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Dynamical Systems IV

Symplectic Geometry and its Applications

With 62 Figures

Springer-Verlag
Berlin Heidelberg New York
London Paris Tokyo
Hong Kong

Encyclopaedia of Mathematical Sciences

Volume 4

Editor-in-Chief: R.V. Gamkrelidze

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Symplectic Geometry

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Foreword

Symplectic geometry is the mathematical apparatus of such areas of physics as classical mechanics, geometrical optics and thermodynamics. Whenever the equations of a theory can be gotten out of a variational principle, symplectic geometry clears up and systematizes the relations between the quantities entering into the theory. Symplectic geometry simplifies and makes perceptible the frightening formal apparatus of Hamiltonian dynamics and the calculus of variations in the same way that the ordinary geometry of linear spaces reduces cumbersome coordinate computations to a small number of simple basic principles.

In the present survey the simplest fundamental concepts of symplectic geometry are expounded. The applications of symplectic geometry to mechanics are discussed in greater detail in volume 3 of this series, and its applications to the theory of integrable systems and to quantization receive more thorough review in the articles of A.A. Kirillov and of B.A. Dubrovin, I.M. Krichever and S.P. Novikov in this volume.

We would like to express our gratitude to Professor G. Wassermann for the excellent and extremely careful translation.

Chapter 1

Linear Symplectic Geometry

§1. Symplectic Space

1.1. The Skew-Scalar Product. By a *symplectic structure* or a *skew-scalar product* on a linear space we mean a nondegenerate skew-symmetric bilinear form. The nondegeneracy of the skew-symmetric form implies that the space must be even-dimensional.

A symplectic structure on the plane is just an area form. The direct sum of n symplectic planes has a symplectic structure: the skew-scalar product of two vectors is equal to the sum of the areas of the projections onto the n coordinate planes of the oriented parallelogram which they span.

The Linear "Darboux Theorem". *Any two symplectic spaces of the same dimension are symplectically isomorphic, i.e., there exists a linear isomorphism between them which preserves the skew-scalar product.*

Corollary. *A symplectic structure on a 2n-dimensional linear space has the form* $p_1 \wedge q_1 + \ldots + p_n \wedge q_n$ *in suitable coordinates* $(p_1, \ldots, p_n, q_1, \ldots, q_n)$.

Such coordinates are called *Darboux coordinates*, and the space $\mathbb{R}^{2n}$ with this skew-scalar product is called the *standard symplectic space*.

Examples. 1) The imaginary part of a Hermitian form defines a symplectic structure. With respect to the coordinates $z_k = p_k + \sqrt{-1}q_k$ on $\mathbb{C}^n$ the imaginary part of the Hermitian form $\sum z_k \bar{z}'_k$ has the form $-\sum p_k \wedge q_k$.

2) The direct sum of a linear space with its dual $V = X^* \oplus X$ is equipped with a canonical symplectic structure $\omega(\xi \oplus x, \eta \oplus y) = \xi(y) - \eta(x)$. If $(q_1, \ldots, q_n)$ are coordinates on X and $(p_1, \ldots, p_n)$ are the dual coordinates on X^*, then $\omega = \sum p_k \wedge q_k$.

The standard symplectic structure on the coordinate space $\mathbb{R}^{2n}$ can be expressed by means of the matrix $\Omega = \begin{pmatrix} 0 & -E_n \\ E_n & 0 \end{pmatrix}$, where E_n is the unit $n \times n$ matrix: $\omega(v, w) = \langle \Omega v, w \rangle$. Here $\langle v, w \rangle = \sum v_k w_k$ is the Euclidean scalar product on $\mathbb{R}^{2n}$. Multiplication by Ω defines a complex structure on $\mathbb{R}^{2n}$, since $\Omega^2 = -E_{2n}$.

1.2. Subspaces. Vectors $v, w \in V$ for which $\omega(v, w) = 0$ are called *skew-orthogonal*. For an arbitrary subspace of a symplectic space the *skew-orthogonal complement* is defined, which by virtue of the nondegeneracy of the skew-scalar product does in fact have the complementary dimension, but, unlike the

Euclidean case, may intersect the original subspace. For example, the skew-scalar square of any vector equals 0, and therefore the skew-orthogonal complement of a straight line is a hyperplane which contains that line. Conversely, the skew-orthogonal complement of a hyperplane is a straight line which coincides with the kernel of the restriction of the symplectic structure to the hyperplane.

While a subspace of a Euclidean space has only one invariant—its dimension, in symplectic geometry, in addition to the dimension, the rank of the restriction of the symplectic structure to the subspace is essential. This invariant is trivial only in the case of a line or a hyperplane. The general situation is described by

The Linear "Relative Darboux Theorem". *In a symplectic space, a subspace of rank $2r$ and dimension $2r+k$ is given in suitable Darboux coordinates by the equations $q_{r+k+1}=\ldots=q_n=0$, $p_{r+1}=\ldots=p_n=0$.*

The skew-orthogonal complement of such a subspace is given by the equations $q_1=\ldots=q_r=0$, $p_1=\ldots=p_{r+k}=0$, and it intersects the original subspace along the k-dimensional kernel of the restriction of the symplectic form.

Subspaces which lie within their skew-orthogonal complements (i.e. which have rank 0) are called *isotropic*. Subspaces which contain their skew-orthogonal complements are called *coisotropic*. Subspaces which are isotropic and coisotropic at the same time are called *Lagrangian*. The dimension of Lagrangian subspaces is equal to half the dimension of the symplectic space. Lagrangian subspaces are maximal isotropic subspaces and minimal coisotropic ones. Lagrangian subspaces play a special rôle in symplectic geometry.

Examples of Lagrangian Subspaces. 1) In $X^*\oplus X$, the subspaces $\{0\}\oplus X$ and $X^*\oplus\{0\}$ are Lagrangian. 2) A linear operator $X\to X^*$ is self-adjoint if and only if its graph in $X^*\oplus X$ is Lagrangian. To a self-adjoint operator A there corresponds a quadratic form $(Ax, x)/2$ on X. It is called the *generating function* of this Lagrangian subspace. 3) A linear transformation of a space V preserves a symplectic form ω exactly when its graph in the space $V\oplus V$ is Lagrangian with respect to the symplectic structure $W=\pi_1^*\omega-\pi_2^*\omega$, where π_1 and π_2 are the projections onto the first and second summands (the area $W(x, y)$ of a parallelogram is equal to the difference of the areas of the projections).

1.3. The Lagrangian Grassmann Manifold. The set of all Lagrangian subspaces of a symplectic space of dimension $2n$ is a smooth manifold and is called the *Lagrangian Grassmann manifold* Λ_n.

Theorem. *Λ_n is diffeomorphic to the manifold of cosets of the subgroup O_n of orthogonal matrices in the group U_n of unitary $n\times n$ matrices* (a unitary frame in $\mathbb{C}^n$ generates a Lagrangian subspace in $\mathbb{C}^n$ considered as a real space).

Corollary. $\dim\Lambda_n=n(n+1)/2$.

On the topology of Λ_n, see chap. 6.

Example. A linear line complex. By a line complex is meant a three-dimensional family of lines in three-dimensional projective space. Below we shall give a construction connecting the so-called linear line complexes with the simplest concepts of symplectic geometry. This connection gave symplectic geometry its name: in place of the adjective "com-plex" (composed of Latin roots meaning "plaited together"), which had introduced terminological confusion, Hermann Weyl [75] in 1946 proposed using the adjective "sym-plectic", formed from the equivalent Greek roots.

The construction which follows further on shows that the Lagrangian Grassmann manifold Λ_2 is diffeomorphic to a nonsingular quadric (of signature $(+++--)$) in four-dimensional projective space.

The points of the projective space $\mathbb{P}^3=P(V)$ are one-dimensional subspaces of the four-dimensional vector space V. The lines in $\mathbb{P}^3$ are two-dimensional subspaces of V. Each such subspace uniquely determines up to a factor an exterior 2-form ϕ of rank 2, whose kernel coincides with this subspace. In the 6-dimensional space $\Lambda^2 V$ of all exterior 2-forms, the forms of rank 2 form a quadratic cone with the equation $\phi\wedge\phi=0$. Thus the manifold of all lines in $\mathbb{P}^3$ is a quadric Q in $\mathbb{P}^5=P(\Lambda^2 V)$. A linear line complex is given by the intersection of the quadric Q with a hyperplane H in $\mathbb{P}^5$. A hyperplane in $P(\Lambda^2 V)$ can be given with the aid of an exterior 2-form ω on V: $H=P(\{\phi\in\Lambda^2 V\,|\,\omega\wedge\phi=0\})$. Nondegeneracy of the form ω is equivalent to the condition that the linear line complex $H\cap Q$ be nonsingular. The equation $\omega\wedge\phi=0$ for a form ϕ of rank 2 means that its kernel is Lagrangian with respect to the symplectic structure ω. Therefore, a nonsingular linear line complex is the Lagrangian Grassmann manifold Λ_2.

§2. Linear Hamiltonian Systems

Here we shall discuss the Jordan normal form of an infinitesimal symplectic transformation.

2.1. The Symplectic Group and its Lie Algebra. A linear transformation G of a symplectic space (V, ω) is called a *symplectic transformation* if it preserves the skew-scalar product: $\omega(Gx, Gy)=\omega(x, y)$ for all $x, y\in V$. The symplectic transformations form a Lie group, denoted by $\mathrm{Sp}(V)$ ($\mathrm{Sp}(2n, \mathbb{R})$ or $\mathrm{Sp}(2n, \mathbb{C})$ for the standard real or complex $2n$-dimensional symplectic space).

Let us consider a one-parameter family of symplectic transformations, and let the parameter value 0 correspond to the identity transformation. The derivative of the transformations of the family with respect to the parameter (at 0) is called a *Hamiltonian operator*. By differentiating the condition for symplecticity of a transformation, we may find the condition for an operator H to be Hamiltonian: $\omega(Hx, y)+\omega(x, Hy)=0$ for all $x, y\in V$. A commutator of Hamiltonian operators is again a Hamiltonian operator: the Hamiltonian operators make up the Lie algebra $\mathrm{sp}(V)$ of the Lie group $\mathrm{Sp}(V)$.

The quadratic form $h(x)=\omega(x, Hx)/2$ is called the *Hamiltonian* of the operator H. A Hamiltonian operator can be reconstructed from its Hamiltonian out of the equation $h(x+y)-h(x)-h(y)=\omega(y, Hx)$ for all x, y. We get an isomorphism of the space of Hamiltonian operators to the space of quadratic forms on the symplectic space V.

Corollary. $\dim \mathrm{Sp}(V)=n(2n+1)$, *where* $2n=\dim V$.

The commutator of Hamiltonian operators defines a *Lie algebra* structure on the space of *quadratic Hamiltonians*: $\{h_1, h_2\}(x)=\omega(x, (H_2H_1-H_1H_2)x)/2 = \omega(H_1x, H_2x)$. The operation $\{\cdot,\cdot\}$ is called the *Poisson bracket*. In Darboux coordinates the Poisson bracket has the form $\{h_1, h_2\}=\sum(\partial h_1/\partial p_k \cdot \partial h_2/\partial q_k - \partial h_2/\partial p_k \cdot \partial h_1/\partial q_k)$.

The matrix of a Hamiltonian operator in Darboux coordinates $H=\begin{pmatrix} A & B \\ C & D\end{pmatrix}$ satisfies the relations[1] $B^*=B$, $C^*=C$, $D^*=-A$. The corresponding Hamiltonian h is the quadratic form whose matrix is $[h]=-\frac{1}{2}\Omega H=\frac{1}{2}\begin{pmatrix} C & D \\ -A & -B\end{pmatrix}$.

A linear *Hamiltonian system* of differential equations $\dot{x}=Hx$ can be written as follows in Darboux coordinates: $\dot{p}=-\partial h/\partial q$, $\dot{q}=\partial h/\partial p$. In particular, the Hamiltonian is a first integral of its own Hamiltonian system: $\dot{h}=\partial h/\partial q\cdot\dot{q} + \partial h/\partial p\cdot\dot{p}=0$. Thus we have conveyed the structure of the Lie algebra of the symplectic group and its action on the space V, in terms of the space of quadratic Hamiltonians.

Examples. 1) To the Hamiltonian $h=\omega(p^2+q^2)/2$ corresponds the system of equations $\dot{q}=\omega p$, $\dot{p}=-\omega q$ of the harmonic oscillator. 2) The group of symplectic transformations of the plane $\mathbb{R}^2$ coincides with the group $\mathrm{SL}(2,\mathbb{R})$ of 2×2 matrices with determinant 1. Its Lie algebra $\mathrm{sl}(2,\mathbb{R})$ has three generators $X=q^2/2$, $Y=-p^2/2$, $H=pq$ with commutators $[X, Y]=H$, $[H, X]=2X$, $[H, Y]=-2Y$.

2.2. The Complex Classification of Hamiltonians. We shall consider two Hamiltonian operators on a symplectic space V to be equivalent if they can be transformed into one another by a symplectic transformation. The corresponding classification problem is entirely analogous to the case of the Jordan normal form of a linear operator. To put it in more erudite terms, it is a question of classifying the orbits of the adjoint action of the symplectic group on its Lie algebra. In the complex case the answer is given by

[1] $*$ = transposition.

Williamson's Theorem [77]. *Hamiltonian operators on a complex symplectic space are equivalent if and only if they are similar (i.e. have the same Jordan structure).*

The symplectic form allows us to identify the space V with the dual space V^* as follows: $x\mapsto\omega(x,\cdot)$. Under this identification the operator $H^*: V^*\to V^*$ dual to a Hamiltonian operator $H: V\to V$ is turned into $-H$. Therefore the Jordan structure of a Hamiltonian operator meets the restrictions

1) if a is an eigenvalue, then $-a$ is also an eigenvalue;
2) the Jordan blocks corresponding to the eigenvalues a and $-a$ have the same structure;
3) the number of Jordan blocks of odd dimension with eigenvalue $a=0$ is even.

Apart from this, the Jordan structure of Hamiltonian operators is arbitrary.

Corollary. *Let $H: V\to V$ be a Hamiltonian operator. Then V decomposes as a direct skew-orthogonal sum of symplectic subspaces, on each of which the operator H has either two Jordan blocks of the same order with opposite eigenvalues, or one Jordan block of even order with eigenvalue* 0.

2.3. Linear Variational Problems. As normal forms for linear Hamiltonian systems, one may take the equations of the extremals of special variational problems. We assume that the reader is familiar with the simplest concepts of the calculus of variations, and we shall make use of the formulas of chap. 3, sect. 1.4, where the Hamiltonian formalism of variational problems with higher derivatives is described.

Let $x=x(t)$ be a function of the variable t, and let $x_k=d^kx/dt^k$. Let us consider the problem of optimizing the functional $\int L(x_0,\ldots,x_n)\,dt$ with the Lagrangian function $L=(x_n^2+a_{n-1}x_{n-1}^2+\ldots+a_0x_0^2)/2$. The equation of the extremals of this functional

$$x_{2n}-a_{n-1}x_{2n-2}+\ldots+(-1)^na_0x_0=0$$

is a linear homogeneous equation with constant coefficients involving only even-order derivatives of the required function x.

On the other hand, the equation of the extremals is equivalent to the Hamiltonian system (see chap. 3, sect. 1.4) with the quadratic Hamiltonian

$$h=\pm\{p_0q_1+\ldots+p_{n-2}q_{n-1}+(p_{n-1}^2-a_{n-1}q_{n-1}^2-\ldots-a_0q_0^2)/2\},$$

where $q_k=x_k$, $p_{n-1}=x_n$, $p_{k-1}=a_kx_k-dp_k/dt$ are Darboux coordinates on the $2n$-dimensional phase space of the equation of the extremals.

We remark that with this construction it is not possible to obtain a Hamiltonian system having a pair of Jordan blocks of odd order n with eigenvalue 0. The Hamiltonian $\pm(p_0q_1+\ldots+p_{n-2}q_{n-1})$ corresponds to such a system. A Hamiltonian Jordan block of order $2n$ with eigenvalue 0 is obtained for $L=x_n^2/2$. The extremals in this case are the solutions of the equation

$d^{2n}x/dt^{2n}=0$, i.e. the polynomials $x(t)$ of degree $<2n$. In general, to the Lagrangian function L with characteristic polynomial $\xi^n+a_{n-1}\xi^{n-1}+\ldots+a_0 = \xi^{m_0}(\xi+\xi_1)^{m_1}\ldots(\xi+\xi_k)^{m_k}$ corresponds a Hamiltonian operator with one Jordan block of dimension $2m_0$ and eigenvalue 0 and k pairs of Jordan blocks of dimensions m_j with eigenvalues $\pm\sqrt{\xi_j}$.

2.4. Normal Forms of Real Quadratic Hamiltonians. An obvious difference between the real case and the complex one is that the Jordan blocks of a Hamiltonian operator split into quadruples of blocks of the same dimension with eigenvalues $\pm a \pm b\sqrt{-1}$, provided $a\neq 0$ and $b\neq 0$. A more essential difference lies in the following. Two real matrices are similar in the real sense if they are similar as complex matrices. For quadratic Hamiltonians this is not always so. For example, the Hamiltonians $\pm(p^2+q^2)$ of the harmonic oscillator have the same eigenvalues $\pm 2\sqrt{-1}$, i.e. they are equivalent over $\mathbb{C}$, but they are not equivalent over $\mathbb{R}$: to these Hamiltonians correspond rotations in different directions on the phase plane oriented by the skew-scalar product. In particular, in the way in which it is formulated there, the Williamson theorem of sect. 2.2 does not carry over to the real case.

We shall give a list of the elementary normal forms of quadratic Hamiltonians in the Darboux coordinates $(p_0,\ldots,p_{n-1},q_0,\ldots,q_{n-1})$ of the standard symplectic space $\mathbb{R}^{2n}$.

1) The case of a pair of Jordan blocks of (odd) order n with eigenvalue 0 is represented by the Hamiltonian

$$h_0=\sum_{k=0}^{n-2} p_k q_{k+1} \quad (h_0=0 \text{ when } n=1).$$

2) The case of a Jordan block of even order $2n$ with eigenvalue 0 is represented by a Hamiltonian of exactly one of the two forms

$$\pm(h_0+p_{n-1}^2/2).$$

3) The case of a pair of Jordan blocks of order n with nonzero eigenvalues $\pm z$ is represented by a Hamiltonian of one of the two forms

$$\pm\left(h_0+p_{n-1}^2/2-\sum_{k=0}^{n-1} c_n^k z^{2(n-k)} q_k^2/2\right)$$

(for real z these two Hamiltonians are equivalent to each other, but for a purely imaginary z they are not equivalent).

4) The case of a quadruple of Jordan blocks of order $m=n/2$ with eigenvalues $\pm a\pm b\sqrt{-1}$ is represented by the Hamiltonian

$$h_0+p_{n-1}^2/2-\sum_{k=0}^{n-1} A_k q_k^2/2,$$

where

$$\sum A_k\xi^k=[\xi^2+2(a^2-b^2)\xi+(a^2+b^2)^2]^m.$$

Theorem ([77]). *A real symplectic space on which a quadratic Hamiltonian h is given decomposes into a direct skew-orthogonal sum of real symplectic subspaces such that the form h can be represented as a sum of elementary forms in suitable Darboux coordinates on these subspaces.*

2.5. Sign-Definite Hamiltonians and the Minimax Principle.[2] In suitable Darboux coordinates, a positive definite quadratic Hamiltonian has the form $h=\sum\omega_k(p_k^2+q_k^2)/2$, where $\omega_n\geq\omega_{n-1}\geq\ldots\geq\omega_1>0$. For the "frequencies" ω_k one has the following minimax principle.

In suitable Cartesian coordinates on the Euclidean space V^N, a skew-symmetric bilinear form Ω can be written as $\lambda_1 p_1\wedge q_1+\ldots+\lambda_n p_n\wedge q_n$, $2n\leq N$, $\lambda_1\geq\lambda_2\geq\ldots\geq\lambda_n>0$. To an oriented plane $L\subset V^N$ let us associate the area $S(L)$ of the unit disk $D(L)=\{x\in L\mid\langle x,x\rangle\leq 1\}$ with respect to the form Ω.

Theorem.

$$\min_{V^{N+1-k}\subset V^N}\ \max_{L\subset V^{N+1-k}} S(L)=\pi\lambda_k,\quad k=1,\ldots,n.$$

Corollary. *The invariants λ'_k of the restriction of the form Ω to a subspace $W^{N-M}\subset V^N$ satisfy the inequalities $\lambda_k\geq\lambda'_k\geq\lambda_{k+M}$ (we set $\lambda_m=0$ for $m>n$).*

For example,

$$\pi\lambda'_1=\max_{L\subset W} S(L)\leq\max_{L\subset V} S(L)=\pi\lambda_1$$

and

$$\pi\lambda'_1=\max_{L\subset W^{N-M}} S(L)\geq\min_{V^{N-M}\subset V^N}\ \max_{L\subset V^{N-M}} S(L)=\pi\lambda_{M+1}.$$

Remarks. 1) Courant's minimax principle (R. Courant) for pairs of Hermitian forms in $\mathbb{C}^n$: $U=\sum z_k\bar z_k$, $U'=\sum\lambda_k z_k\bar z_k$, $\lambda_1\leq\ldots\leq\lambda_n$, states that $\lambda_k=\min_{\mathbb{C}^k\subset\mathbb{C}^n}\max_{\mathbb{C}\subset\mathbb{C}^k}(U'/U)=\max_{\mathbb{C}^{n+1-k}\subset\mathbb{C}^n}\min_{\mathbb{C}\subset\mathbb{C}^{n+1-k}}(U'/U)$. From this it is easy to deduce our theorem.

2) If we take a symplectic structure on $\mathbb{R}^{2n}$ for Ω and a positive definite Hamiltonian h for the Euclidean structure, we obtain the minimax principle and analogous corollaries for the frequencies $\omega_k=2/\lambda_k$. In particular, under an increase of the Hamiltonian the frequencies grow: $\omega'_k\geq\omega_k$.

[2] The results in this section were obtained by V.I. Arnol'd in 1977 in connection with the conjecture (now proved by Varchenko and Steenbrink) that the spectrum of a singularity is semicontinuous. (See Varchenko, A. N.: On semicontinuity of the spectrum and an upper estimate for the number of singular points of a projective hypersurface. Dokl. Akad. Nauk SSSR 270 (1983), 1294–1297 (English translation: Sov. Math., Dokl. 27 (1983), 735–739) and Steenbrink, J.H.M.: Semicontinuity of the singularity spectrum. Invent. Math. 79 (1985), 557–565.)

§3. Families of Quadratic Hamiltonians

The Jordan normal form of an operator depending continuously on a parameter is, in general, a discontinuous function of the parameter. The miniversal deformations introduced below are normal forms for families of operators which are safe from the deficiency mentioned.

3.1. The Concept of the Miniversal Deformation. It has to do with the following abstract situation. Let a Lie group G act on a smooth manifold M. Two points of M are considered equivalent if they lie in one orbit, i.e. if they go over into each other under the action of this group. A family with parameter space (base space) V is a smooth mapping $V \to M$. A deformation of an element $x \in M$ is a germ of a family $(V, 0) \to (M, x)$ (where 0 is the coordinate origin in $V \simeq \mathbb{R}^n$).One says that the deformation ϕ: $(V, 0) \to (M, x)$ is induced from the deformation ψ: $(W, 0) \to (M, x)$ under a smooth mapping of the base spaces v: $(V, 0) \to (W, 0)$, if $\phi = \psi \circ v$. Two deformations ϕ, ψ: $(V, 0) \to (M, x)$ are called equivalent if there is a deformation of the identity element g: $(V, 0) \to (G, \mathrm{id})$ such that $\phi(v) = g(v)\psi(v)$.

Definition. A deformation ϕ: $(V, 0) \to (M, x)$ is called *versal* if any deformation of the element x is equivalent to a deformation induced from ϕ. A versal deformation with the smallest base-space dimension possible for a versal deformation is called *miniversal.*

The germ of the manifold M at the point x is obviously a versal deformation for x, but generally speaking it is not miniversal.

Example. Let M be the space of quadratic Hamiltonians on the standard symplectic space $\mathbb{R}^{2n}$ and let $G = \mathrm{Sp}(2n, \mathbb{R})$ be the group of symplectic linear transformations on $\mathbb{R}^{2n}$. The following deformation (λ are the parameters)

$$\begin{aligned} H_\lambda = \sum_{k=1}^{s} \, [(b_k + \lambda_{2k-1})(p_{2k-1} q_{2k} - q_{2k-1} p_{2k}) \\ -(a_k + \lambda_{2k})(p_{2k-1} q_{2k-1} + p_{2k} q_{2k})] \\ + \sum_{k=2s+1}^{r} (c_k + \lambda_k) p_k q_k + \sum_{k=r+1}^{n} (d_k + \lambda_k)(p_k^2 + q_k^2)/2 \qquad (1) \end{aligned}$$

is a miniversal deformation of the Hamiltonian H_0 if its spectrum $\{\pm a_k \pm \sqrt{-1} b_k, \pm c_k, \pm \sqrt{-1}\, d_k\}$ is nonmultiple.

Let $X \subset M$ be a submanifold. One says that a deformation ϕ: $(V, 0) \to (M, x)$ of a point $x \in X$ is transversal to X if $\phi_*(T_0 V) + T_x X = T_x M$ (Fig. 1).

The Versality Theorem. *A deformation of the point $x \in M$ is versal if and only if it is transversal to the orbit Gx of the point x in M.*

Corollary. *The number of parameters of a miniversal deformation is equal to the codimension of the orbit.*

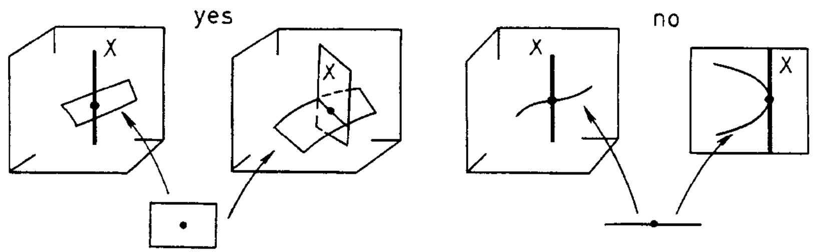

Fig. 1. Transversality

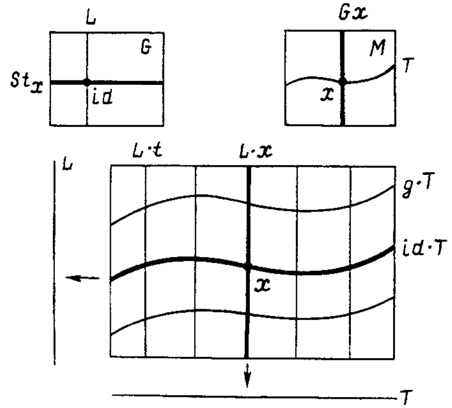

Fig. 2. The proof of the versality theorem

Sketch of the proof of the theorem (see Fig. 2). Let us choose a submanifold L transversal at the identity element of the group G to the isotropy subgroup $\mathrm{St}_x = \{g \mid gx = x\}$ of the point $x \in M$ and a submanifold T transversal at the point x to its orbit in M. The action by the elements of L on the points of T gives a diffeomorphism of a neighbourhood of the point x in M to the direct product $L \times T$. Now every deformation ϕ: $(V, 0) \to (M, x)$ automatically takes on the form $\phi(v) = g(v)t(v)$, where g: $(V, 0) \to (G, \mathrm{id})$, t: $(V, 0) \to (T, x)$.

3.2. Miniversal Deformations of Quadratic Hamiltonians. Let M once again be the space of quadratic Hamiltonians in $\mathbb{R}^{2n}$ and $G = \mathrm{Sp}(2n, \mathbb{R})$. We shall identify M with the space of Hamiltonian matrices of order $2n \times 2n$. Let us introduce in the space of such matrices the elementwise scalar product $\langle , \rangle$. It can be represented in the form $\langle H, F \rangle = \mathrm{tr}(HF^*)$, where $*$ denotes transposition. We note that the transposed matrix of a Hamiltonian matrix is again Hamiltonian. From the properties of the trace we obtain: $\langle [X, Y], Z \rangle + \langle Y, [X^*, Z] \rangle = 0$, where $[X, Y] = XY - YX$ is the commutator.

Lemma. *The orthogonal complement in M to the tangent space at the point H of the orbit of the Hamiltonian H coincides with the centralizer* $Z_{H^*}=\{X\in M \mid [X, H^*]=0\}$ *of the Hamiltonian* H^* *in the Lie algebra of quadratic Hamiltonians.*

Proof. If $\langle [H, F], X\rangle = 0$ for all $F\in M$, then $\langle F, [H^*, X]\rangle = 0$, i.e. $[H^*, X]=0$, and conversely. □

Corollary. *The deformation* $(Z_H, 0)\to(M, H)$: $X\mapsto H+X^*$ *is a miniversal deformation of the quadratic Hamiltonian H.*

For a Hamiltonian H let us denote by $n_1(z)\ge n_2(z)\ge \dots \ge n_s(z)$ the dimensions of the Jordan blocks with eigenvalue $z\ne 0$, and by $m_1\ge \dots \ge m_u$ and $\tilde m_1\ge \dots \ge \tilde m_v$ the dimensions of its Jordan blocks with eigenvalue 0, where the m_j are even and the $\tilde m_j$ are odd (out of every pair of blocks of odd dimension only one is taken into account).

Theorem ([29]). *The dimension d of the base space of the miniversal deformation of a Hamiltonian H equals*

$$d=\frac{1}{2}\sum_{z\neq 0}\sum_{j=1}^{s(z)}(2j-1)n_j(z)+\frac{1}{2}\sum_{j=1}^{u}(2j-1)m_j$$

$$+\sum_{j=1}^{v}[2(2j-1)\tilde m_j+1]+2\sum_{j=1}^{u}\sum_{k=1}^{v}\min(m_j, \tilde m_k).$$

The paper [29] gives the explicit form of the miniversal deformations for all normal forms of quadratic Hamiltonians.

3.3. Generic Families. Let us divide up the space of quadratic Hamiltonians into classes according to the existence of eigenvalues of different types (but not numerical values) and according to the dimensions of the Jordan blocks. Such a classification, in contrast to the classification by G-orbits, is discrete (even finite). One says that the Hamiltonians of a given class are not encountered in generic l-parameter families if one can remove them by an arbitrarily small perturbation of the family. For example, a generic Hamiltonian has no multiple eigenvalues; it also does not have any preassigned spectrum of eigenvalues, but nevertheless does have some other spectrum.

The importance of studying generic phenomena is explained by the fact that in applications the object being investigated is often known only approximately or is subject to perturbations because of which exceptional phenomena are not observed directly.

The codimension c of a given class is the smallest number of parameters of families in which Hamiltonians of this class are encountered unremovably.

Let us denote by v half the number of different nonzero eigenvalues of the Hamiltonians of the given class.

Theorem. $c=d-v$ *(so that the formula for c can be obtained out of the formula for d of the preceding theorem by diminishing each term of the form* $\sum(2j-1)n_j(z)$ *by one).*

The proof of this theorem is based on the intuitively obvious fact that a generic family is transversal to every class (Fig. 3) (for more details on this see [4], [9]), and on the fact that the number of parameters indexing the G-orbits of the given class is equal to v.

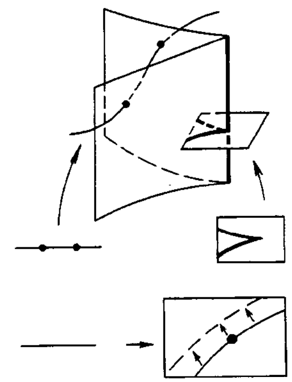

Fig. 3. Generic families

Corollary 1. *In one and two-parameter families of quadratic Hamiltonians one encounters as irremovable only Jordan blocks of the following twelve types:*

$$c=1:\quad (\pm a)^2,\ (\pm ia)^2,\ 0^2$$

(here the Jordan blocks are denoted by their determinants, for example, $(\pm a)^2$ *denotes a pair of Jordan blocks of order 2 with eigenvalues a and* $-a$ *respectively);*

$$c=2:\quad (\pm a)^3,\ (\pm ia)^3,\ (\pm a\pm ib)^2,\ 0^4,\ (\pm a)^2(\pm b)^2,$$
$$(\pm ia)^2(\pm ib)^2,\ (\pm a)^2(\pm ib)^2,\ (\pm a)^2 0^2,\ (\pm ia)^2 0^2$$

(the remaining eigenvalues are simple).

Corollary 2. *Let* F_t *be a smooth generic family of quadratic Hamiltonians which depends on one parameter. In the neighbourhood of an arbitrary value* $t=t_0$ *there exists a system of linear Darboux coordinates, depending smoothly on t, in which* a) *for almost all* t_0, F_t *has the form* H_λ *(see formula* (1)), b) *for isolated values of* t_0, F_t *has one of the following forms*

$$(\pm a)^2:\quad P_1Q_2+P_2^2/2-(a^4+\mu_1)Q_1^2/2-(a^2+\mu_2)Q_2^2+H_\lambda(p, q);$$

$$(\pm ia)^2:\quad \pm[P_1Q_2+P_2^2/2-(a^4+\mu_1)Q_1^2/2+(a^2+\mu_2)Q_2^2]+H_\lambda(p, q);$$

$$0^2:\quad \pm[P^2/2-\mu Q^2/2]+H_\lambda(p, q)$$

(here (P, Q, p, q) are Darboux coordinates on $\mathbb{R}^{2m}$, (λ, μ) are smooth functions of the parameter t, $(\lambda(t_0), \mu(t_0)) = (0, 0)$).

Proof. The formulas listed are miniversal deformations of representatives of the classes of codimension 1. □

3.4. Bifurcation Diagrams. The bifurcation diagram of a deformation of a Hamiltonian is the germ of the partition of the parameter space into the preimages of the classes. The bifurcation diagrams of generic families reflect (in view of the condition of transversality to the classes, see Fig. 3) the class partition structure in the space of quadratic Hamiltonians itself.

In Figs. 4 and 5 are presented the bifurcation diagrams of generic deformations for the classes of codimension 1 and the first four classes of codimension 2 in the order of their being listed in the statement of corollary 1.

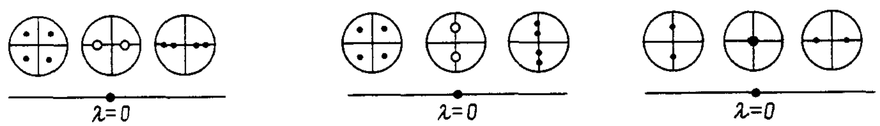

Fig. 4. Bifurcation diagrams of quadratic Hamiltonians, $c=1$

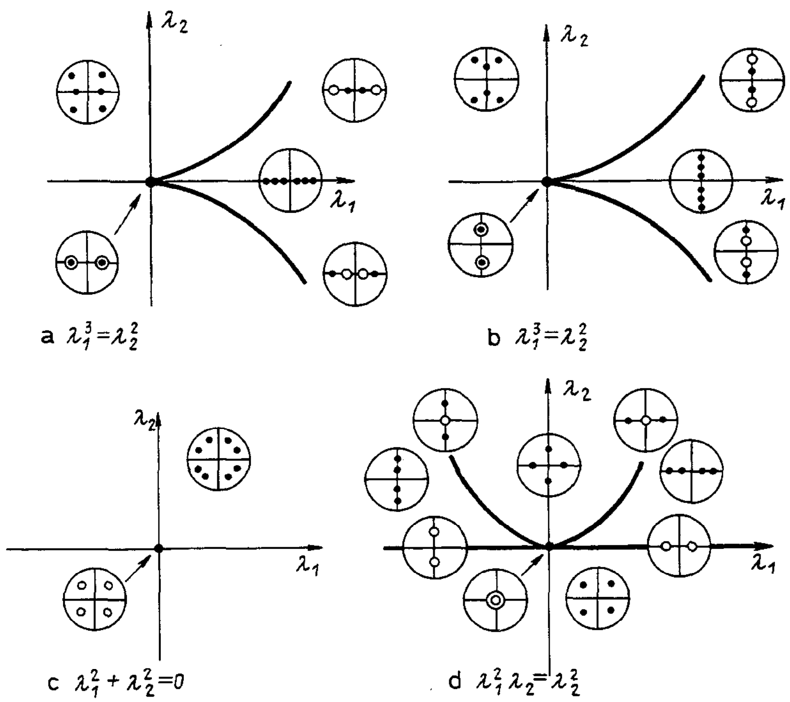

Fig. 5. Bifurcation diagrams of quadratic Hamiltonians, $c=2$

Remark. One should not think that the bifurcation diagram of a real quadratic Hamiltonian depends only on the Jordan structure. Let us look at the following important example. The Hamiltonian operators with a nonmultiple purely imaginary spectrum form an open set in the space of Hamiltonian operators. The Hamiltonian operators with a purely imaginary spectrum with multiple eigenvalues, but without Jordan blocks, form a set of codimension 3 in the space of Hamiltonian operators. If such an operator H has the spectrum $\{\pm\sqrt{-1}\omega_k\}$, then in suitable Darboux coordinates the corresponding Hamiltonian has the form $h = [\omega_1(p_1^2+q_1^2) + \ldots + \omega_n(p_n^2+q_n^2)]/2$. Suppose, say, $\omega_1^2 = \omega_2^2 \neq 0$. If the invariants ω_1 and ω_2 of the Hamiltonian h are of the same sign, then the bifurcation diagram is a point (the class of h) in the space $\mathbb{R}^3$—all Hamiltonians near h have a purely imaginary spectrum and have no Jordan blocks. If the invariants ω_1 and ω_2 are of different signs, then the bifurcation diagram is a quadratic cone (Fig. 6), to the points of which correspond the operators with Jordan blocks of dimension 2.

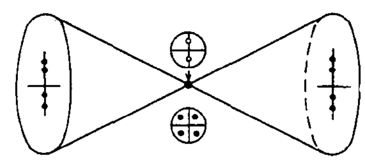

Fig. 6. The bifurcation diagram of the Hamiltonian $p_1^2+q_1^2-p_2^2-q_2^2$

§4. The Symplectic Group

The information we bring below on real symplectic groups is applied at the end of the section to the theory of linear Hamiltonian systems of differential equations with periodic coefficients.

4.1. The Spectrum of a Symplectic Transformation. The *symplectic group* $\mathrm{Sp}(2n, \mathbb{R})$ consists of the linear transformations of the space $\mathbb{R}^{2n}$ which preserve the standard symplectic structure $\omega = \sum p_k \wedge q_k$. The matrix G of a symplectic transformation in a Darboux basis therefore satisfies the defining relation $G^*\Omega G = \Omega$.

Theorem. *The spectrum of a real symplectic transformation is symmetric with respect to the unit circle and the real axis. The eigenspaces in the weaker sense (i.e., where $G-\lambda E$ is nilpotent) corresponding to symmetric eigenvalues have the same Jordan structure.*

In fact, the defining relation shows that the matrices G and G^{-1} are similar over $\mathbb{C}$. This leads to the invariance of the Jordan structure of a symplectic transformation and its spectrum with respect to the symmetry $\lambda\mapsto\lambda^{-1}$. Realness gives the second symmetry $\lambda\mapsto\bar{\lambda}$.

4.2. The Exponential Mapping and the Cayley Parametrization. The exponential of an operator gives the exponential mapping $H\mapsto\exp(H)=\sum H^k/k!$ of the space of Hamiltonian operators to the symplectic group. The symplectic group acts by conjugation on itself and on its Lie algebra. The exponential mapping is invariant with respect to this action: $\exp(G^{-1}HG)=G^{-1}\exp(H)G$.

The mapping exp is a diffeomorphism of a neighbourhood of 0 in the Lie algebra onto a neighbourhood of the identity element in the group. The inverse transformation is given by the series $\ln G=-\sum(E-G)^k/k$. The mapping exp: $\mathrm{sp}(2n,\mathbb{R})\to\mathrm{Sp}(2n,\mathbb{R})$ is neither injective nor surjective. Therefore, for the study of the symplectic group the Cayley parametrization is more useful: $G=(E+H)(E-H)^{-1}$, $H=(G-E)(G+E)^{-1}$. These formulas give a diffeomorphism ca of the set of Hamiltonian operators H all of whose eigenvalues are different from ± 1, 0, onto the set of symplectic transformations G all of whose eigenvalues are different from ± 1.

Using the mappings ca, exp, $-\exp$ and the results of § 2, we may obtain the following result.

Theorem. *A symplectic space on which is given a symplectic transformation G splits into a direct skew-orthogonal sum of symplectic subspaces on each of which the transformation G has, in suitable Darboux coordinates, the form $\pm\exp(H)$, where H is an elementary Hamiltonian operator from sect.* 2.4.

4.3. Subgroups of the Symplectic Group. The symplectic transformations $\pm E$ commute with all elements of the group $\mathrm{Sp}(V^{2n})$ and form its center.

Every compact subgroup of $\mathrm{Sp}(V^{2n})$ lies in the intersection of $\mathrm{Sp}(V^{2n})$ with the orthogonal group $\mathrm{O}(V^{2n})$ of mappings which preserve some positive definite quadratic Hamiltonian $h=\sum\omega_k(p_k^2+q_k^2)/2$. If all the ω_k are different, the intersection $\mathrm{Sp}(V^{2n})\cap\mathrm{O}(V^{2n},h)$ is an n-dimensional torus T^n and is generated by transformations $\exp(\lambda H_k)$, where H_k has the Hamiltonian $(p_k^2+q_k^2)/2$. Every compact commutative subgroup of $\mathrm{Sp}(V^{2n})$ lies in some torus T^n of the kind described above. All such tori are conjugate in the symplectic group.

Let us look at the normalizer $N(T^n)=\{g\in\mathrm{Sp}(V^{2n})\,|\,gT^ng^{-1}=T^n\}$ of the torus T^n in the symplectic group. The factor group $W=N(T^n)/T^n$ is called the Weyl group. It is finite, isomorphic to the permutation group on n letters and acts on the torus by permuting the one-parameter subgroups $\exp(\lambda H_k)$. Two elements of the torus are conjugate in the symplectic group if and only if they lie in the same orbit of this action.

If all the ω_k are equal to each other, the Hamiltonian h together with the symplectic form endows V^{2n} with the structure of an n-dimensional complex Hermitian space. The intersection $\mathrm{Sp}(V^{2n})\cap\mathrm{O}(V^{2n},h)$ coincides with the unitary group U_n of this space. All the subgroups U_n are conjugate. Every compact subgroup of $\mathrm{Sp}(V^{2n})$ lies in some unitary subgroup of this type. In particular, a torus T^n and its normalizer $N(T^n)$ lie in a (unique) subgroup U_n.

Remarks. 1) In the complex symplectic group $\mathrm{Sp}(2n,\mathbb{C})$ a maximal compact subgroup is isomorphic to the *compact symplectic group* Sp_n of transformations of an n-dimensional space over the skew field of quaternions.

2) The torus T^n is a maximal torus in the complex symplectic group as well: $T^n\subset\mathrm{Sp}(2n,\mathbb{R})\subset\mathrm{Sp}(2n,\mathbb{C})$, but its normalizer $N_{\mathbb{C}}(T^n)$ in $\mathrm{Sp}(2n,\mathbb{C})$ differs from the normalizer $N(T^n)$ in $\mathrm{Sp}(2n,\mathbb{R})$. The Weyl group $W_{\mathbb{C}}=N_{\mathbb{C}}(T^n)/T^n$ acts on T^n via compositions of permutations of the subgroups $\exp(\lambda H_k)$ and reflections $\exp(\lambda H_k)\mapsto\exp(-\lambda H_k)$.

Example. The group $\mathrm{Sp}_1\subset\mathrm{Sp}(2,\mathbb{C})$ of unit quaternions coincides with the group SU(2). As a maximal torus in $\mathrm{Sp}(2,\mathbb{C})$ one may take the group SO(2) of rotations of the plane. In this case the maximal torus coincides with the maximal compact subgroup U_1 of $\mathrm{Sp}(2,\mathbb{R})$. The Weyl group W is trivial. The complex Weyl group $W_{\mathbb{C}}$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Its action on SO(2) is given by conjugation by means of the matrix diag $(\sqrt{-1},-\sqrt{-1})$.

4.4. The Topology of the Symplectic Group

Theorem. *The manifold* $\mathrm{Sp}(2n,\mathbb{R})$ *is diffeomorphic to the Cartesian product of the unitary group* U_n *with a vector space of dimension* $n(n+1)$.

The key to the proof is given by the polar decomposition: an invertible operator A on a Euclidean space can be represented uniquely in the form of a product $S\cdot U$ of an invertible symmetric positive operator $S=(AA^*)^{1/2}$ and an orthogonal operator $U=S^{-1}A$. For symmetric operators A acting on the underlying real space $\mathbb{R}^{2n}$ of the Hermitian space $\mathbb{C}^n$, the operators U turn out to be unitary, and the logarithms $\ln S$ of the operators S fill out the $n(n+1)$-dimensional space of symmetric Hamiltonian operators.

Corollary. 1) *The symplectic group* $\mathrm{Sp}(2n,\mathbb{R})$ *can be contracted to the unitary subgroup* U_n.

2) *The symplectic group* $\mathrm{Sp}(2n,\mathbb{R})$ *is connected. The fundamental group* $\pi_1(\mathrm{Sp}(2n,\mathbb{R}))$ *is isomorphic to* $\mathbb{Z}$.

The latter follows from the properties of the manifold U_n: $\mathrm{U}_n\simeq\mathrm{SU}_n\times S^1$ (the function $\det_{\mathbb{C}}$: $\mathrm{U}_n\to\{z\in\mathbb{C}\,|\,|z|=1\}$ gives the projection onto the second factor); the group SU_n is connected and simply connected (this follows from the exact homotopy sequences of the fibrations $\mathrm{SU}_n\xrightarrow{\mathrm{SU}_{n-1}}S^{2n-1}$).

Example. The group $\mathrm{Sp}(2,\mathbb{R})$ is diffeomorphic to the product of an open disk with a circle.

4.5. Linear Hamiltonian Systems with Periodic Coefficients [30]. Let h be a quadratic Hamiltonian whose coefficients depend continuously on the time t and are periodic in t with a common period. To the Hamiltonian h corresponds the linear Hamiltonian system with periodic coefficients

$$\dot{q}=\partial h/\partial p, \quad \dot{p}=-\partial h/\partial q. \tag{2}$$

Such systems are encountered in the investigation of the stability of periodic solutions of nonlinear Hamiltonian systems, in automatic control theory, and in questions of parametric resonance.

We shall call the system (2) stable if all of its solutions are bounded as $t\to\infty$, and strongly stable if all nearby linear Hamiltonian systems with periodic coefficients are also stable (nearness is to be understood in the sense of the norm $\max_t \|h(t)\|$).

Two strongly stable Hamiltonian systems will be called homotopic if they can be deformed continuously into one another while remaining within the class of strongly stable systems of the form (2).

The homotopy relation partitions all strongly stable systems (2) of order $2n$ into classes. It turns out that the homotopy classes are naturally indexed by the 2^n collections of n $\pm$ signs and by one more integer parameter. Here the number 2^n shows up as the ratio of the orders of the Weyl groups $W_{\mathbb{C}}$ and W, and the rôle of the integer parameter is played by the element of the fundamental group $\pi_1(\mathrm{Sp}(2n, \mathbb{R}))$.

Let us consider the mapping G_t: $\mathbb{R}^{2n}\to\mathbb{R}^{2n}$ which associates to an initial condition $x(0)$ the value of the solution $x(t)$ of equation (2), with this initial condition, at the moment of time t. We obtain a continuously differentiable curve G_t in the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$, which uniquely determines the original system of equations. The curve G_t begins at the identity element of the group: $G_0=E$, and if t_0 is the period of the Hamiltonian h, then $G_{t+t_0}=G_tG_{t_0}$. The transformation $G=G_{t_0}$ is called the monodromy operator of the system (2). Stability and strong stability of the system (2) are properties of its monodromy operator.

Theorem A. *The system* (2) *is stable if and only if its monodromy operator is diagonalizable and all of its eigenvalues lie on the unit circle.*

In fact, the stability of the system (2) is equivalent to the boundedness of the cyclic group $\{G^m\}$ generated by the monodromy operator. The latter condition means that the closure of this group in $\mathrm{Sp}(2n, \mathbb{R})$ is compact, i.e. the monodromy operator lies in some torus $T^n\subset \mathrm{Sp}(2n, \mathbb{R})$.

We may consider T^n as the diagonal subgroup of the group of unitary transformations of the space $\mathbb{C}^n$. Then the monodromy operator of a stable system takes the form $G=\mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, $|\lambda_k|=1$.

Theorem B. *The system* (2) *is strongly stable if and only if there are no relations of the form $\lambda_k\lambda_l=1$ between the numbers λ_k.*

Remark. If the monodromy operator of a stable system has a nonmultiple spectrum, then the system is strongly stable. Multiplicity of the spectrum means that $\lambda_k=\lambda_l$ or $\lambda_k\lambda_l=1$. These equations single out the stationary points of the transformations belonging to the Weyl group $W_{\mathbb{C}}$ on the torus T^n. The stationary points of the transformations belonging to the Weyl group W are given by the equations $\lambda_k=\lambda_l$. Using the Cayley parametrization one may verify that the partition into conjugacy classes in the neighbourhood of the monodromy operator $G\in T^n$ is organized in the same way as is the partition into equivalence classes of quadratic Hamiltonians in the neighbourhood of the Hamiltonian $h=\sum\omega_k(p_k^2+q_k^2)/2$. The relations $\lambda_k=\lambda_l\neq\pm1$ correspond in this connection to multiple invariants of the same sign: $\omega_k=\omega_l\neq0$, and the relations $\lambda_k\lambda_l=1$ to invariants of different signs: $\omega_k+\omega_l=0$. The remark in sect. 3.4 explains why strong stability is violated only in the second case.

