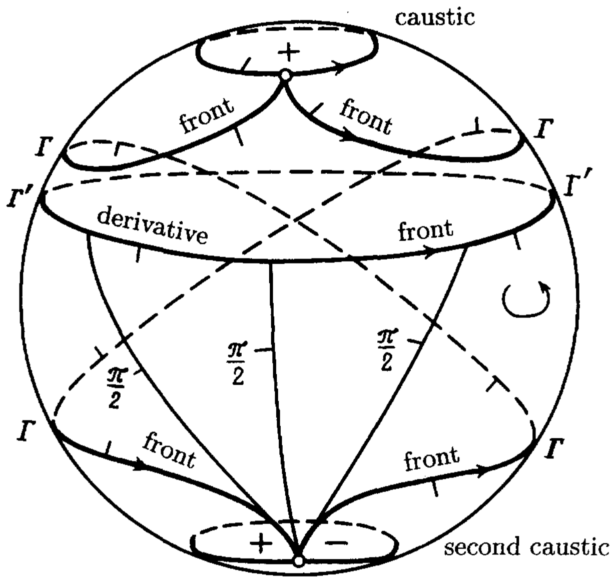

Figure 20

(of length $2\pi p \cos\theta$) is equal to

$$\int_K \varkappa\, ds = -2\pi p \cos\theta \tan\theta = -2\pi p \sin\theta = -2\pi$$

(since the original parallel has length $2\pi/p = 2\pi \sin\theta$).

Our theorem consequently gives $\mu = +2$, as it should.

Remark. For a parallel of latitude K with $z = \cos\theta = k/\pi$, described p times, one would obtain $\mu = 2k$.

§ 24. The area–length duality

Consider an oriented and co-oriented front on the standard oriented two-dimensional sphere of radius 1. The dual front is also oriented and co-oriented.

Theorem 1. *The area of the characteristic 2-chain bounded by the front dual to the given one is equal to the oriented length of the original front.*

Corollary. *The area of the characteristic 2-chain bounded by the given front is opposite to the oriented length of the dual front.*

In fact, the oriented length of the second dual of the front is opposite to the oriented length of the original front, since the antipodal map of the sphere changes its orientation.

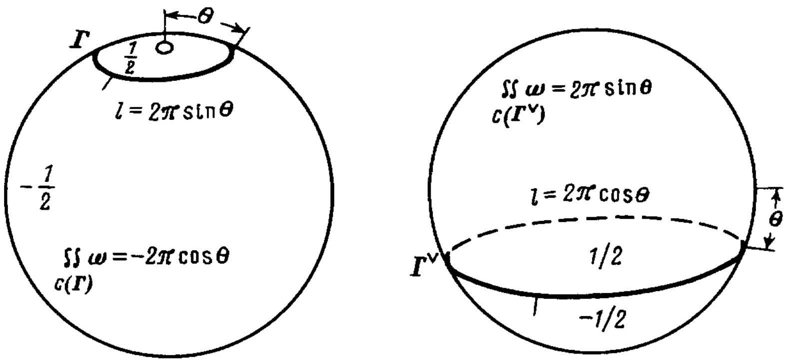

Figure 21

Example. Let the original front be the parallel of latitude at a distance $\theta < \pi/2$ along a meridian from the North pole, co-oriented by the vector directed from the pole (Fig. 21). Take the orientation of this front to be correct (so that the frame (co-orienting vector, orienting vector) positively orients the sphere).

The dual front is the parallel of latitude at a distance $\theta + (\pi/2)$ with the same co-orientation (from the Pole) and with the correct orientation.

The area of the characteristic chain of the dual front is $2\pi \sin\theta$.

The length of the original front is $2\pi \sin\theta$.

The area of the characteristic chain of the original front is $-2\pi\cos\theta$.

The length of the dual front is $2\pi\cos\theta$.

Proof of Theorem 1. Let the original front Γ be the front of an i-Legendrian curve L in $ST^*S^2_j$. Then the dual front $\Gamma^\vee$ is the front of the (also i-Legendrian) curve $L^\vee = Le^{i\pi/4}$ (by Lemma 1 of § 7).

Multiplication by $e^{i\pi/4}$ on the right sends the field $\mathbf{k}$ to the field $\mathbf{j}$ and the field $\mathbf{j}$ to the field $-\mathbf{k}$ (by the theorem of § 8). Therefore

$$\int_L \alpha_k = \int_{L^\vee} \alpha_j, \qquad \int_L \alpha_j = -\int_{L^\vee} \alpha_k.$$

On the other hand, by the main theorem of § 21

$$\int_L \alpha_j = \iint_c \omega, \qquad \int_{L^\vee} \alpha_j = \iint_{c^\vee} \omega.$$

But the oriented length of the projection on S^2_j of the i-Legendrian vector ξ is exactly $\alpha_k(\xi)$ (see the proof of Lemma 2 of § 22). Therefore

$$\int_L \alpha_k = \text{the oriented length of the front } \Gamma,$$
$$\int_{L^\vee} \alpha_k = \text{the oriented length of the front } \Gamma^\vee.$$

The theorem is proved.