Among the eigenvalues $\lambda_k^{\pm1}$ of the monodromy operator of a strongly stable system, let us choose those which lie on the upper semicircle $\mathrm{Im}\,\lambda>0$. We obtain a well-defined sequence of n exponents ±1. Under deformations of the system (2) within the class of strongly stable systems this sequence does not change: because of the relations $\lambda_k\lambda_l\neq1$ the eigenvalues can neither come down from the semicircle, nor change places if they have exponents of different signs.

Theorem C. *The monodromy operators of strongly stable systems* (2) *form an open set* St_n *in the symplectic group* $\mathrm{Sp}(2n, \mathbb{R})$, *consisting of 2^n connected components corresponding to the 2^n different sequences of exponents.*

In Fig. 7 the set St_1 in the group $\mathrm{Sp}(2, \mathbb{R})$ is depicted. In the general case nearly the entire boundary of the set St_n consists of nonstable operators. The stable but not strongly stable monodromy operators also lie on the boundary and form a set of codimension 3 in the symplectic group. At such points the boundary has a singularity (in the simplest case, a singularity like that of the quadratic cone in $\mathbb{R}^3$, Fig. 6). The singularities of the boundary of the set of strong stability along strata of codimension 2 can be seen in Fig. 5b, d.

Theorem D ([30]). *Each component of the set* St_n *is simply connected.*

Under a homotopy of the system (2) the curve G_t, $t\in[0, t_0]$, with its beginning at the identity element and its end at the point corresponding to the monodromy

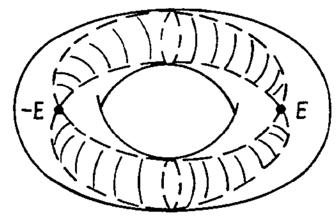

Fig. 7. The subset of stable operators in $\mathrm{Sp}(2, \mathbb{R})$

operator, is deformed continuously in the symplectic group. Conversely, to homotopic curves G_t correspond homotopic systems (2).

Theorem E ([30]). *The homotopy classes of systems* (2) *whose monodromy operators lie in the same component of the set* St_n *as the monodromy operator* G *of a given system are in one-to-one correspondence with the elements of the fundamental group* $\pi_1(\mathrm{Sp}(2n, \mathbb{R})) = \mathbb{Z}$ (to a system (2) with the same monodromy operator G is associated in the fundamental group the class of the closed curve formed in $\mathrm{Sp}(2n, \mathbb{R})$, Fig. 8).

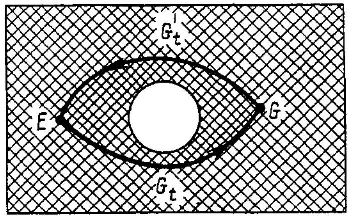

Fig. 8. Nonhomotopic systems with the same monodromy operator

Remark. Essentially we have described the relative homotopy "group" $\pi = \pi_1(\mathrm{Sp}(2n, \mathbb{R}), \mathrm{St}_n)$ in terms of the exact homotopy sequence

$$\pi_1(\mathrm{St}_n) \to \pi_1(\mathrm{Sp}(2n, \mathbb{R})) \to \pi \to \pi_0(\mathrm{St}_n) \to \pi_0(\mathrm{Sp}(2n, \mathbb{R})),$$

where $\pi_0(\mathrm{Sp}(2n, \mathbb{R})) = \{0\}$ (the symplectic group is connected), $\pi_1(\mathrm{St}_n) = \{0\}$ (Theorem D), $\pi_1(\mathrm{Sp}(2n, \mathbb{R})) = \mathbb{Z}$, $\# \pi_0(\mathrm{St}_n) = 2^n$ (Theorem C).

Chapter 2

Symplectic Manifolds

§ 1. Local Symplectic Geometry

1.1. The Darboux Theorem. By a *symplectic structure* on a smooth even-dimensional manifold we mean a closed nondegenerate differential 2-form on it. A manifold equipped with a symplectic structure is called a *symplectic manifold.* A diffeomorphism of symplectic manifolds which takes the symplectic structure of one over into the symplectic structure of the other is called a symplectic transformation or a *symplectomorphism.*[3]

The tangent space at each point of a symplectic manifold is a symplectic vector space. The closedness condition in the definition of symplectic structure connects the skew-scalar products in the tangent spaces at neighbouring points in such a way that the local geometry of symplectic manifolds turns out to be universal.

The Darboux Theorem. *Symplectic manifolds of the same dimension are locally symplectomorphic.*

Corollary. *In the neighbourhood of an arbitrary point, a symplectic structure on a smooth manifold has the form* $dp_1 \wedge dq_1 + \ldots + dp_n \wedge dq_n$ *under a suitable choice of local coordinates* $p_1, \ldots, p_n, q_1, \ldots, q_n$.

The condition of nondegeneracy is worthy of special discussion. Its absence in the definition of a symplectic structure would make the local classification of such structures boundless. Nevertheless in the case of degeneracies of constant rank the answer is simple: a closed differential 2-form of constant corank k has, in suitable local coordinates $p_1, \ldots, p_m, q_1, \ldots, q_m, x_1, \ldots, x_k$, the form $dp_1 \wedge dq_1 + \ldots + dp_m \wedge dq_m$.

1.2. Example. The Degeneracies of Closed 2-Forms on $\mathbb{R}^4$. Let ω be a generic closed differential 2-form on a 4-dimensional manifold.

a) At a generic point of the manifold the form ω is nondegenerate and can be reduced in a neighbourhood under a suitable choice of coordinates to the Darboux form $dp_1 \wedge dq_1 + dp_2 \wedge dq_2$.

b) At the points of a smooth three-dimensional submanifold the form ω has rank 2. At a generic point this submanifold is transversal to the two-dimensional kernel of the form ω. In a neighbourhood of such a point ω can be reduced to the form $p_1 dp_1 \wedge dq_1 + dp_2 \wedge dq_2$.

c) The next degeneracy of the generic form ω occurs at the points of a smooth curve on our three-dimensional submanifold. At a generic point of the curve the two-dimensional kernel of the form ω is tangent to the three-dimensional manifold, but transversal to this curve. In a neighbourhood of such a point the form ω can be reduced to one of the two forms $d(x - z^2/2) \wedge dy + d(xz \pm ty - z^3/3) \wedge dt$. The field of kernels of the form ω cuts out a field of directions on the three-dimensional manifold. The field lines corresponding to the $+$ sign in the normal form are depicted in Fig. 9. With the $-$ sign the spiral rotation is replaced by a hyperbolic turning (see [51], [62], [7]).

d) The hyperbolic and elliptic sections of our curve are separated by parabolic points, at which the two-dimensional kernel of the form ω is tangent both to the three-dimensional manifold and to the curve itself. Here is known only that there

[3] In the literature the traditional name "canonical transformation" is also used.

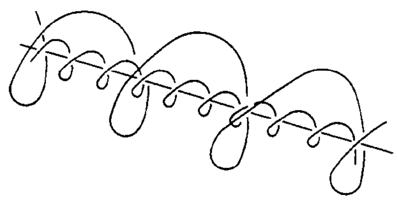

Fig. 9. A "magnetic field" connected with a degenerate structure on $\mathbb{R}^4$

exists at least one modulus—a continuous numeric parameter which distinguishes inequivalent degeneracies of the form in a neighbourhood of the parabolic point [35].

e) All more profound degeneracies of the form ω (for example, its turning to zero at isolated points) are removable by a small perturbation within the class of closed 2-forms.

1.3. Germs of Submanifolds of Symplectic Space. Here we shall discuss under what conditions two germs of smooth submanifolds of symplectic space can be moved into one another by a local diffeomorphism of the ambient space which preserves the symplectic structure. Germs of submanifolds for which this is possible will be called equivalent. The restriction of the symplectic structure of the ambient space to a submanifold defines on it a closed 2-form, possibly a degenerate one. For equivalent germs these degeneracies are identical, in other words, their intrinsic geometry coincides. If this requirement is fulfilled, then there exists a local diffeomorphism of the ambient space taking two submanifold germs over into one another together with the restrictions onto them of the symplectic structure of the ambient space, but not necessarily preserving this structure itself. Thus we may consider that we have one submanifold germ and two symplectic structures in a neighbourhood of the submanifold which coincide upon restriction to it. Two germs of submanifolds of Euclidean space with the same intrinsic geometry may have a different exterior geometry. In symplectic space this is not so.

The Relative Darboux Theorem I. *Let there be given a germ of a smooth submanifold at the coordinate origin of the space $\mathbb{R}^{2n}$ and two germs of symplectic structures ω_0 and ω_1 in a neighbourhood of the origin whose restrictions to this submanifold coincide. Then there exists a germ of a diffeomorphism of the space $\mathbb{R}^{2n}$ which is the identity on the submanifold and which takes ω_0 over into ω_1.*

If the submanifold is a point, we obtain the Darboux theorem of sect. 1.1.

Proof. We apply the homotopic method. One may assume that the submanifold is a linear subspace X and that the differential forms ω_0 and ω_1 coincide at the origin (the latter follows out of the linear "relative Darboux theorem", sect. 1.2 of chap. 1). Then the $\omega_t = (1-t)\omega_0 + t\omega_1$ are symplectic structures in a neighbourhood of the origin for all $t \in [0, 1]$. We shall look for a family of diffeomorphisms taking ω_t into ω_0 and being the identity on X, or, what is equivalent, for a family V_t of vector fields equal to zero on X and satisfying the homological equation $L_{V_t}\omega_t + (\omega_1 - \omega_0) = 0$ (here L_V is the Lie derivative). Since the forms ω_t are closed, we may pass over to the equation $i_{V_t}\omega_t + \alpha = 0$, where $i_V\omega$ is the inner product of the field and the form and α is a 1-form defined by the condition $d\alpha = \omega_1 - \omega_0$ uniquely up to addition of the differential of a function. In view of the nondegeneracy of the symplectic structures ω_t this equation is uniquely solvable for an arbitrary 1-form α. Therefore it remains for us to show that the form α can be taken to be zero at the points of the subspace X. Let $x_1 = \ldots = x_k = 0$ be the equations of X and let $y_1, \ldots, y_{2n-k}$ be the remaining coordinates on $\mathbb{R}^{2n}$. Since the form $\omega_1 - \omega_0$ is equal to zero on X, then $\alpha = \sum (x_i\alpha_i + f_i dx_i) + df$, where the α_i are 1-forms and the f_i and f are functions, f depending only on y. Consequently, we may replace α by the form $\sum x_i(\alpha_i - df_i) = \alpha - d(f + \sum f_i x_i)$, equal to zero at the points of X. □

1.4. The Classification of Submanifold Germs. The relative Darboux theorem allows us to transform information on the degeneracies of closed 2-forms into results on the classification of germs of submanifolds of symplectic space. Thus, the degeneracies of closed 2-forms on $\mathbb{R}^4$ enumerated in sect. 1.2 can be realized as the restriction of the standard symplectic structure on $\mathbb{R}^6$ to the germs at 0 of the following 4-dimensional submanifolds ($p_1, p_2, p_3, q_1, q_2, q_3$ are the Darboux coordinates in $\mathbb{R}^6$):

a′) $p_3 = q_3 = 0$;
b′) $q_3 = 0,\ p_1 = p_3^2/2$;
c′) $p_2 = q_1 q_2,\ p_3 = p_1 q_2 \pm q_1 q_3 - q_2^3/3$.

Theorem. *The germ of a generic smooth 4-dimensional submanifold in 6-dimensional symplectic space can be reduced by a local symplectic change of coordinates to the form in* a′) *at a general point, in* b′) *at the points of a smooth 3-dimensional submanifold, in* c′) *at points on a smooth curve, and is unstable at isolated (parabolic) points.*

Here we have run into the realization question: what is the smallest dimension of a symplectic space in which a given degeneracy of a closed 2-form can be realized as the restriction of the symplectic structure onto a submanifold? The answer is given by

The Extension Theorem. *A closed 2-form on a submanifold of an even-dimensional manifold can be extended to a symplectic structure in a neighbourhood of some point if and only if the corank of the form at this point does not exceed the codimension of the submanifold. The operation of extension can be made continuous in the C^∞ topology (i. e., to nearby forms one may associate nearby extensions).*

1.5. The Exterior Geometry of Submanifolds. We shall cite global analogues of the preceding theorems.

The Relative Darboux Theorem II. *Let M be an even-dimensional manifold, N a submanifold, and let ω_0 and ω_1 be two symplectic structures on M whose restrictions to N coincide. Let us suppose that ω_0 and ω_1 can be continuously deformed into one another in the class of symplectic structures on M which coincide with them on N. Then there exist neighbourhoods U_0 and U_1 of the submanifold N in M and a diffeomorphism $g: U_0 \to U_1$ which is the identity on N and which takes $\omega_1|_{U_1}$ over into $\omega_0|_{U_0}$: $g^*\omega_1 = \omega_0$.*

The distinctive difficulty in the global case consists in clarifying whether the structures ω_0 and ω_1 are homotopic in the sense indicated above: the linear combination $t\omega_1 + (1-t)\omega_0$ can become degenerate along the way. There exist examples which show that this condition can not be neglected, even if one does not require the diffeomorphism g to be the identity on N. The homotopy exists if the symplectic structures ω_0 and ω_1 coincide not only on vectors tangent to N, i.e. on TN, but on all vectors tangent to M and applied at points of N, i.e. on $T_N M$. In this case the linear combination $t\omega_1 + (1-t)\omega_0$ will for all $t \in [0, 1]$ be nondegenerate at the points of N and, consequently, in some neighbourhood of N in M.

The proof of the global theorem is completely analogous to the one cited above for the local version. It is only necessary that in place of the "integration by parts" — the coordinate argument in the concluding part of the proof — one use the following lemma:

The Relative Poincaré Lemma. *A closed differential k-form on M equal to 0 on TN can be represented in a tubular neighbourhood of N in M as the differential of a $k-1$-form equal to 0 on $T_N M$.*

The basis of the proof of this lemma is the conical contraction of the normal bundle onto the zero section (see [73]).

The Extension Theorem II. *Let N be a submanifold of M and on the fibres of the bundle $T_N M \to N$ let there be given a smooth field of nondegenerate exterior 2-forms whose restriction to the subbundle TN defines a closed 2-form on N. Then this field of forms can be extended to a symplectic structure on a neighbourhood of the submanifold N in M.*

The question remains open of the dimension of a symplectic manifold M in which a manifold N given together with a closed 2-form can be realized as a submanifold. In this direction we cite the following result.

The Extension Theorem III. *Any manifold N together with a closed differential 2-form ω can be realized as a submanifold in a symplectic manifold M of dimension* $2\dim N$.

For the manifold M it is sufficient to take the cotangent bundle T^*N. The projection $\pi: T^*N \to N$ determines a closed 2-form $\pi^*\omega$ on T^*N, but obviously a degenerate one. It turns out that on the cotangent bundle there exists a canonical symplectic structure which is equal to 0 on the zero section and in sum with $\pi^*\omega$ is again nondegenerate. We shall begin the next section with the description of the canonical structure.

1.6. The Complex Case. The definition of a symplectic structure and the Darboux theorem can be carried over verbatim to the case of complex analytic manifolds. The same applies to the content of sect. 1.3 and extension theorem III. Whether the remaining results of § 1 are true in the complex analytic category is not known.

§ 2. Examples of Symplectic Manifolds

In this section we discuss three sources of examples of symplectic manifolds—cotangent bundles, complex projective manifolds and orbits of the coadjoint action of Lie groups[4].

2.1. Cotangent Bundles. Let us define a *canonical symplectic structure* on the space T^*M of the cotangent bundle of an arbitrary (real or complex) manifold M. First we shall introduce on T^*M the differential *1-form of action* α. A point of the manifold T^*M is defined by giving a linear functional $p \in T_x^* M$ on the tangent space $T_x M$ to M at some point $x \in M$. Let ξ be a tangent vector to T^*M applied at the point p (Fig. 10). The projection $\pi: T^*M \to M$ determines a tangent vector $\pi_* \xi$ to M, applied at the point x. Let us now set $\alpha_p(\xi) = p(\pi_* \xi)$. In local

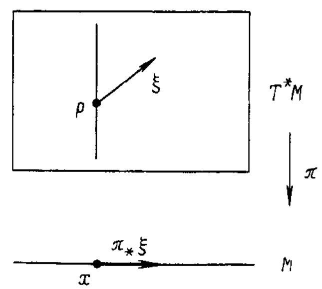

Fig. 10. The definition of the action 1-form

[4] For other constructions which lead to symplectic manifolds (for example, manifolds of geodesics) see sects. 1.5 and 3.2 of chap. 3.

coordinates (q, p) on T^*M, where $p_1, \ldots, p_n$ are coordinates on T_x^*M dual to the coordinates $dq_1, \ldots, dq_n$ on T_xM, the 1-form α has the form $\alpha = \sum p_k dq_k$. Therefore the differential 2-form $\omega = d\alpha$ gives a symplectic structure on T^*M. And just this is our canonical structure.

Cotangent bundles are constantly exploited by classical mechanics as phase spaces of Hamiltonian systems[5]. The base manifold M is called the configuration space, and the functional p is called the generalized momentum of a mechanical system having the "configuration" $x = \pi(p)$. An example: the configurations of a rigid body fixed at a point form the manifold SO(3). The generalized momentum is the three-dimensional vector of angular momentum of the body relative to the fixed point.

2.2. Complex Projective Manifolds. The points of complex projective space $\mathbb{CP}^n$ are one-dimensional subspaces in $\mathbb{C}^{n+1}$. A complex projective manifold is a nonsingular subvariety in $\mathbb{CP}^n$ consisting of the common zeroes of a system of homogeneous polynomial equations in the coordinates $(z_0, \ldots, z_n)$ of $\mathbb{C}^{n+1}$. It turns out that on an arbitrary k-dimensional complex projective manifold, regarded as a $2k$-dimensional real manifold, there exists a symplectic structure. It is constructed in this way. First a symplectic structure is introduced on $\mathbb{CP}^n$ as the imaginary part of a Hermitian metric on $\mathbb{CP}^n$ (compare sect. 1.1 of chap. 1). The restriction of this symplectic structure to a projective manifold $M \subset \mathbb{CP}^n$ gives a closed 2-form on M. It is the imaginary part of the restriction to M of the original Hermitian metric on $\mathbb{CP}^n$, from which its nondegeneracy follows. Therefore all that remains is to produce the promised Hermitian metric on $\mathbb{CP}^n$.

For this let us consider the Hermitian form $\langle\, ,\, \rangle$ on the space $\mathbb{C}^{n+1}$. The tangent space at a point of the manifold $\mathbb{CP}^n$ can be identified with the Hermitian orthogonal complement of the corresponding line in $\mathbb{C}^{n+1}$ uniquely up to multiplication by $e^{i\phi}$. Therefore the restriction of the form $\langle\, ,\, \rangle$ onto this orthogonal complement uniquely determines a Hermitian form on the tangent space to $\mathbb{CP}^n$. The Hermitian metric constructed in this way on $\mathbb{CP}^n$ is invariant with respect to the unitary group U_{n+1} of the space $\mathbb{C}^{n+1}$. Therefore the imaginary part ω of our Hermitian metric and its differential $d\omega$ are U_{n+1}-invariant. In particular, the form $d\omega$ is invariant with respect to the stabilizer U_n of an arbitrary point in $\mathbb{CP}^n$, which acts on the tangent space to this point. Since the group U_n contains multiplication by -1, any U_n-invariant exterior 3-form on the realification of the space $\mathbb{C}^n$ is equal to zero, from which it follows that the differential 2-form ω is closed.

In explicit form, let $\langle z, z \rangle = \sum z_k \bar{z}_k$ and let $w_k = z_k / z_0$ be an affine chart on $\mathbb{CP}^n$. Then the form ω is proportional to

$$\frac{\sqrt{-1}}{2\pi}\, \partial\bar{\partial} \ln \sum_{k=0}^{n} |w_k|^2,$$

[5] See volume 3 of the present publication.

where ∂, $\bar{\partial}$ are the differentials with respect to the holomorphic and anti-holomorphic coordinates respectively. The coefficient is chosen so that the integral over the projective line $\mathbb{CP}^1 \subset \mathbb{CP}^n$ will be equal to 1.

2.3. Symplectic and Kähler Manifolds. The complex projective manifolds form a subclass of the class of Kähler manifolds. By a Kähler structure on a complex manifold is meant a Hermitian metric on it whose imaginary part is closed, i.e. is a symplectic structure. Just as in the preceding item, a complex submanifold of a Kähler manifold is itself Kähler. Kähler manifolds have characteristic geometric properties. In particular, the Hodge decomposition holds in the cohomology groups of a compact Kähler manifold M:

$$H^k(M, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}, \qquad \bar{H}^{p,q} = H^{q,p},$$

where $H^{p,q}$ consists of the classes representable by complex-valued closed differential k-forms on M of type (p, q). The latter means that in the basis $dz_1, \ldots, dz_n, d\bar{z}_1, \ldots, d\bar{z}_n$ of the complexified cotangent space the form appears as a linear combination of exterior products of p of the dz_i and q of the $d\bar{z}_j$.

On the other hand, a symplectic structure on a manifold can always be strengthened to a quasi-Kähler structure, i.e. a complex structure on the tangent bundle and a Hermitian metric whose imaginary part is closed. This follows from the contractibility of the structure group $Sp(2n, \mathbb{R})$ of the tangent bundle of a symplectic manifold to the unitary group U_n.

Example (W. Thurston, [73]). There exist compact symplectic manifolds which do not possess a Kähler structure. Let us consider in the standard symplectic space $\mathbb{R}^4$ with coordinates p_1, q_1, p_2, q_2 the action of the group generated by the following symplectomorphisms:

$$a\colon q_2 \mapsto q_2 + 1, \qquad b\colon p_2 \mapsto p_2 + 1, \qquad c\colon q_1 \mapsto q_1 + 1,$$
$$d\colon (p_1, q_1, p_2, q_2) \mapsto (p_1 + 1, q_1, p_2, q_2 + p_2).$$

In another way this action may be described as the left translation in the group G of matrices of the form (1) by means of elements of the discrete subgroup $G_{\mathbb{Z}}$ of integer matrices.

$$\left(\begin{array}{ccc|cc} 1 & p_1 & q_2 & & \\ 0 & 1 & p_2 & \multicolumn{2}{c}{0} \\ 0 & 0 & 1 & & \\ \hline & & & 1 & q_1 \\ & 0 & & 0 & 1 \end{array}\right) \qquad (1)$$

Therefore the quotient space $M = G/G_{\mathbb{Z}}$ is a smooth symplectic manifold. The functions q_1, p_2 (mod $\mathbb{Z}$) give a mapping $M \to T^2$ which is a fibration over the torus T^2 with fibre T^2, therefore M is compact.

The fundamental group $\pi_1(M)$ is isomorphic to $G_{\mathbb{Z}}$, and the group $H_1(M, \mathbb{Z}) \cong G_{\mathbb{Z}}/[G_{\mathbb{Z}}, G_{\mathbb{Z}}]$. The commutator group $[G_{\mathbb{Z}}, G_{\mathbb{Z}}]$ is generated by the element $bdb^{-1}d^{-1} = a$; therefore $\dim_{\mathbb{C}} H^1(M, \mathbb{C}) = 3$—the dimension of the one-dimensional cohomology space of M is odd! But from the Hodge decomposition it follows that for a Kähler manifold the cohomology spaces in the odd dimensions are even-dimensional.

2.4. The Orbits of the Coadjoint Action of a Lie Group. Let G be a connected Lie group, $\mathfrak{g} = T_eG$ its Lie algebra. The action of the group on itself by conjugations has the fixed point $e \in G$—the identity element of the group. The differential of this action defines the adjoint representation Ad: $G \to \mathrm{GL}(T_eG)$ of the group on its Lie algebra. The dual representation Ad*: $G \to \mathrm{GL}(T_e^*G)$ on the dual space of the Lie algebra is called the *coadjoint representation of the group*. The corresponding adjoint representation ad: $\mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ and coadjoint representation ad*: $\mathfrak{g} \to \mathrm{End}(\mathfrak{g}^*)$ of the Lie algebra are given explicitly by the formulas

$$\mathrm{ad}_x y = [x, y], \qquad x, y \in \mathfrak{g},$$

$$(\mathrm{ad}_x^* \xi)|y = \xi([y, x]), \qquad x, y \in \mathfrak{g},\ \xi \in \mathfrak{g}^*,$$

where [,] is the commutator in the Lie algebra $\mathfrak{g}$.

Theorem. *Every orbit of the coadjoint action of a Lie group possesses a symplectic structure.*

It is constructed in the following manner. The mapping $x \mapsto \mathrm{ad}_x^* \xi$ identifies the tangent space to the orbit of the coadjoint representation at the point $\xi \in \mathfrak{g}^*$ with the space $\mathfrak{g}/\mathfrak{g}_\xi$, where $\mathfrak{g}_\xi = \{x \in \mathfrak{g} \mid \mathrm{ad}_x^* \xi = 0\}$ is the annihilator of the functional ξ. On the space $\mathfrak{g}/\mathfrak{g}_\xi$ there is well defined the nondegenerate skew-scalar product $\xi([x, y])$. The closedness of the thus obtained symplectic form on the orbit follows from the Jacobi identity on the Lie algebra.

Corollary. *The orbits of the coadjoint action are even-dimensional.*

Example. Let $G = \mathrm{U}_{n+1}$. The adjoint representation is in this case isomorphic to the coadjoint one; therefore the orbits of the adjoint action have a symplectic structure. The Lie algebra of the unitary group consists of the skew-Hermitian operators on $\mathbb{C}^{n+1}$. The orbit of a skew-Hermitian operator of rank 1 is isomorphic to $\mathbb{CP}^n$, and we get a new definition of the symplectic structure on complex projective space introduced in sect. 2.2.

The symplectic structure on the orbits of the coadjoint action plays an important rôle in the theory of Lie groups and their representations (see [41], [44]).

§ 3. The Poisson Bracket

A symplectic structure on a manifold allows one to introduce a Lie-algebra structure—the Poisson bracket—on the space of smooth functions on this manifold. In sect. 3.1 this construction is set forth and the properties of the "conservation laws" of classical mechanics are formulated in the language of the Poisson bracket. The remainder of the section is devoted to an important generalization of the concept of a symplectic structure, which takes as its foundation the properties of the Poisson bracket.

3.1. The Lie Algebra of Hamiltonian Functions. Let M be a symplectic manifold. The skew-scalar product ω gives an isomorphism I: $T^*M \to TM$ of the cotangent and tangent bundles, i.e. a correspondence between differential 1-forms and vector fields on M, according to the rule $\omega(\cdot, I\xi) = \xi(\cdot)$.

Let H be a smooth function on M. The vector field IdH is called the *Hamiltonian field with Hamilton function* (or *Hamiltonian*) H. This terminology is justified by the fact that in Darboux coordinates the field IdH has the form of a *Hamiltonian system of equations*: $\dot p = -\partial H/\partial q$, $\dot q = \partial H/\partial p$.

Theorem A. *The phase flow of a smooth vector field on M preserves the symplectic structure if and only if the vector field is locally Hamiltonian.*

Corollary (Liouville's Theorem). *A Hamiltonian flow preserves the phase volume* $\omega \wedge \ldots \wedge \omega$.

The vector fields on a manifold form a Lie algebra with respect to the operation of commutation. Theorem A signifies that the Lie subalgebra of vector fields whose flows preserve the symplectic structure goes over into the space of closed 1-forms under the isomorphism I^{-1}.

Theorem B. *The commutator in the Lie algebra of closed 1-forms on M has the form* $[\alpha, \beta] = d\omega(I\alpha, I\beta)$.

Corollary 1. *The commutator of two locally Hamiltonian fields v_1 and v_2 is a Hamiltonian field with the Hamiltonian* $\omega(v_1, v_2)$.

Let us define the *Poisson bracket* { , } on the space of smooth functions on M by the formula $\{H, F\} = \omega(IdH, IdF)$. In Darboux coordinates $\{H, F\} = \sum(\partial H/\partial p_k \partial F/\partial q_k - \partial F/\partial p_k \partial H/\partial q_k)$.

Corollary 2. *The Hamiltonian functions form a Lie algebra with respect to the Poisson bracket, i.e. bilinearity holds, as do anticommutativity* $\{H, F\} = -\{F, H\}$, *and the* Jacobi identity $\{F, \{G, H\}\} + \{G, \{H, F\}\} + \{H, \{F, G\}\} = 0$. *The Leibniz formula is valid:* $\{H, F_1F_2\} = \{H, F_1\}F_2 + F_1\{H, F_2\}$.

The application of these formulas to Hamiltonian mechanics is based on the following obvious fact.

Theorem C. *The derivative of a function F along a vector field with a Hamiltonian H is equal to the Poisson bracket* $\{H, F\}$.

Corollaries. a) The law of conservation of energy: *a Hamiltonian function is a first integral of its Hamiltonian flow.*

b) The theorem of E. Noether: *A Hamiltonian F whose flow preserves the Hamiltonian H is a first integral of the flow with Hamiltonian H.*

c) The Poisson theorem: *The Poisson bracket of two first integrals of a Hamiltonian flow is again a first integral.*

Example. If in a mechanical system two components M_1, M_2 of the angular momentum vector are conserved, then also the third one $M_3=\{M_1, M_2\}$ will be conserved.

3.2. Poisson Manifolds. A *Poisson structure* on a manifold is a bilinear form $\{\ ,\ \}$ on the space of smooth functions on it satisfying the requirement of anticommutativity, the Jacobi identity and the Leibniz rule (see corollary 2 of the preceding item). This form we shall still call a *Poisson bracket*. The first two properties of the Poisson bracket mean that it gives a Lie algebra structure on the space of smooth functions on the manifold. From the Leibniz rule it follows that the Poisson bracket of an arbitrary function with a function having a second-order zero at a given point vanishes at this point; therefore a Poisson structure defines an exterior 2-form on each cotangent space to the manifold, depending smoothly on the point of application. Conversely, the value of such a 2-form on a pair df, dg of differentials of functions defines a skew-symmetric biderivation $(f, g)\mapsto W(f, g)$ of the functions, that is, a bilinear skew-symmetric operation satisfying the Leibniz identity with respect to each argument.

On a manifold let there be given two smooth fields V and W of exterior 2-forms on the cotangent bundle. We designate as their *Schouten bracket* $[V, W]$ the smooth field of trilinear forms on the cotangent bundle defined by the following formula

$$[V, W](f, g, h)=V(f, W(g, h))+W(f, V(g, h))+\dots,$$

where the dots denote the terms with cyclically permuted f, g and h.

Lemma. *A smooth field W of exterior 2-forms on the cotangent spaces to a manifold gives a Poisson structure on it if and only if* $[W, W]=0$.

In coordinates a Poisson structure is given by a tensor $\sum w_{ij}(\partial/\partial x_i)\wedge(\partial/\partial x_j)$, where the w_{ij} are smooth functions satisfying the conditions: for all i, j, k $w_{ij}=-w_{ji}$ and

$$\sum_l (w_{lj}\partial w_{ik}/\partial x_l+w_{li}\partial w_{kj}/\partial x_l+w_{lk}\partial w_{ji}/\partial x_l)=0.$$

A Poisson structure, like a symplectic one, defines a homomorphism of the Lie algebra of smooth functions into the Lie algebra of vector fields on the manifold: the derivative of a function g along the field of the function f is equal to $\{f, g\}$. Such fields are called Hamiltonian; their flows preserve the Poisson structure. Hamiltonians to which zero fields correspond are called *Casimir functions* and they form the centre of the Lie algebra of functions. In contrast to the nondegenerate symplectic case, the centre need not consist only of locally constant functions.

The following theorem aids in understanding the structure of Poisson manifolds and their connection with symplectic ones. Let us call two points of a Poisson manifold equivalent if there exists a piecewise smooth curve joining them, each segment of which is a trajectory of a Hamiltonian vector field. The vectors of Hamiltonian fields generate a tangent subspace at each point of the Poisson manifold. Its dimension is called *the rank of the Poisson structure* at the given point and is equal to the rank of the skew-symmetric 2-form defined on the cotangent space.

The Foliation Theorem ([42], [74]). *The equivalence class of an arbitrary point of a Poisson manifold is a symplectic submanifold of a dimension equal to the rank of the Poisson structure at that point.*

Thus, a Poisson manifold breaks up into *symplectic leaves*, which in aggregate determine the Poisson structure: the Poisson bracket of two functions can be computed over their restrictions to the symplectic leaves. A transversal to a symplectic leaf at any point intersects the neighbouring symplectic leaves transversally along symplectic manifolds and inherits a Poisson structure in a neighbourhood of the original point.

The Splitting Theorem ([74]). *The germ of a Poisson manifold at any point is isomorphic (as a Poisson manifold) to the product of the germ of the symplectic leaf with the germ of the transversal Poisson manifold of this point. The latter is uniquely determined up to isomorphism of germs of Poisson manifolds.*

This theorem reduces the study of Poisson manifolds in the neighbourhood of a point to the case of a point of rank zero.

3.3. Linear Poisson Structures. A Poisson structure on a vector space is called *linear* if the Poisson bracket of linear functions is again linear. A linear Poisson structure on a vector space is precisely a Lie algebra structure on the dual space; the symplectic leaves of the linear structure are the orbits of the coadjoint action of this Lie algebra, the Casimir functions are the invariants of this action.

At a point x of rank 0 on a Poisson manifold there is well defined the *linear approximation of the Poisson structure*—a linear Poisson structure on the tangent space of this point (or a Lie algebra structure on the cotangent space): $[d_x f, d_x g]=d_x\{f, g\}$.

The Annihilator Theorem. *Let $\mathfrak{g}$ be a Lie algebra, let $\xi \in \mathfrak{g}^*$, and let $\mathfrak{g}_\xi = \{x \in \mathfrak{g} \mid \mathrm{ad}_x^* \xi = 0\}$ be the annihilator of ξ in $\mathfrak{g}$. The linear approximation of the transversal Poisson structure to the orbit of the coadjoint action at the point ξ is canonically isomorphic to the linear Poisson structure on the dual space of the annihilator $\mathfrak{g}_\xi$.*

The isomorphism is given by the mapping $\mathfrak{g}^*/\mathrm{ad}_{\mathfrak{g}}^*\xi \to \mathfrak{g}_\xi^*$ of the spaces of definition of the linear Poisson structures being regarded which is dual to the inclusion $\mathfrak{g}_\xi \subsetneq \mathfrak{g}$.

Corollary 1. *Each element of a semisimple Lie algebra of rank r is contained in an r-dimensional commutative subalgebra.*

Proof. 1°. One of the possible definitions of a semisimple Lie algebra consists in the nondegeneracy of its Killing form $\langle x, y\rangle = \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y)$. This form is invariant with respect to the adjoint action; therefore the adjoint representation of a semisimple Lie algebra is isomorphic to the coadjoint one.

2°. By the rank of a Lie algebra is meant the codimension of a generic orbit of the coadjoint action (i.e. the corank of its Poisson structure at a generic point). It follows from the annihilator theorem that the rank of the annihilator $\mathfrak{g}_\xi$ of an arbitrary $\xi \in \mathfrak{g}^*$ is not less than the rank of $\mathfrak{g}$. Indeed, upon linearization the codimension of a generic symplectic leaf can only increase!

3°. **Duflo's Theorem.** *The annihilator of a generic element $\xi \in \mathfrak{g}^*$ is commutative.* In fact, the symplectic leaves of the transversal Poisson structure at a generic point ξ are zero-dimensional, and the theorem follows from 2°. Let us note that the dimension of such an annihilator is equal to the rank of the Lie algebra $\mathfrak{g}$.

4°. Corollary 1 is obtained by an application of 3° to the annihilator $\mathfrak{g}_x$ of the element $x \in \mathfrak{g}$ with respect to the adjoint action, in view of the fact that x lies in the centre of the algebra $\mathfrak{g}_x$. □

Corollary 2. *A linear Hamiltonian system on $\mathbb{R}^{2n}$ has n linearly independent quadratic first integrals.*

Indeed, $\mathrm{sp}(2n, \mathbb{R})$ is a simple Lie algebra of rank n. □

3.4. The Linearization Problem. In connection with Poisson structures this problem can be stated thus: is a Poisson structure isomorphic in a neighbourhood of a point of rank 0 to its linear approximation at that point? All the previous results of the section were equally true both in the smooth and in the holomorphic case, while the answer to this question may depend on the category in which the linearization is carried out.

We shall call a Lie algebra $\mathfrak{g}$ (analytically, C^∞-) sufficient if any Poisson structure whose linear approximation at a point of rank 0 is isomorphic to $\mathfrak{g}^*$ is itself (analytically, C^∞-) isomorphic to $\mathfrak{g}^*$ in a neighbourhood of this point.

This definition is in keeping with the general approach to linearization problems in analysis: linearizability is treated as a property of the linear approximation.

The description of sufficient Lie algebras is an open problem. It is easy to convince oneself that commutative Lie algebras are not sufficient in either of the two indicated senses. The solvable two-dimensional Lie algebra of the group of affine transformations of the line is sufficient in each of them. The semisimple Lie algebra $\mathrm{sl}_2(\mathbb{R})$ of the group of symplectomorphisms of the plane is analytically sufficient, but is not C^∞-sufficient.

Theorem ([17]). *A semisimple (real or complex) Lie algebra is analytically sufficient. A semisimple Lie algebra of a compact group is C^∞-sufficient.*

Let us observe that linearization of a germ of a Poisson structure with a semisimple linear approximation is equivalent to the selection, in the Lie algebra of Hamiltonian functions, of a semisimple complement to its radical—the nilpotent ideal consisting of the functions with a second-order zero at the coordinate origin. Therefore linearizability modulo terms of sufficiently high order follows from the theorem of Levi–Mal'tsev, which asserts the existence of such a complement in the case of finite-dimensional Lie algebras.

§4. Lagrangian Submanifolds and Fibrations

A submanifold of a symplectic manifold (M^{2n}, ω) is called *Lagrangian* if it has dimension n and the restriction to it of the symplectic form ω is equal to 0. Using the terminology of §1 of Chapter 1, one may say that Lagrangianity of a submanifold is just Lagrangianity of its tangent spaces. More generally, a submanifold of a symplectic manifold is called *(co)isotropic*[6] if its tangent spaces are so.

4.1. Examples of Lagrangian Manifolds. 1) A smooth curve on a symplectic surface is Lagrangian. A smooth curve on a symplectic manifold is isotropic and a hypersurface is coisotropic.

2) Let $M = T^*X$, let α be the action 1-form on M and $\omega = d\alpha$ the canonical symplectic structure. A 1-form on a manifold associates to a point of the manifold a covector at that point. We shall speak of the graph of the 1-form to mean the totality of these covectors. The graph of a closed 1-form on X is a Lagrangian submanifold of M. Indeed, the restriction of the symplectic form $\omega = d(pdq)$ to the graph of the 1-form $\xi: q \mapsto p(q)$ is equal to $d(p(q)\,dq) = d\xi = 0$ if ξ is closed. Conversely, a Lagrangian submanifold in M which projects one-to-one onto the base is the graph of a closed 1-form on X. If this 1-form is exact, $\xi = d\phi$, then its potential ϕ is called a *generating function* of the Lagrangian submanifold. This example suggests definition of a generalized function on X as an arbitrary Lagrangian submanifold in T^*X. A fibre of the cotangent bundle is also Lagrangian and corresponds to the "delta function" of the point of application.

[6] Coisotropic submanifolds are also called involutive.

More generally, the covectors applied at the points of a submanifold $Y \subset X$ and vanishing on vectors tangent to Y form a Lagrangian submanifold in T^*X—the "delta function" of the submanifold Y.

3) A locally Hamiltonian vector field on a symplectic manifold M is a Lagrangian submanifold in the tangent bundle TM, whose symplectic structure is given by its identification $I: T^*M \to TM$ with the cotangent bundle. The generating function of such a Lagrangian manifold is the Hamiltonian of the field.

4) Let there be given a symplectomorphism $\gamma: (M_1, \omega_1) \to (M_2, \omega_2)$. Denoting by π_1 and π_2 the projections of the direct product $M_1 \times M_2$ onto the first and second factors, let us define on it the symplectic structure $\pi_1{}^*\omega_1 - \pi_2{}^*\omega_2$. Then the graph $\Gamma \subset M_1 \times M_2$ of the symplectomorphism γ is a Lagrangian submanifold.

These examples illustrate a general principle: the objects of symplectic geometry are represented by Lagrangian manifolds.

From the relative Darboux theorem II (sect. 1.5) we obtain the

Corollary. *A sufficiently small neighbourhood of a Lagrangian submanifold is symplectomorphic to a neighbourhood of the zero section in its cotangent bundle.*

The corank of the restriction of the symplectic form to a (co)isotropic submanifold is equal to its (co)dimension and is therefore constant. Consequently (see sect. 1.1, chap. 2), a germ of a (co)isotropic submanifold reduces in suitable Darboux coordinates to the linear normal form of sect. 1.2, chap. 1.

The symplectic type of an isotropic submanifold $N^k \subset M^{2n}$ in its tubular neighbourhood is determined in a one-one fashion by the equivalence class of the following $2(n-k)$-dimensional symplectic vector bundle with base space N^k: the fibre of this bundle at a point x is the quotient space of the skew-orthogonal complement to the tangent space T_xN by the space T_xN itself. This follows from the results of sect. 1.5.

If a Hamiltonian function is constant on an isotropic submanifold, then its displacements by the flow of this function sweep out an isotropic submanifold. A coisotropic submanifold on the level surface of a Hamilton function is invariant with respect to its flow. These assertions follow from the fact that the one-dimensional kernel of the restriction of the symplectic form to the level hypersurface of a Hamilton function coincides with the direction of the vector of the Hamiltonian field. Properties of this kind will often be used in the sequel without special mention.

4.2. Lagrangian Fibrations. By a *Lagrangian fibration* is meant a smooth locally trivial fibration of a symplectic manifold all fibres of which are Lagrangian.

Example. A cotangent bundle is a Lagrangian fibration.

Let us recall that an affine structure on a manifold is given by an atlas whose transition functions from one chart to another are affine transformations of the coordinate space (parallel translations, linear transformations or compositions of them).

The Affine Structure Theorem. *The fibres of a Lagrangian fibration have a canonical affine structure.*

Proof. Let $\pi: M \to X$ be a Lagrangian fibration. Then the functions on X, considered as Hamiltonians on M, give commuting Hamiltonian flows there. A function with a zero differential at a point $x \in X$ defines a flow which is stationary on the fibre $\pi^{-1}(x)$. Thus, a neighbourhood of any point of the fibre is the local orbit of a fixed-point free action of the additive group of the space T_x^*X. □

The Darboux Theorem for Fibrations. *In suitable local Darboux coordinates (p, q) a Lagrangian fibration is given as the projection onto the q-space along the p-space.*

Proof. The choice of a Lagrangian section of the Lagrangian fibration identifies it (by the construction out of the proof of the preceding theorem) with the cotangent bundle of the base space. It is not hard to verify that this identification is a symplectomorphism. □

The assignment of a cotangent bundle structure on a Lagrangian fibration is called a *polarization.* A Lagrangian fibration together with a polarization is determined uniquely by its action 1-form α in a neighbourhood of a point where α vanishes: the symplectic structure is $d\alpha$, the zero section consists of the points where α vanishes, and the Euler field of homogeneous dilations in the fibres of the cotangent bundle is $-I\alpha$.

We have seen that in a neighbourhood of a point of the total space of the fibration a Lagrangian fibration is of standard structure—equivalent to a cotangent bundle in a neighbourhood of a point of the zero section. The global picture is entirely different. First, the fibres of a Lagrangian fibration are not obliged to be isomorphic as manifolds with an affine structure, second, fibrations with the same fibres can have different global structure, finally, isomorphic affine fibrations are not necessarily isomorphic as Lagrangian fibrations (i.e. with preservation of the symplectic structure).

We shall call a Lagrangian fibration complete if its fibres are complete as affine manifolds[7]. In particular, a Lagrangian fibration with a compact fibre is complete.

The Classification of Lagrangian Fibres. *A connected component of a fibre of a complete Lagrangian fibration is affinely isomorphic to the product of an affine space with a torus.*

[7] That is, affine lines defined locally can be indefinitely extended.

Indeed, the completeness of the fibration means that every component of the fibre is a quotient space of the group of translations $\mathbb{R}^n$ by a discrete subgroup. The only such subgroups are the lattices $\mathbb{Z}^k \subset \mathbb{R}^n$, $k \leq n$.

Corollary. *A connected compact fibre of a Lagrangian fibration is a torus.*

Let us now describe the complete Lagrangian fibrations with an affine fibre. A *twisted cotangent bundle* is a cotangent bundle $\pi: T^*X \to X$ with a symplectic structure on T^*X equal to the sum of the canonical symplectic form and a form $\pi^*\phi$, where ϕ is a closed 2-form on the base space X. We have already encountered this construction at the end of § 1.

Theorem. *A twisted cotangent bundle is determined by the cohomology class of the form ϕ in $H^2(X, \mathbb{R})$ uniquely up to an isomorphism of Lagrangian fibrations (which is the identity on the base space). Any complete Lagrangian fibration with a connected, simply connected fibre is isomorphic to a twisted cotangent bundle.*

Proof. Let $\pi: (M, \omega) \to X$ be a Lagrangian fibration whose fibres are affine spaces. A section $s: X \hookrightarrow M$ defines a closed form $\phi = s^*\omega$ and $\omega - \pi^*\phi$ is a symplectic structure on M in which $s(X)$ is Lagrangian. Taking s as the zero section, by the construction of the affine structure theorem we identify M with T^*X. A change of the zero section by s' changes ϕ by ds'; therefore the class $[\phi] \in H^2(X, \mathbb{R})$ is the only invariant of the twisted cotangent bundle. □

We have already come across an example of a topologically nontrivial Lagrangian fibration with a compact fibre in sect. 2.3 on Kähler structures: we presented a symplectic manifold M^4, a Lagrangian fibration over the torus T^2 with fibre T^2, not homeomorphic to $T^2 \times T^2$. It turns out the compactness of the fibre of a Lagrangian fibration imposes very stringent conditions on its base space.

Theorem. *The base space of a Lagrangian fibration with a connected compact fibre has a canonical integral affine structure (in other words, in some atlas on the base space the transition functions are compositions of translations and integral linear transformations on $\mathbb{R}^n$).*

Proof. The identification of the fibre T^n over a point x of the base space X with an orbit of the group T_x^*X gives on T_x^*X a lattice of maximal rank. A continuous basis of such a lattice is a set of 1-forms $\alpha_1, \ldots, \alpha_n$ on X. The symplecticity of translations by the lattice vectors in T^*X means that these forms are closed. The local potential $(\phi_1, \ldots, \phi_n)$: $X \to \mathbb{R}^n$ of this basis, $d\phi_i = \alpha_i$, defines a chart of the desired atlas. □

4.3. Intersections of Lagrangian Manifolds and Fixed Points of Symplectomorphisms. The problem of the existence of periodic motions of dynamical systems led H. Poincaré to the following theorem.

Poincaré's Geometric Theorem. *A homeomorphism of a plane circular annulus onto itself which preserves areas and which moves the boundary circles in different directions has at least two fixed points.*

The boundary condition in the theorem means that the mapping has the form $X = x + f(x, y)$, $Y = y + g(x, y)$, where X, x are radial coordinates and Y, y are angular coordinates on the annulus, and the functions f, g are continuous and 2π-periodic in y, and moreover g has different signs on the different boundaries of the annulus.

The proof of this theorem is due to G.D. Birkhoff. H. Poincaré succeeded in proving it only under certain restrictions, but his method is less special than Birkhoff's proof and lends itself to generalization. Poincaré's argument is based on the fact that the fixed points of a symplectomorphism of the annulus are precisely the critical points of the function $F(u, v) = \int (f dv - g du)$, where $u = (X + x)/2$, $v = (Y + y)/2$, true under the assumption that the Jacobian $\partial(u, v)/\partial(x, y)$ is different from zero. This condition is automatically fulfilled if the symplectomorphism is not too far from the identity.

Carrying over Poincaré's arguments to the general symplectic situation leads to the following concepts and results.

Let M be a symplectic manifold. A symplectomorphism $\gamma: M \to M$ is given by a Lagrangian graph $\Gamma \subset M \times M$. A fixed point of γ is an intersection point of Γ with the graph Δ of the identity symplectomorphism. A tubular neighbourhood of Δ in $M \times M$ has the structure of the Lagrangian bundle $T^*\Delta$. If the symplectomorphism γ is sufficiently close to the identity, then it is given by a generating function (generally speaking, by a many-valued one) of the Lagrangian section $\Gamma \subset T^*\Delta$. Its critical points and the fixed points of γ coincide.

We shall say that a symplectomorphism γ is *homologous to the identity* if it can be connected with the identity by a family of symplectomorphisms whose velocity field for each value of the parameter of the family is globally Hamiltonian. The symplectomorphisms homologous to the identity form the commutator subgroup of the identity component of the group of symplectomorphisms of the manifold M.