I now recall that the derivative curve of the front is dual to the caustic.

Applying Theorem 1 to this pair, we are able to find the length of the caustic. We orient and co-orient the caustic in such a way that the derivative curve of the front (with its orientation defined by the orientation of the front and with the correct co-orientation) is dual to the caustic. Then from Theorem 1 one obtains the following result.

Theorem 2. *The oriented length of the caustic on the standard sphere is equal to an integral multiple of 2π, in fact equal to $-\pi\mu$, where μ is the Maslov index of the front.*

In fact, by Theorem 1 this length is equal to the area of the characteristic chain of the derivative of the front, which is equal to $-\pi\mu$ by the theorem of § 23.

Example. In the situation of Fig. 20 the caustic is the parallel of latitude at a distance θ from the North pole (and described p times). It is co-oriented by the direction towards the pole and oriented incorrectly. Therefore its oriented length is $-2\pi p \sin\theta = -2\pi$, while the Maslov index of the front is $\mu = +2$.

I note also another useful formulation of the area–length duality.

Let $\partial\sigma = L_1 - L_0$ be the homology between two i-Legendrian curves on $\mathbb{R}P^3 \approx ST^*S^2_j$ with fronts Γ_1 and Γ_0 on S^2_j.

Theorem 3. *The area of the projection of the chain σ on the sphere (that is, the difference of the areas bounded by the fronts Γ_1 and Γ_0) is equal to the difference of the oriented lengths of the fronts whose duals are the fronts Γ_1 and Γ_0:*

$$\iint_{\widetilde{\pi}_{j*}\sigma} \omega_j = S(\Gamma_1) - S(\Gamma_0) = l(-\Gamma_1^\vee) - l(-\Gamma_0^\vee).$$

The proof is the same as for Theorem 1.

§ 25. The parities of fronts and caustics

Definition. A closed curve in $\mathbb{R}\mathrm{P}^3$ is said to be *even* if it is homologous to zero and *odd* otherwise.

Wave fronts, their derivatives and caustics, equipped with various framings, determine closed curves in $ST^*S^2 \approx \mathbb{R}\mathrm{P}^3$. These curves may be even or odd.

Example 1. All the equidistants of a front, framed with co-orienting normals, have the same parity, since they are mutually homotopic.

Example 2. The derivative curve of a closed front, framed with the directions of the great circles which are parallel along it, has the same parity as the original front.

For moving a distance $\pi/2$ along a tangent can be interpolated by a family of moves by t from the identity map.

A caustic has a natural framing by normals (for which the dual curve of the caustic is the correctly framed derivative curve of the front).

Theorem 1. *The parity of the caustic coincides with the parity of the front if* $\mu \equiv 0 \mod 4$, *and is opposite to it if* $\mu \equiv 2 \mod 4$.

Example 3. In Fig. 20 the caustic is a parallel of latitude described p times. Its parity is the parity of the number p.

The parity of a front (as it is not hard to convince oneself) is the parity of the number $p-1$. In this example $\mu = 2$.

Theorem 1 follows at once from the following fact.

Theorem 2. *The parity of a front coincides with the parity of its derivative curve, framed with the parallel circles, while the parity of the caustic coincides with the parity of the same derivative of the front framed by its normals.*

Both these facts have already been proved above, the first in Example 2 and the second in Example 1.

Proof of Theorem 1. The normal framing of the derivative curve differs from the parallel framing by $\mu/2$ turns (by the theorem of § 23). Each circuit of the framing changes the parity of the curve in ST^*S^2 by 1, since it adds to the homology class of the curve in $\mathbb{R}\mathrm{P}^3$ the generator of the group $H_1(\mathbb{R}\mathrm{P}^3, \mathbb{Z}) \approx \mathbb{Z}_2$. Therefore the difference of the parities of the caustic and the front is $\mu/2$ (mod 2), as was asserted.

The Maslov index and the parity are (the only) invariants of the regular homotopy of the Legendrian immersion of a circle in the contact manifold ST^*S^2. The Maslov index of the curve is also preserved under the contactomorphisms of this manifold (the author thanks S. L. Tabachnikov and E. Zhir for showing him that this follows from the uniqueness of trivialization of the bundle of contact planes on S^3).