Lemma. *A symplectomorphism which is homologous to and close (together with its first derivatives) to the identity has a single-valued generating function.*

Theorem A. *A symplectomorphism of a compact symplectic manifold which is homologous to and close to the identity has fixed points. Their number is not less than the number of critical points which a smooth function on this manifold must have at the very least.*

Conjecture A. The preceding theorem is true without the condition of closeness of the symplectomorphism to the identity.

The conjecture has been proved in the two-dimensional case: a symplectomorphism of a compact surface homologous to the identity has for the sphere not less than 2, and in the case of the sphere with handles not less than 3 fixed points. The proof of these theorems[8] bears an essentially two-dimensional character. Thus, in the case of the sphere the proof is based on the fact that the index of an isolated singular point of a Hamiltonian vector field on the plane is equal to one minus one-half the number of components into which the critical level curve of the Hamiltonian divides a neighbourhood of the singular point and hence does not exceed 1.

Let us observe that Poincaré's theorem on the symplectomorphism of the annulus follows from the (proven) conjecture A for the two-dimensional torus. Out of two copies of the annulus one can glue together a torus. The symplectomorphisms of the torus $\mathbb{R}^2/\mathbb{Z}^2$ homologous to the identity are precisely those which leave in place the centre of gravity of the unit square in $\mathbb{R}^2$ [2]. The latter is not difficult to achieve if one adds connective strips between the glued-together annuli and extends the symplectomorphism so as not to have fixed points on these strips (here it will be necessary to use the rotation of the boundaries of the annulus in different directions).

We shall call two Lagrangian submanifolds of a symplectic manifold N (Lagrangianly) isotopic if one goes over into the other under the action of a symplectomorphism homologous to the identity. In particular, isotopic Lagrangian submanifolds are diffeomorphic.

Theorem B. *A compact Lagrangian submanifold isotopic to and close to a given one has as many points of intersection with it as some smooth function on this manifold has critical points.*

Conjecture B. Theorem B is true without the assumption of closeness of the isotopic Lagrangian submanifolds.

Conjecture (theorem) A follows out of conjecture (theorem) B: the graph $\Gamma \subset M \times M$ of a symplectomorphism γ: $M \to M$ homologous to the identity is a Lagrangian submanifold isotopic to the diagonal Δ.

Without the isotopy assumption intersection points of a compact Lagrangian submanifold with a Lagrangian perturbation of it (of which it is the question in theorem B) may be absent. Nevertheless intersection points necessarily exist if the Euler characteristic of the Lagrangian manifold is different from 0 or if each closed 1-form on this manifold is the differential of a function. It turns out that the absence of intersection points of a compact Lagrangian submanifold with its Lagrangian perturbation means that this manifold is fibred over a circle.

By a new method progress was recently achieved in the proof of conjectures A and B in the many-dimensional case. Namely, conjecture A has been proved for symplectomorphisms of the torus $T^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ with the standard symplectic structure, just as conjecture B has been proved for the Lagrangian torus T^n embedded in the standard symplectic torus T^{2n} as a subgroup and for an arbitrary Lagrangian isotopy of the torus T^n [19]. Let us note that a smooth

[8] The first is due to A.I. Shnirel'man and N.A. Nikishin, the second to Ya.M. Ehliashberg.

function on the n-dimensional torus has no fewer than $n+1$ critical points and no fewer than 2^n, counting them with multiplicities [24].

To demonstrate the essence of the method, we shall give a sketch of the proof of the following assertion.

Theorem. *On the standard symplectic 2n-dimensional torus let there be given a Hamiltonian depending periodically on time. Then the symplectic transformation of the torus after a period by the flow of this Hamiltonian has at least one fixed point.*

1°. Representing T^{2n} as the quotient space $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ of the standard symplectic space $\mathbb{R}^{2n}$, let us consider the vector space Ω of loops—smooth mappings of $\mathbb{R}/\mathbb{Z} \to \mathbb{R}^{2n}$—with the linear structure of pointwise addition and scalar multiplication and with the Euclidean structure $\int_{\mathbb{R}/\mathbb{Z}} (p^2+q^2)/2dt$. On the space Ω let us introduce the action functional $F = \int_{\mathbb{R}/\mathbb{Z}} [(pdq - qdp)/2 - Hdt]$, where $H(p, q, t)$ is the given periodic Hamiltonian with period 1. To the various critical points of the functional $F: \Omega \to \mathbb{R}$ correspond the various periodic trajectories with period 1 of the flow of the Hamiltonian H, i.e. the various fixed points of the symplectomorphism we are interested in.

2°. The gradient ∇F: $\Omega \to \Omega$ has the form $\nabla F = -Jd/dt - \nabla H$, where J: $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is the operator "multiplication by $\sqrt{-1}$", $J(p, q) = (-q, p)$. We regard the mapping ∇F as a perturbation of the linear operator $A = -Jd/dt$. The spectrum of the operator A is $2\pi\mathbb{Z}$, and the eigenspace with the eigenvalue λ is the $2n$-dimensional space of solutions with period 1 of the linear Hamiltonian system with Hamiltonian $\lambda(p^2+q^2)/2$. The expansion of a loop $\mathbb{R}/\mathbb{Z} \to \mathbb{R}^{2n}$ by eigenfunctions of the operator A is simply its Fourier series expansion. The perturbation $B = \nabla H$ has a bounded image ($\|B(\omega)\| < C$) and a bounded norm: $\|B(\omega_1) - B(\omega_2)\| \le D\|\omega_1 - \omega_2\|$.

3°. Let us represent the space Ω in the form of a direct sum of a finite-dimensional space V, spanned by the "lower harmonics" with frequency $|\lambda| < 2D$, and an infinite-dimensional space W of "higher harmonics". The equation $\nabla F = 0$ splits into two: $\partial F/\partial V = 0$ and $\partial F/\partial W = 0$. The second has the form $Aw + PB(v+w) = 0$, where P is the projector of Ω onto W. Since the operator $w \mapsto -A^{-1}PB(v+w)$ is contracting, the second equation has for each v a unique solution $w = w(v)$. Therefore the search for extrema of the functional F reduces to the determination of the critical points of the function $f(v) = F(v, w(v))$ on the finite-dimensional space.

4°. The constant loops, which form the kernel of the operator A, correspond to the zero solution of the equation $\partial F/\partial W = 0$; therefore the function f is periodic on the subspace of such loops. Thus the function f, essentially, is defined on the manifold $T^{2n} \times \mathbb{R}^N_{\lambda>0} \times \mathbb{R}^N_{\lambda<0}$. Its behaviour at infinity is determined by the unperturbed functional and has hyperbolic character (Fig. 11), from which the presence of critical points follows (for example, the set of points where the value

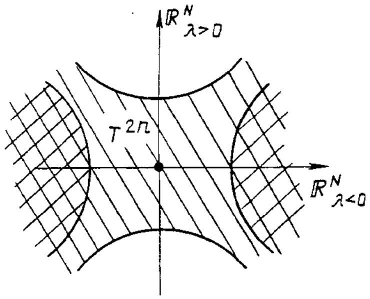

Fig. 11. The change of structure of the level topology of the function f

of the function is less than a given value undergoes a topological change of structure).

Remark. Further analysis of this finite-dimensional situation with tools developed in Morse theory ([55]) leads to an exact estimate of the number of fixed points ($\geq 2n+1$). The fundamental difference of the argument cited to the standard formalism of the global variational calculus consists in the fact that the unperturbed quadratic functional on the loop space is not elliptic but hyperbolic.

Chapter 3

Symplectic Geometry and Mechanics

Here we shall examine the connection of symplectic geometry with the variational calculus, in particular, with Lagrangian mechanics, we shall give a geometric introduction to the theory of completely integrable systems, and we shall describe a procedure for reducing the order of Hamiltonian systems having a continuous symmetry group. For a systematic exposition of the questions of classical mechanics see volume 3, and for the theory of integrable systems see the article by B.A. Dubrovin, I.M. Krichever and S.P. Novikov in this volume.

§ 1. Variational Principles

The motions of a mechanical system are the extremals of a suitable variational principle. On the other hand, any problem of the calculus of variations can be formulated in the language of symplectic geometry.

1.1. Lagrangian Mechanics. A *natural mechanical system* is given by a kinetic and a potential energy. The *potential energy* is a smooth function on the *manifold of configurations* (states) of the system, the *kinetic energy* is a Riemannian metric on the configuration manifold, i.e. a positive definite quadratic form on each tangent space to the manifold of configurations depending smoothly on the point of application.

Example. A system of mass points in Euclidean space has the kinetic energy $T=\sum m_k \dot{r}_k^2/2$ and the potential energy $U=\sum V_{kl}(r_k-r_l)$, where r_k is the radius vector of the kth point and V_{kl} is the potential of the pairwise interaction of the mass points, say the Newtonian gravitational potential $V_{kl}(r)=-\gamma m_k m_l/|r|$.

Out of the kinetic and potential energies one composes the *Lagrangian* or *Lagrange function* $L=T-U$ on the total space of the tangent bundle of the configuration manifold. A motion $t\mapsto q(t)$ of a natural system in the configuration space is an extremal of the functional

$$\int_{t_0}^{t_1} L(q(t), \dot{q}(t))\,dt \tag{1}$$

More generally, if we consider an arbitrary Lagrange function $L\colon TM\to\mathbb{R}$ on the tangent bundle of the configuration manifold and if we designate as motions the extremals of the functional (1), we obtain the definition of a *Lagrangian mechanical system.* One may also assume an explicit dependence of the Lagrangian on time. The extremals of the functional (1) are locally described by the system of *Euler–Lagrange differential equations* $(d/dt)\partial L/\partial\dot{q}=\partial L/\partial q$. For a system of mass points in Euclidean space the Euler–Lagrange equations take the form of the system of Newton's equations $m_k\ddot{r}_k=-\partial U/\partial r_k$. Thus, Lagrangian mechanics generalizes Newtonian mechanics, admitting into consideration, for example, systems of mass points with holonomic (rigid) constraints—the configuration manifolds of such systems are no longer domains of coordinate spaces. At the same time, the Lagrangian approach to mechanics permits considering it as a special case of the variational calculus. For example, the problem of the "free" motion of a natural system ($U\equiv 0$) is equivalent to the description of the geodesic flow on a configuration manifold with Riemannian metric T (see sect. 1.3).

Example. Let us consider an absolutely rigid body one of whose points is fastened at the coordinate origin of the space $\mathbb{R}^3$. The configuration manifold of such a system is the rotation group SO_3. The tangent space to the configuration manifold at each point can be identified with the space $\mathbb{R}^3$: the direction of the vector $\omega\in\mathbb{R}^3$ indicates the axis and the direction of the infinitesimal rotation of the body, and the length of the vector indicates the angular velocity of the rotation. The kinetic energy of the rotation is $T=I\omega^2/2$, where I is the moment of inertia of the body with respect to the axis of rotation, i.e. the kinetic energy is given in the internal coordinates of the body by the inertia quadratic form. Thus the free rotation of a rigid body fixed at a point is described by the geodesic flow on the group SO_3 of a Riemannian metric (left-)invariant with respect to the translations on the group.

1.2. Hamiltonian Mechanics. A *Hamiltonian* mechanical *system* is given by a smooth function—the *Hamiltonian*—on a symplectic manifold (the *phase space*). The motion in the Hamiltonian system is described by the phase flow of the corresponding Hamiltonian vector field (see sect. 3.1, chap. 2). A Hamiltonian H depending explicitly on time gives a nonautonomous Hamiltonian system. In Darboux coordinates the system of Hamilton's equations has the form $\dot{p} = -H_q, \dot{q} = H_p$.

Hamiltonian mechanics generalizes Lagrangian mechanics.

Example 1. The system of Euler–Lagrange equations of a natural system with configuration manifold M, kinetic energy T and potential energy U is converted, under the isomorphism of the tangent and cotangent bundles of the manifold M defined by the Riemannian metric $2T$, into a system of Hamilton's equations with Hamiltonian $T+U$ with respect to the canonical symplectic structure on the total space of the cotangent bundle.

In the general case let us define the Hamiltonian $H: T^*M \to \mathbb{R}$ as the fibrewise Legendre transformation of the Lagrangian $L: TM \to \mathbb{R}$. The *Legendre transformation* of a convex function f of the vector argument v is defined as a function f^* of the dual argument p by the formula $f^*(p) = \max_v [\langle p, v\rangle - f(v)]$ (Fig. 12, compare sect. 1.1, chap. 5). For example, the Legendre transformation of the Euclidean form $\langle Av, v\rangle/2$ is $\langle p, A^{-1}p\rangle/2$.

We shall suppose that the mapping $T_qM \to T_q^*M$: $\dot{q} \mapsto p = L_{\dot{q}}$ is a diffeomorphism for each $q \in M$. Then the Hamiltonian H is a smooth function on the total space of the cotangent bundle, $H(p, q) = p\dot{q} - L(q, \dot{q})$, where $\dot{q}$ is determined from the equation $p = L_{\dot{q}}(q, \dot{q})$.

Theorem. *Under the indicated identification of the total spaces of the tangent and cotangent bundles the mechanical system with Lagrangian function L goes over into the Hamiltonian system with Hamiltonian H.*

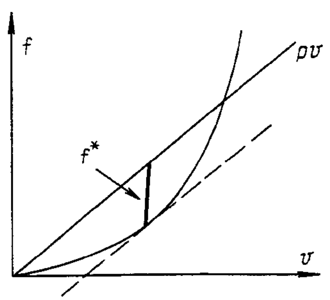

Fig. 12. The Legendre transformation

Example 2. A natural system in a magnetic field. Let the Lagrangian be the sum of the Lagrangian of a natural system and a differential 1-form A on the configuration manifold M, regarded as a function on TM linear with respect to the velocities: $L = T - U + A$. The corresponding system of Euler–Lagrange equations on TM is Hamiltonian with Hamiltonian function $H = T + U$ with respect to the symplectic structure $\Omega + dA$, where Ω is the symplectic structure of example 1. If the 1-form A is a many-valued "vector-potential of a magnetic field" dA defined single-valuedly on M, then the phase space turns out to be a twisted cotangent bundle (sect. 4.2, chap. 2).

1.3. The Principle of Least Action. The fact that the problems of the calculus of variations have a Hamiltonian character is explained by the presence of a variational principle in the Hamiltonian formalism itself. At the basis of this principle lies the following observation: the field of directions of a Hamiltonian vector field on a nonsingular level hypersurface of its Hamiltonian coincides with the *field of characteristic directions of this hypersurface*—the field of skew-orthogonal complements of its tangent hyperplanes.

Let the symplectic manifold M be polarized: $M = T^*B$, and let $\alpha = \sum p_k dq_k$ be the action 1-form on M.

Theorem (*The principle of least action*). *The integral curves of the field of characteristic directions of a nonsingular hypersurface $\Gamma \subset T^*B$ transversal to the fibres of the cotangent bundle $T^*B \to B$ are extremals of the action integral $\int \alpha$ in the class of curves lying on Γ and joining the fibres $T_{q_0}^*B$ and $T_{q_1}^*B$ of the points q_0 and q_1 of the base space B.*

Proof. The increment of the action integral $\int_{\gamma'} \alpha - \int_{\gamma} \alpha$ (Fig. 13) is equal to the symplectic area of the sheet joining two curves γ and γ', and in the case that γ is an

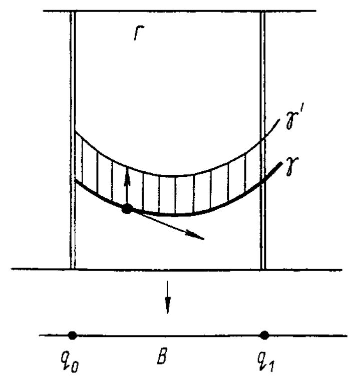

Fig. 13. The proof of the principle of least action

integral curve, it is infinitesimal to a higher order than the difference of the curves γ and γ'. □

Remark. The integral curves of a nonautonomous system of equations with the Hamiltonian function $H(p, q, t)$ in the extended phase space $\mathbb{R}^{n*} \times \mathbb{R}^n \times \mathbb{R}$ are extremals of the action integral $\int_{t_0}^{t_1} (pdq - Hdt)$ in the class of curves $t \mapsto (p(t), q(t), t)$ with the boundary conditions $q(t_0) = q_0$, $q(t_1) = q_1$.

Corollary. *A mass point forced to stay on a smooth Riemann manifold moves on geodesic curves (i.e. on extremals of the length* $\int ds$*).*

In fact, in the case of free motion with kinetic energy $T = (ds/dt)^2/2$ the parameter t ensuring a fixed value of the energy $H = T = h$ must be proportional to the length, $dt = ds/\sqrt{2h}$, and the action integral takes the form $\int pdq = \int p\dot{q}dt = \int 2Tdt = \sqrt{2h}\int ds$.

In the case where the potential energy is different from zero, the trajectories of a natural system are also geodesics of a certain Riemannian metric: in the region of the configuration space where $U(q) < h$ the trajectories of a system with kinetic energy $T = (ds/dt)^2/2$, potential energy $U(q)$ and total energy h will be geodesic curves of the metric $(h - U(q))\,ds^2$.

As an application let us consider the rotation of a rigid body around a fixed point in a potential field. For sufficiently large h the Riemannian metric $(h - U)ds^2$ is defined on the whole compact configuration space SO_3. The space SO_3 is not simply connected (it is diffeomorphic to $\mathbb{RP}^3$ and has a simply connected double covering by S^3).

In the class of all noncontractible closed curves on SO_3 let us choose a curve of minimal length (this is possible [55]) with respect to the Riemann metric introduced above. We obtain the

Corollary. *A rigid body in an arbitrary potential field has at least one periodic motion for each sufficiently large value of the total energy.*

One can show [55] that on a compact Riemannian manifold each element of the fundamental group is represented by a closed geodesic. From this one can obtain an analog of the preceding corollary for an arbitrary natural system with a compact non-simply connected configuration space.

1.4. Variational Problems with Higher Derivatives. Let us describe the Hamiltonian formalism of the problem of minimizing the functional

$$\int_a^b L(x^{(0)}, \ldots, x^{(n+1)})dt \tag{2}$$

within the class of smooth curves $x: \mathbb{R} \to \mathbb{R}^l$ with a given Taylor expansion at the ends of the interval $[a, b]$ up to order n inclusive, where the Lagrangian L depends on the derivatives $x^{(k)} = d^k x/dt^k$ of the curve $x(t)$ up to order $n+1$. The extremals of the functional (2) satisfy the *system of Euler–Poisson equations*

$$L_{x^{(0)}} - \frac{d}{dt}L_{x^{(1)}} + \frac{d^2}{dt^2}L_{x^{(2)}} - \ldots + (-1)^{n+1}\frac{d^{n+1}}{dt^{n+1}}L_{x^{(n+1)}} = 0, \tag{3}$$

which expresses the vanishing of the first variation of the functional (2). Now let us regard the Lagrangian $L(x, y, \ldots, z, w)$ as a function of a point on a curve $(x, y, \ldots, z, w): \mathbb{R} \to \mathbb{R}^{(n+2)l}$ satisfying the restrictions $dx = ydt, \ldots, dz = wdt$, and let us put together the action form according to the Lagrange multiplier rule:

$$\alpha = p_x(dx - ydt) + \ldots + p_z(dz - wdt) + Ldt$$
$$= [p_x(\dot{x} - y) + \ldots + p_z(\dot{z} - w) + L(x, y, \ldots, z, w)]\,dt.$$

The extremals of the functional $\int\alpha$ satisfy the system of Euler–Lagrange equations in $\mathbb{R}^{(2n+3)l}$:

$$\dot{x} = y, \ldots, \dot{z} = w;\ \dot{p}_x = \frac{\partial L}{\partial x},$$
$$\dot{p}_y = -p_x + \frac{\partial L}{\partial y}, \ldots, \dot{p}_z = \ldots, 0 = -p_z + \frac{\partial L}{\partial w}. \tag{4}$$

This system is equivalent to the system of Euler–Poisson equations (3). Let us introduce the symplectic form $\omega = dp_x \wedge dx + \ldots + dp_z \wedge dz$ and with the aid of a Legendre transformation with respect to the variable w let us define the Hamiltonian function $H(x, y, \ldots, z, p_z, \ldots, p_y, p_x) = p_x y + \ldots + p_z W - L(x, y, \ldots, z, W)$ ($W(x, y, \ldots, z, p_z)$ is determined from the equation $p_z = \partial L/\partial w$). The system of Hamilton's equations with the Hamiltonian H and the symplectic structure ω on $\mathbb{R}^{(2n+2)l}$ together with the equation $p_z = \partial L/\partial w$ coincides with (4). Thus, under the condition of convexity of the Lagrangian $L(x^{(0)}, \ldots, x^{(n+1)})$ with respect to the variable $x^{(n+1)}$, the system of Euler–Poisson equations (3) is equivalent to the Hamiltonian system (H, ω).

Having thus written out the coordinate formulas, we shall now impart an invariant sense both to the variational problem (2) itself as well as to its Hamiltonian version.

Upon replacement of the space $\mathbb{R}^l$ by an arbitrary l-dimensional manifold M it is natural to give the functional of type (2) by means of a Lagrange function $L: J^{n+1} \to \mathbb{R}$ on the manifold of $n+1$-jets at 0 of curves $x: \mathbb{R} \to M$. The manifold J^{n+1} is defined by induction together with the projection $J^{n+1} \to J^n$ (Fig. 14) as the affine subbundle of the tangent bundle TJ^n consisting of those tangent vectors $\xi \in T_{j^n}J^n$ which under the differential $\pi_*: T_{j^n}J^n \to T_{\pi(j^n)}J^{n-1}$ of the projection $\pi: J^n \to J^{n-1}$ go over into their point of application: $\pi_*(\xi) = j^n \in TJ^{n-1}$. In addition $J^0 = M$, $J^1 = TM$.

The phase space of the Hamiltonian system corresponding to the Lagrangian $L: J^{n+1} \to \mathbb{R}$ is the total space T^*J^n of the cotangent bundle with the canonical symplectic structure. Let the function L be convex on each affine fibre of the fibre

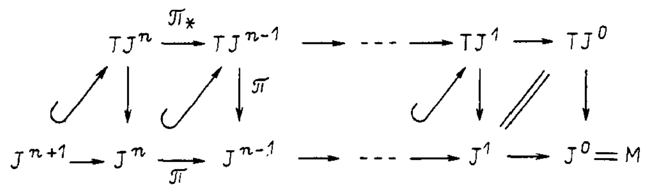

Fig. 14. The definition of the jet spaces of curves

bundle $J^{n+1} \to J^n$. Let us construct a function $H: T^*J^n \to \mathbb{R}$ as follows. A covector p applied at the point $j \in J^n$ defines a linear nonhomogeneous function on the fibre $W \subset T_j J^n$ of the bundle $J^{n+1} \to J^n$. Let us set $H(p) = \max_{w \in W}(p(w) - L(w))$.

Ostrogradskij's Theorem. *The system of Euler–Poisson equations which is satisfied by the extremals of the Lagrangian $L: J^{n+1} \to \mathbb{R}$, where L is a smooth function, strictly convex*[9] *on each fibre of the fibre bundle $J^{n+1} \to J^n$, is equivalent to the Hamiltonian system with phase space T^*J^n and Hamiltonian function H.*

Remark. In the case of explicit dependence of the Lagrangian in the functional (2) on time the system of Euler–Poisson equations is equivalent to the nonautonomous Hamiltonian system on the extended phase space $\mathbb{R} \times T^*J^n$.

1.5. The Manifold of Characteristics. Let us suppose that the integral curves of the field of characteristic directions on a smooth hypersurface in a symplectic manifold form a smooth manifold (locally this is always so). We shall call it the *manifold of characteristics*.

Theorem. *The manifold of characteristics has a symplectic structure (it is well defined by the condition: the skew-scalar product of vectors tangent to the hypersurface is equal to the skew-scalar product of their projections along the characteristics).*

Let the Hamiltonian system with Hamiltonian H have a first integral F, and let M be the manifold of characteristics of a hypersurface $F = \text{const}$. The function H is constant on the characteristics of this hypersurface and defines a smooth function $\tilde{H}$ on M. The field of the Hamiltonian H on the hypersurface $F = \text{const}$ defines, upon projection onto M, a Hamiltonian vector field on M with Hamiltonian $\tilde{H}$.

Corollary 1. *A first integral of a Hamiltonian system allows one to reduce its order by 2.*

---

[9] That is, $d^2(L|_W) > 0$.

The parametrized extremals of a variational problem (possibly a non-autonomous one) form a symplectic manifold, namely the phase space of the corresponding Hamiltonian system (for example, T^*J^n for problem (2)). From the theorem we get

Corollary 2. *If the oriented geodesics (the unparametrized ones) of a Riemannian manifold form a smooth manifold, then it is symplectic.*

Example. The rays (i.e. the oriented straight lines) in Euclidean space $\mathbb{R}^n$ form a symplectic manifold—the manifold of characteristics of the hypersurface $\langle p, p \rangle = 1$ in $T^*\mathbb{R}^n$. Up to the sign of the symplectic structure it is symplectomorphic to the cotangent bundle of the unit sphere in $\mathbb{R}^n$. Figure 15 shows how to associate to a ray a (co)tangent vector to the sphere.

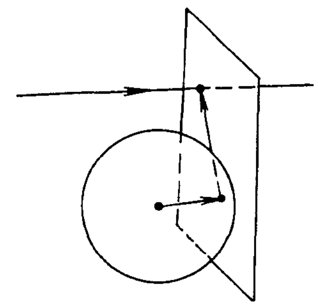

Fig. 15. The ray space

1.6. The Shortest Way Around an Obstacle. Let us regard a smooth surface in space as the boundary of an obstacle. The shortest path between two points q_0 and q_1 avoiding the obstacle (Fig. 16) consists of straight-line segments and a geodesic segment on its surface. The length of the extremals is a many-valued function of the point q_1 with singularities along the rays breaking loose from the obstacle surface in an asymptotic direction. The rays on which the extremals issuing from the source break away from the obstacle surface form a Lagrangian variety with singularities in the symplectic manifold of all the rays of the space (compare sect. 1.5).

The symplectic analysis of the problem of going around an obstacle leads to the notion of a triad in symplectic space. A triad (L, l, H) consists of a smooth Lagrangian manifold L, a smooth hypersurface l in L (l is an isotropic manifold) and a smooth hypersurface H in the ambient symplectic space, tangent to the Lagrangian manifold at the points of the isotropic one. The projection of the isotropic manifold along the characteristics of the hypersurface is a Lagrangian subvariety in the manifold of characteristics and has singularities at those places where the characteristics are tangent to l.

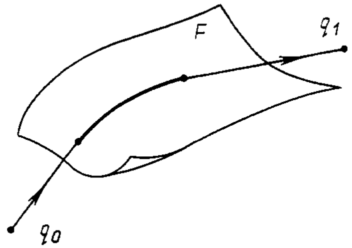

Fig. 16. An extremal in the problem of going around an obstacle

Returning to the problem of going around an obstacle in Euclidean (more generally, in a Riemannian) space, let us associate with a pencil of geodesics on the obstacle boundary a triad in the symplectic space $T\mathbb{R}^n = T^*\mathbb{R}^n$. Motion along straight lines in $\mathbb{R}^n$ is given by the Hamiltonian $h = \langle p, p \rangle$; let $H = h^{-1}(1) \subset T\mathbb{R}^n$ be its unit level hypersurface. The extremals issuing from the source form a pencil of geodesics on the boundary hypersurface F of the obstacle. The manifold $\lambda \subset H$ of unit vectors tangent to the geodesics of the pencil is Lagrangian in $TF = T^*F$ (the length of the extremal is its generating function). Let L consist of all possible extensions of covectors $\xi \in \lambda$ on F to covectors η on $\mathbb{R}^n$ applied at the same point. Let $l = H \cap L$. It is not hard to check (Fig. 17) that H is strictly quadratically tangent to L along l, i.e. (L, l, h) is a triad.

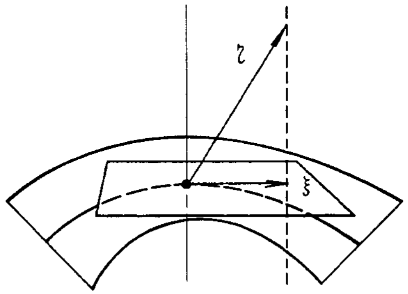

Fig. 17. Quadratic tangency of H and L

Let us denote by $\tau_{n,m}$ the germ at 0 of the following triad in the symplectic space $\mathbb{R}^{2n}$ with Darboux coordinates

$$(p_1, \ldots, p_m, q_1, \ldots, q_m, \bar{p}_1, \ldots, \bar{p}_{n-m}, \bar{q}_1, \ldots, \bar{q}_{n-m}):$$

$$L = \{p = \bar{p} = 0\},\ l = L \cap \{q_m = 0\},\ H = \{q_m^2/2 + p_m q_{m-1} + \ldots + p_2 q_1 + p_1 = 0\}.$$

It is helpful to compare the equation of the hypersurface H with the quadratic Hamiltonian of the functional $\int (d^m x/dt^m)^2 dt$ (sect. 2.3, 2.4 of chap. 1 or sect. 1.4 of

chap. 3). The extremals of the functional are polynomials $x(t)$. Therefore a natural symplectic structure arises in the space of polynomials. The singular Lagrangian variety of the triad $\tau_{m,m}$ is diffeomorphic to the open swallowtail Σ_m — the variety of polynomials of degree $2m-1$ with a fixed leading coefficient and zero root sum which have a root of multiplicity $\geq m$. The variety Σ_m is Lagrangian in the natural symplectic structure on the space of polynomials.

Theorem ([8]). *The germ of a generic triad at a point of quadratic tangency of the hypersurface with the Lagrangian manifold is symplectomorphic to one of the germs* $\tau_{n,m}$, $m \leq n$.

Corollary. *The germ of the Lagrangian variety of rays breaking loose from a generic pencil of geodesics on the boundary of a generic obstacle is symplectomorphic to the Cartesian product of a smooth manifold with the open swallowtail.*

Example. The tangent at a point of simple inflection of a curve bounding an obstacle in the plane is a cusp point of the Lagrangian curve formed by the tangents to the obstacle boundary. The curve Σ_2 on the parameter plane of the family of cubic polynomials $t^3 + qt + p$, formed by the polynomials with a multiple root, has the same kind of singularity.

The triad example shows that the symplectic version of variational problems can be nontrivial. For more details on the problem of the shortest way around an obstacle see sect. 3.5, chap. 5, and also [6], [7], [8], [54], [66].

§2. Completely Integrable Systems

The integrability of a Hamiltonian dynamical system is ensured by a sufficient supply of first integrals. We discuss the geometric effects and causes of integrability. For an investigation of actually integrated systems see the article by B.A. Dubrovin, I.M. Krichever and S.P. Novikov in this volume.

2.1. Integrability According to Liouville. A function F on a symplectic manifold is a first integral of a Hamiltonian system with Hamiltonian H if and only if the Poisson bracket of H with F is equal to zero (see sect. 3.1, chap. 2). One says of functions whose Poisson bracket is equal to zero that they are in involution.

Definition. A Hamiltonian system on a symplectic manifold M^{2n} is called *completely integrable* if it has n first integrals in involution which are functionally independent almost everywhere on M^{2n}.

Examples. 1) A Hamiltonian system with one degree of freedom ($n = 1$) is completely integrable.

2) A linear Hamiltonian system is completely integrable. In article 3.3 of chap. 2 we showed that each quadratic Hamiltonian on $\mathbb{R}^{2n}$ is contained in

an n-dimensional commutative subalgebra of the Lie algebra of quadratic Hamiltonians. In fact the subalgebra may be chosen so that its generators are functionally independent almost everywhere on $\mathbb{R}^{2n}$.

Liouville's Theorem. *On the* $2n$*-dimensional symplectic manifold* M *let there be given* n *smooth functions in involution*

$$F_1, \ldots, F_n; \quad \{F_i, F_j\}=0, \quad i,j=1,\ldots,n.$$

Let us consider a level set of the functions F_i

$$M_f=\{x \in M \mid F_i(x)=f_i, i=1,\ldots,n\}.$$

Let us suppose that on M_f *the* n *functions* F_i *are independent (i.e.* $dF_1 \wedge \ldots \wedge dF_n \neq 0$ *at each point of* M_f*).*

Then:

1) M_f *is a smooth manifold invariant with respect to the phase flow of the Hamiltonian* $H=H(F_1,\ldots,F_n)$ *(say,* $H=F_1$*).*

2) M_f *has a canonical affine structure in which the phase flow straightens out, i.e. in the affine coordinates* $\phi=(\phi_1,\ldots,\phi_n)$ *on* M_f *one has* $\dot{\phi}=$const.

Proof. Under the premises of Liouville's theorem the mapping $F=(F_1,\ldots,F_n)$: $M \to \mathbb{R}^n$ is a Lagrangian fibration in a neighbourhood of the manifold M_f. By the affine structure theorem (sect. 4.2, chap. 2) M_f can locally be identified with a domain in the cotangent space of the base space $\mathbb{R}^n$ at the point f; moreover the field on M_f defined by the Hamiltonian F^*H, $H=H(f_1,\ldots,f_n)$, goes over into the covector $d_f H$ under this identification, i.e. is constant. □

Remarks. 1) The first integrals $F_1,\ldots,F_n$ are independent on M_f for almost all $f \in \mathbb{R}^n$ (the case is not excluded that M_f might be empty here). This follows from Sard's theorem (see [9]).

2) Let M_f be compact. Then (under the assumptions of Liouville's theorem) each connected component of M_f is an n-dimensional torus (see sect. 4.2, chap. 2). The Hamiltonian flow on such a torus is either periodic or conditionally periodic. In the latter case the phase curves are parallel straight-line windings of the torus (Fig. 18). Invariant tori are often encountered in mechanical integrable systems, since for the compactness of M_f it is sufficient that the energy level

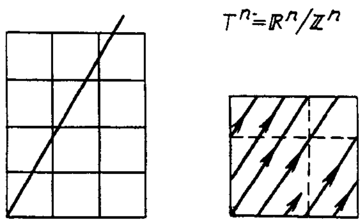

Fig. 18. A winding of the torus

manifolds $H=$const be compact. This is the case, for example, for natural systems with a compact configuration space.

2.2. The "Action-Angle" Variables. Let $M=\mathbb{R}^{2n}$ be the standard symplectic space and let the fibre M_f for $f=0$ be compact and satisfy the conditions of Liouville's theorem. Then in a neighbourhood of M_0 the fibres M_f are n-dimensional Lagrangian tori. Let us choose a basis $(\gamma_1,\ldots,\gamma_n)$ of one-dimensional cycles on the torus M_f depending continuously on f (Fig. 19).

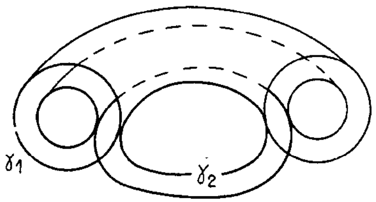

Fig. 19. Families of cycles on the invariant tori

We shall set

$$I_k(f)=\frac{1}{2\pi}\int_{\gamma_k(f)} p\,dq, \quad k=1,\ldots,n.$$

The functions $I_k=I_k(F(p,q))$ are called the action variables.

Theorem. *In a neighbourhood of the torus* M_0 *one can introduce the structure of a direct product* $(\mathbb{R}^n/2\pi\mathbb{Z}^n)\times\mathbb{R}^n$*, with the action coordinates* $(I_1,\ldots,I_n)$ *on the factor* $\mathbb{R}^n$ *and angular coordinates* $(\phi_1,\ldots,\phi_n)$ *on the torus* $\mathbb{R}^n/2\pi\mathbb{Z}^n$*, in which the symplectic structure* $\sum dp_k \wedge dq_k$ *on* $\mathbb{R}^{2n}$ *has the form* $\sum dI_k \wedge d\phi_k$.

Proof. In sect. 4.2, chap. 2, we constructed an integral affine structure on the base space of a Lagrangian fibration with fibre a torus: the identification of the tangent spaces to the affine torus M_f with the cotangent space of the base space $T_f^*\mathbb{R}^n$ introduces an integral lattice $\mathbb{Z}_f^n \subset T_f^*\mathbb{R}^n$ there, and the basis cycles $\gamma_1,\ldots,\gamma_n$ give n differential 1-forms on $\mathbb{R}^n$—the differentials of the affine coordinates on the base space.

In fact, up to a factor 2π and the addition of constants, the action variables are just these affine coordinates. The Lagrangian fibration itself can by a choice of a Lagrangian section be identified locally with $(\mathbb{R}^n)^*/(2\pi\mathbb{Z})^n \times \mathbb{R}^n \to \mathbb{R}^n$ (the group $\mathbb{Z}^n$ acts on $T^*\mathbb{R}^n$ by translations in each cotangent space). The Darboux coordinates on $T^*\mathbb{R}^n$ turn into the action-angle coordinates after factorization. □

Examples. 1) In the case of one degree of freedom the action is equal to the area divided by 2π of the region bounded by the closed component of the level line of the Hamiltonian.

2) For the linear oscillator $H=\sum(p_k^2+\omega_k^2 q_k^2)/2$ the action variables have the form $I_k=(p_k^2+\omega_k^2 q_k^2)/2\omega_k$ (the ratio of the energy of the characteristic oscillation to its frequency), and the angular coordinates are the phases of the characteristic component oscillations.

In action-angle variables the system of Hamilton's equations with Hamiltonian $H(I_1, \ldots, I_n)$ takes the form $\dot{I}_k=0$, $\dot{\phi}_k=\partial H/\partial I_k$ and can immediately be integrated:

$$I_k(t)=I_k(0), \qquad \phi_k(t)=\phi_k(0)+\partial H/\partial I_k|_{I(0)}\cdot t.$$

In the construction of the action-angle variables, apart from differential and algebraic operations on functions only the inversion of diffeomorphisms and the integration of known functions—"quadratures"—were employed. In such a case one says that one has managed to integrate the original system of equations by quadratures.

Corollary. *A completely integrable system can be integrated by quadratures.*

Liouville's theorem covers practically all problems of Hamiltonian mechanics which have been integrated to the present day. But it says nothing about how to find a full set of first integrals in involution. Until recent times, essentially, the only profound means of integration was the method of Hamilton–Jacobi (see sect. 4.4, chap. 4). After the discovery of infinite-dimensional integrable Hamiltonian systems (starting with the Korteweg–de Vries equation) many new integration mechanisms came to light. They are all connected with further algebraic-geometric properties of actually integrated systems, not at all reflected in Liouville's theorem. We shall cite below a number of illustrative examples.

2.3. Elliptical Coordinates and Geodesics on an Ellipsoid. Let $E: V\to V^*$ be a linear operator giving a Euclidean structure on the space V, and let $A: V\to V^*$ be another symmetric operator, $A^*=A$. By a Euclidean pencil of quadrics is meant the one-parameter family of degree two hypersurfaces $\langle A_\lambda x, x\rangle=2$, where $A_\lambda=A-\lambda E$. By a confocal family of quadrics in Euclidean space is meant the family of quadrics dual to the quadrics of some Euclidean pencil, i.e. the family $\langle A_\lambda^{-1}\xi, \xi\rangle=2$, $\xi\in V^*$.

Example. The plane curves confocal to a given ellipse are all the ellipses and hyperbolas with the same foci (Fig. 20).

The elliptical coordinates of a point are the values of the parameter λ for which the quadrics of a given confocal family pass through this point.

Let us fix some ellipsoid in Euclidean space, all of whose axes have unequal lengths.

Jacobi's Theorem. *Through each point of n-dimensional Euclidean space pass n quadrics which are confocal to the chosen ellipsoid. The smooth confocal quadrics intersect at right angles.*

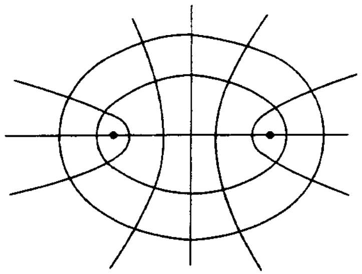

Fig. 20. Elliptical coordinates on the plane

Proof. In terms of the dual space the theorem signifies that the hyperplane $\langle l, x\rangle=1$ is tangent to exactly n quadrics of the Euclidean pencil, where the radius vectors of the points of tangency are pairwise orthogonal. The property stated follows from the fact that these vectors define the principal axes of the quadric $\langle Ax, x\rangle=2\langle l, x\rangle^2$. □

Chasles's Theorem. *A generic straight line in n-dimensional Euclidean space is tangent to $n-1$ different quadrics of a family of confocal quadrics; moreover the planes tangent to the quadrics at their points of tangency with the straight line are pairwise orthogonal.*

Proof. The visible contours of the quadrics of a confocal family under projection along a straight line form a family of quadrics dual to a family of quadrics in a hyperplane of the dual space passing through zero. The latter family is simply the section by a hyperplane of the original Euclidean pencil and therefore forms a Euclidean pencil in the hyperplane. Thus, the visible contours form a confocal family of quadrics in the $n-1$-dimensional space of straight lines parallel to the given one. Chasles's theorem now follows out of Jacobi's theorem applied to this family. □

The Jacobi–Chasles Theorem. *The tangent lines to a geodesic curve of a quadric in n-dimensional space, drawn at all points of the geodesic, are tangent, apart from this quadric, to $n-2$ more quadrics confocal with it, and to the same ones for all points of the geodesic.*

Proof. The manifold of oriented straight lines in a Euclidean space V has a natural symplectic structure and up to the sign of this structure is symplectomorphic to the cotangent bundle of the unit sphere S (see sect. 1.5). Let F be a smooth hypersurface in V.

Lemma A. *The mapping ρ which associates to a point of a geodesic curve on F its tangent line at that point takes the geodesics of F over into the characteristics of the hypersurface $P\subset T^*S$ of straight lines tangent to F within the space of all straight lines.*

Indeed, the geodesics on F are the characteristics on the hypersurface $G \subset T^*F$ of all unit (co)vectors on F. Identifying V with V^* with the aid of the Euclidean structure, let us regard G as the submanifold $\tilde{G}$ of codimension 3 in T^*V of all unit vectors on V tangent to F. In the commutative diagram of Fig. 21 π_2 is the projection along the characteristics of the hypersurface $\{(p, q) \mid \langle p, p \rangle = 1\}$ of all unit vectors on V, and π_1 is the projection along the characteristics of the hypersurface $\{(p, q) \mid q \in F\}$ of all vectors applied at points of F. The mappings $G \xleftarrow{\pi_1} \tilde{G} \xrightarrow{\pi_2} P$ take characteristics over into characteristics, since the characteristics on $\tilde{G}$, G and P are determined only by the symplectic structures of the ambient spaces. Therefore the mapping ρ transfers the geodesics on F into the characteristics of P. □

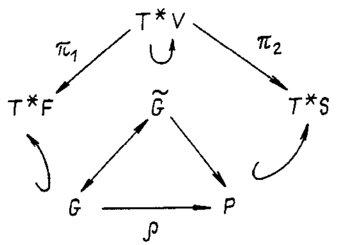

Fig. 21. The proof of Lemma A

In the Euclidean space V let there now be given a smooth function and let some straight line be quadratically tangent to a level surface at some point. Then nearby straight lines are tangent to nearby level surfaces of the function. Let us define an induced function on the space of straight lines, equal to the value of the function at the point of tangency of the straight line with its level surface.

Lemma B. *If two functions on Euclidean space are such that the tangent planes to their level surfaces at the points of tangency with some fixed straight line are orthogonal, then the Poisson bracket of the induced functions is equal to zero at the point which represents the straight line under consideration* (Fig. 22).

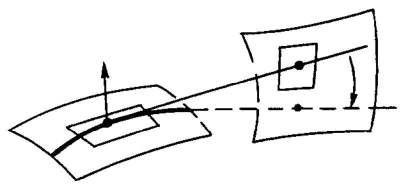

Fig. 22. Involutivity of the induced functions

Indeed, under movement along the geodesic of the first level surface which is tangent to our straight line, the tangent line turns in the direction of the normal to this surface and by the same token, up to second-order small terms, continues to be tangent to the same level surface of the second function. Therefore the derivative of the second induced function along the Hamiltonian flow of the first is equal to zero at the point under consideration. □

Now let us consider a generic straight line in V. By Chasles's theorem it is tangent to $n-1$ quadrics of a confocal family. Let us construct in the neighbourhood of the tangency points $n-1$ functions whose level surfaces are the quadrics of our family. By Lemma B the induced functions on the space of straight lines are in involution. A characteristic on the level surface of one of the induced functions consists (by Lemma A) of the tangent lines to some geodesic of the corresponding quadric. Insomuch as all the induced functions are constant on this characteristic, the theorem is proved. □

Corollary. *A geodesic flow on a quadric in Euclidean space is a completely integrable Hamiltonian system.*

Remarks. 1) Strictly speaking, we proved the Jacobi–Chasles theorem for generic quadrics and straight lines, but by continuity the result can easily be extended to degenerate cases.

2) The coordinate-free nature of the arguments cited allows one to extend them to the infinite-dimensional situation. We obtain a large stock of completely integrable systems—the geodesic flows on the infinite-dimensional ellipsoids defined by self-adjoint operators on Hilbert spaces. It would be interesting to clarify what these systems are for the concrete self-adjoint operators encountered in mathematical physics.

2.4. Poisson Pairs. Let there be given on a manifold M two Poisson structures V and W (see sect. 3.2, chap. 2). One says that they form a *Poisson pair* if all of their linear combinations $\lambda V + \mu W$ are also Poisson structures. Using the Schouten bracket [,] (sect. 3.2, chap. 2) we find that two skew-symmetric bivector fields V, W on M form a Poisson pair if and only if $[V, V] = [W, W] = [V, W] = 0$. In the following theorem we assume for simplicity's sake that M is simply connected and the Poisson structures V and W are everywhere non-degenerate. On the simply connected manifold M two symplectic structures V^{-1}, W^{-1} are thereby defined, whose Poisson brackets $V(f, g)$ and $W(f, g)$ are coordinated via the identity $V(W(f, g)h) + W(V(f, g)h) + (\text{cyclic permutations}) = 0$ for any smooth functions f, g, h on M.

Theorem ([20]). *On the manifold M let there be given a vector field v whose flow preserves both Poisson structures of the Poisson pair V, W. Then there exists a sequence of smooth functions $\{f_k\}$ on M such that* a) *f_0 is a Hamiltonian of the field v*

with respect to V; b) *the field of the V-Hamiltonian f_k coincides with the field of the W-Hamiltonian f_{k+1}*; c) *the functions $\{f_k\}$ are in involution with respect to both Poisson brackets.*

Proof. By the condition, the field v is Hamiltonian for both symplectic structures. Let f_0 and f_1 be Hamiltonians of it with respect to V and W respectively. A formal calculation in application of the identity $[V, W]=0$ shows that the flow of the V-Hamiltonian field with Hamiltonian f_1 preserves the Poisson bracket of W. Let f_2 be a W-Hamiltonian of it. Continuing by induction we obtain a sequence of functions $\{f_k\}$ satisfying a) and b). Let $r>s$. Then $V(f_r, f_s)=W(f_r, f_{s+1})=V(f_{r-1}, f_{s+1})$ etc. At the end we shall obtain either $V(f_t, f_t)$ or $W(f_t, f_t)$, which proves c). □

Example. The Toda lattice (M. Toda). Let us consider a natural system on $\mathbb{R}^N$ with Hamiltonian $H=\sum p_k^2/2+\sum e^{q_k-q_{k+1}}$, $q_{N+1}=q_1$. It describes the dynamics of N identical point masses with one degree of freedom each, joined in a circle, like a benzene molecule, by elastic bonds with a potential e^u-u, where $u=q_k-q_{k+1}$ is the difference of the coordinates of the coupled neighbours. Going over to the system of variables $u_k=q_k-q_{k+1}$, we have the following equations of evolution of the Toda lattice: $\dot{u}_k=p_k-p_{k+1}$, $\dot{p}_k=e^{u_{k-1}}-e^{u_k}$. With the notation $\partial_k=\partial/\partial u_k$, $\nabla_k=\partial/\partial p_k$, let us set $W=\sum(\partial_k\wedge\partial_{k+1}+p_k\nabla_k\wedge(\partial_k-\partial_{k-1})+e^{u_k}\nabla_{k+1}\wedge\nabla_k)$. One immediately checks that W is a Poisson structure on $\mathbb{R}^{2N}$. Let us set $V=\sum\nabla_k\wedge(\partial_k-\partial_{k-1})$. W, V is a Poisson pair. Indeed, $W+\lambda V$ is obtained from W by the translation $p_k\mapsto p_k+\lambda$. As the flow preserving both structures of the Poisson pair let us consider the flow of the field $v\equiv 0$. The total momentum $f_0=\sum p_k$ is a Casimir function for V and therefore f_0 is a V-Hamiltonian of the field v. The function f_0, considered as a W-Hamiltonian, generates the system of equations of the Toda lattice. In accordance with the theorem, this system is V-Hamiltonian with the Hamiltonian $f_1=\sum(p_k^2/2+e^{u_k})$. The system with W-Hamiltonian f_1 is V-Hamiltonian with the Hamiltonian $f_2=\sum[p_k^3/3+p_k(e^{u_{k-1}}+e^{u_k})]$ etc. The arising series $f_0, f_1, f_2, \ldots$ of first integrals in involution provides for the complete integrability of the Toda lattice (see the article by B.A. Dubrovin, I.M. Krichever, S.P. Novikov in this volume).