CHAPTER VI

THE BENNEQUIN INVARIANT AND THE SPHERICAL INVARIANT J^+

For immersions of a circle in the plane there is an invariant J^+, defined in [6], that counts positive self-tangencies that occur during perestroikas (see also [5]). It is uniquely defined by the following conditions:

1) on passing through a positive self-tangency (when both the tangent branches have the same orientation) the value of J^+ changes in the same way that the number of double points changes (that is, it increases the value of J^+ or decreases it by 2), while on passing through the remaining perestroikas of codimension 1 (negative self-tangencies and triple intersections) it does not change its value;

2) the invariant is additive for the connected sum:

$$J^+(X+Y) = J^+(X) + J^+(Y);$$

3) the value of J^+ does not depend either on the orientation of the curve or on the orientation of the plane.

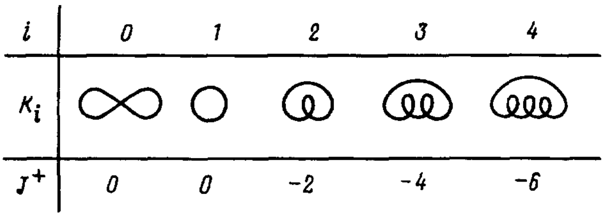

Figure 22

Example. The values of J^+ for the simplest curves are shown in Fig. 22.

The invariant J^+ is not conformally invariant.

Example. The frist and third curves in Fig. 22 are conformally equivalent, but have different values of the invariant J^+.

It turns out that after a small correction the invariant J^+ can be made spherical, that is, conformally invariant.

§ 26. The spherical invariant J^+

Consider on the plane a curve of index i.

Theorem. *The invariant $SJ^+ = J^+ + (i^2/2)$ of the plane curve is conformally invariant (that is, does not depend on which point of the Riemann sphere is chosen as the point at infinity).*

Definition. The quantity SJ^+ is called the *spherical invariant* J^+ of the spherical curve (obtained from the given plane curve by adding to the plane a point at infinity).

Proof of the theorem. Since (by a theorem of Whitney [7]) the space of immersed curves of fixed index i is connected, it is enough to do the following.

1) For one curve of each index to construct a curve conformally equivalent to it but of different index and to verify that the increments in J^+ and $i^2/2$ on passage to the new curve are of opposite sign.

2) To produce just so many such examples that the indices connected by them fall into two classes: even and odd numbers (the parity of the index under conformal transformations does not change, since it is opposite to the parity of the number of points of self-intersection of the curve).

The required examples are shown in Fig. 23.

The verification of the conformal invariance of SJ^+ follows from the identity

$$-2n + (n+1)^2/2 = (n-1)^2/2.$$

The proof is complete.

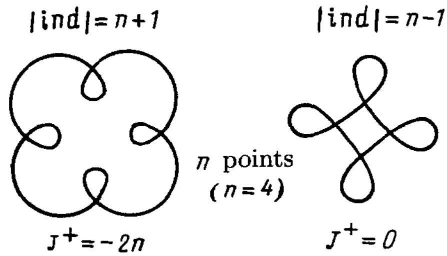

Figure 23

§ 27. The topological meaning of the invariant SJ^+

For each co-oriented front Γ on the sphere S^2 one can define a Legendrian curve $L \subset ST^*S^2(\approx \mathbb{R}P^3)$; its double cover $\hat{L} \subset S^3$ is a Legendrian curve in the standard contact sphere S^3.

The curve $\hat{L}$ consists either of two components or of one.

Definition. The *Bennequin invariant* $\beta(\hat{L})$ is the linking coefficient $\beta(\hat{L}) = l(\hat{L}, \hat{L}_\varepsilon)$ of the Legendrian curve $\hat{L}$ and its small shift $\hat{L}_\varepsilon$ in the direction transverse to the contact structure.

The orientation of S^3 is chosen here in the ordinary way (that is, in such a way that the linking coefficient of the covering fibres of the bundle $ST^*S^2 \to S^2$ is equal to $+1$).

The Bennequin invariant is an invariant of Legendrian knots but not of immersions: it is defined only for *embedded* Legendrian curves (see [8]).

Theorem 1. *The Bennequin invariant is connected with the spherical invariant J^+ by the relation*

$$\beta(\hat{L}) = 2(1 - SJ^+(\Gamma)).$$

In [6] the *linking polynomial* $L(t)$ is defined for an embedded Legendrian curve of index i in $ST^*\mathbb{R}^2$. It belongs to the group ring of the group $\mathbb{Z}_i$. The sum of its coefficients is connected with the invariant J^+ by the relation

$$L(1) + J^+ = 1.$$

In these notations Theorem 1 can be written in the form

$$\beta(\hat{L}) = 2L(1) - i^2.$$

Suppose now that the index $i = 2k$ is even. In this case it is possible to take separately the sums of the coefficients of the polynomial L of even and odd degrees,

$$L_{\text{ev}} = (L(1) + L(-1))/2, \qquad L_{\text{odd}} = (L(1) - L(-1))/2.$$

The covering Legendrian curve $\hat{L}$ in S^3 in this case consists of two components, $\hat{L}_1$ and $\hat{L}_2$.

Theorem 2. *There are two sharpenings of the formula of Theorem* 1:

$$\beta(\hat{L}_1) = \beta(\hat{L}_2) = L_{\mathrm{ev}} - k^2,$$
$$l(\hat{L}_1, \hat{L}_2) = L_{\mathrm{odd}} - k^2.$$

The proofs of Theorems 1 and 2 are obtained by rather direct but heavy calculations of indices of intersection, and I do not dwell on them. The role of the number $i^2/2$ has been discussed before by Tabachnikov in [15].

CHAPTER VII

PSEUDO-FUNCTIONS

The motivation behind all the above research was the wish to promote some generalizations of Morse theory relevant to 'multivalued functions' (see [9], [2], [3], [4], [5], [6], [16], [17]).

We begin with the simplest example: *a function on the circle has no fewer than two critical points.*

What we shall be concerned with is the generalization of this 'Morse inequality' to objects that are more general than functions—the so-called *quasi-functions of Chekanov* (which are, in fact, Legendrian submanifolds, whose Lagrangian curves lie on the surface of a cylinder, and according to the inequality, have no fewer than two points of intersection with its equator) and the pseudo-functions of Givental' (the Lagrangian curves of which lie on the surface of a sphere, and, according to the inequality, intersect each great circle at least twice).

Other generalizations of elementary Morse theory for functions on the circle are provided by the theorem of Sturm (Kellogg, Tabachnikov and others), according to which *the number of zeros of the Fourier series*

$$\sum_{n \geqslant N} a_n \cos(nx) + b_n \sin(nx)$$

on the circle is not less than the number $(2N)$ *of zeros of its first harmonic* (see [2], [3], [4], [5]).

For $N = 1$ this is precisely the theorem of Morse, since a function with mean value 0 is the derivative of a function on the circle. In this way the theorem of Sturm and others is a generalization of the Morse inequality to the case $N > 1$.

In the case $N = 1$ the Morse inequality has been transferred to the 'multivalued' cases of the quasi-functions and pseudo-functions of Chekanov and Givental'.

In the case $N > 1$ only the first steps have been taken in the direction of such transfer.

These first steps, and the conjectures to which they lead, are given below. The generalization of Sturm's theory to functions of more than one variable is a yet more interesting consequence of our constructions, but I do not concern myself with that here.[7]

[7]There is, by the way, an interesting preprint by M. E. Kazaryan on umbilical characteristic classes (Bochum, 1994).

§ 28. The quasi-functions of Chekanov

For any function $f : M \to \mathbb{R}$ there is an exact Lagrangian submanifold $L \subset T^*M$ (formed from the differentials of f at all points). The critical points of the function are intersections of L with the zero section. Suppose that M is compact. The Morse inequalities estimate from below the number of points of intersection of L with the zero section.

The first attempts to generalize the Morse inequalities are related to replacing the Lagrangian section L by Lagrangian submanifolds that need not be sections.

Example. Let $M = S^1$ be the circle. Then $T^*M = \mathbb{R} \times S^1 = \{p, q \mod 2\pi\}$—the cylinder, and L is a curve homotopic to the zero section. The exactness condition consists in the fact that the area between L and the zero section is zero:

$$\int_L p\,dq = 0.$$

If such a curve L has no self-intersections, then it must intersect the zero section twice (for example, if it were above the zero section everywhere then the integral would be positive).

However, if the curve can intersect itself, then the intersection with the zero section can vanish. I described the simplest example of such a thing in 1965 in [9] (Fig. 24).

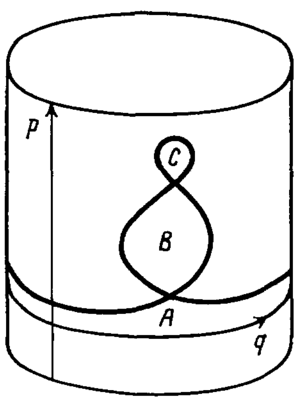

Figure 24

The exactness condition in this example takes the form of a relation between the areas, $A + C = B$. It is easy to construct a curve with $p > 0$ satisfying this condition. For such a curve the 'generalized Morse inequality' (the number of points of intersecton with the zero section $\geqslant 2$) is not true.