Another method of constructing functions in involution with respect to a Poisson pair consists in the following. Let f_V, g_W be Casimir functions of the Poisson structures V, W respectively (here it is assumed that the Poisson structures V, W which form the Poisson pair are degenerate—otherwise f_V and g_W are necessarily constant).

Lemma. *The functions f_V and g_W are in involution with respect to the Poisson structure $\lambda V+\mu W$.*

We shall apply this lemma in the next item.

2.5. Functions in Involution on the Orbits of a Lie Coalgebra. Let $\mathfrak{g}$ be a Lie algebra. On the dual space $\mathfrak{g}^*$ there exists a linear Poisson structure (see sect. 3.3, chap. 2): the Poisson bracket of two linear functions x, y on $\mathfrak{g}^*$ is equal to their commutator $[x, y]$ in $\mathfrak{g}$. The symplectic leaves of this Poisson structure are the orbits of the coadjoint action of the Lie algebra $\mathfrak{g}$ on $\mathfrak{g}^*$, the Casimir functions are the invariants of the coadjoint action. The following method of constructing functions in involution on the orbits is called the *method of translation of the argument*.

Theorem. *Let f, g: $\mathfrak{g}^*\to\mathbb{R}$ be invariants of the coadjoint action of the Lie algebra $\mathfrak{g}$ and let $\xi_0\in\mathfrak{g}^*$. Then the functions $f(\xi+\lambda\xi_0)$, $g(\xi+\mu\xi_0)$ of the point $\xi\in\mathfrak{g}^*$ are in involution for any λ, $\mu\in\mathbb{R}$, on each orbit of the coadjoint action.*

The proof is based on the following lemma.

Lemma. *Let ω be an exterior 2-form on $\mathfrak{g}$. The constant Poisson structure on $\mathfrak{g}^*$ defined by the form ω forms a Poisson pair with the linear Poisson structure on $\mathfrak{g}^*$ if and only if ω is a 2-cocycle on $\mathfrak{g}$, i.e. $\forall x, y, z\in\mathfrak{g}$*

$$\omega([x, y], z)+\omega([y, z], x)+\omega([z, x], y)=0.$$

If the 2-cocycle ω is a coboundary, i.e. $\omega(x, y)=\xi_0([x, y])$, where $\xi_0\in\mathfrak{g}^*$ is a linear function on $\mathfrak{g}$, then the Poisson structure $\{\ ,\ \}_\lambda=[\cdot,\ ]+\lambda\omega(\ ,\ )$ is obtained from the linear Poisson structure $[\ ,\ ]$ by means of a translation in $\mathfrak{g}^*$ by $\lambda\xi_0$. The functions $f(\xi+\lambda\xi_0)$, $g(\xi+\mu\xi_0)$ are in this case Casimir functions for the Poisson structures $\{\ ,\ \}_\lambda$ and $\{\ ,\ \}_\mu$ respectively. If we apply the lemma of the preceding article, we shall obtain the assertion of the theorem for $\lambda\neq\mu$ and by continuity, for arbitrary λ, $\mu\in\mathbb{R}$. □

2.6. The Lax Representation. One says that a *Lax representation* of a system of differential equations $\dot{x}=v(x)$ on a manifold M is given (v is a vector field on M), if

1) there are given two mappings, L, A: $M\to\mathfrak{g}$ of the manifold M into a Lie algebra $\mathfrak{g}$ (for example, into a matrix algebra), where L is an embedding;

2) the *Lax equation* $\dot{L}=[L, A]$ holds, where $\dot{L}$ is the derivative of L along the vector field v and $[\ ,\ ]$ is the commutator in the Lie algebra $\mathfrak{g}$.

The Lax equation $\dot{L}=[L, A]$ means that L, as it changes in time, remains on the same orbit of the adjoint action of the Lie algebra $\mathfrak{g}$. Therefore the invariants of the orbit (for example, the coefficients of the characteristic polynomial or the eigenvalues of L, if $\mathfrak{g}$ is a matrix algebra) are first integrals of the system $\dot{x}=v(x)$.

Example 1. Let $H(p, q)$ be a polynomial Hamiltonian on the standard symplectic space $\mathbb{R}^{2n}$ with the singular point 0. Let us decompose H as a sum $\sum H_k$ of homogeneous components of degree k ($k\neq 1$) and let us set $G=\sum H_k/(k-1)$. Let us consider the following matrices of size $(2n+1)\times(2n+1)$

(E is the unit matrix of size n):

$$\Lambda = \left(\begin{array}{cc|c} 0 & E & \\ -E & 0 & 0 \\ \hline & 0 & 1 \end{array}\right),$$

$$L = \left(\begin{array}{cc|c} & & p \\ \multicolumn{2}{c|}{0} & q \\ \hline p & q & 0 \end{array}\right) \quad \Lambda = \left(\begin{array}{cc|c} & & p \\ \multicolumn{2}{c|}{0} & q \\ \hline -q & p & 0 \end{array}\right),$$

$$A = \Lambda \left(\begin{array}{c|c} d^2 G & 0 \\ \hline 0 & 0 \end{array}\right) = \left(\begin{array}{cc|c} G_{qp} & G_{qq} & \\ -G_{pp} & -G_{pq} & 0 \\ \hline & 0 & 0 \end{array}\right).$$

Then $\dot{L} = [L, A]$ is the Lax representation for the system of Hamilton's equations with Hamiltonian H (we make use of the Euler formulas $G_{pp}p + G_{pq}q = H_p$, $G_{qp}p + G_{qq}q = H_q$).

In this example $L^3 = 0$, and no first integrals arise at all. Usually integrable systems are connected with nontrivial one-parameter families of Lax representations.

Example 2. In example 1 let H be a quadratic Hamiltonian. Then A is a constant matrix. We may set $L_\lambda = L + \lambda A$, where λ is a parameter, and we obtain the Lax representation $\dot{L}_\lambda = [L_\lambda, A]$ of the linear Hamiltonian system. Now let S be the matrix of the quadratic Hamiltonian $\langle Sz, z\rangle/2$ in the Darboux coordinates $z = (p, q)$, and let $\Omega = \begin{pmatrix} 0 & -E \\ E & 0 \end{pmatrix}$ be the matrix of the symplectic form. Then the characteristic polynomial of the matrix L_λ has the form: $\det(\mu E_{2n+1} - L_\lambda) = \det(\lambda S - \mu\Omega)[\mu + \langle(\lambda S - \mu\Omega)^{-1}\Omega z, \Omega z\rangle]$. The coefficients in this polynomial of the μ^{2k}, $k = 0, \ldots, n-1$, are first integrals, quadratic in z, of our linear Hamiltonian system. They are in involution. Indeed, setting $w_\mu = \Omega^*(\lambda S - \mu\Omega)^{-1}\Omega$ and making use of the identity $w_\alpha - w_\beta = (\alpha - \beta)w_\beta \Omega w_\alpha$, for the quadratic forms $I_\alpha = \langle w_\alpha z, z\rangle$ and $I_\beta = \langle w_\beta z, z\rangle$ we get

$$\begin{aligned} \{I_\alpha, I_\beta\} &= \langle \Omega(w_\alpha + w_{-\alpha})z, (w_\beta + w_{-\beta})z\rangle \\ &= (\alpha - \beta)^{-1}\langle[(w_\alpha - w_{-\alpha}) - (w_\beta - w_{-\beta})]z, z\rangle \\ &\quad + (\alpha + \beta)^{-1}\langle[(w_\alpha - w_{-\alpha}) + (w_\beta - w_{-\beta})]z, z\rangle = 0, \end{aligned}$$

since the $w_\mu - w_{-\mu}$ are skew-symmetric matrices.

Practically all completely integrable systems known at the present day can be integrated with the aid of a suitable Lax representation in which L and A are matrices with coefficients which are polynomial in the parameter λ.

Example 3. The free rotation of a multidimensional rigid body. The system under consideration is equivalent to the geodesic flow of a particular left-invariant Riemannian metric on the group SO_n. The metric is given by the inertia quadratic form "in the internal coordinates of the body" (see sect. 1.1), i.e. on the Lie algebra so_n. As we shall see in §3, the investigation of such a system reduces to the study of Hamiltonian flows on the orbits of the coadjoint action on so_n^* with a quadratic Hamilton function. The inertia quadratic form on the algebra so_n of skew-symmetric $n \times n$ matrices has the form $-\mathrm{tr}(\omega D\omega)$, where $\omega \in so_n$, $D = \mathrm{diag}(d_1, \ldots, d_n)$, $d_k = \frac{1}{2}\int \rho(x)x_k^2\,dx$, and $\rho(x)$ is the density of the body at the point $x = (x_1, \ldots, x_n)$. Denoting by M the operator of the inertia form, M: $so_n \to so_n^*$, we obtain for the angular momentum $M(\omega)$ the *Euler equation* $\dot{M} = \mathrm{ad}_\omega^* M$. In matrix form $M(\omega) = D\omega + \omega D$, and the Euler equation has the Lax form $\dot{M} = [M, \omega]$. Setting $M_\lambda = M + \lambda D^2$, $\omega_\lambda = \omega + \lambda D$, we obtain a Lax representation with parameter for the Euler equation: $\dot{M}_\lambda = [M_\lambda, \omega_\lambda]$. This representation guarantees the complete integrability of the free rotation of an n-dimensional rigid body about an immovable point. The involutivity of the first integrals $H_{\lambda,\mu} = \det(M + \lambda D^2 + \mu E)$ can be proved using the theorem on translation of the argument out of the preceding item (see [24]).

§3. Hamiltonian Systems with Symmetries

The procedure described in sect. 1.5 for reducing the order of a Hamiltonian system invariant with respect to a Hamiltonian flow is generalized below to the case of an arbitrary Lie group of symmetries.

3.1. Poisson Actions and Momentum Mappings. Let the Lie group G act on the connected symplectic manifold (M, ω) by symplectomorphisms. Then to each element of the Lie algebra $\mathfrak{g}$ of the group G there corresponds a locally Hamiltonian vector field on M. We shall assume in the following that all these vector fields have single-valued Hamiltonians. If we choose such Hamiltonians for a basis in $\mathfrak{g}$, we get a linear mapping $\mathfrak{g} \to C^\infty(M)$ which associates to an element $a \in \mathfrak{g}$ its Hamiltonian H_a. The Poisson bracket $\{H_a, H_b\}$ may differ from $H_{[a,b]}$ by a constant: $\{H_a, H_b\} = H_{[a,b]} + C(a, b)$.

Definition. An *action of a connected Lie group G* by symplectomorphisms on a connected symplectic manifold is called a *Poisson action* if the basis Hamiltonians are chosen so that $C(a, b) = 0$ for all $a, b \in \mathfrak{g}$.

Remark. In the general case the function $C(a, b)$ is bilinear, skew-symmetric and satisfies the identity $C([a, b], c) + C([b, c], a) + C([c, a], b) = 0$, that is, it is a

2-cocycle of the Lie algebra $\mathfrak{g}$. A different choice of the constants in the Hamiltonians H_a leads to the replacement of the cocycle C by the cohomologous $C'(a, b) = C(a, b) + p([a, b])$, where p is a linear function on $\mathfrak{g}$. Thus a symplectic action determines a cohomology class in $H^2(\mathfrak{g}, \mathbb{R})$ and is a Poisson action if and only if this class is zero. In the latter case the basis of Hamiltonians for which $C(a, b) \equiv 0$ is determined uniquely up to the addition of a 1-cocycle $p: \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] \to \mathbb{R}$ and gives a homomorphism of the Lie algebra $\mathfrak{g}$ into the Lie algebra of Hamiltonian functions on M.

A Poisson action defines a *momentum mapping* $P: M \to \mathfrak{g}^*$ whose components are the basis Hamiltonians: to the point $x \in M$ corresponds the functional $P(x)|_{a \in \mathfrak{g}} = H_a(x)$.

The Poisson action of the group G on M goes over under the momentum mapping into the coadjoint action of the group G: $P(gx) = \mathrm{Ad}_g^* P(x)$. On this consideration is based the classification of homogeneous symplectic manifolds (see the article of A.A. Kirillov in the present volume).

Let the connected Lie group G act on the connected manifold V and let $M = T^*V$ be its cotangent bundle.

Theorem. *The natural action of the group G on M is Poisson (with the choice of Hamiltonians indicated below).*

In fact, let v_a be the vector field on M of the one-parameter subgroup of the element $a \in G$. The action form α on M is G-invariant, and therefore $L_{v_a}\alpha = di_{v_a}\alpha + i_{v_a}d\alpha = 0$. This equality means that $H_a = i_{v_a}\alpha$ is the Hamiltonian of the field v_a. Since the function H_a is linear in the momenta, $\{H_a, H_b\}$ and $H_{[a,b]}$ are also linear and, consequently, equal. □

Corollary. *The value of the Hamiltonian H_a on a covector $p \in T_x^* V$ is equal to the value of the covector p on the velocity vector of the one-parameter subgroup of the element $a \in \mathfrak{g}$ at the point x.*

The momentum mapping in this case can be described as follows. Let us consider the mapping $G \to M$ defined by the action on some stipulated point $x \in M$. The preimage of the 1-form α on M under this mapping is a 1-form on G. Its value at the identity element of the group is just the momentum $P(x)$ of the point x.

Examples. 1) The group SO_3 of rotations of the Euclidean space $\mathbb{R}^3$ is generated by the one-parameter subgroups of the rotations with unit velocity about the q_1, q_2, q_3 coordinate axes. The corresponding Hamiltonians are the components of the angular momentum vector: $M_1 = q_2p_3 - q_3p_2$ etc.

2) The action of the group by left translations on its cotangent bundle is Poisson. The corresponding momentum mapping $P: T^*G \to \mathfrak{g}^*$ coincides with the right translation of covectors to the identity element of the group.

3.2. The Reduced Phase Space and Reduced Hamiltonians. Let us suppose that the Hamiltonian H on the symplectic manifold M is invariant with respect to the Poisson action of the group G on M. The components of the momentum mapping are first integrals of such a Hamiltonian system.

Let us denote by M_p the fibre above the point $p \in \mathfrak{g}^*$ of the momentum mapping $P: M \to \mathfrak{g}^*$. Let G_p be the stabilizer of the point p in the coadjoint representation of the group G. The group G_p acts on M_p. The quotient space $F_p = M_p/G_p$ is called the *reduced phase space.*

In order for F_p to be a smooth manifold certain assumptions are necessary. For example, it is sufficient to suppose that a) p is a regular value of the momentum mapping, so that M_p is a smooth manifold; b) G_p is a compact Lie group; c) the elements of the group G_p act on M_p without fixed points. Condition b) can be weakened: it is enough to suppose that the action of G_p on M_p is proper, i.e. under the mapping $G_p \times M_p \to M_p \times M_p$: $(g, x) \mapsto (gx, x)$ the preimages of compacta are compact. For example, the action of a group on itself by translations is always proper.

Let us suppose that the conditions we have formulated are fulfilled.

Theorem (Marsden-Weinstein, see [2]). *The reduced phase space has a natural symplectic structure.*

The skew-scalar product of vectors on F_p is defined as the skew-scalar product of their preimages under the projection $M_p \to F_p$ applied at one point of the fibre of the projection. One can show that the tangent space $T_x M_p$ to the fibre of the momentum mapping and the tangent space $T_x(Gx)$ to the orbit of the group G are skew-orthogonal complements of each other in the tangent space $T_x M$ and intersect along the isotropic tangent space $T_x(G_p x)$ to the orbit of the stabilizer G_p. From this follows the well-definedness of the skew-scalar product and its nondegeneracy. □

An invariant Hamiltonian H defines a *reduced Hamilton function* H_p on F_p. The Hamiltonian vector field on M corresponding to the function H is tangent to the fibre M_p of the momentum mapping and is invariant with respect to the action of the group G_p on M_p. Therefore it defines a *reduced vector field* X_p on F_p.

Theorem. *The reduced field on the reduced phase space is Hamiltonian with the reduced Hamilton function.*

Example. In the case of the action of a Lie group by left translations on its cotangent bundle the fibre M_p of the momentum mapping is the right-invariant section of the cotangent bundle equal to p at the identity element of the group. The stationary subgroup G_p coincides with the stabilizer of the point p in the coadjoint representation. The reduced phase space F_p is symplectomorphic to the orbit of the point p.

3.3. Hidden Symmetries. One speaks of hidden symmetries when a Hamiltonian system possesses a nontrivial *Lie algebra of first integrals* not connected *a priori* with any action of a finite-dimensional symmetry group. A generalization of the momentum mapping in such a situation is the concept of a *realization of a Poisson structure* [74]—a submersion $M \to N$ of a symplectic manifold onto the Poisson manifold under which the Poisson bracket of functions on N goes over into the Poisson bracket of their pull-backs on M. We have the following obvious

Lemma. *A submersion $M \to N$ is a realization if and only if the inverse images of the symplectic leaves are coisotropic.*

Let the Hamiltonian $H\colon M \to \mathbb{R}$ commute with the Lie algebra $\mathscr{A}$ of functions lifted from N under the realization. All such Hamiltonians form a Lie algebra $\mathscr{A}'$ connected with a realization $M \to N'$ of a different Poisson structure. This realization is called *dual* to the original one and can be constructed as follows. Let us consider on M the distribution of skew-orthogonal complements to the fibres of the submersion $M \to N$. It is generated by the fields of the Hamiltonians in the Lie algebra $\mathscr{A}$. Therefore this distribution is integrable and is tangent to the coisotropic inverse images of the symplectic leaves. The projection $M \to N'$ along its integral manifolds (which is defined at least locally) is (by the lemma) a realization of a Poisson structure arising on N'. The symplectic leaves of the latter are reduced phase spaces on which the Hamiltonian $H \in \mathscr{A}'$, considered as a function on N', defines a reduced motion.

The momentum mappings of the actions of a Lie group by left and right translations on its cotangent bundle furnish an important example of dual realizations.

In the general case there is a close connection between the Poisson manifolds N and N' of dual realizations. For example, they share common Casimir functions—considered as functions on M, they form the subalgebra $\mathscr{A} \cap \mathscr{A}'$. There is a correspondence between the symplectic leaves in N and N', bijective if the inverse images of these leaves in M are connected: leaves with intersecting inverse images correspond to one another.

Theorem ([74]). *The germs of the transversal Poisson structures to corresponding symplectic leaves of dual realizations are anti-isomorphic* (*i.e. isomorphic up to the sign of the Poisson bracket*).

Let us define an equivalence of realizations $M_1 \to N$, $M_2 \to N$ as a symplectomorphism $M_1 \to M_2$ commuting with them, and a stabilization of the realization $M \to N$ as its composition with the projection onto a factor $M \times \mathbb{R}^{2k} \to M$ of the product of symplectic manifolds.

Theorem ([74]). *A germ $(\mathbb{R}^n, 0)$ of a Poisson structure at a point of corank r possesses a realization $P\colon (\mathbb{R}^{n+r}, 0) \to (\mathbb{R}^n, 0)$. Any realization of it is equivalent to a stabilization of P.*

The construction of the realization P cited in [74] is a non-linear generalization of the momentum mapping $T^*G \to \mathfrak{g}^*$ of a Lie group G.

3.4. Poisson Groups. The Poisson bracket of two functions on a symplectic manifold which are invariant under a symplectic action of a Lie group is again an invariant function. The converse of this assertion is false—the algebra of invariants' being closed under the Poisson bracket does not imply symplecticity of the action. This circumstance led V.G. Drinfel'd to generalize the procedure of reduction of Hamiltonian systems to a broader class of actions.

Let us consider the category whose objects are the Poisson manifolds and whose morphisms are the Poisson mappings, that is, the smooth mappings which transform the Poisson bracket of two functions into the Poisson bracket of their pull-backs. A product $M \times N$ of Poisson manifolds is endowed with a Poisson structure for which the projections onto the factors are Poisson mappings and the pull-backs of functions from different factors are in involution. By a Poisson group one means a Poisson manifold endowed with a Lie group structure for which multiplication $G \times G \to G$ is a Poisson mapping and inversion $G \to G$ is an anti-automorphism (changes the sign of the Poisson bracket). An example is the additive group of a Lie coalgebra.

On the Lie algebra $\mathfrak{g}$ of a Poisson group G there is defined a linear Poisson structure—the linearization of the Poisson structure on G at the identity. Therefore a Lie algebra structure is defined on $\mathfrak{g}^*$ (the double structure arising here is a Lie bialgebra structure in the sense of [21]). In the example cited above it coincides with the original Lie algebra structure.

An action of a Poisson group G on a Poisson manifold M is called Poisson if $G \times M \to M$ is a Poisson mapping. In the case of an action of a group G with the trivial Poisson structure on a symplectic manifold M this condition is equivalent to the action's being symplectic (but not to its being Poisson in the old sense!).

It is not hard to verify that the invariants of a Poisson action of a Poisson group on a Poisson manifold form a Lie subalgebra in the Lie algebra of functions on it. The same is true for the invariants of a connected subgroup $H \subset G$ if the orthogonal complement $\mathfrak{h}^\perp \subset \mathfrak{g}^*$ of its Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subalgebra in $\mathfrak{g}^*$.

For such a subgroup there is on the manifold M/H (if it exists) a unique Poisson structure under which the projection $M \to M/H$ is a Poisson mapping. On the symplectic leaves in M/H an H-invariant Hamiltonian defines a reduced motion. Let us note that in the construction described the condition imposed on H does not mean that H is a Poisson subgroup—the latter is true if $\mathfrak{h}^\perp$ is an ideal in $\mathfrak{g}^*$.

As an example let us consider the action of a connected subgroup H of the additive group of a Lie coalgebra G by translations on G. If $H^\perp \subset G^*$ is a Lie subalgebra, then the linear Poisson structure on its dual space is just the sought-for Poisson structure on the orbit space $G/H = (H^\perp)^*$.

Poisson groups have come to occupy an important place in the theory of completely integrable systems. Thus, the example analyzed above is closely connected with the method of translation of the argument (sect. 2.5). This direction is developing rapidly. Details can be found in the papers of M.A. Semenov-Tyan-Shanskij, for example in [63].

3.5. Geodesics of Left-Invariant Metrics and the Euler Equation. On the connected Lie group G let there be given a left-invariant Riemannian metric. It is determined by its value at the identity of the group, i.e. by a positive definite quadratic form Q on the space $\mathfrak{g}^*$. A left-invariant geodesic flow on the group defines a reduced Hamiltonian flow on each orbit of the coadjoint representation—a reduced phase space. The reduced Hamiltonian of this flow coincides with the restriction to the orbit of the quadratic form Q. Let Ω: $\mathfrak{g}^* \to \mathfrak{g}$ be the operator of the quadratic form Q. Then the reduced motion of a point $P \in \mathfrak{g}^*$ is described by the *Euler equation* $\dot{P} = \mathrm{ad}^*_{\Omega(P)} P$.

In the special case $G = SO_3$ we obtain the classical Euler equation describing the free rotation of a rigid body in the internal coordinates of the body. In vector notation it has the form $\dot{P} = P \times \Omega$, where Ω is the angular velocity vector and P is the angular momentum vector, connected with the vector Ω by a linear transformation—the inertia operator of the body. The equations of the hydrodynamics of an ideal fluid [2] and the system of Maxwell–Vlasov equations describing the dynamics of a plasma ([74]) have the form of the Euler equation. In these cases the group G is infinite-dimensional.

Let us consider in greater detail the flow of an ideal (i.e. homogeneous, incompressible, inviscid) fluid in a domain $D \subset \mathbb{R}^3$. Let G be the group of volume-preserving diffeomorphisms of the domain D. The Lie algebra $\mathfrak{g}$ consists of the smooth divergence-free vector fields on D which are tangent to the boundary of the domain D. The kinetic energy $\int v^2/2dx$ of the flow with velocity field v is a right-invariant Riemannian metric on the group G. The flow of an ideal fluid is a geodesic of this metric. The Euler equation can be written down in the form $\partial \operatorname{rot} v/\partial t = [v, \operatorname{rot} v]$, where [,] is the commutator of vector fields. Let us note that the "inertia operator" $v \mapsto \operatorname{rot} v$ maps the space $\mathfrak{g}$ bijectively onto the space of smooth divergence-free fields on D under certain restrictions on the domain D (it is sufficient that D be a contractible bounded domain with a smooth boundary).

3.6. Relative Equilibria. The phase curves of a system with a G-invariant Hamiltonian function which project into an equilibrium position of the reduced Hamiltonian function on a reduced phase space are called *relative equilibria*.

For example, the stationary rotations of a rigid body fixed at its centre of mass, but also the rotations of a heavy rigid body with constant velocity about the vertical axis are relative equilibria.

Theorem ([2]). *A phase curve of a system with a G-invariant Hamiltonian function is a relative equilibrium if and only if it is the orbit of a one-parameter subgroup of the group G in the original phase space. (Let us recall that the action of the group G_p on M_p is assumed to be free).*

Now let $G = \mathbb{R}/\mathbb{Z}$ be the circle. Let us suppose that the group G acts on the configuration manifold V without fixed points. A reduced phase space F_p of the Poisson action of G on T^*V is symplectomorphic to a twisted cotangent bundle of the factored configuration manifold V/G. The reduction of a natural Hamiltonian system on T^*V with a G-invariant potential and kinetic energy leads to a natural system in a magnetic field (see sect. 1.2), which is equal to zero only when $p = 0$.

Let an asymmetrical rigid body, fixed at a point, be subject to the action of the force of gravity or of another potential force which is symmetric with respect to the vertical axis. The reduced configuration space in this case is the two-dimensional sphere $S^2 = SO_3/S^1$.

Corollary 1. *An asymmetrical rigid body in an axis-symmetric potential field, attached at a point on the axis of the field, has at least two stationary rotations (for each value of the angular momentum with respect to the symmetry axis).*

Corollary 2. *An axis-symmetrical rigid body fixed at a point on its symmetry axis has at least two stationary rotations (for each value of the angular momentum with respect to the symmetry axis) in an arbitrary potential force field.*

Both corollaries are based on the fact that a function on the sphere—the potential of the reduced motion—has at least two critical points.

3.7. Noncommutative Integrability of Hamiltonian Systems. Let us suppose that the Hamiltonian H of a system is invariant with respect to the Poisson action of a Lie group G on the phase manifold M, and that $p \in \mathfrak{g}^*$ is a regular value of the momentum mapping P: $M \to \mathfrak{g}^*$.

Theorem. *If the dimension of the phase manifold is equal to the sum of the dimension of the algebra $\mathfrak{g}$ and its rank, then the level set M_p of a generic regular level of the momentum mapping is nonsingular and has a canonical affine structure. In this affine structure the phase flow of the invariant Hamiltonian H becomes straight. Each compact connected component of the set M_p is a torus on which the phase flow is conditionally periodic.*

Remarks. 1) We recall that the rank of a Lie algebra is the codimension of a generic orbit in the coadjoint representation.

2) The theorem just formulated generalizes Liouville's theorem on complete integrability: there the group G was a commutative group ($\mathbb{R}^n$) of rank n which acted on a symplectic manifold of dimension $2n = \dim G + \operatorname{rk} G$.

Proof. The premise $\dim M = \dim G + \operatorname{rk} G$ together with the regularity of the generic value p of the momentum mapping implies that $\dim M_p = \operatorname{rk} G = \dim G_p$, i.e. each connected component K of the level set M_p is a quotient space of (a connected component of) the group G_p by a discrete subgroup. By Duflo's theorem (sect. 3.3, chap. 2), the algebra $\mathfrak{g}_p$ (for a generic $p \in \mathfrak{g}^*$) is commutative, i.e.

$K = \mathbb{R}^n/\mathbb{Z}^k$ and in the compact case is a torus. The straightening of the flow is easily deduced from the invariance of the Hamiltonian H. □

Example. Let us consider the Kepler problem of the motion of a mass point in the Newtonian gravitational potential of a fixed centre: $H = p^2/2 - 1/r$, where r is the distance to the centre and p is momentum. Then the Hamiltonian H is invariant with respect to the group of rotations SO_3 and its flow together with the action of the group SO_3 makes up a Poisson action of the four-dimensional group $G = \mathbb{R} \times SO_3$ of rank 2 on the space $T^*\mathbb{R}^3$ of dimension $6 = 4 + 2$. Therefore the Kepler problem is integrable in the noncommutative sense. The same relates to an arbitrary natural system on the Euclidean space $\mathbb{R}^3$ with a spherically symmetric potential: the phase flow of such a system straightens out on the two-dimensional combined level sets of the angular momentum vector and the energy.

It is evident from the formulation of the theorem (and from the example) that motion in a system which is integrable in the noncommutative sense takes place on tori of dimension less than one-half the dimension of the phase space, that is, such systems are degenerate in comparison with general completely integrable systems.

Theorem ([26]). *If a Hamiltonian system on the compact symplectic manifold M^{2n} possesses a Lie algebra $\mathfrak{g}$ of almost everywhere independent first integrals, where* $\dim \mathfrak{g} + \operatorname{rk} \mathfrak{g} = 2n$, *then there exists another set of n almost everywhere independent integrals in involution.*

Geometrically this means that the invariant tori of the small dimension $\operatorname{rk} \mathfrak{g}$ can be united into tori of the half dimension.

For a Lie algebra $\mathfrak{g}$ of first integrals on an arbitrary symplectic manifold the assertion of the theorem follows from the statement: on the space $\mathfrak{g}^*$ there exist $d = (\dim \mathfrak{g} - \operatorname{rk} \mathfrak{g})/2$ smooth functions in involution which are independent almost everywhere on generic orbits in $\mathfrak{g}^*$ (their dimension is equal to $2d$).

This statement has been proved (on the basis of the method of translation of the argument, sect. 2.5) for a broad class of Lie algebras, including the semisimple ones (see [25]); its correctness for all Lie algebras would allow one to prove the analogous theorem for arbitrary and not only for compact phase manifolds.

3.8. Poisson Actions of Tori. A set of k functions in involution on a symplectic manifold gives a Poisson action of the commutative group $\mathbb{R}^k$. The compact orbits of this action inevitably are tori.

Here we shall consider the case of a Poisson action of the torus $T^k = \mathbb{R}^k/\mathbb{Z}^k$ on a compact symplectic manifold M^{2n}. For $k = n$ the geometry of such an action may be looked at like the geometry of completely integrable systems, although of a fairly special class.

Example. A Hamiltonian $H: M^2 \to \mathbb{R}$ on a compact symplectic surface gives a Poisson action of the additive group of $\mathbb{R}$. If this action is actually an action of the

group $T^1 = \mathbb{R}/\mathbb{Z}$, then the function H necessarily has as its critical points only a nondegenerate maximum and minimum, and in particular, M^2 is a sphere (Fig. 23). If this property of the function H is realized, then its product with a suitable non-vanishing function is the Hamiltonian of a Poisson action of the group T^1 on the sphere.

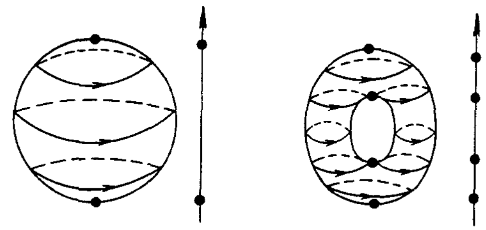

Fig. 23. Hamiltonian flows on surfaces

Theorem ([10]). *Let there be given a Poisson action of the torus $T^k = \mathbb{R}^k/\mathbb{Z}^k$ on the compact connected symplectic manifold M^{2n}. Then the image of the momentum mapping $P: M^{2n} \to (\mathbb{R}^k)^*$ is a convex polyhedron. What is more, the image of the set of fixed points of the action of the group T^k on M^{2n} consists of a finite number of points in $(\mathbb{R}^k)^*$ (called the vertices), and the image of the whole manifold coincides with the convex hull of the set of vertices. The closure of each connected component of the union of the orbits of dimension $r \le k$ is a symplectic submanifold of M^{2n} of codimension $\le 2(k-r)$, on which the quotient group $T^r = T^k/T^{k-r}$ of the torus T^k by the isotropy subgroup T^{k-r} acts in a Poisson manner. The image of this submanifold in $(\mathbb{R}^k)^*$ under the momentum mapping (a face of the polyhedron) is the convex hull of the image of its fixed points, has dimension r and lies in a subspace of dimension r parallel to the (integral) subspace of covectors in $(\mathbb{R}^k)^*$ which annihilate the tangent vectors to the stabilizer T^{k-r} in the Lie algebra $\mathbb{R}^k$ of the torus T^k.*

Remark. Under the conditions of the theorem the fibres of the momentum mapping are connected. The convexity of the image can be deduced from this by induction on the dimension of the torus.

Example. The classical origin of this theorem are the Shur inequalities for Hermitian matrices: the vector of diagonal entries of a Hermitian matrix lies in the convex hull of the vectors obtained out of the set of its eigenvalues by permutations (see Fig. 24).

Indeed, let us consider the coadjoint action of the group SU_{n+1} of special unitary matrices. It is isomorphic to the adjoint action on the Lie algebra of skew-Hermitian matrices with trace zero. The space of such matrices can by multiplication with $\sqrt{-1}$ be identified with the space of Hermitian $(n+1) \times (n+1)$ matrices with trace zero, and we may reckon that on the latter

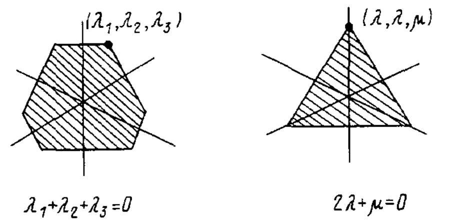

Fig. 24. The Shur inequalities

space there is given an action of the group SU_{n+1} whose orbits are compact symplectic manifolds. The maximal torus $T^n = \{\mathrm{diag}(e^{i\phi_0}, \ldots, e^{i\phi_n}) | \sum \phi_k = 0\}$ of SU_{n+1} acts in a Poisson manner on each such orbit. The momentum mapping associates to the Hermitian matrix (ω_{kl}) the vector of its diagonal entries $(\omega_{00}, \ldots, \omega_{nn})$ in the space $\mathbb{R}^n = \{(x_0, \ldots, x_n) | \sum x_k = 0\}$. The fixed points of the action of the torus on the orbit are the diagonal matrices $\mathrm{diag}(\lambda_0, \ldots, \lambda_n)$ of this orbit.

Another characteristic property of Poisson actions of tori is the *integration formula* [22]. In the simplest case of a Poisson action of the circle T^1 on a symplectic manifold (M^{2n}, ω) it has the following form. Let H: $M^{2n} \to \mathbb{R}^*$ be the Hamiltonian of the action. With each of its critical values $p \in \mathbb{R}^*$ let us connect an integer $E(p)$, equal to the product of the nonzero eigenvalues, each divided by 2π, of the quadratic part of the Hamiltonian H at a critical point $m \in H^{-1}(p)$. Then

$$\int_M e^{itH} \omega^n = \frac{n!}{(it)^n} \sum_p \frac{e^{-itp}}{E(p)},$$

where the sum is taken over all critical values. By means of the Fourier transformation one obtains from this that the function $f(h) = \int_{H=h} \omega^n / dH$ (the volume of the fibre over $h \in \mathbb{R}^*$) is a polynomial of degree $\leq n-1$ on every interval of the set of regular values of the Hamiltonian H.

As another corollary of the integration formula we find an expression for the volume of the manifold M via the characteristics of the fixed-point set of the action: $\int_M \omega^n = (-1)^n n! \sum_p p^n / E(p)$ and a series of relations on the critical values of the function H: $\sum_p p^k / E(p) = 0$ for $0 \leq k < n$.

Analogous results are true also for the actions of tori of greater dimension. The subject of this article has turned out to be connected with the theory of residues, the method of stationary phase, with characteristic classes, equivariant cohomology, Newton polyhedra, toroidal embeddings, and with the computation of characters of irreducible representations of Lie groups (see [11], [38]).

Chapter 4

Contact Geometry

Contact geometry is the odd-dimensional twin of symplectic geometry. The connection between them is similar to the relation of projective and affine geometry.

§ 1. Contact Manifolds

1.1. Contact Structure. One says that a field of hyperplanes is given on a smooth manifold if in the tangent space to every point a hyperplane is given which depends smoothly on the point of application. A field of hyperplanes is defined locally by a differential 1-form α which does not vanish: $\alpha|_x = 0$ is the equation of the hyperplane of the field at the point x. A field of hyperplanes on a $2n+1$-dimensional manifold is called a *contact structure* if the form $\alpha \wedge (d\alpha)^n$ is nondegenerate. This requirement's independence of the choice of the defining 1-form α can instantly be verified. The meaning of the definition of a contact structure becomes clearer if one considers the problem of the existence of integral manifolds of the field of hyperplanes, i.e. submanifolds which at each of their points are tangent to the hyperplane of the field. If an integral hypersurface passes through the point x, then $\alpha \wedge d\alpha|_x = 0$. Therefore a contact structure may be called a "maximally nonintegrable" field of hyperplanes. In fact the dimension of integral manifolds of a contact structure on a $2n+1$-dimensional manifold does not exceed n. For the proof let us observe that the form $d_x\alpha$ gives a symplectic structure on the hyperplane of the field at the point x, in which the tangent space to the integral submanifold passing through x is isotropic.

Integral submanifolds of dimension n in a $2n+1$-dimensional contact manifold are called *Legendre submanifolds*. A smooth fibration of a contact manifold, all of whose fibres are Legendre, is called a *Legendre fibration*.

Diffeomorphisms of contact manifolds which preserve the contact structure we shall call *contactomorphisms*.

Darboux's Theorem for Contact Manifolds. *Contact manifolds of the same dimension are locally contactomorphic.*

Corollary. *In the neighbourhood of each point of a contact $2n+1$-dimensional manifold there exist coordinates $(z, q_1, \ldots, q_n, p_1, \ldots, p_n)$ in which the contact structure has the form $dz = \sum p_k dq_k$.*

In fact, $dz = \sum p_k dq_k$ is a contact structure on $\mathbb{R}^{2n+1}$. We shall call this structure the *standard one*, and the coordinates (z, p, q) the *contact Darboux coordinates*.

Darboux's Theorem for Legendre Fibrations. *Legendre fibrations of contact manifolds of the same dimension are locally isomorphic, i.e. there exists a local contactomorphism of the total spaces of the fibrations which takes fibres into fibres.*

Corollary. *In the neighbourhood of each point of the total space of a Legendre fibration there exist contact Darboux coordinates (z, q, p) in which the fibration is given by the projection $(z, q, p) \mapsto (z, q)$.*

Indeed, the fibres $(z, q) = \text{const}$ are Legendre subspaces of the standard contact space.

1.2. Examples. A. *Projective space.* Let V be a $2n+2$-dimensional symplectic linear space, $P(V)$ its projectivization. $P(V)$ is provided with a contact structure in the following way: the hyperplane of the contact structure at a point $l \in P(V)$ is given by the hyperplane $P(H) \subset P(V)$ passing through l, where H is the skew-orthogonal complement to the straight line $l \subset V$. In Darboux coordinates $(q_0, \ldots, q_n, p_0, \ldots, p_n)$ on V and in the affine chart $q_0 = 1$ on $P(V)$ this structure has the form $dp_0 = \sum p_k dq_k - q_k dp_k$, $k \geq 1$, from which follows its maximal nonintegrability. The Legendre subspaces of $P(V)$ are the projectivizations of the Lagrangian subspaces of V.

The contact structure introduced on $P(V)$ gives an isomorphism between $P(V)$ and the dual projective space $P(V^*)$ of hyperplanes in $P(V)$, under which each point lies in the hyperplane corresponding to it. Conversely, every isomorphism of $\mathbb{P}^{2n+1}$ and $\mathbb{P}^{*2n+1}$ with this incidence property is given by a symplectic structure on the underlying vector space and consequently defines a contact structure on $\mathbb{P}^{2n+1}$. Indeed, an isomorphism of $P(V)$ and $P(V^*)$ can be lifted to an isomorphism of V and V^*, i.e. to a nondegenerate bilinear form on V; the incidence condition which was formulated is equivalent to the skew-symmetry of this form.

B. *The manifold of contact elements.* By a *contact element* on a manifold M, applied at a given point, is meant a hyperplane in the tangent space at that point. All the contact elements on M form the total space PT^*M of the *projectivized cotangent bundle.* The following rule defines a contact structure on PT^*M: the velocity vector of a motion of a contact element belongs to the hyperplane of the contact field if the velocity vector of the point of application of the contact element belongs to the contact element itself (Fig. 25). In Darboux coordinates $(q_0, \ldots, q_n, p_0, \ldots, p_n)$ on T^*M and in the affine chart $p_0 = 1$ on PT^*M this structure is given by the vanishing of the action form: $dq_0 + p_1 dq_1 + \ldots + p_n dq_n = 0$.

Let X be a smooth submanifold of M. Let us consider the set $L(X)$ of contact elements on M which are applied at points of X and are tangent to X. $L(X)$ is a Legendre submanifold of PT^*M. In the special case when X is a point, $L(X)$ is the projective space of all contact elements on M which are applied at that point. Thus, the bundle $PT^*M \to M$ is Legendrian.

C. *The space of* 1*-jets of functions.* The 1-jet of the smooth function f at the point x (notation $j^1_x f$) is just $(x, f(x), d_x f)$. The space $J^1 M = \mathbb{R} \times T^*M$ of 1-jets of functions on the manifold M has the contact structure $du = \alpha$, where u is the coordinate on the axis $\mathbb{R}$ of values of the functions and $\alpha = \sum p_k dq_k$ is the action 1-form on T^*M. The 1-graph of the function f (notation $j^1 f$) consists of the 1-jets of the function at all points. $j^1 f$ is a Legendre submanifold of $J^1 M$. The projection $J^1 M \to \mathbb{R} \times M$ along the fibres of the cotangent bundle of M is a Legendre fibration.

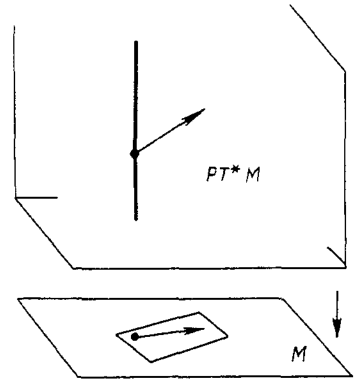

Fig. 25. The definition of the contact structure on PT^*M

One can analogously define the contact structure and the Legendre fibration of the space of 1-jets of sections of a one-dimensional vector bundle over M (not necessarily the trivial one) over the total space of this bundle.

D. *The phase space of thermodynamics.* Let us quote the beginning of the article of J.W. Gibbs "Graphical Methods in the Thermodynamics of Fluids" [31]: "We have to consider the following quantities:— v, the volume, p, the pressure, t, the (absolute) temperature, ε, the energy, η, the entropy, of a given body in any state, also W, the work done, and H, the heat received, by the body in passing from one state to another. These are subject to the relations expressed by the following differential equations:— . . . $d\varepsilon = dH - dW$, $dW = pdv$, $dH = td\eta$. Eliminating dW and dH, we have

$$d\varepsilon = td\eta - pdv. \qquad (1)$$

The quantities v, p, t, ε and η are determined when the state of the body is given, and it may be permitted to call them *functions of the state of the body.* The state of a body, in the sense in which the term is used in the thermodynamics of fluids, is capable of two independent variations, so that between the five quantities v, p, t, ε and η there exist relations expressible by three finite equations, different in general for different substances, but always such as to be in harmony with the differential equation (1)."

In our terminology the states of a body form a Legendre surface in the five-dimensional phase space of thermodynamics equipped with the contact structure (1).

1.3. The Geometry of the Submanifolds of a Contact Space. A submanifold of a contact manifold carries an induced structure. Locally this structure is given by the restriction of the defining 1-form to its tangent bundle. Forms obtained from one another by multiplication with a non-vanishing function give the same induced structure. The thus defined induced structure is finer than simply the field of tangent subspaces cut out on the submanifold by the hyperplanes of the contact structure. For example, the contact structure $du = pdq$ induces non-diffeomorphic structures on the curves $u = p - q = 0$ and $u = p - q^2 = 0$ in a neighbourhood of the point 0.

Examples. 1) In the neighbourhood of a generic point of a generic even-dimensional submanifold of a contact space there exists a coordinate system $x_1, \ldots, x_k, y_1, \ldots, y_k$ in which the induced structure has the form $dy_1 + x_2 dy_2 + \ldots + x_k dy_k = 0$. The non-generic points form a set of codimension ≥ 2.

2) In the neighbourhood of a generic point of an odd-dimensional submanifold of a contact space the induced structure is a contact structure, but in the neighbourhood of the points of some smooth hypersurface it reduces to one of the two (not equivalent) normal forms $\pm du^2 + (1 + x_1) dy_1 + x_2 dy_2 + \ldots + x_k dy_k = 0$ [51].

The induced structure defines the "exterior" geometry of the submanifold at least locally:

A. The Relative Darboux Theorem for Contact Structures. *Let N be a smooth submanifold of the manifold M and let γ_0 and γ_1 be two contact structures which coincide on TN. Then for an arbitrary point x in N there exists a diffeomorphism $U_0 \to U_1$ of neighbourhoods of the point x in M which is the identity on $N \cap U_0$ and takes $\gamma_0|_{U_0}$ over into $\gamma_1|_{U_1}$.*

In the special case $N = \{\text{point}\}$ we get the Darboux theorem for contact manifolds of sect. 1.1.

A differential 1-form on a manifold which gives a contact structure on it we shall call a *contact form*. A contact form α defines a field of directions—the field of kernels of the 2-form $d\alpha$. Thus to the form $du - \sum p_k dq_k$ corresponds the direction field $\partial/\partial u$. We shall call the contact form α transversal to a submanifold if the field of kernels of the form $d\alpha$ is nowhere tangent to it.

B. The Relative Darboux Theorem for Contact Forms. *Let α_0 and α_1 be two contact forms on the manifold M which are transversal to the submanifold N, coincide on TN, and lie in one connected component of the set of contact forms with these properties. Then there exists a diffeomorphism of neighbourhoods of the submanifold N in M which is the identity on N and takes α_1 over into α_0.*

Corollary. *A contact form reduces locally to the form $du - \sum p_k dq_k$.*

Let us pass on to the proof of theorems A and B.

Lemma. *Theorem A follows from theorem B.*

We may assume that $M = \mathbb{R}^{2n+1}$, $x = 0$, N is a linear subspace in M, and γ_0 and γ_1 are given by the contact forms α_0 and α_1 respectively. By multiplication of α_1 with an invertible function we may achieve the coincidence of α_0 and α_1 on TN and by multiplication of the forms α_0 and α_1 with the same invertible function we may make them transversal with respect to N at 0.

There exists a linear transformation of the space M which is the identity on N and which takes $\alpha_0|_x$ over into $\alpha_1|_x$ and $d_x\alpha_0$ over into $d_x\alpha_1$. Indeed, by a linear transformation $A: M \to M$ we may take $\alpha_0|_x$ over into $\alpha_1|_x$ and $\ker_x d\alpha_0$ over into $\ker_x d\alpha_1$, moreover in such a way that $\pi A|_N = \pi|_N$, where $\pi: M \to \ker_x \alpha_1$ is the projection along $\ker_x d\alpha_1$. The forms $d_x\alpha_0$ and $d_x\alpha_1$ give two symplectic structures on $\ker_x \alpha_1$ which coincide on $\pi(N)$. They can be identified by means of a linear transformation which is the identity on $\ker_x d\alpha_1$ and on $\pi(N)$ (compare §1, chap. 1). Since $N \subset \ker_x d\alpha_1 \oplus \pi(N)$ is the graph of the function $\alpha_1|_N$, the resulting transformation possesses the required properties.

Now $t\alpha_0 + (1-t)\alpha_1$, $t \in [0, 1]$, is a family of contact forms which coincide on TN and are transversal with respect to N at the point x, and consequently also in some neighbourhood of it. The lemma is proved. □

Proof of Theorem B. Following the homotopic method (see sect. 1.3, chap. 2), we arrive at the equation

$$L_{V_t}\alpha_t + \partial\alpha_t/\partial t = 0,$$

where α_t is a smooth family of contact forms which are transversal to N and coincide on TN. This equation we want to solve with respect to the family of vector fields V_t, equal to zero on N.

We shall allow the reader to look after the smoothness in t of the subsequent contructions.

A contact form α gives a trivialization of the fibration $\ker d\alpha$. If α is transversal to N, then we may consider that a neighbourhood of the manifold N in M is the trivial fibration $\mathbb{R} \times P \to P$: $(u, x) \mapsto x$ by the integral curves of the field of directions $\ker d\alpha$, where the coordinate u on the fibres is chosen so that $i_{\partial/\partial u}\alpha \equiv 1$, $N \subset \{0\} \times P$.

We want to represent the 1-form $\partial\alpha/\partial t$, equal to zero on TN, in the form of a sum $\beta + df$, where β does not depend on du, $\beta|_{T_N M} = 0$ and $f|_N = 0$. After this it will be possible to set $V = W - (f + i_V \alpha)\,\partial/\partial u$, where the field W does not depend on $\partial/\partial u$ and is determined from the equation $i_W d\alpha + \beta = 0$.