I conjectured in 1965 that, for embedded exact Lagrangian submanifolds in the space of the cotangent bundle of a manifold that are Hamiltonially isotopic to the zero section, the number of points of intersection with the zero section is bounded below by the sum of the Betti numbers of the manifold. This conjecture (after the proof by Conley and Zehnder [10] of an analogous conjecture on the fixed points of exact symplectomorphisms of the torus) has been proved by many people: Chaperon [11], Laudenbach and Sikorav [12], and others.

Chekanov [13] noted that the result remains true even for some self-intersecting immersed exact Lagrangian submanifolds, which he called quasi-functions.

Definition. A *quasi-function* on a compact manifold M^m is a Legendrian embedding of M^m in the contact space $J^1(M, \mathbb{R}) = (T^*M) \times \mathbb{R}$ of 1-jets of functions on M^m, which can be joined to the zero section by a regular homotopy consisting of Legendrian embeddings.

The projection of a Legendrian quasi-function embedded in $J^1(M, \mathbb{R})$ is an exact Lagrangian immersion of M in T^*M. This immersion is homotopic to the zero section, but in general has self-intersections.

Chekanov's theorem [13], [16]. *The number of points of intersection of the Lagrangian projection of a quasi-function to the space of the cotangent bundle T^*M with the zero-section (the number of quasi-critical points of the quasi-function) is not less than the sum of the Betti numbers of the manifold M (if one takes multiplicities of points into account).*[8]

Example. Let $M = S^1$. Then the number of points of a quasi-function where $p = 0$ is not less than two.

In particular, the curve of Fig. 24 cannot be the Lagrangian projection of a quasi-function.

This means that for any regular homotopy of the zero section to the curve of Fig. 24 there is a moment when the curve has a double point, breaking it into two loops, along each of which $\int pdq = 0$.

For the simplest homotopy the proof of Chekanov's theorem even for curves on a cylinder is not known.

Remark. Of course, in Chekanov's theorem for the number of quasi-critical points of a quasi-function there were obtained more precise lower estimates than by the sum of the Betti numbers. For many manifolds M this number is not less than the Morse number; the usual lower estimates in this situation for the number of geometrically distinct quasi-critical points have also been obtained.

However, the conjecture that these numbers are always not less than the minimal number of critical points of a function on M (taking account of multiplicities or not) has been neither proved nor disproved.

§ 29. From quasi-functions on the cylinder to pseudo-functions on the sphere, and conversely

The following useful simple lemma is due to Archimedes.

Consider the cylinder $x^2 + y^2 = 1$ wrapped round the sphere $x^2 + y^2 + z^2 = 1$. The *projection of Archimedes* projects the complement to the poles of the sphere onto the surface of the cylinder by horizontal radii $x = e^t x_0$, $y = e^t y_0$, $z = z_0$).

[8]The proof of this important theorem was for a long time missing from the literature, until in 1994 there appeared a preprint by M. Chaperon on this theme (Paris-7).

Lemma. *The Archimedean projection $S^2 \setminus S^0 \to S^1 \times (-1, 1)$ is a symplectomorphism.*

This lemma enables us to transfer quasi-functions to the sphere (less its poles). Adjoining the poles, we arrive at the following concept.

Definition. A *pseudo-function* (on the circle) is an exact i-Lagrangian immersion, which is the projection of an i-Legendrian embedding of a circle in $ST^*S_i^2$, belonging to the component of a Legendrian embedding that projects to the embedding of the equator, in the space of all Legendrian embeddings.

In other words, we deform the equator of the sphere to a smoothly immersed curve so that

1) the curve at all times bounds an area of half the sphere ($2\pi \bmod 4\pi$),
2) during the deformations neither of the loops of the curve into which a double point divides it bounds the area of half the sphere.

Givental's theorem [14]. *A pseudo-function intersects a great circle of the sphere in no fewer than two points.*

Remark. If a pseudo-function has no self-intersections, then what has to be proved is obvious. The example of Fig. 24 allows us to construct an immersed circle in the Northern hemisphere, bounding an area of half the sphere. From Givental's theorem it follows that the immersed curve of Fig. 24 not only does not determine a quasi-function, but is not even a pseudo-function.

Example. If a pseudo-function is obtained from the equator by a homotopy not passing through the poles, then this homotopy projects to the cylinder and there one obtains a quasi-function. For this quasi-function one has, therefore, an assertion close to Chekanov's theorem. If, on the contrary, the pseudo-function passes through the poles during the deformation, then the projection to the cylinder fails to determine a quasi-function. For example, in Fig. 25 (see p. 64) we show one such perestroika. As a result of this perestroika on the surface of the cylinder $S^1 \times (-1, 1)$ one obtains a closed contractible curve, bounding the area 2π.

Such a curve is far from being a quasi-function, but it must nevertheless intersect the zero section at least twice. If the curve is embedded, then this is obvious, since the area of the cylinder whether above or below the zero-section is equal to 2π. Givental's theorem asserts that the intersection points cannot vanish even for curves that have self-intersections, if in the process of deformation none of the double points forms on the curve a loop bounding an area of half the sphere.

Remark. The theorem of Givental' presented here was proved by him in the much more general setting of exact Lagrangian embeddings of $\mathbb{R}\mathrm{P}^n$ in $\mathbb{C}\mathrm{P}^n$ (our case of spherical curves corresponds to taking $n = 1$).

§ 30. Conjectures concerning pseudo-functions

Symplectic and contact generalizations of the classical four-vertex theorem lead to the following result (see [4]).

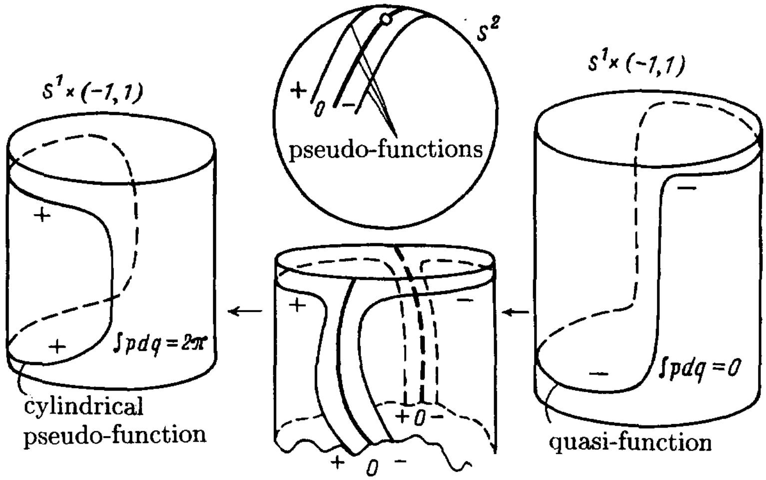

Figure 25

The tennis-ball theorem. *Consider a curve embedded in the standard sphere and bounding an area of half the sphere. Such a curve has no fewer than four points of spherical inflection.*

Using Fig. 24 it is easy to construct an example of a curve, immersed in the standard sphere and obtained from the equator by regular homotopy in the class of immersions bounding the area of half the sphere, that has only two points of spherical inflection.

We have already seen that the curve of Fig. 24 cannot be a pseudo-function.

Conjecture 1. *The number of points of spherical inflection of a pseudo-function is not less than four.*

One can formulate an analogous conjecture concerning quasi-functions on the cylinder.

Definition. A *point of spherical inflection of a quasi-function* is a point of spherical inflection of the projection of its Lagrangian curve to the sphere by rays from the centre.

If a quasi-function is given locally by the equation $p = f(q)$, then the point q of spherical inflection is determined from the equation $f'' + f = 0$.

Conjecture 2. *A quasi-function has no fewer than four points of spherical inflection.*

Theorem. *If the Lagrangian projection of a quasi-function has no points of self-intersection, then it has no fewer than four points of spherical inflection.*

Proof (the author is grateful to B. A. Khesin, whose idea is used here).

Suppose that the spherical projection has fewer than four points of inflection. Then this curve on the sphere can be confined to a hemisphere (see [4]). This hemisphere cannot be bounded by a vertical meridian, since a quasi-function has points with arbitrary values of q.

Thus the Lagrangian curve lies in the half-cylinder $z > ax + by$ (or $z < ax + by$). But $\{(x, y, z - ax - by)\}$ is also a quasi-function. Since everywhere $z \neq ax + by$ we obtain a contradiction to Chekanov's theorem (in its elementary form, since our Lagrangian curve has no points of self-intersection). The theorem is proved.

These conjectures and results can be applied, for example, to the study of the vertices of curves on the Euclidean plane.

Consider a plane co-oriented and smoothly immersed curve of index 1 with no double oriented normals. We associate with it a quasi-function in the following (usual) way.

With the point (x, y) with the co-orienting normal direction φ we associate the 1-jet at the point φ of the function with the value $z = x\cos\varphi + y\sin\varphi$ and the value of its derivative $p = -x\sin\varphi + y\cos\varphi$. If locally $p = A(\varphi)$ and the radius of curvature of the original curve is $R(\varphi)$, then

$$R' = A + A'',$$

according to § 13. This means that the vertices of the original curve (where $R' = 0$) are the points of spherical inflection of the qausi-function so obtained. If the original curve has no double normals with the same orientation, then the Lagrangian curve does not intersect itself. According to the preceding theorem the original plane curve has no fewer than four vertices.

Consider an immersed curve of index 1 on the Euclidean plane.