Let us set

$$\mathscr{F}(u, x) = \int_0^u [i_{\partial/\partial u}(\partial\alpha/\partial t)](\xi, x)\, d\xi, \quad \xi \in \mathbb{R}.$$

Then $\mathscr{F}|_N = 0$ and $\partial\alpha/\partial t = \beta' + d\mathscr{F}$, where β' does not depend on du and $\beta'|_{T_N M} = 0$. Using the relative Poincaré lemma out of sect. 1.5, chap. 2, we may represent β' in the form $\beta' = \beta + d\phi$, where β and $f = \mathscr{F} + \phi$ satisfy the requirements stated above. Theorem B is proved. □

1.4. Degeneracies of Differential 1-Forms on $\mathbb{R}^n$. In the neighbourhood of a generic point a generic differential 1-form reduces by means of a diffeomorphism to the Darboux normal form $du + x_1 dy_1 + \ldots + x_m dy_m$ $(n = 2m+1)$ or $(1+x_1)dy_1 + x_2 dy_2 + \ldots + x_m dy_m$ $(n = 2m)$, but in the neighbourhood of a point on some smooth hypersurface it reduces to the Martinet normal form (J. Martinet) $\pm du^2 + (1+x_1)dy_1 + x_2 dy_2 + \ldots + x_m dy_m$ $(n = 2m+1)$ or $(1 \pm x_1^2)dy_1 + x_2 dy_2 + \ldots + x_m dy_m$ $(n = 2m)$ ([51], compare sect. 1.3).

Theorem ([78]). *In the neighbourhood of a point where it does not turn to zero, a differential 1-form is either equivalent to one of the Darboux and Martinet normal forms, or its equivalence class is not determined by any finite-order jet (i.e. by a finite section of the Taylor series at the point under consideration).*

Remark. The equivalence class of the Darboux form is determined by its 1-jet, and of the Martinet form, by its 2-jet.

Example. In the neighbourhood of a point where it does not turn to zero, a generic differential 1-form on the plane reduces to the form $F(x, y)dy$ and gives the field of directions $dy = 0$. On the integral curves $y = \text{const}$ of this field let us consider the family of functions $F(\cdot, y)$. If at the point under consideration two critical points of the functions of the family merge (Fig. 26), we may choose the parameter y in such a way that the sum of the critical values of the functions $F(\cdot, y)$ will be equal to 1. Then the difference of the critical values, considered as a function of the parameter, will be a functional invariant of the equivalence class of our 1-form. In particular, a finite number of coefficients of the Taylor series does not determine the equivalence class.

For the investigation of 1-forms in the neighbourhood of singular points see [50]. It leads to the following problem. On a symplectic space with the structure ω, let v be a vector field such that $L_v\omega = \omega$. Does there exist a symplectomorphism of a neighbourhood of a singular point of the field which takes v over into its linear part at that point? The connection of this problem with the original

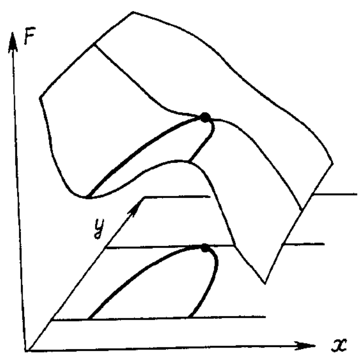

Fig. 26. A functional modulus of a 1-form on $\mathbb{R}^2$

one is as follows. A generic differential 1-form α on an even-dimensional space gives a symplectic structure in a neighbourhood of a singular point. For the field v defined by the condition $i_v\omega = \alpha$ we get $L_v\omega = di_v\omega = d\alpha = \omega$.

Theorem ([50]). *For an arbitrary vector field v on the symplectic space $(\mathbb{R}^{2n}, \omega)$, with a given linear part V at a singular point, and having the property $L_v\omega = \omega$, to be symplectically C^∞-equivalent to V, it is necessary and sufficient that among the eigenvalues $\lambda_1, \ldots, \lambda_{2n}$ of the field V there be no relations of the form $\sum m_k\lambda_k = 1$, $0 \le m_k \in \mathbb{Z}$, $\sum m_k \ge 3$.*

We note that both the vector field v and its linear part are a sum of the Euler field $E = (1/2)\sum x_k \partial/\partial x_k$ and a Hamiltonian field with a singular point at the coordinate origin. Therefore the spectrum of the field V is symmetric with respect to $\lambda = 1/2$. In Darboux coordinates the 1-form $i_E\omega$ has the form $\sum (p_k dq_k - q_k dp_k)/2$.

Corollary. *A generic hypersurface in a contact space, in a neighbourhood of a point of tangency with the hyperplane of the contact field, reduces, by means of a suitable choice of coordinates, in which the contact structure has the form $dt = \sum (p_k dq_k - q_k dp_k)$, to the normal form $t = Q(p, q)$, where Q is a nondegenerate quadratic Hamiltonian.*

Remark. The Hamiltonian Q can be taken in the normal form H_0 of sect. 3.1, chap. 1, since the indicated contact structure is $\mathrm{Sp}(2n, \mathbb{R})$-invariant.

§ 2. Symplectification and Contact Hamiltonians

Symplectification associates to a contact manifold a symplectic manifold of dimension one greater. We shall bring a description of the Lie algebra of infinitesimal contactomorphisms based on the properties of this operation. The dual operation of contactification will be discussed.

2.1. Symplectification. Let M be a contact manifold. Let us consider the total space L of the one-dimensional bundle $L \to M$ whose fibre over a point $x \in M$ is formed by all nonzero linear functions on the tangent space T_xM which vanish on the hyperplane of the contact field at the point x. We shall call such functions contact functionals. Giving L as a subbundle of the cotangent bundle T^*M is equivalent to the introduction of a contact structure on M. On the manifold L a differential 1-form α is canonically defined: the value of α on a tangent vector v applied at the point $\xi \in L$ is equal to the value of the contact functional ξ on the image of the vector v under the projection $L \to M$ (Fig. 27).

Example. Let $M = PT^*B$ be the projectivized cotangent bundle with the canonical contact structure. Then $L = T^*B \setminus B$ is the cotangent bundle with the zero section removed and α is the action 1-form on T^*B.

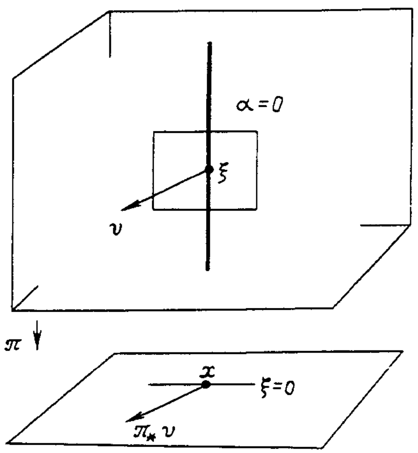

Fig. 27. Symplectification

In the general case the 1-form α on the manifold L defines the symplectic structure $d\alpha$. Its nondegeneracy follows from the example cited, in view of the local uniqueness of the contact structure.

Definition. The symplectic manifold $(L, d\alpha)$ is called the *symplectification* of the contact manifold M.

The multiplicative group $\mathbb{R}^\times$ of nonzero scalars acts on L by multiplication of the contact functionals with constants. This action turns $L \to M$ into a principal bundle. The symplectic structure $d\alpha$ is homogeneous of degree 1 with respect to this action. Conversely, a principal $\mathbb{R}^\times$-fibration $N \to M$ of a symplectic manifold with a homogeneous degree-1 symplectic structure gives a contact structure on M for which N is the symplectification.

The Properties of the Symplectification. **A.** The inclusion $L \subsetneq T^*M$ and the projection $L \to M$ establish a one-to-one correspondence between the contactomorphisms of the manifold M and the symplectomorphisms of the manifold L which commute with the action of the group $\mathbb{R}^\times$.

B. The projection of the symplectification $L \to M$ gives a one-to-one correspondence between the $\mathbb{R}^\times$-invariant ("conical") Lagrangian submanifolds of L and the Legendre submanifolds of M.

C. The composition of the projection $L \to M$ and a Legendre fibration $M \to B$ defines an $\mathbb{R}^\times$-invariant Lagrangian fibration $L \to B$, and vice versa. Using an $\mathbb{R}^\times$-invariant version of Darboux's theorem for Lagrangian fibrations, it is easy to deduce from this Darboux's theorem for Legendre fibrations.

D. The fibres of a Lagrangian fibration carry a canonical affine structure (see sect. 4.2, chap. 2). Together with the $\mathbb{R}^\times$-action on the space of the symplectification this allows one to introduce a canonical projective structure on the fibres

of a Legendre fibration. This projective structure may be described explicitly thus. The hyperplane of the contact field at a point $x \in M$ contains the tangent space to the fibre of the fibration π: $M \to B$ passing through x and consequently projects into a contact element on B, applied at the point $\pi(x)$. We obtain a local contactomorphism $M \to PT^*B$ mapping the Legendre fibres into fibres.

Corollary. *A Legendre fibration with a compact fibre is canonically contactomorphic to the projectivized cotangent bundle of the base space or to its fibrewise covering, i.e. to* the sphericalized cotangent bundle, *if the dimension of the fibre is greater than one (in the case of a one-dimensional fibre there is a countable number of different coverings).*

2.2. The Lie Algebra of Infinitesimal Contactomorphisms. Vector fields on a contact manifold whose local flows preserve the contact structure are called *contact vector fields.* Such fields obviously form a Lie subalgebra of the Lie algebra of all vector fields on the contact manifold.

We shall define in the following way the symplectification of a contact vector field: it is the vector field on the symplectification of the contact manifold whose flow is the symplectification of the flow of the original contact field.

Theorem. *The symplectification of contact vector fields gives an isomorphism of the Lie algebra of such fields with the Lie algebra of locally Hamiltonian $\mathbb{R}^\times$-invariant vector fields on the symplectification of the contact manifold. The Hamiltonian of such a field can be made homogeneous of degree* 1 *by the addition of a locally constant function.*

Now let us suppose that the contact structure on the manifold is given by a globally defined differential 1-form α. The contact form α defines a section of the bundle $L \to M$ of contact functionals. Thus the existence of the form α is equivalent to the triviality of this bundle. As soon as a section has been chosen, it gives a one-to-one correspondence between the homogeneous degree-1 Hamiltonians on L and the functions on M.

Definition. The *contact Hamiltonian* of a contact vector field on M is the function on M which at a point x is equal to the value of the homogeneous Hamiltonian of the symplectification of this vector field on the contact functional $\alpha|_x$, considered as a point of the fibre above x in the bundle $L \to M$.

Let us cite the coordinate formulas for the contact field V_K of a function K. Let $\alpha = du - pdq$ (we drop the summation sign). Then (with the notation $dK = K_u du + K_p dp + K_q dq$)

$$V_K = (K - pK_p)\partial/\partial u + (K_q + pK_u)\partial/\partial p - K_p \partial/\partial q.$$

Corollaries. 1. *The contact Hamiltonian K of a contact field V is equal to the value of the form α on this field: $K = i_V \alpha$.*

2. *The correspondence $V \mapsto i_V\alpha$ maps the space of contact fields bijectively onto the space of smooth functions. In particular, triviality of the bundle $L \to M$ implies the global Hamiltonicity of all $\mathbb{R}^\times$-invariant locally Hamiltonian fields on L.*

The Lie algebra structure introduced in this manner on the space of smooth functions on M is called the *Lagrange bracket*. The explicit description of this operation looks like this. A contact diffeomorphism, preserving the contact structure, multiplies the form α by an invertible function. Therefore we may associate to a contact Hamiltonian K a new function ϕ_K by the rule $L_{V_K}\alpha = \phi_K\alpha$. Then the Lagrange bracket $[F, G]$ of two functions will take the form $[F, G] = (L_{V_F}G - L_{V_G}F + F\phi_G - G\phi_F)/2$. With the previous coordinate notations

$$\phi_F = F_u,$$

$$[F, G] = FG_u - F_uG - p(F_pG_u - F_uG_p) - F_pG_q + F_qG_p,$$

from which, of course, the cited invariant formula for the Lagrange bracket follows. It follows from the intrinsic definition of the Lagrange bracket that the expression on the right-hand side satisfies the Jacobi identity (this is a non-obvious formula).

The Lagrange bracket does not give a Poisson structure (§3, chap. 2), inasmuch as it does not satisfy the Leibniz rule. We shall denote as a *Lie structure* on a manifold a bilinear operation [,] on the space of smooth functions which gives a Lie algebra structure on this space and has the property of localness, i.e. $[f, g]|_x$ depends only on the values of the functions f, g and of their partial derivatives of arbitrary order at the point x. One can show [42] that a Lie manifold canonically breaks up into smooth symplectic and contact manifolds. This result is a generalization of the theorem on symplectic leaves for Poisson manifolds. The analogues of the other properties of Poisson structures (transversal structures, linearization, and the like) for Lie structures have not been studied.

2.3. Contactification. It is defined in the case when a symplectic structure on a manifold N is given as the differential of a 1-form α. By definition the contactification of the manifold (N, α) is the manifold $\mathbb{R} \times N$ with the contact structure $du = \alpha$, where u is the coordinate on $\mathbb{R}$.

Example. Let N be the symplectification of a contact manifold. Since the symplectic structure on N is given as the differential of the canonical 1-form α, the contactification of N is defined. In the special case $N = T^*B$ the contactification of the manifold N is the space of 1-jets of functions on B.

For a given symplectic manifold (N, ω) with an exact symplectic structure a different choice of the potential α $(d\alpha = \omega)$ leads to different contact structures on $\mathbb{R} \times N$. Nevertheless if the difference of two potentials α_1, α_2 is exact $(\alpha_1 - \alpha_2 = d\phi)$, then the corresponding structures are equivalent in the following sense: the translation $(u, x) \mapsto (u + \phi(x), x)$ is a contactomorphism of $(\mathbb{R} \times N, du - \alpha_1)$ onto $(\mathbb{R} \times N, du - \alpha_2)$. If the closed form $\alpha_1 - \alpha_2$ is not a total differential, then these contact manifolds might not be contactomorphic.

The situation described is typical. The symplectification of contact objects always exists and leads to a topologically trivial symplectic object. The contactification exists only under certain conditions of topological triviality and may give a nonunique result. Here is yet another example of this sort. Let there be given a contactification $\mathbb{R} \times N \to N$. By a contactification of a Lagrangian manifold $\Lambda \subset N$ is meant a Legendre submanifold $L \subset \mathbb{R} \times N$ which projects diffeomorphically onto Λ. It is not difficult to convince oneself that a contactification of the Lagrangian manifold exists precisely in the case when the closed 1-form $\alpha|_\Lambda$ on Λ is exact. If $\alpha|_\Lambda = d\phi$, then one may set $L = \{(\phi(\lambda), \lambda) \in \mathbb{R} \times N \mid \lambda \in \Lambda\}$. The function ϕ is defined uniquely up to the addition of a locally constant function on Λ, and we see that the contactification is nonunique. Lagrangian embeddings which admit a contactification will be called *exact*.

2.4. Lagrangian Embeddings in $\mathbb{R}^{2n}$. An embedded circle in the symplectic plane does not possess a contactification: the integral $\int p\,dq$ is equal to the area of the region bounded by this circle and is different from zero. In other words, the projection of a Legendre circle in $\mathbb{R}^3$ onto the symplectic plane has self-intersection points. The question of the existence of exact Lagrangian embeddings is non-trivial already for the two-dimensional torus.

Theorem ([7]). *An orientable compact Lagrangian submanifold in the symplectic space $\mathbb{R}^{2n}$ has vanishing Euler characteristic.*

Proof. The self-intersection number of an orientable submanifold in a tubular neighbourhood of it is the same as in the containing space. The self-intersection number in Euclidean space is zero. The self-intersection number in a tubular neighbourhood is equal to the Euler characteristic of the normal bundle. For a Lagrangian submanifold the normal bundle is isomorphic to the tangent bundle. □

In particular, the sphere cannot be embedded Lagrangianly in $\mathbb{R}^4$. The torus admits Lagrangian embeddings in $\mathbb{R}^4$. There exist exact Lagrangian embeddings of the torus in the space $\mathbb{R}^4$ with a nonstandard symplectic structure. Exact Lagrangian embeddings of the torus in the standard symplectic space do not exist.

Theorem (M. Gromov, 1984). *A closed n-dimensional manifold has no exact Lagrangian embeddings into the standard 2n-dimensional symplectic space.*

This theorem, applied to the two-dimensional torus, implies the existence of a symplectic manifold diffeomorphic to $\mathbb{R}^4$ but not symplectomorphic to any region in the standard symplectic space $\mathbb{R}^4$.

The theorem may be reformulated as follows (see [7]): *a compact hypersurface of a front*[10] *in* $J^0\mathbb{R}^n$ *with an everywhere non-vertical tangent space has a vertical chord with parallel tangent spaces at the ends* (Fig. 28).

This result is closely connected with the generalizations of Poincaré's geometric theorem which were discussed in sect. 4.3 of chap. 2. However Gromov's arguments are different from the variational methods described there and are based on the study of quasi-Kähler structures on a symplectic manifold. The possibilities of the methods he developed extend far beyond the scope of the theorem stated above.

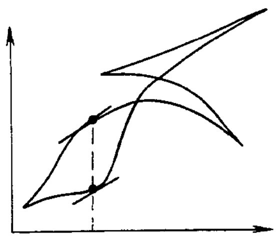

Fig. 28. The geometric meaning of Lagrangian self-intersections

§ 3. The Method of Characteristics

The partial differential equation $\sum a_i(x)\,\partial u/\partial x_i = 0$ expresses the fact that the sought-for function u is constant on the phase curves of the vector field $\sum a_i\,\partial/\partial x_i$. It turns out that an arbitrary first-order partial differential equation $F(u, \partial u/\partial x, x) = 0$ admits a reduction to a system of ordinary differential equations on the hypersurface $F(u, p, x) = 0$ in the contact space of 1-jets of functions of x.

3.1. Characteristics on a Hypersurface in a Contact Space. Let $\Gamma \subset M^{2n+1}$ be a hypersurface in a contact manifold. By the characteristic direction $l(x)$ at a point $x \in \Gamma$ is meant the kernel of the restriction of the differential $d_x\alpha$ of the contact 1-form α to the (generally $2n-1$-dimensional) intersection $\Pi(x) \cap T_x\Gamma$ of the hyperplane of the contact field Π with the tangent space to the hypersurface. An equivalent definition: on the inverse image of the hypersurface Γ under the symplectification $L^{2n+2} \to M^{2n+1}$ there is defined an $\mathbb{R}^\times$-invariant field of directions—the skew-orthogonal complements to the tangent hyperplanes. The projection of this field to M^{2n+1} defines the field of characteristic directions on Γ. It has singular points where Γ is tangent to the hyperplanes of the contact field Π.

[10] For the definition of a front see sect. 1.1 of chap. 5.

The integral curves of the field of characteristic directions are called the *characteristics of the hypersurface* Γ.

Proposition. *Let $N \subset \Gamma$ be an integral submanifold of the contact structure, not tangent to the characteristic of the hypersurface Γ passing through the point $x \in N$. Then the union of the characteristics of Γ passing through N in the neighbourhood of the point x is again an integral submanifold.*

Corollary 1. *If N is Legendrian then the characteristics passing through N lie in N.*

This property of characteristics may also be taken as their definition.

Corollary 2. *If N has dimension $n-1$, then a Legendre submanifold of Γ containing a neighbourhood of the point x in N exists and is locally unique.*

3.2. The First-Order Partial Differential Equation. Such an equation on an n-dimensional manifold B is given by a hypersurface Γ in the space J^1B of 1-jets of functions on B. A solution of the equation Γ is a smooth function on B whose 1-graph (see sect. 1.2) lies on Γ. By corollary 1 of the preceding item, the 1-graph of a solution consists of characteristics of the hypersurface Γ.

Let D be a hypersurface in B and ϕ a smooth function on D. By a solution of the Cauchy problem for the equation Γ with the initial condition (D, ϕ) is meant a solution of the equation Γ which coincides with ϕ on D. We note that the initial condition defines an $(n-1)$-dimensional submanifold $\Phi = \{(\phi(x), x) \mid x \in D\}$ in the space $J^0B = \mathbb{R} \times B$. This submanifold, just like any submanifold of the base space of a Legendre fibration, defines a Legendre submanifold $\Psi \subset J^1B$, which consists of all possible extensions of the 1-jets of the function ϕ on D to 1-jets of functions on B: $\Psi = \{(u, p, x) \mid x \in D, u = \phi(x), p|_{T_xD} = d_x\phi\}$. The intersection $N = \Psi \cap \Gamma$ is called the initial manifold of the Cauchy problem. A point of the initial manifold is called noncharacteristic, if at this point the intersection of Ψ with Γ is transversal (see Fig. 29). We note that the points of tangency of the characteristics with the initial manifold do not satisfy this requirement.

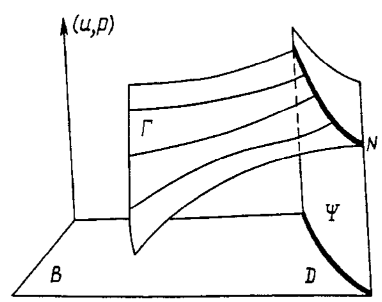

Fig. 29. The solution of the Cauchy problem by the method of characteristics

Theorem. *A solution of the Cauchy problem in the neighbourhood of a noncharacteristic point of the initial manifold exists and is locally unique.*

The 1-graph of the solution consists of the characteristics which intersect the initial manifold in a neighbourhood of this point.

Coordinate Formulas. Let $\Gamma \subset J^1(\mathbb{R}^n)$ be given by the equation $F(u, p, x) = 0$. Then the equation of the characteristics has the form

$$\dot{x} = F_p, \qquad \dot{p} = -F_q - pF_u, \qquad \dot{u} = pF_p.$$

Noncharacteristicity of a point (u, p, x) of the initial manifold is equivalent to the condition that the vector $F_p(u, p, x)$ not be tangent to D at the point x. In other words, noncharacteristicity permits one to find from the equation the derivative of the desired function at the points of D along the normal to D, after the derivatives in the tangent directions and the value of the function have been determined by the initial condition ϕ.

3.3. Geometrical Optics. The equivalence well-known in geometric optics of descriptions of the propagation of light in terms of rays and of fronts served as the prototype of the method of characteristics. The movement of "light corpuscules" along straight lines in $\mathbb{R}^n$ is described by the Hamiltonian $H(p, q) = p^2$. The eikonal equation $(\partial u/\partial q)^2 = 1$ describes the propagation of short light waves: the solution of it equal to zero on a hypersurface D in $\mathbb{R}^n$ is the optical length of the shortest path from the light source D to the point q. The projections to $\mathbb{R}^n$ of the characteristics composing the 1-graph of the function u are the normal lines (rays) to the level surfaces of the function u (the fronts).

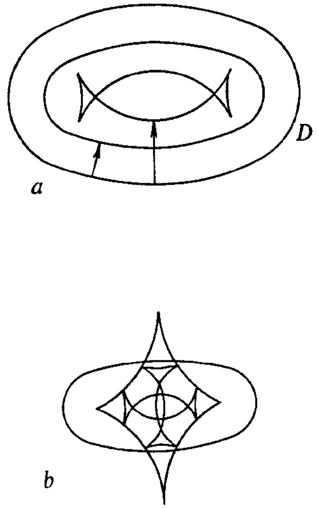

Fig. 30. The fronts (a) and the caustic (b) of an elliptical source

The solutions of the eikonal equation may be many-valued, and the fronts may have singularities. For example, upon propagation of light inside an elliptical source on the plane the front acquires semicubical cusp points (Fig. 30a). Upon movement of the front its singularities slide along a caustic (Fig. 30b). The caustic may be defined as the set of the centres of curvature of the source or as the envelope of the family of rays. In the neighbourhood of the caustic the light becomes concentrated. Singularities of wave fronts and caustics will be studied in chap. 5.

3.4. The Hamilton–Jacobi Equation. A *Hamilton–Jacobi equation* is an equation of the form $H(\partial u/\partial x, x) = 0$. It differs from the general first-order equation in that it does not contain the required function u explicitly. Integration of the equations of the characteristics reduces essentially to the integration of the Hamiltonian system with Hamiltonian $H(p, q)$. The eikonal equation is a special case of the Hamilton–Jacobi equation.

The path, inverse to the method of characteristics, of integration of Hamiltonian systems by a reduction to the solution of a Hamilton–Jacobi equation has proved to be very effective in mechanics.

Jacobi's Theorem. *Let $u(Q, q)$ be a solution of the Hamilton–Jacobi equation $H(\partial u/\partial q, q) = h$ depending on n parameters $Q = (h, \lambda_1, \ldots, \lambda_{n-1})$ and on the n variables q. Let us suppose that the equation $\partial u(Q, q)/\partial q = p$ is solvable for Q, in particular, that* $\det(\partial^2 u/\partial q \partial Q) \neq 0$. *Then the functions $Q(p, q)$ are n involutive first integrals of the Hamiltonian H.*

Indeed, the Lagrangian manifolds Λ_Q with the generating functions $u(Q, .)$, $\Lambda_Q = \{(p, q) \mid p = \partial u(Q, q)/\partial q\}$, are the fibres of a Lagrangian fibration over the space of parameters Q. The Hamilton–Jacobi equation means that the restriction of H onto Λ_Q is equal to h, i.e. that the Hamiltonian of the system is a function of Q. □

Success in applying Jacobi's theorem is always tied up with a felicitous choice of the system of coordinates in which the separation of the variables in the Hamilton–Jacobi equation takes place. It was by just such a method that Jacobi integrated the equation of the geodesics on a triaxial ellipsoid. One says that in the equation $H(\partial u/\partial q, q) = h$ the variable q_1 is separable, if $\partial u/\partial q_1$ and q_1 enter into H only in the form of a combination $\phi(\partial u/\partial q_1, q_1)$. Then, in trying to find a solution in the form $u = u_1(q_1) + U(q_2, \ldots, q_n)$, we arrive at the system $\phi(\partial u/\partial q_1, q_1) = \lambda_1$, $H(\lambda_1, \partial u/\partial q_2, \ldots, \partial u/\partial q_n, q_2, \ldots, q_n) = h$. If in the second equation variables again separate etc., then we finally arrive at a solution of the original equation of the form $u_1(q_1, \lambda_1) + u_2(q_2, \lambda_1, \lambda_2) + \ldots + u_n(q_n, \lambda_1, \ldots, \lambda_{n-1}, h)$ and we shall be able to apply Jacobi's theorem.

Let us illustrate this approach with the example of Euler's problem of the attraction of a point on the plane by two fixed centres. Let r_1, r_2 be the distance from the moving point to the centres O_1, O_2 (Fig. 31). The Hamiltonian of the

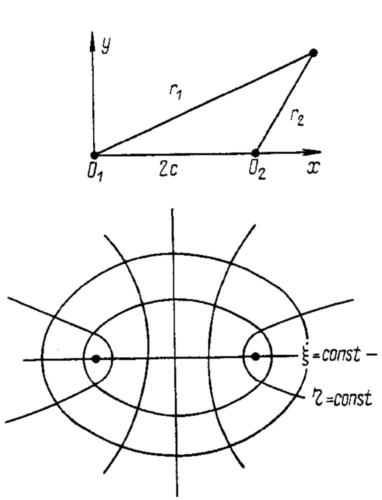

Fig. 31. Integration of the Euler problem

problem has the form $(p_x^2+p_y^2)/2-k(1/r_1+1/r_2)$. Let us pass over to elliptical coordinates (ξ, η) on the plane: $\xi=r_1+r_2$, $\eta=r_1-r_2$. The level lines of the functions ξ, η are mutually orthogonal families of curves—ellipses and hyperbolas with the foci O_1, O_2. In the canonical coordinates $(p_\xi, p_\eta, \xi, \eta)$ on $T^*\mathbb{R}^2$ the Hamiltonian (after some computations) takes on the form

$$H=2p_\xi^2\frac{\xi^2-4c^2}{\xi^2-\eta^2}+2p_\eta^2\frac{4c^2-\eta^2}{\xi^2-\eta^2}-\frac{4k\xi}{\xi^2-\eta^2}.$$

In the Hamilton–Jacobi equation

$$(\partial u/\partial\xi)^2(\xi^2-4c^2)+(\partial u/\partial\eta)^2(4c^2-\eta^2)=h(\xi^2-\eta^2)+2k\xi$$

we may separate the variables, setting

$$(\partial u/\partial\xi)^2(\xi^2-4c^2)-2k\xi-h\xi^2=\lambda$$

$$(\partial u/\partial\eta)^2(4c^2-\eta^2)+h\eta^2=-\lambda$$

From this we find a two-parameter family of solutions of the Hamilton–Jacobi equation in the form

$$u(h,\lambda,\xi,\eta)=\int\sqrt{\frac{\lambda+h\xi^2+2k\xi}{\xi^2-4c^2}}\,d\xi+\int\sqrt{\frac{-\lambda-h\eta^2}{4c^2-\eta^2}}\,d\eta.$$

The endeavour to extract explicit expressions for the trajectories of Hamiltonian systems from a similar kind of formulas led Jacobi to the problem of the inversion of hyperelliptic integrals, whose successful solution constitutes today one of the best achievements of algebraic geometry.

Chapter 5

Lagrangian and Legendre Singularities

In this section we set forth the foundations of the mathematical theory of caustics and wave fronts. The classification of their singularities is connected with the classification of regular polyhedra. The proofs take contributions from the theory of critical points of functions, from reflection groups, and from Lie groups and Lie algebras. Perhaps this explains why the final results, elementary in their form, were not already obtained in the last century.

§ 1. Lagrangian and Legendre Mappings

These are constructions which formalize in the language of symplectic geometry the concepts of a caustic and of a wave front of geometrical optics.

1.1. Fronts and Legendre Mappings. A *Legendre mapping* is a diagram consisting of an embedding of a smooth manifold as a Legendre submanifold in the total space of a Legendre fibration, and the projection of the total space of the Legendre fibration onto the base. By abuse of language we shall call the composition of these maps the Legendre mapping, too. The image of a Legendre mapping is called its *front*.

Examples. A. The *equidistant mapping* is the mapping which associates to each point of an oriented hypersurface in Euclidean space the end point of the unit vector of the normal at this point. The image of the equidistant mapping is called the equidistant (compare sect. 3.3, chap. 4). More generally, let B be a Riemannian manifold and $X \subset B$ a smooth submanifold. The flow of the contact Hamiltonian $H=\|p\|$ in the contact space ST^*B of transversally oriented contact elements moves (in time t) the Legendre submanifold Λ_0 of elements tangent to X into a Legendre submanifold Λ_t. The projection of Λ_t to the base space B is an equidistant of the submanifold X—the set of free ends of segments of geodesics (extremals of the Lagrangian $\|p\|$) of length t sent out from X along normals. Thus, the equidistant mapping is Legendrian and the equidistant is its front.

B. Projective duality. The *tangential mapping* is the mapping which associates to each point of a hypersurface in a projective space the hyperplane tangent at this point. Let us consider in the product $\mathbb{P} \times \mathbb{P}^*$ of the projective space and its dual the submanifold F of pairs (p, p^*) satisfying the incidence condition: the point $p \in \mathbb{P}$ lies in the hyperplane $p^* \subset \mathbb{P}$, and also the submanifold F^* picked out by the dual condition: the point $p^* \in \mathbb{P}^*$ lies in the hyperplane $p \subset \mathbb{P}^*$.

1°. The two incidence conditions coincide: $F^* = F$.

The projection $F \to \mathbb{P}$ $(F^* \to \mathbb{P}^*)$ is a Legendre fibration of the manifold of contact elements $F = PT^*\mathbb{P}$ $(F^* = PT^*\mathbb{P}^*$ respectively).

2°. The two contact structures on $F = F^*$ coincide.

This follows from 1° and the definition of the contact structure on PT^*B. □

The tangential mapping is just the projection of the Legendre submanifold Λ of $PT^*\mathbb{P}$, formed by the contact elements of the submanifold X of $\mathbb{P}$, to the base space of the second Legendre fibration $PT^*\mathbb{P}^* \to \mathbb{P}^*$. Therefore the tangential mapping of a smooth hypersurface is Legendrian. Its front X^* in $\mathbb{P}^*$ is called the *dual hypersurface*.

3°. The Legendre submanifolds Λ and Λ^* coincide (Fig. 32).

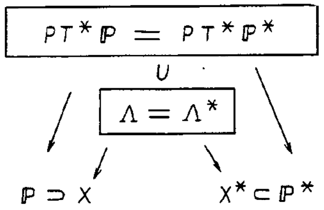

Fig. 32. Projective duality

Indeed, a Legendre manifold is determined by its projection to the base space of a Legendre fibration.

Corollary. $(X^*)^* = X$.

C. *The Legendre transformation.* Let us consider the two Legendre fibrations of the standard contact space $\mathbb{R}^{2n+1}$ of 1-jets of functions on $\mathbb{R}^n$: $(u, p, q) \mapsto (u, q)$ and $(u, p, q) \mapsto (pq - u, p)$.

The projection of the 1-graph of a function $u = S(q)$ onto the base of the second fibration gives a Legendre mapping $q \mapsto (q\partial S/\partial q - S(q), \partial S/\partial q)$. In the case that the function S is convex, the front of this mapping is again the graph of a convex function $v = S^*(p)$—the Legendre transform of the function S (compare sect. 1.2, chap. 3).

The fronts of Legendre mappings are in general not smooth. The problem of classifying the singularities of fronts reduces to the study of Legendre singularities (i.e. singularities of Legendre mappings). Generic Legendre singularities are different from the singularities of generic mappings of n-dimensional manifolds into $n+1$-dimensional ones. Thus the projection of a generic space curve into the plane has as its singularities only self-intersection points, while the projection of a generic Legendre curve in a Legendre fibration has cusp points as well.

By a *Legendre equivalence* of two Legendre mappings is meant a contactomorphism of the corresponding Legendre fibrations which takes the Legendre manifold of the first Legendre mapping over into the Legendre manifold of the second.

Remarks. 1. A contactomorphism of Legendre fibrations is uniquely determined by a diffeomorphism of the base spaces. A smooth front of a Legendre mapping uniquely determines the original Legendre submanifold. In this sense the effect of a Legendre equivalence is reduced to the effect of the diffeomorphism of the base on the front. This remark is also applicable to singular fronts whose set of points of regularity is dense in the original Legendre manifold. The last condition is violated only for germs of Legendre mappings forming a set of infinite codimension in the space of all germs.

2. One can show that (up to equivalence) all Legendre singularities can already be realized in the case of equidistants of hypersurfaces in Euclidean space. In this sense the investigation of Legendre singularities coincides with the investigation of equidistants (one can show that to nearby Legendre singularities correspond equidistants of nearby hypersurfaces and conversely, so that the generic singularities for fronts of Legendre mappings are the same as for equidistants). The same may be asserted for the singularities of hypersurfaces projectively dual to smooth ones, or for the singularities of Legendre transforms of graphs of smooth functions.

1.2. Generating Families of Hypersurfaces. A nonsingular Legendre mapping is determined by its front—the *generating hypersurface* of the Legendre manifold. An arbitrary Legendre mapping can be given by a generating family of hypersurfaces. Let us consider an auxiliary trivial fibration $\mathbb{R}^{k+l} \to \mathbb{R}^l$ of a "big space" $\mathbb{R}^{k+l}$ over the base space $\mathbb{R}^l$. The contact elements in $\mathbb{R}^{k+l}$ which are tangent to the fibres of the fibration form the mixed submanifold $P \subset PT^*\mathbb{R}^{k+l}$ of codimension k. The mixed manifold is fibred over the manifold of contact elements of the base space (Fig. 33). A Legendre submanifold of $PT^*\mathbb{R}^{k+l}$ is called regular if it is transversal to the mixed space P.

Lemma. 1. *The image of the projection of the intersection of a regular Legendre manifold with the mixed space P into the space of contact elements of the base is an immersed Legendre submanifold.*

2. *Every germ of a Legendre submanifold of $PT^*\mathbb{R}^l$ can be obtained by this construction from some regular Legendre submanifold, generated by a generating hypersurface, of a suitable auxiliary fibration.*

Let us give the proof of the second assertion of the lemma. In the fibration $PT^*\mathbb{R}^l \to \mathbb{R}^l$ let us introduce contact Darboux coordinates $(u, p, q) = (u, p_I, p_J, q_I, q_J)$, $I \cup J = \{1, \ldots, l-1\}$, $I \cap J = \varnothing$, so that the Legendre submanifold germ being investigated projects injectively to the space of the coordinates (p_J, q_I) along the (u, p_I, q_J)-space. Then from the relations

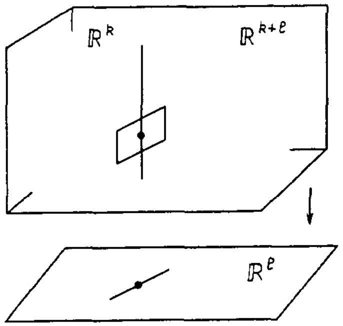

Fig. 33. The mixed space

$du = pdq = d(p_J q_J) - q_J dp_J + p_I dq_I$ we get: there exists a germ of a smooth function $S(p_J, q_I)$ so that our Legendre submanifold is given by the equations

$$p_I = \partial S/\partial q_I, \quad q_J = -\partial S/\partial p_J, \quad u = p_J q_J + S(p_J, q_I).$$

Now let us consider the hypersurface $u = F(x, q)$ in the "big" space $\mathbb{R}^{k+l}$, where $k = |J|$ and $F(x, q) = xq_J + S(x, q_I)$, as the generating hypersurface of the Legendre manifold $u = F$, $y = F_x$, $p = F_q$. An application of the construction of the first part of the lemma leads to the original Legendre germ in $PT^*\mathbb{R}^l$. The regularity condition has the form $\det(F_{xq_J}) \neq 0$ and is fulfilled. □

The hypersurface of the big space, through which the germ of the Legendre mapping is given by the construction described in the lemma, is called a *generating family of hypersurfaces* of this Legendre mapping (the elements of the family are in general singular intersections of the hypersurface with the fibres of the fibration $\mathbb{R}^{k+l} \to \mathbb{R}^l$).

Remark. The "physical meaning" of the generating family consists in the following. Let us consider the propagation of light in $\mathbb{R}^l$ from a source $X \subset \mathbb{R}^l$ of dimension k. According to Huygens' principle, each point x of the source radiates a spherical wave. Let us denote by $F(x, q)$ the propagation time of this wave to the point q of the space $\mathbb{R}^l$. Then the condition that the least time of motion of the light from the source X to the point q is equal to u gives the equations:

$$\exists x \in X \colon u = F(x, q),\ \partial F(x, q)/\partial x = 0. \qquad (1)$$

But this is just the equation of the front for the projection to $\mathbb{R}^l$ of the intersection of the Legendre manifold $\{u = F, y = F_x, p = F_q\}$ with the mixed space $\{y = 0\}$.

By a fibred equivalence of generating families of hypersurfaces Γ_1 and Γ_2 in the total space of the fibration $\mathbb{R}^{k+l} \to \mathbb{R}^l$ is meant a fibred diffeomorphism $(x, q) \mapsto (h(x, q), \phi(q))$ which transfers Γ_1 into Γ_2. Let $\Gamma \subset M$ be a smooth hypersurface with a simple equation $f = 0$. The doubling of M with branching along Γ is the hypersurface in the direct product $\mathbb{R} \times M$ with the equation $u^2 = f(v)$, $u \in \mathbb{R}$. In the complex case the doubling is a double covering of M ramified along Γ. The real type of the equation depends on the choice of the side of Γ. Two families of hypersurfaces in auxiliary fibrations with a common base space are called stably fibred-equivalent if they become fibred-equivalent after a series of fibrewise doublings.

Theorem ([9]). *Two germs of generating families of hypersurfaces give equivalent germs of Legendre mappings if and only if these families of hypersurfaces are fibred stably equivalent.*

The reason for the appearance of stable equivalence here will become clear in §2.

1.3. Caustics and Lagrangian Mappings. A *Lagrangian mapping* is a diagram consisting of an embedding of a smooth manifold as a Lagrangian submanifold in the total space of a Lagrangian fibration and the projection to the base space of this fibration.

Examples. A. A *gradient mapping* $q \mapsto \partial S/\partial q$ is Lagrangian.

B. The *Gauss mapping* of a transversally oriented hypersurface in Euclidean space $\mathbb{R}^n$ to the unit sphere is Lagrangian. In fact, it is a composition of two maps. The first associates to a point of the hypersurface the oriented normal to the hypersurface at that point; its image is a Lagrangian submanifold in the space of all straight lines in $\mathbb{R}^n$, which is isomorphic to the (co)tangent bundle of the sphere (Fig. 34). The second is the Lagrangian projection $T^*S^{n-1} \to S^{n-1}$.

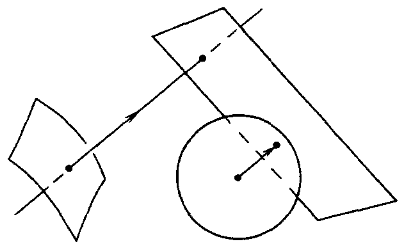

Fig. 34. The Gauss mapping is Lagrangian

C. The *normal mapping*, which associates to a vector $\overrightarrow{uv}$ of a normal to a submanifold in Euclidean space, applied at the point u, the point v of the space itself, is Lagrangian.

The set of critical values of a Lagrangian mapping is called a *caustic* (Fig. 35).

Example. The caustic of the normal mapping of a submanifold in Euclidean space is the set of its centres of curvature: to construct the caustic, one must lay out along each normal the respective radii of the principal curvatures.

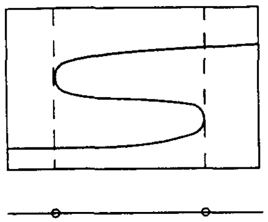

Fig. 35. The caustic of a Lagrangian mapping

A *Lagrangian equivalence* of Lagrangian mappings is a symplectomorphism of the Lagrangian fibrations which takes the Lagrangian manifold of the first mapping over into the Lagrangian manifold of the second.

Remarks. 1. An automorphism of the Lagrangian bundle $T^*M \to M$ factors into the product of an automorphism induced by a diffeomorphism of the base space and a translation automorphism $(p, q) \mapsto (p + \phi(q), q)$, where ϕ is a closed 1-form on M.

2. The caustics of equivalent Lagrangian mappings are diffeomorphic. The converse is in general not true.

3. Every germ of a Lagrangian mapping is equivalent to a germ of a gradient (Gauss, normal) mapping. All germs of Lagrangian mappings which are close to a given gradient (Gauss, normal) mapping are themselves gradient (Gauss, normal). Therefore the generic local phenomena in these classes of Lagrangian mappings are the same as in the class of all Lagrangian mappings.

1.4. Generating Families of Functions. A Lagrangian section of a cotangent bundle, i.e. a closed 1-form on the base space, is given by its generating function—a potential of this 1-form. Let us consider an auxiliary fibration $\mathbb{R}^{k+l} \to \mathbb{R}^l$. The covectors in $T^*\mathbb{R}^{k+l}$ which vanish on the tangent space to the fibre at their point of application form the mixed submanifold $Q \subset T^*\mathbb{R}^{k+l}$. The mixed manifold is fibred over $T^*\mathbb{R}^l$ with k-dimensional isotropic fibres. A Lagrangian submanifold of $T^*\mathbb{R}^{k+l}$ transversal to Q is called regular. The projection to $T^*\mathbb{R}^l$ of the intersection $\Lambda \cap Q$ of a regular Lagrangian manifold is an immersed Lagrangian submanifold in $T^*\mathbb{R}^l$. Conversely, any germ of a Lagrangian submanifold of $T^*\mathbb{R}^l$ can be obtained as a projection of the intersection with the mixed submanifold Q of a regular Lagrangian submanifold Λ which is a section of the cotangent bundle of the "big space", for a suitable choice of the auxiliary fibration.

A *generating family of functions* of a germ of a Lagrangian mapping is a germ of a function on a "big" space which generates a regular Lagrangian section of the cotangent bundle of the "big" space and by means of the construction described above defines the given Lagrangian mapping. (The function on $\mathbb{R}^{k+l}$ is here regarded as a family of functions on the fibres of the projection $\mathbb{R}^{k+l} \to \mathbb{R}^l$, depending on the point of the base space as a parameter).

In Darboux coordinates (p_I, p_J, q_I, q_J) for the Lagrangian fibration $T^*\mathbb{R}^l \to \mathbb{R}^l$ such that the Lagrangian submanifold of $T^*\mathbb{R}^l$ projects nonsingularly along the (p_I, q_J)-space, let us give this submanifold by the equations $p_I = \partial S/\partial q_I$, $q_J = -\partial S/\partial p_J$, where $S(p_J, q_I)$ is some smooth function. Then the family of functions $F(x, q) = xq_J + S(x, q_I)$ may be taken as a generating family of the Lagrangian mapping of our Lagrangian submanifold of $T^*\mathbb{R}^l$ to the base space $\mathbb{R}^l$.

In the general case the family of functions $F(x, q)$ is generating for some Lagrangian mapping if and only if $\mathrm{rk}(F_{xx}, F_{xq}) = k$. In this case it defines

a) a regular Lagrangian submanifold Λ in $T^*\mathbb{R}^{k+l}$:

$$\Lambda = \{(y, p, x, q) \mid y = F_x, p = F_q\};$$

b) the intersection of Λ with the mixed space $Q = \{y = 0\}$:

$$\Lambda \cap Q = \{(p, x, q) \mid F_x = 0, p = F_q\};$$

c) a Lagrangian submanifold $L \subset T^*\mathbb{R}^l$—the projection of $\Lambda \cap Q$:

$$L = \{(p, q) \mid \exists x: F_x(x, q) = 0, p = F_q(x, q)\};$$

d) the caustic K in $\mathbb{R}^l$:

$$K = \{q \mid \exists x: F_x(x, q) = 0, \det|F_{xx}(x, q)| = 0\}; \tag{2}$$

We shall call two families of functions $F_1(x, q)$, $F_2(x, q)$ fibred R_+-equivalent if there exists a fibred diffeomorphism $(x, q) \mapsto (h(x, q), \phi(q))$ and a smooth function $\Psi(q)$ on the base space, such that $F_2(x, q) = F_1(h(x, q), \phi(q)) + \Psi(q)$. Two families of functions $F_1(x_1, q)$, $F_2(x_2, q)$, in general of a different number of variables, will be called stably fibred R_+-equivalent if they become fibred R_+-equivalent after adding to them nondegenerate quadratic forms $Q_1(z_1)$, $Q_2(z_2)$ in new variables

$$F_1(x_1, q) + Q_1(z_1) \overset{R_+}{\sim} F_2(x_2, q) + Q_2(z_2).$$

Example: the family $x^3 + yz + qx$ (q is the parameter) is stably fibred R_+-equivalent to the family $x^3 + qx$.

Theorem ([9]). *Two germs of Lagrangian mappings are Lagrangianly equivalent if and only if the germs of their generating families are stably fibred R_+-equivalent.*

1.5. Summary. The investigation of singularities of caustics and wave fronts led to the study of generating families of functions and hypersurfaces. The formulas (1) and (2) mean that the front of a generating family of hypersurfaces consists of those points of the parameter space for which the hypersurface of the family is singular, and the caustic of a generating family of functions consists of those points of the parameter space for which the function of the family has degenerate critical points, i.e. points at which the differential of the function turns to zero and the quadratic form of the second differential is degenerate.

All the definitions and results of this section can be carried over verbatim to the holomorphic or real-analytic case.

§ 2. The Classification of Critical Points of Functions

The theory considered below of deformations of germs of functions and hypersurfaces is analogous in principle to the finite-dimensional theory of deformations, developed in § 3 of chap. 1 for quadratic Hamiltonians.

2.1. Versal Deformations: An Informal Description. A generating family of hypersurfaces of a Legendre mapping is the family of zero levels of some family of smooth functions. We regard the family of functions as a mapping of the (finite-dimensional) base space of the family into the infinite-dimensional space of smooth functions. The functions with a singular zero level form a set of codimension one in this space. The front of a Legendre mapping is just the inverse image of the set of such functions in the base space of the generating family (Fig. 36). A generating family of functions of a Lagrangian mapping can by subtraction of a family of constants (this is an R_+-equivalence!) be turned into a family of functions equal to zero at the coordinate origin. Therefore a Lagrangian mapping can be given by a mapping of the base space into the space of such functions. The functions with degenerate critical points form a set of codimension one in this space. The inverse image of this set in the base is the caustic of the Lagrangian mapping being generated by the family.

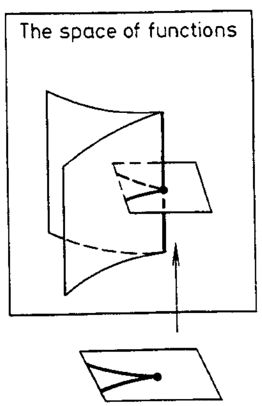

Fig. 36. The front as a bifurcation diagram

Our programme is as follows. We shall call a germ of a Lagrangian (Legendre) mapping stable, if all nearby germs of Lagrangian (Legendre) mappings are equivalent to it. In the language of generating families this means that germs near to a given generating family are fibred-equivalent to it. Fibred equivalence is just the equivalence of families of points of the function space with respect to the action on it of a suitable (pseudo)group (see sect. 3.1, chap. 1). We obtain the following result: the germ of a Lagrangian (Legendre) mapping at a point is stable if and only if the germ of its generating family at that point is versal with respect to the equivalence in the corresponding function space[11].

Further on we shall cite results on the classification of germs of functions and we shall see that the beginning part of this classification is discrete. This means that almost all of the space of functions is filled out by a finite number of orbits (Fig. 37), and the continuous families of orbits form a set of positive codimension l in the space of functions. Since generic families of functions, considered as mappings of the base space into the function space, are transversal to this set and to each orbit out of the finite list, we obtain for Lagrangian and Legendre mappings implications such as:

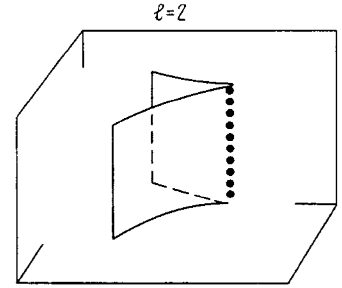

Fig. 37. The stratification of the space of functions

the germ of a generic Lagrangian (Legendre) mapping with a base space of dimension less than $l = 6$ (resp. 7) is stable and equivalent to one of the germs of a finite list.

Finally, when we have studied the miniversal deformations of representatives of the finite number of orbits, we shall obtain an explicit description, up to a diffeomorphism, of the singularities of caustics (wave fronts) in spaces of fewer than $l = 6$ (resp. 7) dimensions.

2.2. Critical Points of Functions. We shall need the following equivalence relations on the space of germs at 0 of holomorphic (smooth) functions on $\mathbb{C}^n$ (resp. on $\mathbb{R}^n$).

[11] For the definition of a (mini)versal deformation, see sect. 3.1, chap. 1.

R-equivalent germs are taken into each other by the germ at 0 of a diffeomorphism of the preimage space;

R_+-equivalent germs become R-equivalent after addition of a suitable constant to one of them;

V-equivalent germs become R-equivalent after multiplication of one of them with the germ of a non-vanishing function (V-equivalence of germs of functions is just equivalence of the germs of their zero-level hypersurfaces).

At a critical (or singular) point of a function its differential turns to 0. The critical point is nondegenerate if the quadratic form of the second differential of the function at that point is so. By the Morse lemma [9], in the neighbourhood of a nondegenerate critical point the function is R-equivalent to a function $\pm x_1^2 \pm \ldots \pm x_n^2 + \text{const}$. By the corank of a critical point is meant the corank of the second differential of the function at that point.

The Morse Lemma with Parameters ([9]). *The germ of a function at a critical point of corank r is R-equivalent to the germ at 0 of a function of the form* $\text{const} + \phi(x_1, \ldots, x_r) \pm x_{r+1}^2 \pm \ldots \pm x_n^2$, *where* $\phi = O(|x|^3)$.

This lemma explains the appearance of the concept of stable equivalence in the theorems on generating families: in fact the germ of a Lagrangian (Legendre) mapping at a point may be given by a germ of a generating family with zero second differential of the function at that point. Fibred equivalence of in this sense minimal generating families means the same as the equivalence of the original mappings. For the construction of minimal generating families it is only necessary that in the constructions of sects. 1.2 and 1.4 one choose the number of "pathological" variables p_j to be minimal.

Germs of functions (possibly of a different number of variables) are called stably $R(R_+, V)$-equivalent if they are $R(R_+, V)$-equivalent to sums of the same germ of rank 0 with nondegenerate quadratic forms of the appropriate number of additional variables.

A degenerate critical point falls apart into nondegenerate ones upon deformation (Fig. 38). If the number of the latter is finite for an arbitrary small deformation, then the critical point is said to be of finite multiplicity. The germ of a function at a critical point of finite multiplicity is R-equivalent to its Taylor polynomial of sufficiently high order. In the holomorphic case finite multiplicity is equivalent to isolatedness of the critical point. The number of nondegenerate

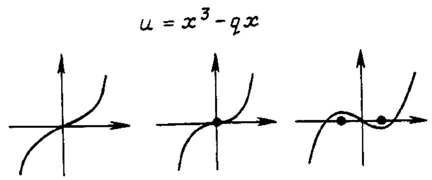

Fig. 38. Morsification

critical points into which such a point falls apart upon deformation does not depend on the deformation and is called the multiplicity or the Milnor number μ of the critical point. The germs of infinite multiplicity form a set of infinite codimension in the space of germs of functions.

2.3. Simple Singularities. A germ of a function at a critical point is called simple if a neighbourhood of it in the space of germs of functions at this point can be covered by a finite number of equivalence classes. The simplicity concept depends in general on the equivalence relation and is applicable to an arbitrary Lie group action on a manifold. The number of parameters (moduli) which are needed for the parametrization of the orbits in the neighbourhood of a given point of the manifold is called the modality of the point. Examples: the modality of an arbitrary quadratic Hamiltonian on $\mathbb{R}^{2n}$ with respect to the action of the symplectic group is equal to n; the critical value is a modulus with respect to R-equivalence on the space of germs of functions at a given point, but is not a modulus for R_+-equivalence on this space.

Theorem ([9]). *A germ of a function at a critical point, which is simple in the space of germs of smooth functions (with value zero at that point), is stably R_+ (resp. R, V)-equivalent to one of the following germs at zero*

$$A_\mu^\pm,\ \mu \geq 1: f(x) = \pm x^{\mu+1};\quad D_\mu^\pm,\ \mu \geq 4: f(x, y) = x^2 y \pm y^{\mu-1};$$
$$E_6^\pm: f(x, y) = x^3 \pm y^4;\quad E_7: f(x, y) = x^3 + xy^3;$$
$$E_8: f(x, y) = x^3 + y^5.$$

The nonsimple germs form a set of codimension 6 in these spaces.

Remarks. 1) The index μ is equal to the multiplicity of the critical point. 2) The enumerated germs are pairwise stably inequivalent, except for the following cases: $A_{2k}^+ \overset{R}{\sim} A_{2k}^-$, $A_\mu^+ \overset{V}{\sim} A_\mu^-$, $D_{2k+1}^+ \overset{V}{\sim} D_{2k+1}^-$, $E_6^+ \overset{V}{\sim} E_6^-$, $A_1^+ \overset{R}{\sim} A_1^-$ (stably).

3) In the holomorphic case the germs which differ only in the sign $\pm$ are equivalent among themselves. Figure 39 depicts the adjacencies of the simple classes and the unimodal classes bordering on them in the space of functions.

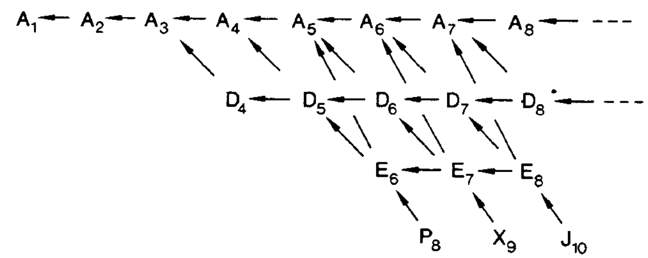

Fig. 39. Adjacencies of simple singularities of functions

2.4. The Platonics. In another context the list of the singularities A_μ, D_μ, E_μ was already known in the last century. Let us consider the finite subgroups of the group SU_2. They may be described as the binary subgroups of the regular polygons, the dihedra (regular polygons in space), the tetrahedron, the cube and the icosahedron. The definition of the binary group is as follows. The group SU_2 maps epimorphically onto the rotation group SO_3 with kernel $\{\pm 1\}$. The group of rotations of a regular polyhedron in space is a finite subgroup of SO_3. It is the inverse image of this group in SU_2 which is the binary group of the polyhedron. To the regular n-gon there corresponds by definition a cyclic subgroup of order n in SU_2.

A finite subgroup $\Gamma \subset SU_2$ acts (together with SU_2) on the plane $\mathbb{C}^2$. The quotient space $\mathbb{C}^2/\Gamma$ is an algebraic surface with one singular point. The algebra of Γ-invariant polynomials on $\mathbb{C}^2$ has three generators x, y, z. They are dependent. The relation $f(x, y, z) = 0$ between them is just the equation of the surface $\mathbb{C}^2/\Gamma$ in $\mathbb{C}^3$. For example, in the case of the cyclic subgroup Γ of order n generated by the unitary transformation of the plane $(u, v) \mapsto (e^{2\pi i/n}u, e^{-2\pi i/n}v)$, the algebra of invariants is generated by the monomials $x = uv$, $y = u^n$, $z = v^n$ with the relation $x^n = yz$.

Theorem ([43]). *All the surfaces $\mathbb{C}^2/\Gamma$ for finite subgroups $\Gamma \subset SU_2$ have singularities of the types A_μ (for polygons), D_μ (for dihedra), E_6, E_7, E_8 (for the tetrahedron, the cube and the icosahedron respectively).*

2.5. Miniversal Deformations. In the theory of deformations of germs of functions one can prove a versality theorem [9]: germs of finite multiplicity have versal deformations (with a finite number of parameters).

An R-miniversal deformation of a germ of finite multiplicity (with respect to the pseudogroup of local changes of the independent variables) can be constructed in the following manner. Let us consider the germ f of a function at the critical point 0 of multiplicity μ. Let $f(0) = 0$.

1) The tangent space to the orbit of the germ f is just its gradient ideal $(\partial f/\partial x)$, consisting of all function germs of the form $\sum h_i(x)\partial f/\partial x_i$ ($h\partial/\partial x$ is the germ at 0 of a vector field, not necessarily equal to zero at the coordinate origin).

2) The quotient algebra $Q = \mathbb{R}\{x\}/(\partial f/\partial x)$ of the algebra of all germs of functions at 0 by the gradient ideal has dimension μ (and is called the local algebra of the germ f; one may conceive of it as the algebra of functions on the set $\{x \mid \partial f/\partial x = 0\}$ of the μ critical points of the function f which have merged at the point 0).

3) Let $e_0(x) = 1$, $e_1(x), \ldots, e_{\mu-1}(x)$ be functions (for example, monomials) which represent a basis of the space Q. Then the deformation

$$F(x, q) = f(x) + q_{\mu-1}e_{\mu-1}(x) + \ldots + q_1 e_1(x) + q_0$$

is R-miniversal for the germ f (F is transversal to the tangent space of the orbit of the germ f).

4) Throwing away the constant term q_0 yields an R_+-miniversal deformation of the germ f.

5) The analogous construction for the local algebra $Q = \mathbb{R}\{x\}/(f, f_x)$ gives a V-miniversal deformation of the germ f ($hf_x + \phi f$ is the general form of a tangent vector to the class of equations of diffeomorphic hypersurfaces).

The simple germs A_μ, D_μ, E_μ lie in their own gradient ideal: $f \in (f_x)$. Indeed, the normal forms of the theorem of sect. 2.3 are quasihomogeneous (i.e. homogeneous of degree 1 after a choice of positive fractional degrees $\alpha_1, \ldots, \alpha_n$ for the variables $x_1, \ldots, x_n$; an example: the function $x^2y + y^{\mu-1}$ is quasihomogeneous with the weights $\alpha_x = (\mu-2)/(2\mu-2)$, $\alpha_y = 1/(\mu-1)$). Therefore $f = \sum \alpha_i x_i \partial f/\partial x_i$. From this it follows that R-miniversal deformations of simple germs of functions are V-miniversal.

Example. $F(x, q) = x^{\mu+1} + q_{\mu-1}x^{\mu-1} + \ldots + q_1 x$ is an R_+-miniversal deformation of the germ A_μ and $F(x, q) + q_0$ is a V-miniversal deformation of it: indeed, $\{1, x, \ldots, x^{\mu-1}\}$ is a basis of the space $\mathbb{R}\{x\}/(x^\mu)$. For the normal forms of sect. 2.3 of the simple singularities of functions a monomial basis of the algebra Q is listed in table 1.

Table 1

A_μ	$1, x, \ldots, x^{\mu-1}$	E_6	$1, y, x, y^2, xy, xy^2$
D_μ	$1, y, \ldots, y^{\mu-2}, x$	E_7	$1, y, x, y^2, xy, x^2, x^2y$
B_μ	$1, x, \ldots, x^{\mu-1}$	E_8	$1, y, x, y^2, xy, y^3, xy^2, xy^3$
C_μ	$1, y, \ldots, y^{\mu-1}$	F_4	$1, y, x, xy$

§ 3. Singularities of Wave Fronts and Caustics

Classificational results will be cited for the singularities of wave fronts, caustics, and their metamorphoses in time. We shall discuss generalizations of the theory of generating families for fronts and caustics originating from a source with boundary and in the problem of going around an obstacle (compare chap. 3, sect. 1.6).

3.1. The Classification of Singularities of Wave Fronts and Caustics in Small Dimensions. By a front of type A_μ, D_μ or E_μ is meant the hypersurface germ in the μ-dimensional base space of a V-miniversal deformation of the corresponding simple function germ, whose points correspond to the functions with a singular zero level.

Example. The front of type A_μ is just the set of polynomials in one variable with multiple roots, in the space of polynomials of degree $\mu+1$ with a fixed leading coefficient and zero sum of the roots. In Fig. 40 are depicted the fronts A_2 and A_3.

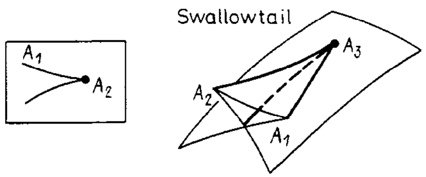

Fig. 40. Singularities of wave fronts

Theorem. *A generic wave front in a space of $l \leq 6$ dimensions is stable and in the neighbourhood of any of its points is diffeomorphic to the Cartesian product of a front of type A_μ, D_μ, E_μ with $\mu \leq l$ and a nonsingular manifold of dimension $l-\mu$, or to a union of such fronts which are transversal* (Fig. 41).

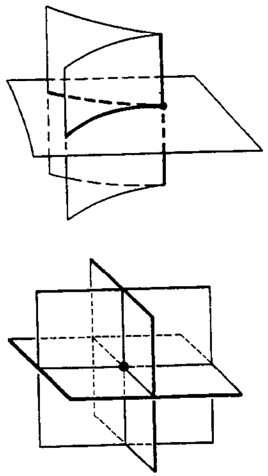

Fig. 41. Transversal fronts (caustics)

By a caustic of type $A_\mu^\pm, D_\mu^\pm, E_\mu^\pm$ is meant the hypersurface germ in the $\mu-1$-dimensional base space of an R_+-miniversal deformation of the corresponding simple function germ, whose points correspond to the functions with degenerate critical points (i.e. the projection of the cuspidal edge A_2 of the front of the same name to the base space of the R_+-miniversal deformation).

Example. The caustic of type A_μ is diffeomorphic to the front of type $A_{\mu-1}$: to a point of this caustic corresponds a polynomial of degree $\mu+1$, whose derivative (i.e. a polynomial of degree μ) has a multiple root.

Theorem. *A generic caustic in a space of $l \leq 5$ dimensions is stable and in the neighbourhood of any of its points is diffeomorphic to the Cartesian product of a*

caustic of type A_μ, D_μ, E_μ with $\mu-1 \leq l$ and a nonsingular manifold of dimension $l-\mu+1$, or to a union of such caustics which are transversal.

In particular, a generic caustic in space is locally diffeomorphic to one of the surfaces of Figs. 41, 42.

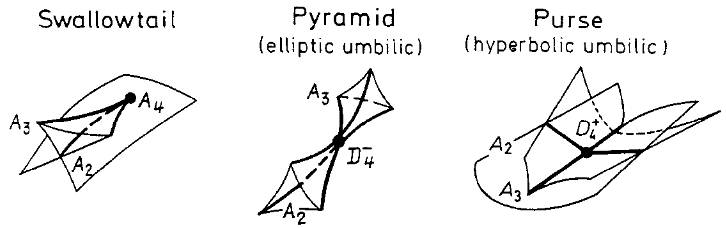

Fig. 42. Singularities of caustics in $\mathbb{R}^3$

Fronts and caustics of type A_μ, D_μ, E_μ are stable in all dimensions. Generic fronts (caustics) in spaces of dimension $l \geq 7$ ($l \geq 6$) may be unstable[12]. This is connected with the existence of nonsimple singularities of functions, the first of which is P_8 (see Fig. 39). The existing classification of unimodal and bimodal critical points of functions [9] carries considerable information about the singularities of generic fronts (caustics) in spaces of $l \leq 11$ ($l \leq 10$) dimensions. Nonetheless a classification of the singularities of generic caustics in $\mathbb{R}^6$, even only up to homeomorphisms, is lacking for the present.

3.2. Boundary Singularities. Let us suppose that the source of a radiation is a manifold with boundary (for example, the solar disk). In this situation the wave front has two components—the front of the radiation of the boundary and the front from the source proper (Fig. 43). The caustic in this case has three components in general—the boundaries of light and umbra, umbra and penumbra, and also of penumbra and light.

The corresponding theory of Lagrangian and Legendre mappings and their generating families leads to the theory of singularities of functions on a manifold with boundary. A boundary should be taken to mean a nonsingular hypersurface on a manifold without boundary. A point is considered singular for a function on the manifold with boundary if it is critical either for the function itself or for its restriction to the boundary. The diffeomorphisms which enter into the definition of equivalence are required to preserve the boundary (in the real case, each half-space of the complement of the boundary). The theory of boundary singularities of functions includes the usual one, since the functions may have singularities also outside the boundary.

[12] and for $l \geq 10$ ($l \geq 6$) they may have functional moduli.

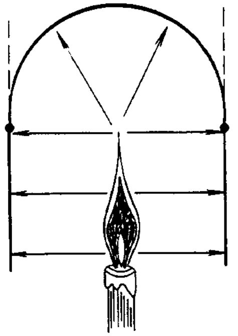

Fig. 43. A source with a boundary

Theorem. *The simple germs of functions on a manifold with boundary are stably equivalent to the germs at 0 (the boundary is $x=0$) either of $\tilde{A}_\mu^{\pm(\pm)}$, $\tilde{D}_\mu^{\pm(\pm)}$, $\tilde{E}_\mu^{\pm(\pm)}$: $\pm x + f(y,z)$, where $f(y,z)$ is the germ $A_\mu^\pm$, $D_\mu^\pm$, $E_\mu^\pm$ of sect. 2.3 on the boundary; or of $B_\mu^\pm$, $\mu \geq 2$: $\pm x^\mu$; $C_\mu^\pm$, $\mu \geq 2$: $xy \pm y^\mu$; $F_4^\pm$: $\pm x^2 + y^3$.*

Remarks. 1) The germs enumerated are pairwise inequivalent, except for the cases: $\tilde{A}_{2k}^+ \overset{R}{\sim} \tilde{A}_{2k}^-$, $C_2^\pm \overset{R}{\sim} B_2^\pm$ (stably), but also, germs which differ in the signs $\pm$ are equivalent in the holomorphic and V-classification, except for the case C_{2k+1}, in which the hypersurfaces $C_{2k+1}^\pm$ are real-inequivalent.

2) The rôle of the nondegenerate singularities is played in the theory with boundary by the germs $\tilde{A}_1$: $\pm x \pm y_1^2 \pm \ldots \pm y_n^2$ and A_1: $\pm(x-x_0)^2 \pm y_1^2 \pm \ldots \pm y_n^2$, so that the singularities $\tilde{A}_\mu, \tilde{D}_\mu, \tilde{E}_\mu$ may be considered as the stabilizations of the A_μ, D_μ, E_μ singularities on the boundary.

3) The simple singularities outside the boundary are $A_\mu^\pm, D_\mu^\pm, E_\mu^\pm$.

4) The theory of boundary singularities is equivalent to the theory of critical points of functions which are even in the variable u on the double covering $x = u^2$, ramified along the boundary $x=0$.

The construction of miniversal deformations of boundary singularities of finite multiplicity is analogous to that described in sect. 2.6 and reduces to finding a monomial basis of the local algebra $\mathbb{R}\{x, y_1, \ldots, y_n\}/(x\partial f/\partial x, \partial f/\partial y)$. For the germs B_μ, C_μ, F_4 such a basis is indicated in table 1. The fronts of the miniversal families of B_μ and C_μ are diffeomorphic to each other and consist of two irreducible hypersurfaces in the space of polynomials of degree μ with a fixed leading coefficient—the set of polynomials with a zero root and the set of polynomials with a multiple root. In the case of B_μ the first component has the meaning of the front of the radiation from the boundary of the source ($\tilde{A}_1$), the second that of the front of the source itself (A_1), but in the case of C_μ everything is the other way around. The caustics of B_μ and C_μ are diffeomorphic to the fronts of $B_{\mu+1}$ and $C_{\mu+1}$. The fronts and the caustics of the germs $\tilde{A}_\mu, \tilde{D}_\mu, \tilde{E}_\mu$ are the same as for A_μ, D_μ, E_μ. Figure 44 depicts the caustics of B_μ, C_μ, F_4 in space and on the plane.

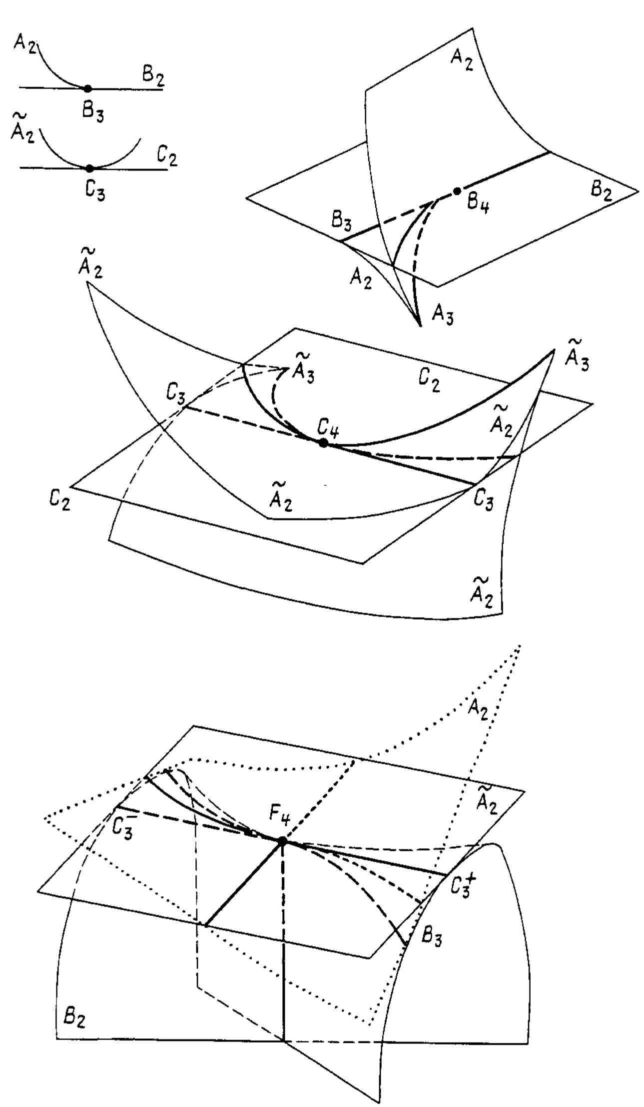

Fig. 44. Boundary caustics

Theorem. *A germ of a generic wave front (caustic) from a "source with boundary" in a space of $l \leq 4$ ($l \leq 3$) dimensions is stable and is diffeomorphic to the Cartesian product of the front (caustic) of a miniversal family of one of the germs $A_\mu, B_\mu, C_\mu, D_\mu, E_\mu, F_4$ with $\mu \leq l$ ($\mu - 1 \leq l$) and a germ of a nonsingular manifold of dimension $l - \mu$ ($l - \mu + 1$), or to a union of such fronts (caustics) which are transversal. Unstable generic fronts (caustics) are encountered in spaces of dimension $l \geq 5$ ($l \geq 4$).*

Example ([64]). In Euclidean space $\mathbb{R}^3$ let there be given a generic surface with boundary. Where the boundary is tangent to a line of curvature of the surface, three things together—the focal points of the surface (A_2), the focal points of the boundary ($\tilde{A}_2$), the normals to the surface at the points of the boundary (B_2)—form the caustic F_4 at the centre of curvature of the surface.

The symplectic version of the theory of caustics from a source with boundary leads to the following object: in the total space of a Lagrangian fibration, two nontangent Lagrangian manifolds which intersect along a hypersurface of each of them. The caustic of such an object consists of three parts: the caustics of both Lagrangian manifolds and the projection of their intersection to the base space of the Lagrangian fibration.

The theory of generating families of such objects reduces to the theory of singularities of functions on a manifold with boundary [58]. Interchanging the two Lagrangian manifolds corresponds to interchanging the stable equivalence classes of the function itself and its restriction to the boundary [65]. This duality generalizes the duality of the B and C series of simple boundary singularities and clears up the classification of unimodal and bimodal critical points of functions on a manifold with boundary [53].

3.3. Weyl Groups and Simple Fronts. The classification of simple germs of functions on a manifold with boundary is parallel to many other classifications of "simple" objects. One of them is the classification of symmetry groups of regular integral polyhedra in multidimensional spaces.

A Weyl group is a finite group of orthogonal transformations of a Euclidean space V which is generated by reflections in hyperplanes and preserves some full-dimensional integral lattice in V. The irreducible pairs (Weyl group, lattice) are classified by Dynkin diagrams [14]; see Fig. 45.

The vertices of the Dynkin diagram correspond to the basis vectors of the lattice $\mathbb{Z}^\mu$, the edges give the scalar product of the basis vectors according to a definite rule (the absence of an edge signifies their orthogonality). The Weyl group corresponding to the diagram is generated by the reflections in the

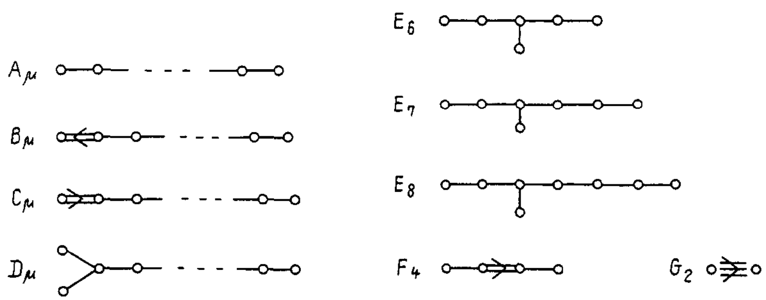

Fig. 45. Dynkin diagrams

hyperplanes orthogonal to the basis vectors of the lattice. Any Weyl group is isomorphic to a direct product of irreducible ones.

Examples. The Weyl group A_1 is just the group $\mathbb{Z}_2$, which acts by reflection on the line. The Weyl groups on the plane (besides $A_1 \oplus A_1$) are just the symmetry groups of the regular triangle, square and hexagon (Fig. 46). To the diagram C_μ corresponds the symmetry group of the μ-dimensional cube, and to B_μ that of its dual, the μ-dimensional "octahedron", so that the corresponding Weyl groups coincide, but the lattices connected with them are different.

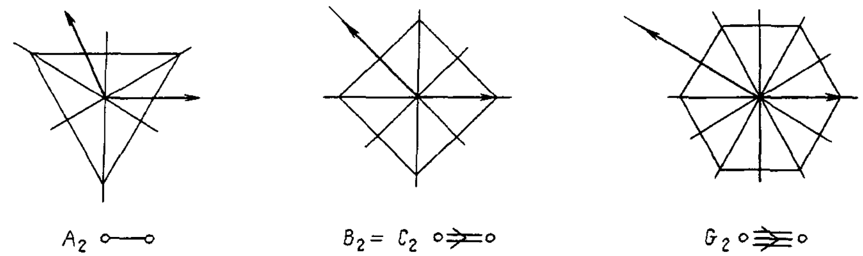

Fig. 46. Weyl groups on $\mathbb{R}^2$

Let us consider the action of a Weyl group W on the complexified μ-dimensional space $V^{\mathbb{C}}$. It turns out that the quotient manifold $V^{\mathbb{C}}/W$ is nonsingular and diffeomorphic to $\mathbb{C}^\mu$.

Example. For the group A_μ of permutations of the roots of a polynomial of degree $\mu + 1$ (with zero root sum) this is a fundamental theorem about symmetric polynomials: every symmetric polynomial can be uniquely represented as a polynomial in the elementary symmetric functions.

Let us consider all reflections in hyperplanes (mirrors) which lie in the Weyl group W. The image of the mirrors under the projection $V^{\mathbb{C}} \to V^{\mathbb{C}}/W$ (the Vieta map) is a singular hypersurface and is called the discriminant of the Weyl group.

Example. The discriminant of the Weyl group A_μ is the variety of polynomials with multiple roots in the μ-dimensional space of polynomials of degree $\mu + 1$ in one variable with a given leading coefficient and zero sum of the roots.

Theorem. *The complex front of a simple boundary germ is diffeomorphic to the discriminant of the irreducible Weyl group of the same name.*

Corollary. *The strata of the complex front of a simple boundary germ are in one-to-one correspondence with the subdiagrams of the corresponding Dynkin diagram (a subdiagram is obtained by discarding a certain number of vertices of the diagram together with the edges adjoining them,* Fig. 47).

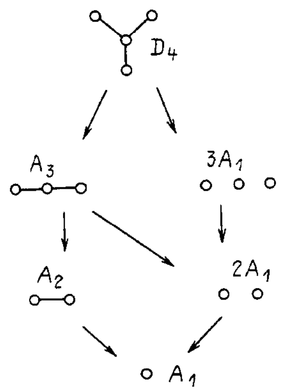

Fig. 47. The stratification of the front D_4

Remarks. 1) The singularity with the front G_2 can be obtained as a simple singularity in the theory of functions on the plane invariant with respect to the symmetry group of a regular triangle.

2) Besides Weyl groups, platonics, fronts and caustics other objects too can be classified by Dynkin diagrams, for example, complex simple Lie groups [24]. We have already pointed out this connection in chap. 1, sect. 4.3 with regard to the example of the symplectic group $\mathrm{Sp}(2\mu, \mathbb{C})$—it corresponds to the diagram C_μ. Direct correspondences between the various such classifications have not been established completely, although many of them are known. Thus, from a simple complex Lie algebra one can construct a simple singularity of surfaces in $\mathbb{C}^3$ together with its miniversal deformation [15]. The general reason for the universality of the *A*, *B*, *C*, *D*, *E*, *F*, *G*-classification remains rather mysterious.

3.4. Metamorphoses of Wave Fronts and Caustics. A spreading wave front will not at all moments of time be a generic front: at isolated moments of time it bifurcates. The investigation of such metamorphoses leads to the problem of generic singularities in a family of Legendre mappings.

Let us consider a family of Legendre mappings depending on one parameter t—the time. We shall call the union in space-time (the product of the base space by the time axis) of the fronts corresponding to the different values of t the big front.

Lemma. *The germ of the big front at each point is a germ of a front of a Legendre mapping in space-time.*

Indeed, it is given by the generating family $F_t(x, q) = 0$ of hypersurfaces in the x-space with the (q, t)-space-time as the base space. □

An equivalence of metamorphoses of fronts is just a diffeomorphism of space-time which takes the big fronts over into each other and preserves the time function on this space up to an additive constant: $t \mapsto t + \mathrm{const}$.

Example. Special metamorphoses. In the space $\mathbb{R}^m \times \mathbb{R}^\mu$ let us consider a big front Σ which is the product of $\mathbb{R}^m$ with the front of a simple germ of multiplicity μ. Let us choose as a miniversal deformation of the simple germ f a monomial deformation of the form $f(x) + q_0 e_{\mu-1}(x) + \ldots + q_{\mu-2} e_1(x) + q_{\mu-1}$, where $e_{\mu-1}(x)$ represents the class of the highest quasihomogeneous degree in the local algebra of the germ (for example, for A_μ: $x^{\mu+1} + q_0 x^{\mu-1} + \ldots + q_{\mu-1}$). Let us denote by $(\tau_1, \ldots, \tau_m)$ the coordinates on $\mathbb{R}^m$ and let us give a special metamorphosis by a time function on $\mathbb{R}^m \times \mathbb{R}^\mu$ of the form $t = \pm q_0 \pm \tau_1^2 \pm \ldots \pm \tau_m^2$ or $t = \tau_1$.

Theorem. *The metamorphoses in generic one-parameter families of fronts in spaces of dimension $l < 6$ are locally equivalent to the germs of special metamorphoses at 0, where $\mu + m = l + 1$ (Fig. 48).*

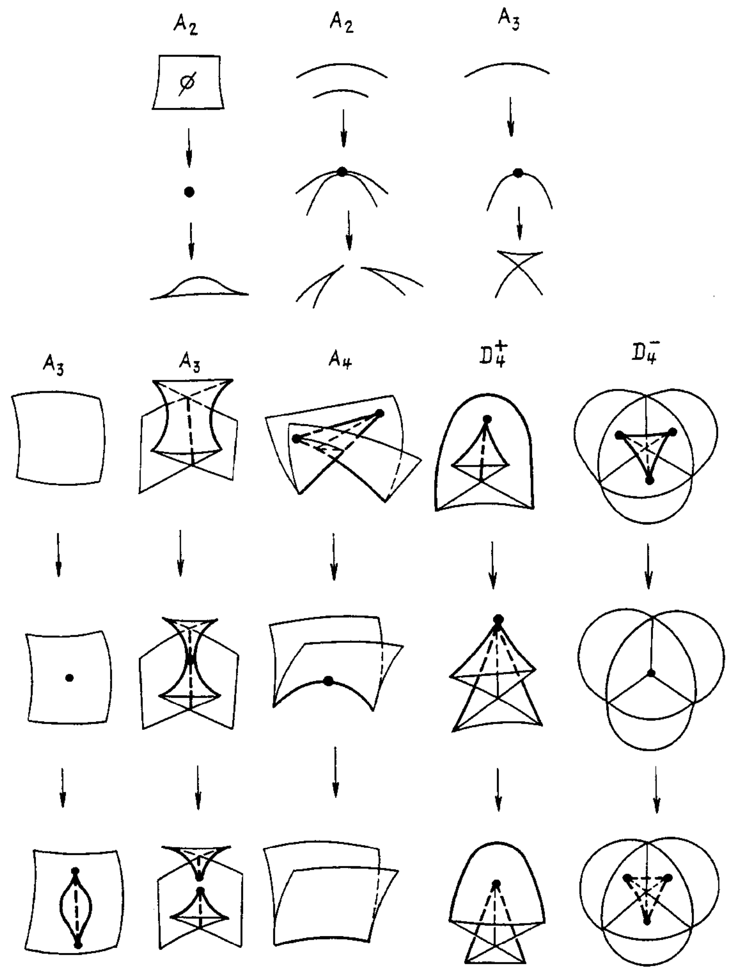

Fig. 48. Metamorphoses of wave fronts

The idea of the proof. Let us consider the (branched) covering of the space of polynomials of the form $x^{\mu+1}+q_0x^{\mu-1}+\ldots+q_{\mu-1}$ $(q\in\mathbb{C}^\mu)$ by the space of their complex roots $\{(x_1,\ldots,x_\mu)\mid\sum x_i=0\}$. The coefficients q_k then turn out to be elementary symmetric polynomials in the roots: $q_k=(-1)^k\sum x_{i_0}\ldots x_{i_{k+1}}$. A generic time function $t(q)$ satisfies the requirement $c=\partial t/\partial q_0|_0\neq 0$. This means that as a function on the space of roots, the time function has a nondegenerate quadratic differential $c\cdot\sum dx_idx_j$, $\sum dx_j=0$. Now in the case of the metamorphosis of a holomorphic front of type A_μ the proof of the theorem is completed by

The Equivariant Morse Lemma [3]. *A holomorphic function on $\mathbb{C}^k$, invariant with respect to a linear representation of a compact (for example, a finite) group G on $\mathbb{C}^k$, and with a nondegenerate critical point at 0, can be reduced to its quadratic part by means of a local diffeomorphism which commutes with the action of G.*

One can show that such a diffeomorphism can be lowered to a diffeomorphism of the space of polynomials.

The general case can be obtained analogously by using the Vieta map $V^{\mathbb{C}}\to V^{\mathbb{C}}/W$ of the other Weyl groups (and for $m>0$, the Morse lemma with parameters of sect. 2.2). □

The metamorphoses of caustics in generic one-parameter families, like the bifurcations of fronts, can be described by the dissections of a big caustic—the union of the momentary caustics in space-time—by the level surfaces of the time function. However, with the lack of an analogue of the Vieta map for caustics, these metamorphoses, even for simple big caustics, do not have such a universal normal form as the metamorphoses of fronts.

The list of normal forms of the time function has been computed in the cases A_μ and D_μ [9]. The big caustic is given by the generating family $F=\pm x^{\mu+1}+q_0x^{\mu-1}+\ldots+q_{\mu-2}x$ in the case of A_μ and $F=x_1^2x_2\pm x_2^{\mu-1}+q_0x_2^{\mu-2}+\ldots+q_{\mu-3}x_2+q_{\mu-2}x_1$ in the case of D_μ. Here the space-time is $\mathbb{R}^m\times\mathbb{R}^{\mu-1}$, $q\in\mathbb{R}^{\mu-1}$, $\tau\in\mathbb{R}^m$. By means of an equivalence of metamorphoses the germ of a generic time function may be reduced in the A_μ case to the form $t=\tau_1$ or $t=\pm q_0\pm\tau_1^2\pm\ldots\pm\tau_m^2$, and in the D_μ case, if one also allows diffeomorphisms of the value axis of the time function, to the form $t=\tau_1$ or $t=\pm q_0-q_{\mu-1}+aq_1\pm\tau_1^2\pm\ldots\pm\tau_m^2$. If in the case D_μ for $m=0$, in reducing the time function to normal form, one allows diffeomorphisms of a punctured neighbourhood of the origin in space-time which can be continuously extended to this point, then the resulting topological classification of generic metamorphoses turns out to be finite (V.I. Bakhtin): for D_4^- $t=q_0+q_1$, for D_4^+ $t=q_0\pm q_1$ or $t=q_0+q_3$, for D_{2k} and $k\geq 3$ $t=q_0\pm q_1$, for D_{2k+1} $t=\pm q_0$.

In general one-parameter families of caustics in spaces of dimension $l\leq 3$ one only encounters metamorphoses equivalent to the enumerated ones of types A_μ and D_μ with $\mu-2+m=l$. In Figs. 49, 50 these metamorphoses are depicted for $l=3$.

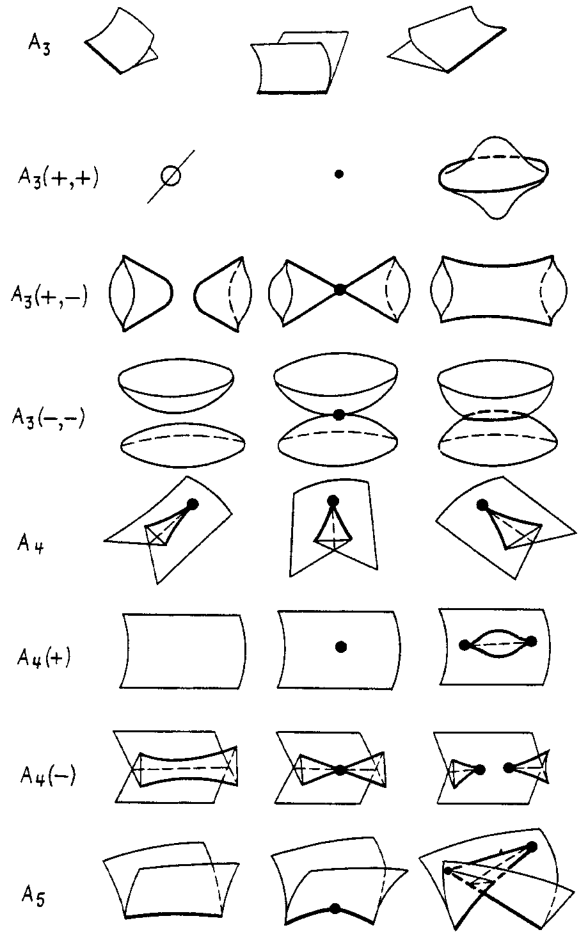

Fig. 49. Metamorphoses of caustics, the A series

Not all metamorphoses of caustics can be realized in geometrical optics. Thus, the caustic of the pencil of straight rays from a smooth source on the plane has no inflection points (this will become clear in sect. 3.5). The "flying saucer" of the metamorphosis $A_3(+,+)$ also can not be realized as the caustic of a pencil of geodesics of a Riemannian metric on a three-dimensional manifold.

3.5. Fronts in the Problem of Going Around an Obstacle. In the problem of the quickest way around an obstacle bounded by a smooth surface in Euclidean space, the extremals are rays breaking away along tangent directions to the pencil of geodesics on the surface of the obstacle. In sect. 1.6 of chap. 3 the

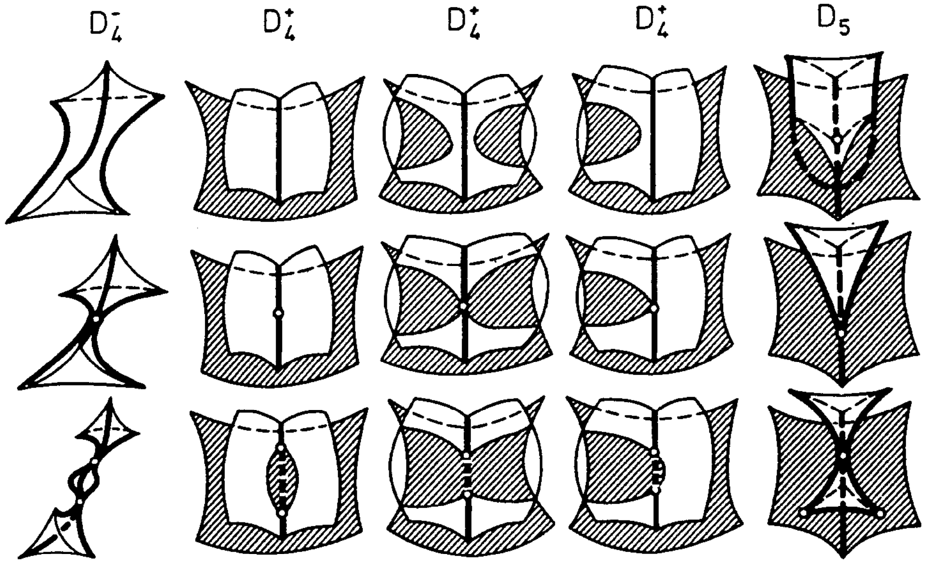

Fig. 50. Metamorphoses of caustics, the *D* series

singularities of a system of rays tangent to a geodesic pencil on a generic surface were studied. Here we shall describe, following O.P. Shcherbak, the singularities of the time function and of its levels—the fronts, in supposition of the single-valuedness of the time function on the obstacle surface. The discriminant of the symmetry group of the icosahedron appears among the normal forms in this problem.

In connection with our problem Huygens' principle is to the effect that each point x of the obstacle surface radiates into space along all rays close to the direction of the pencil geodesic on the surface at the point x. The optical length of the path consisting of the segment of the pencil geodesic from the source to the point x and the segment of the ray from the point x to a point q of the space (Fig. 51) is $F(x, q) = \phi(x) + G(x, q)$, where $\phi(x)$ is the time function on the surface and $G(x, q)$ is the distance between x and q in space. The extremals of the problem of going around the obstacle which pass through the point q break away from the obstacle surface at critical points of the function $F(\cdot, q)$. What is decisive for our subsequent considerations is the fact that all the critical points of the generating family F are of even multiplicity. The proof is depicted in Fig. 52: to a generic extremal corresponds a critical point of type A_2, and the multiplicity of a more complicated critical point is equal to twice the number of points of type A_2 into which it breaks up under perturbation of the parameter q.

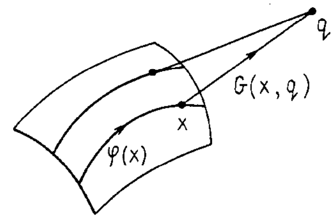

Fig. 51. The generating family in the problem of going around an obstacle

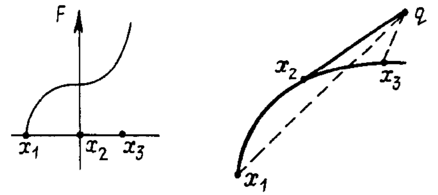

Fig. 52. The even multiplicity of the critical points

The reduction of the generic generating families $F(x, q)$ to normal form amounts to enumerating the maximal subfamilies (with a nonsingular base space) of R-miniversal families belonging to function germs with even multiplicity, in which all the functions have critical points only of even multiplicity. Such deformations of the simple germs are enumerated below.

$$A'_{2k}:\ y^2 + \int_0^x (u^k + q_1 u^{k-2} + \ldots + q_{k-1})^2 du + q_k;$$

$$D'_{2k}:\ \int_0^y (u^{k-1} + q_1 u^{k-2} + \ldots + q_{k-2} u + x)^2 du + q_{k-1} x + q_k;$$

$$E'_6:\ x^3 + y^4 + q_1 y^2 + q_2 y + q_3;$$

$$E'_8:\ x^3 + y^5 + q_1 y^3 + q_2 y^2 + q_3 y + q_4;$$

$$E''_8:\ x^3 + \int_0^y (u^2 + q_1 x + q_2)^2 du + q_3 x + q_4;$$

The front of the family consists of the points in the parameter space corresponding to functions with a critical point on the zero level.

Theorem. *The functions of a generating family in the problem of going around a generic obstacle in space and in the case of a generic pencil of geodesics on the obstacle surface have only simple critical points. The graph of the time function in four-dimensional space-time (the big front in the terminology of sect. 3.4) is in the neighbourhood of any point diffeomorphic to the Cartesian product of the front of one of the families* A'_2, A'_4, A'_6, D'_4, D'_6, D'_8, E'_6, E'_8, E''_8 *by a nonsingular manifold.*

Examples. 1) To the family A'_2 correspond the nonsingular points of the front.

2) The self-intersection line of the swallowtail—the latter is the caustic of an R_+-versal deformation of the germ A_4—corresponds to the functions with two critical points of type A_2. This is how one obtains the family A'_4 of the preceding list. Its front $q_2 \sim \pm q_1^{5/2}$ is diffeomorphic to the discriminant of the symmetry group H_2 of the regular pentagon.

3) In Fig. 53a is shown the metamorphosis of a front in the neighbourhood of an inflection point of an obstacle on the plane. The graph of the time function (Fig. 53b) is diffeomorphic to the discriminant of the symmetry group H_3 of the icosahedron[13]. The discriminant of H_3 is the front of the family D'_6. It is also diffeomorphic to the front in the neighbourhood of the points on an obstacle surface in space at which the pencil geodesic has an asymptotic direction.

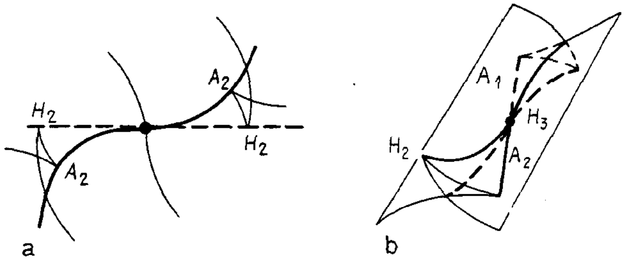

Fig. 53. Going around an obstacle in $\mathbb{R}^2$ and the icosahedron discriminant

4) After breaking away from the obstacle surface in a nonasymptotic direction, the rays may become focused far away from it, forming a caustic. The metamorphoses of the front in the neighbourhood of a nonsingular point, a point on a cusp ridge, or a swallowtail vertex of such a caustic are described by the families D'_4 $(x^3 - y^3 + q_1 y + q_2)$, E'_6 and E'_8. These families can be obtained from the R-versal families of the germs A_2, A_3, A_4 by addition of the cube of a new variable. This operation turns all critical points of the functions of the original family into ones of even multiplicity, but does not change its front.

5) In the classification of regular polytopes (see [14]), just after the pentagon and the icosahedron comes a body with 120 vertices in four-dimensional space. It can be described as follows. The Lie group SU_2 of quaternions of unit length contains the binary group of the icosahedron (see sect. 2.4). Its 120 elements are just the vertices of our polytope in the space of quaternions. The discriminant of the symmetry group H_4 of this polytope is diffeomorphic to the front of the

[13] A different description: the union of the tangents to the curve (t, t^3, t^5) in $\mathbb{R}^3$.

family E''_8. In the problem of going around an obstacle this front is encountered as the graph of the time function in the neighbourhood of a point of intersection of an asymptotic ray with a cusp ridge of the caustic far away from the obstacle surface[14].

If we now turn to the classification of the irreducible Coxeter groups—finite groups generated by reflections but not necessarily preserving an integral lattice (see [14]), then we will discover that among the wave fronts in the various problems of geometrical optics, we have encountered the discriminants of all such groups except for the symmetry groups of the regular n-gons with $n \geq 6$.

Chapter 6

Lagrangian and Legendre Cobordisms

Cobordism theory studies the properties of a smooth manifold which do not change when it is replaced by another manifold of the same dimension which together with the first forms the boundary of a manifold of dimension one greater (Fig. 54). In this chapter the manifolds and the sheets bounded by them will be Lagrangian or Legendre submanifolds. The corresponding cobordism theories reflect, for example, the global properties of wave fronts which are preserved under metamorphoses.[15]

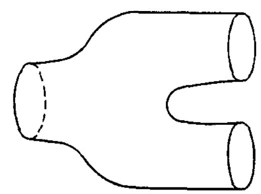

Fig. 54. Cobordism

§ 1. The Maslov Index

A Lagrangian submanifold of a phase space describes the phase of a short-wave oscillation. The Maslov index associates integers to the curves on the

[14] Such a ray breaks away from the obstacle at a parabolic point.

[15] Beginning with § 2, we are compelled to abandon all concern for the inexperienced reader. By way of compensation, § 1 includes a completely elementary exposition of the theory of cobordisms of wave fronts on the plane.

Lagrangian submanifold. These numbers enter into an asymptotic expression for the solutions of the wave equations at the short-wave limit. In the following sections the Maslov index will appear in the rôle of the simplest characteristic class of the theory of Legendre and Lagrangian cobordisms.

1.1. The Quasiclassical Asymptotics of the Solutions of the Schrödinger Equation. Let us consider the Schrödinger equation

$$ih\frac{\partial \Psi}{\partial t} = -\frac{h^2}{2}\Delta\Psi + U(q)\Psi \tag{1}$$

(h is the Planck constant, Δ is the Laplace operator on $\mathbb{R}^n$), which is satisfied by the probability amplitude $\Psi(q, t)$ of a quantum-mechanical particle moving in the potential U.

To equation (1) corresponds the classical mechanical system with Hamiltonian $H = p^2/2 + U(q)$ on the standard symplectic space $\mathbb{R}^{2n}$. To an initial condition of the form $\Psi(q, 0) = \phi(q)\exp(if(q)/h)$ (ϕ is a function of compact support) there corresponds the function ϕ on the Lagrangian manifold L in $\mathbb{R}^{2n}$ with generating function f: $L = \{(p, q) \in \mathbb{R}^{2n} \mid p = f_q\}$. The flow g^t of the Hamiltonian H defines a family of Lagrangian manifolds $L_t = g^t L$, which for large t may, unlike L_0, project noninjectively into the configuration space $\mathbb{R}^n$ (Fig. 55). There arises a family of Lagrangian mappings (see sect. 1.2, chap. 5) of the configuration space into itself $Q_t\colon \mathbb{R}^n \to \mathbb{R}^n$. The following asymptotic formula holds for the solution of equation (1) with the given initial condition [52]. Let q_j be the points in $\mathbb{R}^n$ such that $Q_t(q_j) = Q$ and let $x_j(p_j, q_j)$ be the corresponding points of L_0. Let us suppose that the Jacobians $|\partial Q_t/\partial q|_{q=q_j}$ are different from zero. Then for $h \to 0$

$$\Psi(q, t) = \sum_j \phi(q_j)|\partial Q_t/\partial q_j|^{-1/2}\exp\left(i\frac{S_j(Q, T)}{h} - \frac{i\pi\mu_j}{2}\right) + O(h),$$

where $S_j(Q, t)$ is the action along the trajectory $g^\tau x_j$: $S_j(Q, t) = f(q_j) + \int_0^t (p\,dq - H\,d\tau)$, and μ_j is the Morse index of the trajectory $g^\tau x_j$, i.e. the number of critical points of the Lagrangian projection $L_\tau \to \mathbb{R}^n$ on this traectory for $0 < \tau < t$.

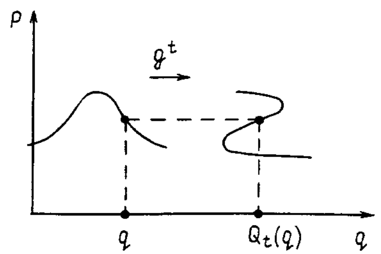

Fig. 55. The family of Lagrangian mappings

1.2. The Morse Index and the Maslov Index. The Morse index is a special case of the Maslov index. In the total space of the cotangent bundle T^*X of a configuration manifold X let there be given a generic Lagrangian submanifold L. The *Maslov index* of an oriented curve on L is its intersection number with the cycle of singular points of the Lagrangian projection $L \to X$. This definition needs to be made more precise.

Lemma. *The set Γ of singular points of the Lagrangian projection $L \to X$ is a hypersurface in L, smooth outside a set of codimension 3 in L. The hypersurface Γ possesses a canonical coorientation, i.e. at each of its points (outside the set of codimension 3) one may state which side of Γ is "positive" and which is "negative".*

The Maslov index of a generic oriented curve on L can now be defined as the number of its crossings from the "positive" side of Γ to the "negative", minus the number of opposite crossings. For an arbitrary curve, whose ends do not lie on Γ, its Maslov index may be taken to be equal to the Maslov index of a perturbation of it, and by the lemma it does not depend on this perturbation.

Proof of the lemma. According to the classification of Lagrangian singularities (chap. 5), the germ of the Lagrangian mapping $L \to X$ at a generic point has type A_1, at the points of some hypersurface, type A_2 (the generic points of Γ), and at the points of a variety of codimension 2, type A_3. A study of the normal form of the germ A_3 shows that at the points of type A_3 the hypersurface Γ is nonsingular (Fig. 56). Singularities of Γ begin with the stratum D_4 and form a set of codimension ≥ 3. For the coorientation of the hypersurface Γ let us consider the action integral $\int p\,dq$. Locally on the manifold L it defines uniquely up to a constant summand a smooth function S—a generating function for L. At a singular point of type A_2 the kernel of the differential of the Lagrangian projection $L \to X$ is one-dimensional and is a tangent line transversal to Γ. On this line the first and second differentials of the function S vanish; therefore the third differential is well defined. It is different from zero. We take as "positive" that side of the hypersurface Γ in the direction of which the third differential of the function S increases. An immediate check (Fig. 56) shows that this coorientation can be extended in a well-defined manner to the points of type A_3. □

The Morse index may be interpreted as a Maslov index. Let us consider the phase space $\mathbb{R}^{2n+2}$ with coordinates (p_0, p, q_0, q), where $(p, q) \in \mathbb{R}^{2n}$. If we set $q_0 = \tau$, $p_0 = -H(p, q)$, and we make the point $(p; q)$ run through the Lagrangian manifold L_τ in $\mathbb{R}^{2n}$, then as τ changes from 0 to t we obtain an $n+1$-dimensional Lagrangian manifold L in $\mathbb{R}^{2n+2}$. The phase curves of the flow of the Hamiltonian H which begin in L_0 may be considered as curves on L. The Maslov index of such a curve on L coincides with the Morse index of the original phase curve in $\mathbb{R}^{2n}$. Indeed, the contribution of a critical point of type A_2 of the Lagrangian projection $L_\tau \to \mathbb{R}^n$ to the Maslov index of the phase curve passing through this point is determined by the sign of the derivative $\partial^3 S/\partial v^2 \partial\tau$, where its index is equal to 2, and the preceding formula becomes the so-called

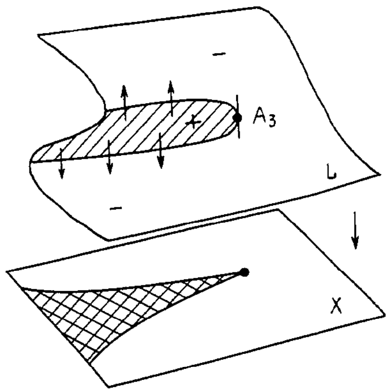

Fig. 56. The coorientation of the cycle Γ

$S=\int(pdq-Hd\tau)$ is a generating function for L, and v is a vector from the kernel of the differential of the projection $L\subsetneq\mathbb{R}^{2n+2}\to\mathbb{R}^{n+1}$ at the critical point under consideration. This sign is always negative in view of the convexity of the function $H=p^2/2+U(q)$ with respect to the momenta. Therefore each critical point lying on the phase curve gives a contribution of $+1$ both to its Maslov index and to the Morse index.

1.3. The Maslov Index of Closed Curves. The intersection number of a closed curve on a Lagrangian manifold $L\subset T^*X$ with the cooriented hypersurface Γ of singular points of the projection $L\to X$ does not change upon replacement of the curve by a homologous one. Therefore Γ defines a *Maslov class* in the cohomology group $H^1(L,\mathbb{Z})$.

The Maslov indices of closed curves enter into the asymptotic formulas for the solutions of stationary problems (characteristic oscillations) [52]. Let us suppose that on the level manifold $H=E$ of the Hamiltonian $H=p^2/2+U(q)$ there lies a Lagrangian submanifold L. If a sequence of numbers $\mu_N\to\infty$ satisfies the conditions

$$\frac{2\mu_N}{\pi}\oint_\gamma pdq\equiv \operatorname{ind}\gamma\ (\mathrm{mod}\ 4)$$

for all closed contours γ on L (for the existence of the sequence μ_N the existence of at least one such number $\mu\neq 0$ is sufficient), then the equation $\Delta\Psi/2=\lambda^2(U(q)-E)\Psi$ has a series of eigenvalues λ_N with the asymptotic behaviour $\lambda_N=\mu_N+O(\mu_N^{-1})$.

In the one-dimensional case the Lagrangian manifold is an embedded circle, its index is equal to 2, and the preceding formula becomes the so-called

"quantization condition" (see the article by A.A. Kirillov in this volume)

$$\mu\oint_{H=E} pdq=2\pi(N+\tfrac{1}{2}).$$

For example, in the case $H=p^2/2+q^2/2$ with the given Planck constant $h=1/\mu$ we obtain the exact values for the characteristic energy levels $E_N=h(N+\frac{1}{2})$, $N=0, 1, \ldots$ of the quantum harmonic oscillator.

The Maslov class of Lagrangian submanifolds of the standard symplectic space $\mathbb{R}^{2n}$ is the inverse image of a universal class of the Lagrangian Grassmann manifold Λ_n under the Gauss mapping. The *Gauss mapping* $G: L\to\Lambda_n$ associates to a point of the Lagrangian submanifold the tangent Lagrangian space at that point, translated to 0. The cohomology group $H^1(\Lambda_n,\mathbb{Z})$ is isomorphic to $\mathbb{Z}$.

Theorem. *The generator of the group $H^1(\Lambda_n,\mathbb{Z})$ goes over into the Maslov class under the homomorphism G^*: $H^1(\Lambda_n,\mathbb{Z})\to H^1(L,\mathbb{Z})$.*

Corollary. *The Maslov class of a Lagrangian submanifold in $\mathbb{R}^{2n}$ does not depend on the Lagrangian projection $\mathbb{R}^{2n}\to\mathbb{R}^n$.*

Below we shall cite two descriptions of the generators in the group $H^1(\Lambda_n,\mathbb{Z})$.

1.4. The Lagrangian Grassmann Manifold and the Universal Maslov Class. The *Lagrangian Grassmann manifold* Λ_n is the manifold of all Lagrangian linear subspaces of the $2n$-dimensional symplectic space. The manifold of all oriented Lagrangian subspaces in $\mathbb{R}^{2n}$ is called the *oriented* Lagrangian Grassmann manifold and is denoted by Λ_n^+. Obviously Λ_n^+ is a double covering of Λ_n.

Examples. 1) $\Lambda_1=\mathbb{RP}^1\cong\Lambda_1^+=S^1$.

2) The manifold Λ_2 is isomorphic to the quadric $x^2+y^2+z^2=u^2+v^2$ in $\mathbb{RP}^4$ as a projective algebraic manifold (see sect. 1.3, chap. 1). The manifold Λ_2^+ is diffeomorphic to $S^2\times S^1$. The covering $\Lambda_2^+\to\Lambda_2$ is the factorization of $S^2\times S^1$ by the antipodal involution $(x,y)\mapsto(-x,-y)$ (in [37] Λ_2 is found incorrectly).

3) $\dim\Lambda_n=n(n+1)/2$: a generic Lagrangian subspace in $\mathbb{R}^{2n}$ is given by a generating quadratic form in n variables.

Theorem. *The Grassmann manifolds of Lagrangian subspaces in $\mathbb{C}^n$ are homogeneous spaces: $\Lambda_n=\mathrm{U}_n/\mathrm{O}_n$, $\Lambda_n^+=\mathrm{U}_n/\mathrm{SO}_n$, where U_n, O_n, SO_n are the unitary, orthogonal and special orthogonal groups respectively.*

Indeed, an orthonormal basis of a Lagrangian subspace in $\mathbb{C}^n$ is a unitary basis in $\mathbb{C}^n$ and conversely, the real linear span of a unitary frame in $\mathbb{C}^n$ is a Lagrangian subspace. Unitary bases which generate the same Lagrangian subspace can be obtained from each other by an orthogonal transformation of this subspace. □

Corollary. $\pi_1(\Lambda_n)=H_1(\Lambda_n,\mathbb{Z})\cong H^1(\Lambda_n,\mathbb{Z})\cong\mathbb{Z}$.

Let us consider the mapping $\det^2$: $U_n \to \mathbb{C}^\times$, which associates to a unitary matrix the square of its determinant. The mapping $\det^2$ well-defines a fibration of the Lagrangian Grassmann manifold Λ_n over the circle $S^1 = \{e^{i\phi}\}$ of complex numbers of modulus 1. The fibre SU_n/SO_n of this fibration is simply connected, and therefore the differential 1-form $\alpha = (1/2\pi)(\det^2)^* d\phi$ on Λ_n represents a generator of the one-dimensional cohomology group $H^1(\Lambda_n, \mathbb{Z})$. We shall call this generator the *universal Maslov class.*

Example. $\det^2$: $\Lambda_1 \to S^1$ is a diffeomorphism, and $\alpha = d\theta/\pi$, where θ is the angular coordinate of the line $l \in \Lambda_1$ on the plane $\mathbb{R}^2$.

Let us define an inclusion j: $\Lambda_{n-1} \subsetneq \Lambda_n$ in the following manner. Let H be a hyperplane in $\mathbb{R}^{2n}$. The projection $H \to H/H^\perp$ into the $2n-2$-dimensional symplectic space of characteristics of the hyperplane H establishes a one-to-one correspondence between the $n-1$-dimensional Lagrangian subspaces in $H/H^\perp$ and the n-dimensional Lagrangian subspaces lying in H.

Lemma. *The inclusion j induces an isomorphism of the fundamental groups.*

Indeed, the sequence of inclusions $\Lambda_1 \subsetneq \Lambda_2 \subsetneq \ldots \subsetneq \Lambda_n$ maps the circle Λ_1 into a circle S over which the integral of the 1-form α on Λ_n is equal to 1, i.e. the inclusion $\Lambda_1 \subsetneq \Lambda_n$ induces an isomorphism $\pi_1(\Lambda_1) \to \pi_1(\Lambda_n)$. □

Let us now present the cycle which is dual to the generator of the group $H^1(\Lambda_n, \mathbb{Z}) = \mathbb{Z}$. Let us fix a Lagrangian subspace $L \subset \mathbb{R}^{2n}$ and let us denote by Σ the hypersurface in Λ_n formed by the Lagrangian subspaces in $\mathbb{R}^{2n}$ which are not transversal to L. The Lagrangian spaces in $\mathbb{R}^{2n}$ which intersect L along subspaces of dimension 2 or more form a set Σ' of codimension 3 in Λ_n (and 2 in Σ). At points $\lambda \in \Sigma \setminus \Sigma'$ the hypersurface Σ is smooth. Let us coorient Σ by choosing as a vector of the positive normal at the point λ the velocity vector of the curve $e^{i\theta}\lambda$ (it is transversal to Σ).

Theorem. *The intersection number of a 1-cycle in Λ_n with the cooriented cycle Σ is equal to the value of the universal Maslov class on this 1-cycle.*

Indeed, the intersection number of the generator S of the group $H_1(\Lambda_n, \mathbb{Z})$ with the cycle Σ is equal to 1. □

The cycle Γ of singularities of the projection of a Lagrangian submanifold in $\mathbb{R}^{2n}$ along the Lagrangian subspace L is the inverse image of the cycle Σ under the Gauss map. From this the theorem of sect. 1.3 follows. □

Remark. The Maslov index has found application in the theory of representations [49]. In this context a series of Maslov indices is used—symplectic invariants of chains of k Lagrangian subspaces in $\mathbb{R}^{2n}$. The simplest of them is the triple index $\tau(\lambda_1, \lambda_2, \lambda_3)$, equal to the signature of the quadratic form $Q(x_1 \oplus x_2 \oplus x_3) = \omega(x_1, x_2) + \omega(x_2, x_3) + \omega(x_3, x_1)$ on the direct sum of the Lagrangian subspaces $\lambda_1 \oplus \lambda_2 \oplus \lambda_3$, where ω is the symplectic form on $\mathbb{R}^{2n}$. It has the cocycle property: $\tau(\lambda_1, \lambda_2, \lambda_3) - \tau(\lambda_1, \lambda_2, \lambda_4) + \tau(\lambda_1, \lambda_3, \lambda_4) - \tau(\lambda_2, \lambda_3, \lambda_4) = 0$.

With the aid of the Maslov index of a quadruple of subspaces $\tau(\lambda_1, \lambda_2, \lambda_3) + \tau(\lambda_1, \lambda_3, \lambda_4)$ one may define on a Lagrangian submanifold in the total space of the cotangent bundle of an arbitrary manifold a Čech cocycle corresponding to the Maslov class of sect. 1.3 (see [37]).

1.5. Cobordisms of Wave Fronts on the Plane. The simplest example of a Legendre cobordism is the relation between the traces of a wave front spreading in a three-dimensional medium on its boundary at the various moments of time. These traces are not necessarily homeomorphic, but their being cobordant imposes a restriction on the types of the singularities (see Corollary 1 below).

Two compact wave fronts F_0, F_1 on the plane are called *cobordant*, if in the direct product of the plane with the interval $0 \le t \le 1$ there exists a compact front K transversal to the planes $t = 0$ and $t = 1$ whose intersection with the first of these is F_0 and with the second, F_1 (Fig. 57). The front K is called a *cobordism.* We distinguish the cases of oriented and nonoriented, *armed* (cooriented) or unarmed fronts and cobordisms. An armament of a cobordism K induces an armament of the boundary $\partial K = F_0 \cup F_1$, which must coincide with the own armament of the fronts F_0, F_1 (i.e. fronts which coincide but which are spreading to different sides are considered different and may be incobordant). The same also relates to orientations. The addition of fronts is defined as their disjoint union. This operation provides the set of classes of cobordant fronts with the structure of a commutative semigroup. In the cases being considered it turns out to be a group. The class of the empty front serves as the zero element.

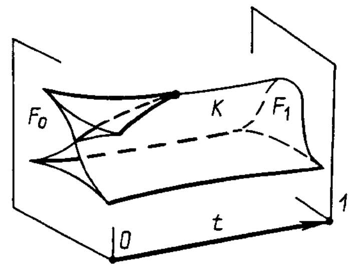

Fig. 57. Cobordism of fronts in $\mathbb{R}^2$

Theorem ([5]). *The group of cobordism classes of armed oriented fronts on the plane is free cyclic (the generator is the class of the "bow", Fig. 58a), of armed nonoriented ones is trivial, and of unarmed ones is finite: $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ for oriented and $\mathbb{Z}_2$ for nonoriented fronts (as generators one may take the classes of the "drops", Fig. 58b, which differ in the orientation in the oriented case).*

What serves as the unique invariant of the cobordism class of an armed oriented front on the plane is its index—the number of cusp points (or the number of inflection points) with signs taken into account, Fig. 58c. The index of a compact front in $\mathbb{R}^2$ is even.

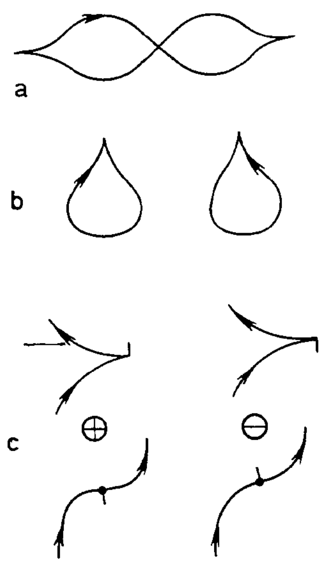

Fig. 58. a) the bow b) the drop c) the Maslov index of an armed front

Corollary 1. *The algebraic number of cusp (inflection) points of the compact trace on the plane of an oriented wave front spreading in space is even and does not change with time.*

The index of a front is connected with the Maslov index in the following way. An armed front in $\mathbb{R}^2$ defines the conical Lagrangian surface L in $T^*\mathbb{R}^2$ of covectors equal to zero on a contact element tangent to the front and positive on the arming normal. For a generic front L is smoothly immersed in $T^*\mathbb{R}^2$. The scalar product on $\mathbb{R}^2$ defines an immersion of the front into L (to a point of the front corresponds the covector equal to 1 on the unit vector of the arming normal). The index of an oriented front in $\mathbb{R}^2$ is equal to the Maslov index of the constructed curve on L.

The computation of the cobordism groups described in the theorem is based on the information on metamorphoses of wave fronts from sect. 3.4 of chap. 5. Thus, the nonoriented cobordance to zero of the armed bow follows from the series of restructurings of Fig. 59, using the local metamorphoses A_2 and A_3 of Fig. 48.

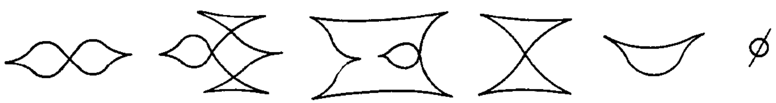

Fig. 59. The unoriented bow is cobordant to zero

A cobordism K of fronts defines the immersed Legendre manifold of its contact elements in the manifold of all contact elements of the product of the plane with the interval.

Corollary 2. *The Klein bottle admits a Legendre immersion into* $\mathbb{R}^5$.

This immersion is induced by the metamorphoses of Fig. 59 together with the reverse series: the fronts being bifurcated in this drawing have a nonvertical tangent everywhere and their union glues the "bow" by means of a Möbius strip. □

Theorem ([5], compare sect. 2.4, chap. 4). *The compact connected two-dimensional manifolds with an even Euler characteristic admit a Legendre immersion into the contact space* $\mathbb{R}^5$, *but those with an odd Euler characteristic do not even admit a Lagrangian immersion into* $\mathbb{R}^4$.

§ 2. Cobordisms

In [5] about a score of different Lagrangian and Legendre cobordism theories are defined and the corresponding groups of classes of cobordant curves are computed. Here we shall consider in the main the cobordisms of exact Lagrangian immersions—the theory in which the most complete results have been obtained.

2.1. The Lagrangian and the Legendre Boundary. In the total space of the cotangent bundle of a manifold with boundary let there be given an immersed Lagrangian submanifold $L \looparrowright T^*M$, transversal to the boundary $\partial(T^*M)$. Under the mapping $\partial(T^*M) \to T^*(\partial M)$ (to a covector at a point of the boundary is associated its restriction to the boundary) the intersection $L \cap \partial(T^*M)$ projects to an immersed Lagrangian submanifold ∂L of the total space of the cotangent bundle of the boundary. ∂L is called the *Lagrangian boundary* of the manifold L.

The *Legendre boundary* ∂L of a Legendre manifold $L \looparrowright J^1M$ immersed in the space of 1-jets of functions on a manifold with boundary and transversal to the boundary $\partial(J^1M)$ is defined analogously with the aid of the projection $\partial(J^1M) \to J^1(\partial M)$ (to the 1-jet of a function at a point of the boundary is associated the 1-jet of the restriction of the function to the boundary). In a similar way one can define the boundary of a Legendre submanifold of the space of (cooriented) contact elements on a manifold with boundary.

A *Lagrangian cobordism* of two compact Lagrangian immersed submanifolds $L_0, L_1 \looparrowright T^*M$ is an immersed Lagrangian submanifold of the space $T^*(M \times [0, 1])$, the cotangent bundle of the cylinder over M, whose Lagrangian boundary is the difference of $L_1 \times 1$ and $L_0 \times 0$ (for oriented cobordisms changing the orientation of a manifold changes its sign). The manifolds L_0, L_1

are called Lagrangianly (orientedly) cobordant, if there exists a Lagrangian (oriented) cobordism between them.

Legendre cobordisms are defined analogously. In this case in place of cobordism of Legendre manifolds one may speak directly of cobordism of fronts. The theory of cobordisms of Legendre immersions in the space of 1-jets of functions is equivalent to the theory of cobordisms of exact Lagrangian immersions: under the projection $J^1M \to T^*M$ the Legendre immersed submanifolds go over into Lagrangian immersed submanifolds on which the action 1-form is exact, and conversely (see sect. 2.3, chap. 4).

2.2. The Ring of Cobordism Classes. Lagrangian (Legendre) immersions of manifolds L_1, L_2 of the same dimension into a symplectic (contact) manifold give an immersion of their disjoint union into this manifold, called the sum of the original immersions.

Lemma ([5]). *The classes of Lagrangianly (Legendrianly) cobordant immersions into T^*M (J^1M, PT^*M or ST^*M) form an abelian group with respect to the summation operation.*

In the simplest and most important case $M = \mathbb{R}^n$ let us define the product of two (exact) Lagrangian immersions $L_1 \looparrowright T^*\mathbb{R}^n$, $L_2 \looparrowright T^*\mathbb{R}^m$ as the (exact) Lagrangian immersion of the direct product $L_1 \times L_2$ in $T^*\mathbb{R}^{n+m} = T^*\mathbb{R}^n \times T^*\mathbb{R}^m$ The corresponding cobordism classes form a skew-commutative graded ring with respect to the operations introduced.

Theorem ([7], [12], [19]). 1) *The graded ring $\mathfrak{N}\mathbb{L}_* = \bigoplus_k \mathfrak{N}\mathbb{L}_k$ of nonoriented Legendre cobordism classes in the spaces of 1-jets of functions on $\mathbb{R}^k$ is isomorphic to the graded ring $\mathbb{Z}_2[x_5, x_9, x_{11}, \ldots]$ of polynomials with coefficients in $\mathbb{Z}_2$ in generators x_k of odd degrees $k \neq 2^r - 1$.*

2) *The graded ring $\mathbb{L}_* = \bigoplus_k \mathbb{L}_k$ oriented Legendre cobordism classes in the spaces of 1-jets of functions on $\mathbb{R}^k$ is isomorphic (after tensor multiplication with $\mathbb{Q}$) to the exterior algebra over $\mathbb{Q}$ with generators of degrees $1, 5, 9, \ldots, 4n+1, \ldots$.*

We are far from being able to prove this theorem, but we shall cite the fundamental results on the way to its proof.

2.3. Vector Bundles with a Trivial Complexification. Every k-dimensional vector bundle with a finite CW base space X can be induced from a universal classifying bundle ξ_k. As the latter one may take the tautological bundle over the Grassmann manifold $G_{\infty,k}$ of all k-dimensional subspaces in a space $\mathbb{R}^N$ of growing dimension N (serving as the fibre of the tautological bundle over a point is the k-dimensional subspace which answers to that point). The induction of a bundle over X under a continuous mapping $X \to G_{\infty,k}$ establishes a one-to-one correspondence between the equivalence classes of k-dimensional vector bundles

with the base space X and the homotopy classes of mappings of X into the classifying space $G_{\infty,k}$. In the category of oriented vector bundles the rôle of the classifying spaces is played by the Grassmann manifolds $G^+_{\infty,k}$ of oriented k-dimensional subspaces.

Let the complexification of a real k-dimensional vector bundle over X be trivial and let a trivialization be fixed.

An example: the complexification of a tangent space L to a Lagrangian manifold immersed in the realification $\mathbb{R}^{2k}$ of the Hermitian space $\mathbb{C}^k$ is canonically isomorphic to $\mathbb{C}^k = L \oplus iL$.

If we associate to a point in X the subspace of $\mathbb{C}^k$ with which the fibre over it is identified under the trivialization, we get a mapping of X into the Grassmann manifold of k-dimensional real subspaces L of $\mathbb{C}^k$ for which $L \cap iL = 0$. This Grassmann manifold is homotopy equivalent to the Lagrangian Grassmann manifold Λ_k.

Theorem ([27]). *The tautological bundle λ_k (λ_k^+) over the (oriented) Lagrangian Grassmann manifold Λ_k (Λ_k^+) is classifying in the category of k-dimensional (oriented) vector bundles with a trivialized complexification.*

2.4. Cobordisms of Smooth Manifolds. In this classical theory two closed manifolds (immersed nowhere) are called cobordant, if their difference is the boundary of some compact manifold with boundary. In the computation of the corresponding cobordism groups the key rôle is played by the following construction. By the Thom space $T\xi$ of a vector bundle ξ with a compact base space is meant the one-point compactification of the total space of this bundle (Fig. 60). For an induction of a bundle ξ from a bundle η under a mapping of the base spaces $X \to Y$ there is a corresponding mapping $T\xi \to T\eta$ of the Thom spaces, which takes the distinguished point (∞) over into the distinguished point. Let the compact n-dimensional manifold M be embedded in a sphere of large dimension $n+k$. Collapsing the complement of a tubular neighbourhood of M in

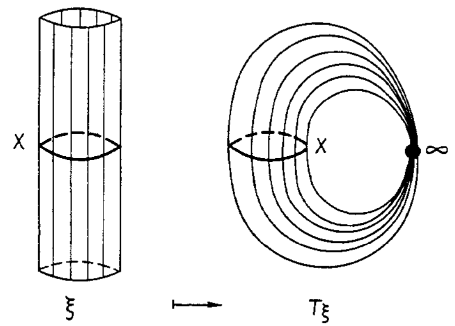

Fig. 60. The Thom space

S^{n+k} to a point gives a mapping $S^{n+k} \to Tv$ of the sphere to the Thom space of the normal bundle of the manifold M. The induction of the normal bundle from the classifying bundle ξ_k furnishes a mapping $Tv \to T\xi_k$ of Thom spaces, which in composition with the first defines a mapping $S^{n+k} \to T\xi_k$. An analogous construction, applied to a cobordism of manifolds which is embedded in $S^{n+k} \times [0, 1]$, shows that for a cobordism class there is a corresponding homotopy class of mappings $S^{n+k} \to T\xi_k$, i.e. an element of the homotopy group $\pi_{n+k}(T\xi_k, \infty)$. Conversely, the inverse image of the zero section $G_{\infty,k} \subsetneq T\xi_k$ under a mapping of the sphere $S^{n+k} \to T\xi_k$ transversal to it is a smooth n-dimensional submanifold in S^{n+k}, and the inverse image of the zero section under a homotopy $S^{n+k} \times [0, 1] \to T\xi_k$ transversal to it is a cobordism of such manifolds.

Theorem ([69]). *The group $\mathfrak{N}_n$ (Ω_n) of (oriented) cobordism classes of closed n-dimensional manifolds is isomorphic to the stable homotopy group $\lim_{k\to\infty} \pi_{n+k}(T\xi_k)$ of the Thom spaces of the classifying (oriented) vector bundles.*

Here the symbol lim denotes the following. A mapping $S^{n+k} \to T\xi_k$ can be suspended to a mapping $S^{n+k+1} \to T(\xi_k \oplus 1)$ of the sphere to the Thom space of the sum of the bundle ξ_k with the one-dimensional trivial bundle. This sum can be induced from ξ_{k+1}, which gives a mapping $S^{n+k+1} \to T\xi_{k+1}$. It is over the so arising sequence of homomorphisms $\pi_{n+k}(T\xi_k) \to \pi_{n+k+1}(T\xi_{k+1})$ that the limit is taken.

This theorem reduces the computation of cobordism groups to a purely homotopic problem, which one can succeed in solving to a significant degree.

Theorem ([69]). 1) *The ring $\mathfrak{N}_* = \bigoplus \mathfrak{N}_n$ is isomorphic to the ring $\mathbb{Z}_2[y_2, y_4, y_5, \ldots]$ of polynomials over $\mathbb{Z}_2$ in generators y_n of degrees $n \neq 2^r - 1$.*

2) *The ring $\Omega_* = \bigoplus \Omega_n$ is isomorphic modulo torsion to the ring of polynomials over $\mathbb{Z}$ with generators of degrees $4k$, $k = 1, 2, \ldots$.*

2.5. The Legendre Cobordism Groups as Homotopy Groups. Let us associate to a Legendre immersion of a manifold L in $J^1\mathbb{R}^n$ a trivialization of the complexified tangent bundle $T^{\mathbb{C}}L$, as explained in the example of sect. 2.3.

Theorem ([36], [46]). *This mapping is a one-to-one correspondence between the set of homotopy classes of Legendre immersions $L \subsetneq J^1\mathbb{R}^n$ and the set of homotopy classes of trivializations of the bundle $T^{\mathbb{C}}L$.*

From this, just as in the theory of cobordisms of smooth manifolds, one can deduce

Theorem (Ya.M. Ehliashberg, see [19]). *The groups of Legendre cobordism classes are isomorphic to the stable homotopy groups of the Thom spaces of the tautological bundles over the Lagrangian Grassmann manifolds:*

$$\mathfrak{N}\mathbb{L}_n = \lim_{k\to\infty} \pi_{n+k}(T\lambda_k), \quad \mathbb{L}_n = \lim_{k\to\infty} \pi_{n+k}(T\lambda_k^+).$$

The passage to the limit is in correspondence with the sequence of inclusions of Lagrangian Grassmann manifolds described in sect. 1.4.

An analogous expression for the cobordism groups through homotopy groups (of more unwieldy spaces, it is true) holds also for the other Lagrangian and Legendre cobordism theories[16], but these homotopy groups have at present not been computed.

The computation of the stable homotopy groups of the spaces $T\lambda_k$ leads to the following refinement of the theorem of sect. 2.2.

Theorem ([12]). *The mapping θ: $\mathfrak{N}\mathbb{L}_* \to \mathfrak{N}_*$ which associates to the cobordism class of an immersed Legendre manifold the nonoriented cobordism class of this manifold is an inclusion of graded rings and $\mathfrak{N}_* = \theta(\mathfrak{N}\mathbb{L}_*) \otimes \mathbb{Z}_2[y_2, \ldots, y_{2k}, \ldots]$.*

Corollary. *The nonoriented Legendre cobordism class of a Legendre immersion $L \subsetneq J^1\mathbb{R}^n$ depends only on the manifold L.*

2.6. The Lagrangian Cobordism Groups. These groups are as a rule too large to be readily visible. Only the Lagrangian cobordism groups of curves on surfaces have been computed [5]. The point is that the action integral over a closed curve on a Lagrangian cobordism-manifold depends only on the homology class of the curve on it. If the action integrals over a basis of 1-cycles on a Lagrangianly immersed closed manifold L are rationally independent, then the space $H_1(L, \mathbb{Q})$ together with the linear real-valued function on it defined by the cohomology class of the action form is an invariant of the Lagrangian cobordism class of the manifold L.

A Lagrangian immersion $L \subsetneq T^*\mathbb{R}^n$, along with the Gauss mapping $L \to \Lambda_n$, gives a mapping $L \to K(\mathbb{R}, 1)$ into the Eilenberg–MacLane space of the additive group of real numbers ($\pi_1(K(\mathbb{R}, 1)) = \mathbb{R}$, $\pi_k(K(\mathbb{R}, 1)) = 0$ for $k \neq 1$), defined by the homomorphism of the fundamental groups $\pi_1(L) \to H_1(L, \mathbb{Q}) \to \mathbb{R}$.[17] Let T_n be the Thom space of the bundle over $K(\mathbb{R}, 1) \times \Lambda_n^+$ induced from the tautological bundle by the projection onto the second factor.

Theorem.[18] *The group of oriented Lagrangian cobordism classes in $T^*\mathbb{R}^n$ is $\lim_{k\to\infty} \pi_{n+k}(T_k)$.*

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[16] See Ya.M. Ehliashberg's article in [19].

[17] The first of these two maps is the Hurewicz homomorphism, and the second is the cohomology class of the restriction to L of the Liouville 1-form pdq on the cotangent bundle $T^*\mathbb{R}^n$, which on L is closed because L is Lagrangian. The homomorphism $H_1(L, \mathbb{Q}) \to \mathbb{R}$ may be thought of as being given by integration of this 1-form along the 1-cycles. (Note added in translation).

[18] See Ya.M. Ehliashberg's article in [19].

For $n=1$ this group is $\mathbb{R} \oplus \mathbb{Z}$—the only (and independent) invariants of the Lagrangian cobordism class of a closed curve on the symplectic plane are its Maslov index and the area of the region bounded by the curve [5].

§ 3. Characteristic Numbers

Here we shall describe discrete invariants of the Lagrangian (Legendre) cobordism class. They arise from cohomology classes of the Lagrangian Grassmann manifolds, but they receive a geometric interpretation in the calculation of Lagrangian and Legendre singularities. This circumstance sheds light on the algebraic nature of the classification of critical points of functions.

3.1. Characteristic Classes of Vector Bundles. When a vector bundle is induced by a mapping of the base space into a classifying space a cohomology class of the classifying space determines a cohomology class of the base space, called a *characteristic class* of the bundle. The original cohomology class of the classifying space is called the universal characteristic class. If two bundles of the same dimension with a common base space have different characteristic classes obtained from one universal class, then these bundles are inequivalent.

The sequences of homomorphisms of the cohomology groups corresponding to the sequences of inclusions

$$G_{\infty,n} \subsetneq G_{\infty,n+1} \subsetneq \ldots, \quad G^+_{\infty,n} \subsetneq G^+_{\infty,n+1} \subsetneq \ldots,$$

$$\Lambda_n \subsetneq \Lambda_{n+1} \subsetneq \ldots, \quad \Lambda^+_n \subsetneq \Lambda^+_{n+1} \subsetneq \ldots,$$

define the stable cohomology groups of the corresponding Grassmann manifolds.

Theorem ([56], [27]). *The graded stable cohomology rings of the Grassmann manifolds are as follows:*

1) $H^*(G^+, \mathbb{Q}) = \mathbb{Q}[p_1, \ldots, p_k, \ldots]$ *is the ring of polynomials with rational coefficients in the integral Pontryagin classes* p_k *of degree* $4k$;

2) $H^*(G, \mathbb{Z}_2) = \mathbb{Z}_2[w_1, \ldots, w_k, \ldots]$ *is the ring of polynomials over the field* $\mathbb{Z}_2$ *in the Stiefel–Whitney classes* w_k *of degree* k;

3) $H^*(\Lambda^+, \mathbb{Q})$ *is an exterior algebra over* $\mathbb{Q}$ *with integral generators of degrees* $4k+1$, $k = 0, 1, 2, \ldots$;

4) $H^*(\Lambda, \mathbb{Z}_2)$ *is the exterior algebra over* $\mathbb{Z}_2$ *in the Stiefel–Whitney classes.*

Remarks. 1) The kernel of the epimorphism $H^*(G, \mathbb{Z}_2) \to H^*(\Lambda, \mathbb{Z}_2)$ defined by the inclusions $\Lambda_n \subsetneq G_{2n,n} \subsetneq G_{\infty,n}$ is the ideal generated by the squares [28]. We note that the dimension of the space $H^k(\Lambda, \mathbb{Z}_2)$ over the field $\mathbb{Z}_2$ is equal to the number of partitions of k into a sum of different natural-number summands.

2) Every nonzero element of the cohomology rings enumerated in the theorem defines a nontrivial universal characteristic class which distinguishes the stable equivalence classes of (oriented) vector bundles and of (oriented) vector bundles with a trivialized complexification respectively.

3.2. The Characteristic Numbers of Cobordism Classes

Lemma. *The value of an n-dimensional stable characteristic class of the tangent bundle of an n-dimensional closed manifold on its fundamental cycle depends only on the cobordism class of this manifold.*

Indeed, the restriction of the tangent bundle of a manifold to its boundary is isomorphic to the sum of the one-dimensional trivial bundle and the tangent bundle of the boundary, i.e. it is stably equivalent to the latter. The value of a stable characteristic class of the tangent bundle of the manifold on the fundamental cycle of the boundary is equal to zero, since this cycle is homologous to zero. □

Thus, each stable universal n-dimensional characteristic class defines a *characteristic number* of a closed n-dimensional manifold—an invariant of the cobordism class of this manifold. Analogously, characteristic numbers of a Lagrangian (Legendre) immersion into $T^*\mathbb{R}^n$ ($J^1\mathbb{R}^n$) are defined by the n-dimensional characteristic classes of a trivialization of the complexified tangent bundle of the immersed manifold and they are invariants of the Lagrangian (Legendre) cobordism class. Obviously, the characteristic number of a sum of cobordism classes is equal to the sum of the characteristic numbers of the summands.

Theorem ([12]). 1) *The group homomorphism* $\mathfrak{N}\mathbb{L}_n \to H_n(\Lambda, \mathbb{Z}_2)$ *given by the Stiefel–Whitney characteristic numbers is a monomorphism.*

2) *The analogous homomorphism* $\mathbb{L}_n \to H_n(\Lambda^+, \mathbb{Z})$ *is a monomorphism and becomes an isomorphism after tensoring with* $\mathbb{Q}$.

Corollary. *The nonoriented Legendre cobordism class of a Legendre immersion in* $J^1\mathbb{R}^n$ *is determined by the Stiefel-Whitney numbers of the immersed manifold.*

Remarks. 1) An analogous theorem is true for the cobordism theory of closed manifolds.

2) The number of partitions of a natural number k into a sum of odd summands—the dimension of the subspace H_k of the graded ring $\bigoplus H_k$ of polynomials over $\mathbb{Z}_2$ in generators of odd degree—is equal to the number of partitions of k into distinct summands (we split each even summand into a sum of 2^r identical odd ones...).

3) Between the Stiefel–Whitney numbers of Legendre immersions there are relations. For example, the Maslov index of a closed Legendre curve in $J^1\mathbb{R}$ is even, i.e. the characteristic number $w_1 = 0$.

4) The multiplication of characteristic classes together with the multiplication of the objects dual to them—the cobordism classes—gives a Hopf algebra structure on $H = \mathbb{L}_* \otimes \mathbb{Q}$: the comultiplication $H \to H \otimes H$ is an algebra homomorphism.

5) The corollary has no analogue in the oriented case: the Pontryagin classes of the tangent bundle of a manifold which admits a Lagrangian immersion in $T^*\mathbb{R}^n$ are zero.

3.3. Complexes of Singularities. Let us partition the space of germs of smooth functions of one variable at a critical point 0 with critical value zero (more precisely—the space of jets of such functions of sufficiently high order) into nonsingular strata—the R-equivalence classes (see § 2, chap. 5) of $A_1^{\pm}$, A_2, $A_3^{\pm}$, $A_4, \ldots$ and the class containing the zero function. We shall call a stratum of finite codimension coorientable if its normal bundle possesses an orientation invariant with respect to the action of the group of germs of diffeomorphisms on the space of germs of functions. Those which prove to be noncoorientable are the strata $A_{4k-1}^{\pm}$ and only they. For example, the transversal to the stratum A_3^+ at the point x^4 may be taken of the form $x^4 + \lambda_1 x^3 + \lambda_2 x^2$; the substitution $x \mapsto -x$ changes the orientation of the transversal. Let us define a complex ω whose group of k-chains consists of the formal integral linear combinations of the cooriented strata of codimension k. Changing the orientation changes the sign of the stratum. The coboundary operator δ is defined using the adjacencies of strata like the usual operator of taking the boundary of chains. We remark that a noncoorientable stratum enters into the boundary of a coorientable one with coefficient zero, and therefore the operator δ is well-defined and $\delta^2 = 0$. The complex ν, which does not take into account the coorientation of the strata, consists of the formal sums with coefficients in the field $\mathbb{Z}_2$ of all strata and is provided with an operator of taking the boundary of a stratum modulo 2.

We have used the classification of critical points of functions on the line only as an illustration. In reality we need universal complexes $\boldsymbol{\omega}$ and $\mathbf{v}$ defined analogously with respect to a discrete stratification of the spaces of germs of functions of an arbitrary number of variables into nonsingular strata which are invariant with respect to stable R-equivalence of germs.

Let there be given a generic Lagrangian immersion $L \looparrowright T^*M^n$. The Lagrangian projection into M^n defines a stratification of the manifold L according to the types of the singularities of the Lagrangian mapping $L \to M^n$ in correspondence with the stratification of the space of germs of functions.

Theorem (V.A. Vasil'ev [71]). *To each (cooriented) cycle of the universal complex* $\mathbf{v}$ $(\boldsymbol{\omega})$ *there corresponds a (cooriented) cycle of the closed (oriented) manifold L with coefficients in* $\mathbb{Z}_2$ $(\mathbb{Z})$. *A homology class of the complex* $\mathbf{v}$ $(\boldsymbol{\omega})$ *defines a cohomology class of the manifold L with coefficients in* $\mathbb{Z}_2$ $(\mathbb{Z})$*—the intersection number of cycles on L with the corresponding (cooriented) cycle of singularities.*

The cohomology classes on L defined by the homology classes of the universal complexes $\boldsymbol{\omega}$ and $\mathbf{v}$ are called the characteristic classes of the Lagrangian immersion. This construction generalizes the construction of the Maslov class of sect. 1.3. The value of a characteristic class of highest dimension on the fundamental cycle of the manifold L defines a characteristic number—an invariant of the Lagrangian cobordism class. The corresponding cycle of singularities is just simply a set of points with signs determined by whether or not the coorientation of the point coincides with the orientation of the manifold, and the characteristic number is the quantity of such points, taking these signs into account in the oriented case, and modulo 2 in the nonoriented case.

There are similar constructions [71] of characteristic classes and numbers in the Legendre cobordism theory. The corresponding universal complexes $\boldsymbol{\omega}$ and $\mathbf{v}$ of cooriented and noncooriented singularities of Legendre mappings are defined according to a nonsingular stratification of the spaces of germs of (cooriented) hypersurfaces at a singular point, invariant with respect to the group of germs of diffeomorphisms of the containing space, for the theory of Legendre cobordisms in PT^*M^n $(ST^*M^n$ or J^1M^n respectively).

The characteristic classes of Lagrangian immersions in $T^*\mathbb{R}^n$, or of Legendre immersions in $J^1\mathbb{R}^n$, defined by the homology classes of the universal complexes, can be induced from suitable cohomology classes of the Lagrangian Grassmann manifolds under the Gauss mapping.

Theorem (M. Audin [19]). *There exist natural homomorphisms* $H^*(\boldsymbol{\omega}) \to H^*(\Lambda^+, \mathbb{Z})$, $H^*(\mathbf{v}) \to H^*(\Lambda, \mathbb{Z}_2)$.

Indeed, let us consider the space J^N of jets of a high order N of germs at 0 of (oriented) Lagrangian submanifolds in $T^*\mathbb{R}^n$. The space J^1 is the Lagrangian Grassmann manifold Λ_n (Λ_n^+). The fibration $J^N \to J^1$ has a contractible fibre, i.e. J^N is homotopy equivalent to J^1. The Lagrangian projection $T^*\mathbb{R}^n \to \mathbb{R}^n$ allows one to stratify the space J^N by types of singularities of germs of Lagrangian mappings. Similarly to the way that the Maslov class of a generic Lagrangian immersion $L \looparrowright T^*\mathbb{R}^n$ is defined as the inverse image of a cycle $\Sigma \subset J^1$ (see sect. 1.4) under the Gauss mapping, each cycle of singularities on L is the inverse image of a corresponding cycle on the space J^N under the mapping which associates to a point on L the N-jet of the Lagrangian immersion at that point.

3.4. Coexistence of Singularities. The cohomology groups of the universal complexes $\boldsymbol{\omega}$ and $\mathbf{v}$ have been computed for the strata of codimension ≤ 6. The corresponding stable R-equivalence classes of germs of functions are the following: $A_{2k-1}^{\pm}$, A_{2k} $(k = 1, 2, 3)$, $D_k^{\pm}$ $(k = 4, 5, 6, 7)$, E_6, E_7, P_8, where the A, D, E classes are simple (see sect. 2.3, chap. 5), and P_8 is the unimodal class of germs $x^3 + ax^2z \pm xz^2 + y^2z$ (a is a modulus, $a^2 \neq 4$ in the case of the $+$ sign; see [72]). The strata $A_{4k-1}^{\pm}$ and $D_k^{\pm}$ turn out to be noncoorientable. The results of the computation of the cohomology groups are listed in Table 2.

Table 2

Theory	k	1	2	3	4	5	6
T^*M, J^1M or ST^*M	$H^k(\omega)$	$\mathbb{Z}$	0	0	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$
	generators	A_2	—	—	A_5	A_6 or E_6	P_8
	zeros	—	—	—	$2A_5$	A_6-E_6	E_7+3P_8
	$H^k(\nu)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$
	generators	A_2	A_3	A_4 or D_4	A_5	A_6 or D_6	A_7, E_7 or P_8
PT^*M	$H^k(\omega)$	0	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}\oplus\mathbb{Z}_2$
	generators	—	—	—	A_5	—	P_8, $3P_8+E_7$
	$H^k(\nu)$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^3$
	generators	A_2	A_3	A_4, D_4	A_5	A_6, D_6, E_6	A_7, E_7, P_8

The cohomological product (cup product) $\cup$ of characteristic classes is also a characteristic class. As it turns out, no new characteristic classes arise in this way.

Theorem. *On an arbitrary closed Lagrangian immersed manifold, the following multiplicative relations between characteristic classes are valid:*

for cooriented classes $A_2 \cup A_2 = 0$, $A_2 \cup A_6 = 3P_8$, $P_8 \cup P_8 = 0$ *modulo torsion,*

for noncooriented classes $A_2 \cup A_3 = A_4 = D_4$, $A_2 \cup A_6 = E_7 = P_8$, $A_2 \cup A_7 = E_8 = D_8 = A_8$, $A_3 \cup A_5 = E_7 = P_8$ *and the remaining products of dimension* ≤ 6 *of the generators out of table 2 are zero.*

The relations between cycles of singularities in the cohomology of the universal complexes impose a restriction on the coexistence of singularities. Thus, on an arbitrary closed oriented Lagrangian manifold of appropriate dimension, brought into general position with respect to the Lagrangian projection, the number of A_5 points taken with regard for signs is equal to zero, as are also the numbers of $A_6 - E_6$ and $E_7 + 3P_8$, and on a nonoriented one, the numbers of $A_4 + D_4$, D_5, D_6, $A_6 + E_6$, D_7, $E_7 + P_8$, $A_8 + D_8$ and $A_8 + E_8$ are even.

Not all cohomology classes of the universal complexes generate nontrivial characteristic classes of Lagrangian or Legendre immersions. For example, the number of A_3 points on a generic closed Legendre surface in J^1M^2 is always even, since the A_3 points of a generic closed wave front are pairwise joined by self-intersection lines of the wave front, of type (A_1A_1), issuing from them. The examination of points of the intersections of different strata of wave fronts leads to new characteristic numbers. For example, the parities of the quantities of (A_1A_2), (A_1A_4), (A_2A_4), (A_1A_6), and $(A_1A_2A_4)$ points on generic fronts of appropriate dimension turn out to be Legendre characteristic numbers. The enumerated classes are cocycles of the universal complex of multisingularities defined in [72]. From computations in this complex many implications may be obtained about the coexistence of multisingularities. For example, on a generic closed front in J^0M^2, the number of (A_1A_2) points where the front is pierced by its cusp ridge is even. It turns out that the product of the cohomology classes of the space containing the front which are dual to the front and to the closure of its cusp ridge is dual to the closure of the (A_1A_2) stratum. From this the assertion we made follows, since for the open three-dimensional manifold J^0M^2 this product is zero. For the many-dimensional generalization of this result see [7].

As was already noted in § 1, the Maslov class A_2 of a Lagrangian immersion in $T^*\mathbb{R}^n$ is induced under the Gauss mapping from a generator of the group $H^1(\Lambda^+)$.

Theorem. *On an oriented compact Lagrangian submanifold of* $T^*\mathbb{R}^n$ *the characteristic classes* A_6 *and* P_8 *coincide modulo torsion with the classes induced from three times the generator of the group* $H^5(\Lambda^+)$ *and from the product of the Maslov class with this generator respectively, and on a nonoriented one, the characteristic classes* A_2, A_3, A_4, A_5, A_6, E_7 *coincide with the Stiefel–Whitney classes* w_1, w_2, w_1w_2, w_1w_3, w_2w_3, $w_1w_2w_3$ *respectively.*

Corollary. *The number of* A_6 *singularities, taken with regard for sign, on a generic Lagrangian oriented manifold in* $T^*\mathbb{R}^5$ *is a multiple of three.*

In conclusion let us note the almost complete parallelism between the beginnings of the hierarchies of degenerate critical points of functions and of Steenrod's "admissible sequences" [69] $(x_1 \geq 2x_2 \geq 4x_3 \geq \ldots)$:

$A_2A_3A_4A_5A_6A_7A_8$	...	1	2	3	4	5	6	7	...
$D_4D_5D_6D_7D_8$	...			2,1	3,1	4,1	5,1	6,1	...
$E_6E_7E_8$	...					?	4,2	5,2	...
P_8	...							4,2,1	...

(the number of terms of the sequence is equal to the corank of the singularity, the sum is the codimension of the orbit; the deficiency of E_6 is explained, perhaps, by the relation $A_6 \sim E_6$ in the complex ω).

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Besides the references which were quoted, the list includes classical works and textbooks on dynamics [16], [23], [31], [39], [45], [48], [60], [61], [76], a few contemporary monographs [1],

* For the convenience of the reader, references to reviews in Zentralblatt für Mathematik (Zbl.), compiled using the MATH database, and Jahrbuch über die Fortschritte der Mathematik (FdM.) have, as far as possible, been included in this bibliography.

[33], [37], [47], [68], [73], and also papers which relate to questions which were completely, or almost, untouched upon in the survey but are connected with it by their subject—[18], [32], [44], [54], [59], [67], [70] (for an important advance in contact topology we refer to D. Bennequin in [19]). The collections [13] and [19] give a good idea of the directions of present research. Detailed expositions of the foundations of symplectic geometry—from various points of view—can be found in [2] and [37], and for isolated parts of it in [4], [9], [24]. Among the bibliographical sources on our topic let us note [1].

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Added in proof: Additional list of new publications

On linear symplectic geometry:

1. Arnol'd V.I.: The Sturm theorems and symplectic geometry. Funkts. Anal. Prilozh. *19*, No. 4, 1–10 (1985) (English translation: Funct. Anal. Appl. *19*, 251–259 (1985))

On Poisson structures:

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Geometric Quantization

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Contents

Introduction

The word "quantization" is used both in physical and in mathematical works in many different senses. In recent times this has come to be reflected explicitly in the terminology: the terms "asymptotic", "deformational", "geometric" quantization, etc., have emerged.

The common basis of all these theories is the premise that classical and quantum mechanics are just different realizations of the same abstract scheme. The fundamental components of this scheme are the algebra of observables (physical quantities) and the space of states (phase space).

In asymptotic quantization one assumes that the observables and the states can be expanded as series in a small parameter h. The constant terms of these series correspond to classical mechanics, and the first-order terms give the so-called quasiclassical approximation.

Deformational quantization investigates the algebraic structure of the algebra of observables and regards the quantum situation as a deformation of the classical one.

Geometric quantization sets as its goal the construction of quantum objects using the geometry of the corresponding classical objects as a point of departure.

The origins of geometric quantization lie, on the one hand, in the attempts of physicists to extend the known quantization procedures for simple mechanical systems to more general configurations and phase spaces, and on the other hand, in the development by mathematicians of the theory of unitary representations, which led to the orbit method.

The merging of these two sources took place at the end of the sixties and turned out to be useful both for physicists and for mathematicians.

The physicists added to their arsenal a new mathematical apparatus (fibre bundles, connections, cohomology), the mathematicians were enriched by new heuristic considerations and formulations of problems.

Although the possibilities of geometric quantization are still far from having been exhausted, it is possible even now to give a description of the foundations of this method and to indicate approximately the scope of its applicability.

§ 1. Statement of the Problem

1.1. The Mathematical Model of Classical Mechanics in the Hamiltonian Formalism (see [2] and volume 3 of the present publication). In the simplest

mechanical systems the phase space is the usual $2n$-dimensional real vector space with coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$ which describe the position and the velocity of the particles composing the system. In more complicated systems the phase space M is the cotangent bundle over a smooth manifold N (the configuration space). The coordinates $q_1, \ldots, q_n$ in this case are defined only locally and map a part U of the manifold N onto a region V in $\mathbb{R}^n$. The corresponding coordinates $p_1, \ldots, p_n$ run through the space $(\mathbb{R}^n)^*$ dual to $\mathbb{R}^n$ and give a trivialization of the cotangent bundle over U, identifying T^*U with $V \times (\mathbb{R}^n)^*$. On $M = T^*N$ the 1-form $\theta = \sum_{k=1}^{n} p_k dq_k$ is well-defined, and hence so is its differential $\omega = d\theta = \sum_{k=1}^{n} dp_k \wedge dq_k$. The form ω is obviously closed and nondegenerate on M.

For the formulation of the Hamiltonian formalism it is sufficient to have a smooth manifold M (not necessarily of the form T^*N) with a closed nondegenerate form ω on it. Such a *manifold* is called *symplectic.*

The Darboux theorem (compare [2] and the article of V.I. Arnol'd and A.B. Givental') asserts that locally in suitable coordinates the form ω can always be written in the form

$$\omega = \sum_{k=1}^{n} dp_k \wedge dq_k. \tag{1.1}$$

However, these canonical coordinates are far from being uniquely defined, and their division into "positional" and "momentum" coordinates bears the character of a convention.

The form ω sets up an isomorphism between the tangent and cotangent spaces at each point of M. The inverse isomorphism is given by a bivector c, which has the form

$$c = \sum_{k=1}^{n} \frac{\partial}{\partial p_k} \wedge \frac{\partial}{\partial q_k} \tag{1.2}$$

in the same system of coordinates in which equality (1.1) holds.

In a general system of coordinates the form ω and the bivector c are written in the form

$$\omega = \sum_{i<j} \omega_{ij} dx^i \wedge dx^j, \quad c = \sum_{i<j} c^{ij} \partial_j \wedge \partial_i$$

with mutually inverse skew-symmetric matrices $\|\omega_{ij}\|$ and $\|c^{ij}\|$.

Physical quantities or *observables* are identified with the smooth functions on M and form the space $C^\infty(M)$. With respect to the usual multiplication $C^\infty(M)$ forms a commutative associative algebra. In addition to this, a *Poisson bracket* is defined in $C^\infty(M)$, giving a Lie algebra structure:

$$\{F, G\} = \sum_{i,j} c^{ij} \partial_j F \partial_i G. \tag{1.3}$$

The Jacobi identity for the Poisson bracket

$$\{\{F, G\}, H\}+\{\{G, H\}, F\}+\{\{H, F\}, G\}=0 \tag{1.4}$$

is equivalent to the condition $d\omega=0$, and also to the vanishing of the *Schouten bracket* (see e.g. [24])

$$[c, c]^{ijk}=\circlearrowleft_{ijk}\sum_{m} c^{im}\partial_m c^{jk}, \tag{1.5}$$

where the sign $\circlearrowleft$ denotes the sum over the cyclic permutations of the indices i, j, k.

A *submanifold* $N \subset M$ is called *isotropic* if the form ω vanishes on TN. The dimension of an isotropic manifold does not exceed n. If it is equal to n, the *manifold* is called *Lagrangian*. Locally a Lagrangian manifold is given by a system of equations

$$F_k=c_k, \quad k=1, 2, \ldots, n \tag{1.6}$$

where $\{F_i\}$ is a system of *functions which are in involution*, i.e., which generate a commutative subalgebra with respect to the Poisson bracket:

$$\{F_i, F_j\}=0.$$

By changing the constants c_k in the equations (1.6), we obtain a partition of the space M into Lagrangian submanifolds. Such a partition is called a (global) *real polarization* if it is a smooth fibration. As a rule a manifold M does not admit global polarizations. If such a polarization does exist, then the manifold M, or some covering of it, can be identified with an open subset in a cotangent bundle T^*N, where N is the base space of the polarization. We note that under this identification the form ω *does not generally go over* into the canonical form (1.1) on T^*N, but differs from it by a summand of the form

$$\sum a_{ij} dq_i \wedge dq_j. \tag{1.7}$$

For a system which represents a mass point in $\mathbb{R}^3$ this summand can be interpreted as an external magnetic field.

A *state of the system* is a linear functional on $C^\infty(M)$ which takes non-negative values on non-negative functions and equals 1 on the function which is identically equal to 1. The general form of such a functional is a probability measure μ on M.

By a *pure state* is meant an extremal point of the set of states. This is a measure of the δ-function type, concentrated at one point of M.

(As we shall see below, mixed classical states which satisfy the classical analogue of the uncertainty principle are always limits of pure quantum states. Of interest in connection with this is A. Weinstein's suggestion [46] to consider as elementary classical states δ-functions concentrated on Lagrangian submanifolds in M.) The dynamics of a system is determined by the choice of a *Hamiltonian function* (W.R. Hamilton) or *energy*, whose rôle can be played by an arbitrary function $H \in C^\infty(M)$. Two equivalent ways of describing the dynamics are possible. In the first of them—the so-called *Hamiltonian picture*—the states do not depend on time, and the physical quantities are functions of the point of the phase space and of time, that is, functions on $M \times \mathbb{R}$. The equations of motion have the form:

$$\dot{F}=\{H, F\}, \tag{1.8}$$

where F is an arbitrary observable. In particular, by applying (1.8) to the canonical variables p_k, q_k, we obtain *Hamilton's equations*

$$\dot{q}_k=\frac{\partial H}{\partial p_k}, \qquad \dot{p}_k=-\frac{\partial H}{\partial q_k}. \tag{1.9}$$

Here the dot denotes the derivative with respect to time. The other method of describing the motion consists in considering the physical quantities to be functions on M not depending on time and having the states of the system change with time in such a manner that a pure state with coordinates $p(t)$, $q(t)$ obeys the Hamilton equations (1.9). It is easy to verify that a mixed state with the density $\rho(p, q, t)$ changes here according to the law

$$\dot{\rho}(p, q, t)=\{\rho, H\}=-\{H, \rho\}. \tag{1.10}$$

This description of the motion is called the *Liouville picture* (J. Liouville) and is usually used in statistical mechanics.

Both pictures are equivalent, since the mean value of the quantity F in the state ρ changes with time in the same way:

$$\frac{d}{dt}\langle F, \rho\rangle=\int_M \{H, F\}\rho\, dp dq=\int_M F\{\rho, H\}\, dp dq.$$

The latter equality is true because the Hamiltonian flow preserves the canonical measure $dpdq=\omega^n$, $n=\frac{1}{2}\dim M$.

A function F is called a *first integral* of the system if it is in involution with H, that is $\{F, H\}=0$. In this case the vector field $c(F)=\sum_{i,j} c^{ij}\partial_j F \partial_i$ generates a Hamiltonian flow which commutes with the evolution of the system. The existence of several first integrals which generate a finite-dimensional Lie algebra $\mathfrak{g}$ with respect to the Poisson bracket (S.D. Poisson) leads to a realization of the corresponding Lie group $G=\exp \mathfrak{g}$ as a group of symmetries of the system under consideration.

A set $F_1, \ldots, F_m$ *of physical quantities* is called *complete* if from the conditions $\{F_i, G\}=0$, $i=1, \ldots, m$ it follows that $G=\text{const}$. It is easy to verify that this condition is equivalent to saying that the functions $F_1, \ldots, F_m$ locally separate points almost everywhere on M (that is, there are enough of these functions for the construction of local coordinate systems on M).

An example of a complete set are the coordinates and the momenta in the case $M=T^*\mathbb{R}^n$.

Example 1.1. Small oscillations. The oscillator.

To a first approximation, every manifold is flat and every function in the neighbourhood of an extremum is quadratic. This explains the large rôle of a particular example of a mechanical system—the so-called oscillator. Here the configuration space can be identified with $\mathbb{R}^n$, the phase space with $\mathbb{R}^n \times (\mathbb{R}^n)^*$. Global coordinates $q_1, \ldots, q_n, p_1, \ldots, p_n$ are defined. The Hamiltonian of the system has the form:

$$H = \frac{1}{2} \sum_{k=1}^{n} (p_k^2 + q_k^2).$$

This system may be regarded as a union of non-interacting systems with one degree of freedom. (In the language of the Hamiltonian formalism a union of systems without interaction corresponds to multiplication of the phase spaces and addition of the Hamiltonians.)

We shall therefore assume that $n=1$. In this case the motion of the system consists of a uniform rotation of the phase plane with unit angular velocity. The trajectory of a point coincides with a level curve of the function H, that is to say, with a circle $p^2+q^2=\text{const}$. Let us note also, that the Poisson brackets between the fundamental observables have the form:

$$\{H, p\} = -q, \quad \{H, q\} = p, \quad \{p, q\} = 1.$$

Example 1.2. The mathematical pendulum.

The configuration space N is the two-dimensional sphere S^2, given by the equation $x_1^2+x_2^2+x_3^2=r^2$. The phase space $M=T^*N$ is the cotangent bundle of the sphere S^2.

One may identify the manifold M with the subset of $\mathbb{R}^6$ with coordinates $x_1, x_2, x_3, y_1, y_2, y_3$, given by the equations

$$x_1^2+x_2^2+x_3^2=r^2, \quad x_1y_1+x_2y_2+x_3y_3=0.$$

As local coordinates on N one may take $q_1=x_1$, $q_2=x_2$. With respect to the quadratic form $T=(m/2)(y_1^2+y_2^2+y_3^2)$, which gives the kinetic energy of the system, the dual coordinates in the fibres have the form $p_1=m(y_1x_3-x_1y_3)/x_3$, $p_2=m(y_2x_3-x_2y_3)/x_3$. In the coordinates p and q the kinetic energy is expressed by the formula

$$T=\frac{1}{2m}\left(p_1^2+p_2^2-\frac{(p_1q_1+p_2q_2)^2}{r^2}\right),$$

where m is the mass of the pendulum.

If the pendulum is considered to be in a constant gravitational field, directed along the x_3 axis, then the potential energy has the form:

$$V=-mgx_3=\pm mg\sqrt{r^2-q_1^2-q_2^2}.$$

The Hamiltonian $H=T+V$ possesses a first integral—the angular momentum with respect to the x_3 axis:

$$P=p_1q_2-p_2q_1.$$

The joint level lines of the functions H and P give a polarization with singularities on M. Namely, the fibres $P=0$, $H=\pm mgr$ degenerate to a point (if $H=-mgr$) or to a pinched torus (if $H=+mgr$), but the remaining fibres are two-dimensional tori on which a conditionally periodic motion takes place (see [2]).

Example 1.3. As the phase manifold M let us take the sphere $x_1^2+x_2^2+x_3^2=r^2$, and for the form ω let us take the usual area, which in the local coordinates $u=x_1$, $v=x_2$ has the form: $\omega=r(du\wedge dv)/\sqrt{r^2-u^2-v^2}$.

(In this case M cannot be represented as a cotangent bundle T^*N—the configuration space does not exist! A well-known theorem of smooth topology asserts that there are no non-vanishing (continuous) vector fields on M. By the same token there are also no real polarizations.)

As we shall see below, in this example there exists a complex polarization, which allows one to construct a quantization of this system under the condition that the surface area of M is an integer. This exotic system turns out to be the classical analogue of a quantum system with one spin degree of freedom. In applications it is encountered not all by itself as a rule, but in the form of a "growth" over the usual phase space.

As the Hamiltonian of this system one usually considers the linear function $H=a_1x_1+a_2x_2+a_3x_3$, whose coefficients can be interpreted as the projections of the field strength vector of a magnetic field. The motion of the system consists in a uniform rotation of the sphere. Thus, the notion of spin as a "hidden rotational degree of freedom" gets an exact classical interpretation here.

1.2. The Mathematical Model of Quantum Mechanics (see [13]). In quantum mechanics the physical quantities or observables are self-adjoint linear operators on some complex Hilbert space $\mathscr{H}$. They form a linear space on which two bilinear operations are defined:

1) Jordan multiplication

$$A\circ B=\frac{1}{2}(AB+BA)=\left(\frac{A+B}{2}\right)^2-\left(\frac{A-B}{2}\right)^2; \tag{1.11}$$

2) the commutator

$$[A, B]_h=\frac{2\pi i}{h}(AB-BA), \tag{1.12}$$

where h is *Planck's constant* (M. Planck).

With respect to the first operation the set of observables forms a commutative, but not associative algebra; with respect to the second it forms a Lie algebra. These two operations are the quantum analogues of the usual multiplication and the Poisson bracket in classical mechanics.

The *state space*, or *phase space*, in quantum mechanics consists of the so-called "density matrices", that is, the non-negative definite operators S with the property that $\operatorname{tr} S=1$. The *pure states* (the extremal points of the set of states) are the one-dimensional projection operators on $\mathscr{H}$. Usually $\mathscr{H}$ can be realized as a space of square-integrable functions; in this case a pure state is given by a function ψ with unit norm (a so-called *wave function*), where functions which differ by a numeric multiplier give the same state.

The *mean value of the quantity* A in the state S is by definition equal to $\operatorname{tr} AS$. For a pure state, given by a function ψ, this quantity is equal to $(A\psi, \psi)$. In quantum mechanics an observable A, even in a pure state ψ, is not obliged to have an exactly defined value. Its probability distribution is given by the monotone function $p(\lambda)=(E_\lambda\psi, \psi)$, where E_λ is the spectral family (resolution of the identity) of the operator A. In particular, if the operator A has a simple discrete spectrum with eigenvalues λ_k and eigenfunctions ψ_k, then in the state ψ it takes on the value λ_k with probability $p_k=|(\psi, \psi_k)|^2$. The dynamics of the system is defined by the *energy operator* $\hat{H}$. (It is customary to place a caret above a letter to distinguish a quantum-mechanical quantity—an operator on $\mathscr{H}$—from the corresponding classical quantity—a function on M). Here, just as in classical mechanics, two means of description are possible. If the states do not depend on time, but the quantities change, then we obtain the *Heisenberg picture*. The motion is described by *Heisenberg's equation* (W. Heisenberg)

$$\dot{\hat{A}}=[\hat{H}, \hat{A}]_h, \tag{1.13}$$

which is the exact analogue of Hamilton's equation (1.6). The integrals of the system are all the operators which commute with $\hat{H}$. In particular, the energy operator itself does not change with time (the law of conservation of energy in quantum mechanics).

The other description, in which the operators corresponding to physical quantities do not change, but the states change, is called the *Schrödinger picture* (E. Schrödinger). It is easy to check that for the equivalence of these two ways of describing the motion it is necessary that a pure state ψ should vary according to the law

$$\dot{\psi}=\frac{2\pi i}{h}\hat{H}\psi, \tag{1.14}$$

called the *Schrödinger equation*. The eigenfunctions of the Schrödinger operator give the stationary states of the system. (Remember that functions differing by a scalar multiplier define the same state).

We shall call *a set of quantum physical quantities* $\hat{A}_1, \ldots, \hat{A}_m$ *complete*, if any operator $\hat{B}$ which commutes with $\hat{A}_i$, $i=1, \ldots, m$, is a multiple of the identity. One can show that this condition is equivalent to the irreducibility of the set $\hat{A}_1, \ldots, \hat{A}_m$. That means that any closed subspace of $\mathscr{H}$ which is invariant with respect to all the $\hat{A}_i$, $1\le i\le m$, is either equal to $\{0\}$ or to $\mathscr{H}$.

Example 1.4. A mass particle in a potential force field on the line.

The Hilbert space $\mathscr{H}$ consists of the complex functions $\psi(x)$ on the real line with a square-integrable modulus. The basic physical quantities are represented by the following operators:

the coordinate operator $\hat{x}$ consists in multiplication of the function $\psi(x)$ by x;

$$\text{the momentum operator } \hat{p}=i\hbar\frac{d}{dx};$$

$$\text{the energy operator } \hat{H}=-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V(x).$$

Here $V(x)$ is the given potential, $\hbar=h/2\pi$ is the *reduced Planck constant*.

In particular, in the state ψ the coordinate of the particle has a probability distribution with density $|\psi(x)|^2$, but its momentum is distributed with density $|\tilde{\psi}(k/\hbar)|^2$, where $\tilde{\psi}(k)=\int\psi(x)e^{-ikx}dx$ is the Fourier transform of the function ψ.

1.3. The Statement of the Quantization Problem. The Connection with the Method of Orbits in Representation Theory. The problem of geometric quantization is, starting from the geometry of a symplectic manifold (M, ω) which gives the model of a classical mechanical system, to construct a Hilbert space $\mathscr{H}$ and a set of operators on it which give the quantum analogue of this system. If the initial classical system had a symmetry group G it is natural to require that the quantum model obtained should also possess this symmetry. That means that on the space $\mathscr{H}$ there should act a unitary representation (possibly a projective one, see below) of the group G.

The maximal symmetry group of the symplectic manifold (M, ω) is the infinite-dimensional group $\operatorname{Symp}(M, \omega)$ of all *symplectomorphisms*, or *canonical transformations* of M which preserve the form ω.

For the quantum system the maximal symmetry group is the infinite-dimensional group $PU(\mathscr{H})$ of all projective unitary transformations.

These two groups are not isomorphic. Therefore there is *a priori* no hope that to each classical symmetry there should correspond a quantum symmetry. In concrete situations one or another finite-dimensional symmetry group might be preserved while others might be broken. In the physical literature one speaks in the latter case of "*quantum anomalies*" in the commutation relations.

A particular interest is presented by the homogeneous symplectic manifolds (M, ω), on which some Lie group G acts transitively. Such systems have no G-invariant subsystems. Therefore in the corresponding quantum systems irreducible representations of the group G must arise. If the thesis is true that every quantum system with a symmetry group G can be obtained by quantization of a classical system with the same symmetry group, then the irreducible representations of the group G must be connected with homogeneous symplectic G-manifolds. The method of orbits in the theory of unitary representations of Lie groups ties together the unitary representations of a Lie group G with the orbits of this group in the coadjoint representation, which acts on the space $\mathfrak{g}^*$ dual to the Lie algebra of the group G.

The connection between quantization and the method of orbits is founded on the following remarkable circumstance.

Theorem 1.1 ([22], [26], [42]). *Every G-orbit in $\mathfrak{g}^*$ is a homogeneous symplectic G-manifold and conversely, every homogeneous symplectic G-manifold is locally isomorphic to an orbit in the coadjoint representation of the group G or a central extension of it.*

The procedure of geometric quantization is the natural generalization to the nonhomogeneous situation of the procedure for constructing an irreducible unitary representation of a group G starting from a G-orbit in the coadjoint representation. More precisely, quantization with the aid of a choice of a real polarization (see below § 3) generalizes the construction of an induced representation; the concept of a complex polarization arises from the construction of a holomorphically induced representation; finally, cohomology representations have also recently found an analogue in the method of geometric quantization (see § 4).

§ 2. Prequantization

2.1. The Koopman–Van Hove–Segal Representation. Let (M, ω) be a symplectic manifold. By $\mathscr{P}(M, \omega)$ we shall denote the Poisson algebra on M, i.e. the space $C^\infty(M)$ equipped with the Poisson bracket (1.3).

Following P.A.M. Dirac, a *quantization* is a linear mapping $F \to \hat{F}$ of the Poisson algebra (or some subalgebra of it) into the set of operators on some (pre-)Hilbert space, having the properties:

1) $\hat{1} = 1$ (here the 1 on the left denotes the function on M which is identically equal to 1, and the 1 on the right is the identity operator);

2) $\{F, G\}^\wedge = [\hat{F}, \hat{G}]_h \left(= \frac{i}{h}(\hat{F}\hat{G} - \hat{G}\hat{F}) \right)$;

3) $\widehat{F^*} = (\hat{F})^*$ (the asterisk on the left denotes complex conjugation, and the asterisk on the right denotes the transition to the adjoint operator);

4) for some complete set of functions $F_1, \ldots, F_m$ the operators $\hat{F}_1, \ldots, \hat{F}_m$ also form a complete set.

A linear mapping $F \to \check{F}$ which possesses the first three properties is called a *prequantization.*

For the case $M = T^*N$, $\omega = d\theta$ (see sect. 1.1) a prequantization was constructed in 1960 by I. Segal [37], who generalized the results of D. Koopman [25] and L. Van Hove [45]. It has the form:

$$\check{F} = F + \frac{h}{2\pi i} c(F) - \theta(c(F)), \tag{2.1}$$

where $c(F)$ is the Hamiltonian vector field on M with *generating function F*, considered as an operator on $C^\infty(M)$. The space of the prequantization consists of the smooth functions of compact support on M with the scalar product

$$(\phi_1, \phi_2) = \int_M \phi_1 \bar{\phi}_2 dv,$$

where $v = \omega^n = d^n p d^n q$ is the Liouville measure on M.

That the mapping is linear and condition 1) is fulfilled is obvious. That condition 2) is fulfilled follows from the well-known identities [3]

$$[c(F), c(G)] = c(\{F, G\})$$

and

$$\{F, G\} = \omega(c(F), c(G)) = c(F)\theta(c(G)) - c(G)\theta(c(F)) - \theta([c(F), c(G)]).$$

Condition 3) is equivalent to the condition that the operators $\check{F}$ be self-adjoint for real functions $F \in C^\infty(M)$. That these operators are symmetric follows from the fact that the field $c(F)$ is Hamiltonian. If the field $c(F)$ is complete (that is, generates a one-parameter group of transformations of M), then the operator $\check{F}$ is essentially self-adjoint. In particular, if the function F has compact support, condition 3) is fulfilled.

The prequantization (2.1) is not a quantization, as can be seen in the simplest example $M = T^*\mathbb{R}$. In this case the coordinate and momentum operators have the form:

$$\check{q} = q + \frac{h}{2\pi i}\frac{\partial}{\partial p}, \quad \check{p} = -\frac{h}{2\pi i}\frac{\partial}{\partial q}$$

on the space of smooth functions on the plane with coordinates p, q and measure $dp \wedge dq$.

It is easy to check that the operators $\partial/\partial p$ and $\partial/\partial q + (2\pi i/h)p$ commute with $\check{q}$ and $\check{p}$. Therefore $\check{q}$ and $\check{p}$ do not form a complete set.

A comparison with example 1.2 shows that in the present case one can construct a quantization if one restricts the action of $\check{q}$ and $\check{p}$ to the space of functions not depending on p. We shall consider a generalization of this method below in § 3.

2.2. Hermitian Bundles with a Connection. The Souriau–Kostant Prequantization. The attempt to carry over the Koopman–Van Hove–Segal construction to general symplectic manifolds leads to complex line (= one-dimensional vector) bundles over M, equipped with a connection and a Hermitian structure on the fibres. The point is that the possibility of representing the operators of the prequantization by means of formula (2.1) is based on the equality $\omega = d\theta$. Therefore, if the form ω is not exact (for example this will be so for an arbitrary compact manifold M), then the representation (2.1) is impossible. However a closed form ω is always exact locally. So one can cover the manifold

M by open sets U_α such that in each U_α the equality $\omega = d\theta_\alpha$ holds for a suitable 1-form θ_α on U_α. By the same token we obtain the possibility of defining operators $\check{F}_\alpha = F + (h/2\pi i)c(F) - \theta_\alpha(c(F))$ on $C^\infty(U_\alpha)$. It turns out that, with the additional condition that the cohomology class given by the form ω be integral, these local operators $\check{F}_\alpha$ can be "glued together" to one global operator $\check{F}$. However this operator acts not on the functions, but on the sections of some line bundle L over M. The form θ_α can be interpreted here as the local expression for a connection on the trivialization of L over the domain U_α. Let us pass over to exact formulations.

Let L be a complex vector bundle over M with a one-dimensional fibre. We shall suppose that on L a Hermitian structure $\langle\cdot,\cdot\rangle$ and a connection ∇ are defined, which are compatible in a natural fashion:

$$\xi\langle s_1, s_2\rangle = \langle\nabla_\xi s_1, s_2\rangle + \langle s_1, \nabla_\xi s_2\rangle. \tag{2.2}$$

Here $\langle s_1, s_2\rangle$ denotes the function on M which at a point $x \in M$ is equal to the scalar product of $s_1(x)$ and $s_2(x)$ in the sense of the Hermitian structure. ∇_ξ is the operator of covariant differentiation of a section along the field ξ on M.

If the bundle L over the domain $U_\alpha \subset M$ admits a nonvanishing section s_α, then the space of sections $\Gamma(L, U_\alpha)$ can be identified with $C^\infty(U_\alpha)$ by the formula

$$C^\infty(U_\alpha) \ni \phi \leftrightarrow \phi \cdot s_\alpha \in \Gamma(L, U_\alpha).$$

Under this identification the operator ∇_ξ takes on the form:

$$\nabla_\xi \phi = \xi\phi - \frac{2\pi i}{h}\theta_\alpha(\xi)\phi, \tag{2.3}$$

where the form θ_α is defined out of the equation

$$\nabla_\xi s_\alpha = -\frac{2\pi i}{h}\theta_\alpha(\xi)\cdot s_\alpha. \tag{2.4}$$

A comparison of (2.3) with (2.1) suggests the following formula of the Souriau–Kostant prequantization:

$$\check{F} = F + \frac{h}{2\pi i}\nabla_{c(F)}. \tag{2.5}$$

That condition 1) is fulfilled is obvious. Condition 3) is fulfilled for F with compact support by virtue of (2.5), if one defines a scalar product on the space of sections of L by the formula

$$(s_1, s_2) = \int_M \langle s_1, s_2\rangle dv \tag{2.6}$$

and observes that the volume $dv = \omega^n = d^n p \wedge d^n q$ is invariant with respect to the Hamiltonian flow generated by the field $c(F)$.

The verification of condition 2) requires some calculations. Let us recall that the curvature form of the connection ∇ is defined as the 2-form Ω on M given by the formula

$$\Omega(\xi, \eta) = \frac{1}{2\pi i}([\nabla_\xi, \nabla_\eta] - \nabla_{[\xi,\eta]}). \tag{2.7}$$

From (2.3) one can derive the following local expression for the curvature form:

$$\Omega = h^{-1} d\theta_\alpha \text{ on } U_\alpha. \tag{2.8}$$

On the other hand, condition 2) can be rewritten in our case in the form

$$\{F, G\} = h\Omega(c(F), c(G)).$$

Thus the following theorem is true:

Theorem 2.1. *The Souriau–Kostant formula* (2.5) *gives a prequantization* $\mathscr{P}(M, \omega)$ *if and only if the curvature form* Ω *of the connection* ∇ *coincides with* $h^{-1}\omega$.

The question arises of which 2-forms on M can serve as curvature forms for a connection on some line bundle L over M, and when is it possible to define a Hermitian structure on L which is compatible with the connection. The answer to these questions is given by

Theorem 2.2. *A form Ω is the curvature form of some line bundle L over M with connection ∇ if and only if the cohomology class defined by the form Ω is integral (that is to say, the integral of the form Ω over an arbitrary 2-cycle in M is an integer). A Hermitian structure on L compatible with ∇ exists if and only if the form Ω is real.*

The proof of the first assertion of the theorem follows from the relations which connect the transition functions of the bundle L with the forms θ_α. Namely, if on the intersection $U_{\alpha\beta} = U_\alpha \cap U_\beta$ the equality

$$s_\alpha = c_{\alpha\beta} \cdot s_\beta$$

holds, where $c_{\alpha\beta} \in C^\infty(U_{\alpha\beta})$, then from (2.4) it follows that

$$\theta_\beta - \theta_\alpha = \frac{h}{2\pi i} d \ln c_{\alpha\beta}. \tag{2.9}$$

The transition functions $c_{\alpha\beta}$ satisfy the relations $c_{\alpha\beta}c_{\beta\gamma}c_{\gamma\alpha} \equiv 1$ on $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$. Therefore, if one writes $c_{\alpha\beta}$ in the form

$$c_{\alpha\beta} = \exp(2\pi i b_{\alpha\beta}),$$

then in the domain $U_{\alpha\beta\gamma}$ there will hold the equality

$$b_{\alpha\beta} + b_{\beta\gamma} + b_{\gamma\alpha} \equiv \text{const} \in \mathbb{Z}. \tag{2.10}$$

Thus we have constructed an integral Čech 2-cocycle (E. Čech) of the manifold M with the covering U_α. By the de Rham theorem (G. de Rham) the cohomology class of this cocycle coincides with the class defined by the form Ω.

Conversely, let Ω define an integral cohomology class. Then the form θ_α, which is defined up to a summand of the form $df_\alpha, f_\alpha \in C^\infty(U_\alpha)$, can be chosen so that the functions $b_{\alpha\beta} \in C^\infty(U_{\alpha\beta})$, given (up to a constant summand) by the equations

$$db_{\alpha\beta} = h^{-1}(\theta_\beta - \theta_\alpha),$$

satisfy condition (2.10). Setting $c_{\alpha\beta} = \exp(2\pi i b_{\alpha\beta})$, we obtain the transition functions of the desired bundle L.

Now let Γ be a piecewise smooth closed path on the manifold M, bounding a two-dimensional surface $D \subset U_\alpha$. From (2.4) it follows that under parallel translation along Γ the section s_α is multiplied by the numeric coefficient

$$Q(\Gamma) = \exp\left(\frac{2\pi i}{h}\int_\Gamma \theta_\alpha\right) = \exp\left(2\pi i \iint_D \Omega\right). \tag{2.11}$$

If a ∇-invariant Hermitian structure exists, then this coefficient must be 1 in modulus, from which it follows that Ω is real.

(Let us note that formula (2.11) can easily be generalized to arbitrary curves $\Gamma = \partial D$ not necessarily lying in a U_α. If we apply this formula to a closed 2-cycle D, we obtain yet another proof that the class of Ω is integral.)

Conversely, let the form Ω be real. Then the forms $\operatorname{Im}\theta_\alpha$ are closed and one can find real functions ρ_α in $C^\infty(U_\alpha)$ such that

$$d\rho_\alpha = \frac{2\pi}{h}\operatorname{Im}\theta_\alpha, \quad \rho_\alpha - \rho_\beta = \ln|c_{\alpha\beta}| \quad \text{in } U_{\alpha\beta}. \tag{2.12}$$

Let us define a Hermitian structure on L by setting

$$\|s_\alpha\| = \langle s_\alpha, s_\alpha\rangle^{1/2} = \exp \rho_\alpha. \tag{2.13}$$

It is easily verified that this structure is compatible with the connection ∇: (2.2) follows from (2.13) and (2.4).

It is recommended to the reader that he retrace how the prequantization (2.1) arises as a special case of the construction described. (As the unique open set U_α one must take the whole manifold $M = T^*N$ and set $\theta_\alpha = \theta$, $L = M \times \mathbb{C}$, $s_\alpha = 1$.)

The natural question also arises whether the Souriau–Kostant prequantization (J.M. Souriau–B. Kostant) is uniquely determined by the symplectic manifold (M, ω). For simply connected manifolds the answer is positive. In the general case we have

Theorem 2.3. *The set of prequantizations of a symplectic manifold (M, ω) for which the form $\Omega = h^{-1}\omega$ defines an integral cohomology class is a principal homogeneous space for the character group Π^* of the fundamental group Π of the manifold M.*

Here two prequantizations are considered to be the same if they are defined with the aid of equivalent line bundles L_i with a connection ∇_i and a Hermitian structure $\langle\cdot,\cdot\rangle_i$, $i = 1, 2$, which go over into each other under a suitable diffeomorphism of L_1 onto L_2. From the two prequantizations corresponding to (L_1, ∇_1) and (L_2, ∇_2) one can construct a character $\chi_{12} \in \Pi^*$ given by the formula

$$\chi_{12}([\Gamma]) = Q_1(\Gamma) Q_2(\Gamma)^{-1}, \tag{2.14}$$

where Γ is a closed path on M, $[\Gamma]$ is its class in the group Π, $Q_i(\Gamma)$ is the coefficient by which a section of the bundle L_i is multiplied upon parallel translation along the path Γ. Formula (2.11) and the fact that the curvature forms for ∇_1 and ∇_2 coincide guarantee that the right-hand side in (2.14) depends only on the class $[\Gamma]$ of the path Γ.

2.3. Examples. Prequantization of the Two-Dimensional Sphere and the Two-Dimensional Torus. For a two-dimensional manifold a symplectic structure is just simply an area element. The condition for the existence of a prequantization reduces to the condition that the area be an integer, when measured in units of h.

According to a well-known result of J. Moser [32], the area is the unique invariant of a compact two-dimensional symplectic manifold (if its topological genus is given).

We shall analyze the two simplest examples here: the sphere S^2 and the torus $T^2 = S^1 \times S^1$.

On the standard sphere $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$ let us introduce the surface element $\omega_N = (Nh/4\pi)(x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy)$. The area of the sphere will then be equal to Nh. We shall construct explicitly a bundle L_N over S^2 with a connection ∇, for which the form $\Omega_N = h^{-1}\omega_N$ will be the curvature form. For this let us cover the sphere by two coordinate neighbourhoods $U_\pm = S^2 \setminus \{P_\pm\}$, where $P_\pm = (0, 0, \pm 1)$. In the neighbourhood $U_\pm$ let us introduce a complex coordinate $w_\pm$:

$$w_\pm = \frac{x \pm iy}{1 \mp z}.$$

On the intersection $U_+ \cap U_-$ the relation $w_+ w_- = 1$ holds. In these coordinates the form Ω_N takes on the form:

$$\Omega_N = \frac{N}{2\pi i}\frac{d\bar{w}_\pm \wedge dw_\pm}{(1 + |w_\pm|^2)^2}.$$

As the 1-form $\theta_\pm$ on the domains $U_\pm$ let us take

$$\theta_\pm = \frac{Nh}{2\pi i}\frac{\bar{w}_\pm dw_\pm}{1 + |w_\pm|^2}.$$

Then for the transition function c on $U_+ \cap U_-$ we obtain the equation

$$d \ln c = \frac{2\pi i}{h}(\theta_- - \theta_+) = N\left(\frac{\bar{w}_- dw_-}{1 + |w_-|^2} - \frac{\bar{w}_+ dw_+}{1 + |w_+|^2}\right) = N\frac{dw_-}{w_-},$$

from which we have

$$c=(w_-)^N=(w_+)^{-N}.$$

Thus, the bundle L_N is the Nth tensor power of the Hopf bundle (H. Hopf).

Now let us consider the two-dimensional torus $T^2=\mathbb{R}^2/\mathbb{Z}^2$. We shall identify the space of functions on the torus with the space of doubly periodic functions on the plane:

$$F(x+m, y+n)=F(x, y), \quad m, n\in\mathbb{Z}.$$

As the area form on T^2 let us take

$$\omega_N=Nhdx\wedge dy.$$

For each pair of real numbers (a, b) one can define a mapping $\phi_{a,b}$ of the open unit square $K\subset\mathbb{R}^2$ into the torus:

$$\phi_{a,b}(x, y)=(x+a, y+b)\bmod\mathbb{Z}^2.$$

Then $\phi^*_{a,b}(\omega_N)=Nhdx\wedge dy$. Let us choose the 1-form $\theta_{a,b}$ so that $\phi^*_{a,b}(\theta_{a,b})=Nhxdy$. It is easy to check that

$$\phi^*_{a,b}(\theta_{c,d}-\theta_{c_1,d_1})=Nh(c-c_1)dy.$$

Therefore the sections of the bundle L_N can be identified with the functions on the plane having the property

$$f(x+m, y+n)=e^{2\pi iNmy}f(x, y), \quad m, n\in\mathbb{Z}. \tag{2.15}$$

The bundles L_N have many interesting properties. Thus, when $N=1$ the space of all smooth sections of L_N admits an isomorphism onto the Schwartz space (L. Schwartz) $\mathscr{S}(\mathbb{R})$. Namely, to each function $f(x, y)$ having the property (2.15) one may associate a function $\phi(x)$ on the line:

$$\phi(x)=\int_0^1 f(x, y)dy. \tag{2.16}$$

Conversely, $f(x, y)$ can be recovered from $\phi(x)$:

$$f(x, y)=\sum_{k\in\mathbb{Z}}\phi(x+k)e^{-2\pi iky}. \tag{2.17}$$

The prequantization operator corresponding to a doubly periodic function $H(x, y)$ has the form:

$$\check{H}=H+\frac{1}{2\pi i}\left(\frac{\partial H}{\partial x}\left(\frac{\partial}{\partial y}-2\pi ix\right)-\frac{\partial H}{\partial y}\frac{\partial}{\partial x}\right). \tag{2.18}$$

Let us note that the operators $\partial/\partial x$ and $\partial/\partial y-2\pi ix$ take the space of functions satisfying condition (2.15) into itself. Under the isomorphism with $\mathscr{S}(\mathbb{R})$ described above they go over into d/dx and $-2\pi ix$ respectively.

2.4. Prequantization of Symplectic Supermanifolds. The Kostant–Souriau prequantization scheme described here can be carried over *mutatis mutandis* to symplectic *supermanifolds* M (see [7], [8]) with an even form ω. A detailed exposition of this construction is given in [28]. Here we shall indicate only the principal changes which must be brought in.

As local coordinates on a supermanifold M both ordinary numerical coordinates $x_1, \ldots, x_n$, called *even coordinates*, play a part, and special *odd coordinates* $\xi_1, \ldots, \xi_m$, which take values in some Grassmann algebra and which satisfy the anticommutation relations

$$\xi_i\xi_j+\xi_j\xi_i=0, \quad 1\le i, j\le m. \tag{2.19}$$

In suitable local coordinates the symplectic form ω reduces to the canonical form

$$\omega=\sum_{k=1}^n dp_k\wedge dq_k+\sum_{j=1}^m \varepsilon_j(d\xi_j)^2, \tag{2.20}$$

where p_k, q_k are the even coordinates, ξ_j are the odd coordinates, and $\varepsilon_j=\pm1$.

The Hamiltonian vector field corresponding to a function F has the form

$$c(F)=\sum_{k=1}^n\left[\frac{\partial F}{\partial p_k}\frac{\partial}{\partial q_k}-\frac{\partial F}{\partial q_k}\frac{\partial}{\partial p_k}\right]+(-1)^{\alpha(F)}\sum_{j=1}^m\varepsilon_j\frac{\partial F}{\partial\xi_j}\frac{\partial}{\partial\xi_j}, \tag{2.21}$$

where $\alpha(F)$ is the parity of the function F. The Poisson bracket looks as follows:

$$\{F, G\}=\sum_{k=1}^n\left[\frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q_k}-\frac{\partial F}{\partial q_k}\frac{\partial G}{\partial p_k}\right]+(-1)^{\alpha(F)}\sum_{j=1}^m\varepsilon_j\frac{\partial F}{\partial\xi_j}\frac{\partial G}{\partial\xi_j}. \tag{2.22}$$

The Poisson bracket operation defines on $C^\infty(M)$ a *Lie superalgebra* structure. It has the properties

$$\begin{gathered}\{F, G\}=-(-1)^{\alpha(F)\alpha(G)}\{G, F\},\\ \{F, \{G, H\}\}=\{\{F, G\}, H\}+(-1)^{\alpha(F)\alpha(G)}\{G, \{F, H\}\}.\end{gathered} \tag{2.23}$$

The definition of the quantization and prequantization of sect. 2.1 can be retained entirely if one merely replaces the commutator on the right-hand side by the supercommutator

$$[A, B]=AB-(-1)^{\alpha(A)\alpha(B)}BA. \tag{2.24}$$

The tie between 1-forms and connections on one-dimensional bundles is also preserved. The analogues of theorems 2.1 and 2.2 are valid.

§ 3. Polarizations

3.1. The Definition of a Polarization. Let (M, ω) be a symplectic manifold, $T^{\mathbb{C}}M$ the complexification of the tangent bundle over M. A subbundle $P\subset T^{\mathbb{C}}M$ is called a *polarization* if it fulfills the conditions

1) The space $P(x)$ is a Lagrangian (= maximal isotropic) subspace of $T_x^{\mathbb{C}}(M)$ for each $x \in M$.

2) The distribution $x \mapsto P(x)$ is integrable.

The latter condition, according to the well-known Frobenius criterium, can be stated in two equivalent forms:

a) Let us call a vector field ξ on M P-admissible if $\xi(x) \in P(x)$ for all $x \in M$. The distribution P is integrable if and only if the commutator of any two P-admissible fields is P-admissible.

b) Let us call a differential form θ on M P-admissible if the form $\theta(x)$ vanishes when one of its arguments belongs to $P(x)$. Obviously, the P-admissible forms form an ideal. The distribution P is integrable if and only if the differential of an arbitrary P-admissible form is P-admissible.

It is clear that if the subbundle P is a polarization, then the complex conjugate subbundle $\bar{P}$ is also a polarization. If $P = \bar{P}$, the *polarization* is called *real*. In this case P is the complexification of some real integrable subbundle $P^{\mathbb{R}} \subset TM$. The manifold M admits a foliation of half its dimension such that the space $P^{\mathbb{R}}(x)$ coincides with the tangent space of the leaf passing through the point x. Each leaf is a Lagrangian submanifold of M equipped with a canonical affine structure [2]. Usually one requires in addition that this foliation be a smooth fibration (compare sect. 1.1).

If $P(x) \cap \bar{P}(x) = \{0\}$ for all $x \in M$, the *polarization P* is called *pseudo-Kählerian*. In this case

$$b(X, Y) = i\omega(X, \bar{Y}) \tag{3.1}$$

is a nondegenerate Hermitian form on $P(x)$. The *polarization* is called *Kählerian* if this form is positive definite.

The Nirenberg–Newlander theorem [33], [17] states that for an arbitrary pseudo-Kähler polarization P one can introduce local complex coordinates $z_1, \ldots, z_n$ in the neighbourhood of an arbitrary point of M so that P is generated by the fields $\partial/\partial\bar{z}_1, \ldots, \partial/\partial\bar{z}_n$.

In the general case let us denote by $E(x)$ and $D(x)$ the subspaces in $T_x(M)$ for which

$$E^{\mathbb{C}}(x) = P(x) + \bar{P}(x), \quad D^{\mathbb{C}}(x) = P(x) \cap \bar{P}(x). \tag{3.2}$$

The dimensions of $E(x)$ and $D(x)$ can change from point to point, in such a way that their sum is always equal to $2n$. If $\dim E(x) \equiv n + k$, $\dim D(x) \equiv n - k$, and if E is an integrable bundle (let us note that D is automatically integrable), then the polarization P is called admissible. In this case, in the neighbourhood of an arbitrary point of M one can introduce local coordinates $x_1, \ldots, x_{n-k}; y_1, \ldots, y_{n-k}; u_1, \ldots, u_k; v_1, \ldots, v_k$ so that $D(x)$ is generated by the fields $\partial/\partial x_j$ and $P(x)$ by the fields $\partial/\partial x_j$ and $\partial/\partial\bar{z}_m$, where $z_m = u_m + iv_m$.

In the case where E is a nonintegrable distribution, the local description of the polarization is not known at present. This question is closely tied up with the problem of classifying real submanifolds of a complex manifold up to holomorphic changes of variables.

Polarizations P_1 and P_2 are called *transverse* if $P_1(x) \cap P_2(x) = \{0\}$ for all $x \in M$. If P_1 and P_2 are arbitrary Kähler polarizations, then P_1 and $\bar{P}_2$ are transverse.

For an arbitrary polarization P let us denote by $\mathscr{F}(P)$ the space of functions on M which are annihilated by all P-admissible vector fields. It is clear that $\mathscr{F}(P)$ is an algebra with respect to the usual multiplication of functions and a maximal abelian Lie subalgebra with respect to the Poisson bracket. For a real polarization P the algebra $\mathscr{F}(P)$ consists of the functions which are constant on the leaves of a