Conjecture 3. *If the curve of normals of our curve belongs to the component of the curve of normals of a circle in the space of Legendrian embeddings $S^1 \to ST^*\mathbb{R}^2$, then the original curve has no fewer than four vertices.*

Example. The curve of Fig. 26 has two vertices in all. But this curve has $J^+ \neq 0$, and therefore its curve of normals does not lie in the component of the curve of normals of a circle.

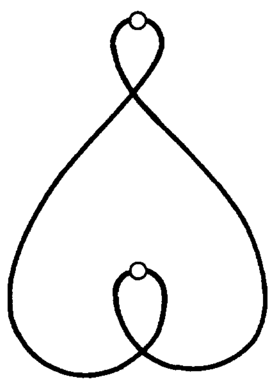

Figure 26

Returning to perestroikas of quasi-functions by passing pseudo-functions through the poles, consider on the surface of the cylinder $|p| \leqslant 1$ a closed contractible smoothly immersed curve, bounding the area 2π and lying in the component of a non-self-intersecting curve in the sense that one can join it to a curve with no self-intersections by a regular homotopy in the class of Lagrangian immersions bounding the area 2π and such that no double point of an intermediate curve forms on it a loop of area 2π (for safety's sake perhaps one should prohibit loops of areas that are multiples of 2π as well).

Is it true that such a Lagrangian curve has no fewer than four points of spherical inflection?

Theorem. *If a contractible Lagrangian curve of area 2π on the cylinder $|p| < 1$ has no points of self-intersection, then it has no fewer than four points of spherical inflection.*

Proof. In the opposite case the projection to the sphere from the centre is confined to an open hemisphere, and the original curve is confined to an open half-cylinder. But since this Lagrangian curve bounds the area 2π it cannot be confined to a half-cylinder of area 2π.

The theorem is proved.

§ 31. Space curves and Sturm's theorem

We now state a geoemtrical problem, leading to the case of an arbitrarily large number of zeros in a theorem of Sturm type.

Consider an immersion of the circle into a Euclidean subspace of dimension $2n$ in a Euclidean space of dimension $2n+1$. The highest torsion of such a curve is identically equal to zero.

Consider an arbitrary small (along with its derivatives) spatial perturbation of this plane immersion. The number of flat points of the generic perturbed immersion (that is, the number of zeros of the highest torsion) is finite. How small can it be?

Theorem. *The least number of flat points is equal to $2n+2$ for all curves in $\mathbb{R}^{2n+1}$ whose projections into $\mathbb{R}^{2n}$ satisfy the following convexity condition: the number of points of intersection with any $(2n-1)$-dimensional hypersurface in $\mathbb{R}^{2n}$ does not exceed $2n$ (taking multiplicities into account).*

Example. We call a *standard curve* (or *generalized ellipse*) in the space of dimension $2n$ a curve given by the equations

$$x_1 = a_1 \cos t, \quad \dots, \quad x_n = a_n \cos(nt);$$
$$y_1 = b_1 \sin t, \quad \dots, \quad y_n = b_n \sin(nt),$$

where $a_1, \dots, b_n \neq 0$. This curve in $\mathbb{R}^{2n}$ is convex.

Theorem. *The number of flat points of a small perturbation of a standard curve in $\mathbb{R}^{2n+1}$ is no fewer than $2n+2$.*

Example. For $n = 1$ the standard curve is an ellipse. The assertion consists in saying that a ***perturbation of the ellipse which is small along with its derivatives has no fewer than four flat points.***

This result is true also for curves in odd-dimensional projective spaces.

Analogous results are true also for curves in even-dimensional projective spaces having convex projections or obtainable by small perturbations from convex curves lying in odd-dimensional hyperplanes.

As an example of a convex curve in $\mathbb{R}\mathrm{P}^{2n-1}$, take the image of the immersion

$$\begin{aligned} x_1 &= \cos t, \quad \ldots, \quad x_n = \cos(2n-1)t; \\ y_1 &= \sin t, \quad \ldots, \quad y_n = \sin(2n-1)t \end{aligned}$$

(the coordinates being homogeneous, with $t \in \mathbb{R} \bmod \pi$). The corresponding theorems on Chebyshev systems of sections of a non-trivial one-dimensional bundle over the circle will be published in another place.

From these theorems it follows, for example, that a curve projectively dual to a convex curve is convex, and that convex curves in $\mathbb{R}\mathrm{P}^m$ form a connected set (M. Shapiro, M. Kazaryan).

Remark. The appearance of numbers of flat points linearly increasing with the dimension of the space leads one to conjecture that the question is related to the Morse theory of a projective space (or more correctly to its product with the circle). How vigorously one can deform a convex curve is not known even for the case of curves in $\mathbb{R}\mathrm{P}^2$, where the question reduces to an extension of the theorem of Möbius (on three points of inflection of a curve non-homologous to zero) to curves with self-intersections. For a discussion of the theorem of Möbius see [18].

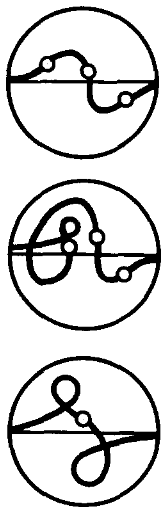

Figure 27

Conjecture. *A non-contractible immersion of a circle in the real projective plane has no fewer than three points of inflection if it can be joined with the standard embedding of the projective line in the class of immersions without dangerous self-tangencies (in which both branches have the same direction).*

Remark. The imposed condition of avoiding dangerous self-tangencies means the preservation under deformation of a type of Legendrian knot. It is easy to construct an immersion with one point of inflection, which can be obtained from the standard embedding of the projective line by once passing through a dangerous self-tangency (Fig. 27).

Bibliography

[1] V. I. Arnold, "Lagrangian and Legendrian cobordisms II", *Funktsional. Anal. i Prilozhen.* **14**:4 (1980), 8–17; English transl. in *Functional Anal. Appl* **14** (1980).
[2] S. L. Tabachnikov, "Around four vertices", *Uspekhi Mat. Nauk* **45**:1 (1990), 229–230; English transl. in *Russian Math. Surveys* **45**:1 (1990).
[3] V. I. Arnol'd, "Sur les propriétés topologiques des projections Lagrangiennes en géométrie symplectique des caustiques", in: Cahiers mathématiques de la decision, vol. 9320, CEREMADE, 1993, pp. 1–9.
[4] V. I. Arnol'd, "The topological properties of Legendrian projections in the contact geometry of wave fronts", *Algebra i Analiz* **6**:3 (1994), 1–16; English transl. in *St. Petersburg Math. J.* **6** (1994).
[5] V. I. Arnol'd, "Topological invariants of plane curves and caustics", University Lecture Series 5, Amer. Math. Soc., Providence, RI 1994, pp. 1–60.
[6] V. I. Arnold, "Plane curves, their invariants, perestroikas and classifications", *Preprint ETH, Zürich, May 1993*; "Singularities and curves", *Adv. Soviet Math.* **21** (1994), 33–91.
[7] H. Whitney, "On regular closed curves in the plane", *Compositio Math.* **4** (1937), 276–284.
[8] D. Bennequin, "Entrelacements et équation de Pfaff", *Astérisque* **107–108** (1983), 87–162.
[9] V. I. Arnol'd, "Sur une propriété topologique des applications globalement chaotiques de la mécanique classique", *C.R. Acad. Sci.* Paris **261** (1965), 3719–3722.
[10] C. Conley and E. Zehnder, "The Birkhoff–Lewis fixed point theorem and a conjecture of V.I. Arnol'd", *Invent. Math.* **73** (1983), 33–49.
[11] M. Chaperon, "Quelques questions de géométrie symplectique d'aprés entre autres, Poincaré, Arnold, Conley et Zehnder", *Astérisque* **105–106** (1983), 231–249.
[12] F. Laudenbach and J.-C. Sikorav, "Persistence d'intersection avec la section null au course d'une isotopie hamiltonienne dans un fibré cotangent", *Invent. Math.* **82** (1985), 349–358.
[13] Yu. V. Chekhanov, "Legendrian Morse theory". (The reference given in the Russian text is incorrect.)
[14] A. B. Givental', "Nonlinear generalization of the Maslov index", in: Theory of singularities and its applications, *Adv. Soviet Math.* **1** (1990), 71–103.
[15] S. L. Tabachnikov, "The computation of the generalised Bennequin invariant by means of fronts", *Funktsional. Anal. i Pilozhen* **22**:3 (1988), 89–90; English transl. in *Functional Anal. Appl.* **22** (1988).
[16] V. I. Arnol'd, "First steps in symplectic topology", *Uspekhi Mat. Nauk* **41**:6 (1986), 3–18; English transl. in *Russian Math. Surveys* **41**:6 (1986).
[17] V. I. Arnol'd, "Invariants and perestroikas of fronts in the plane", *Trudy Mat. Inst. Steklov.* **209** (1995), 1–60.
[18] V. I. Arnol'd, "The branched covering $\mathbb{C}\mathrm{P}^2 \to S^4$, hyperbolicity and projective topology", *Sibirsk. Mat. Zh.* **29**:4 (1988), 36–47; English transl. in *Siberian Math. J.* **29** (1988).

Steklov Institute of Mathematics

Received 25/OCT/94
Translated by I. R. PORTEOUS
Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX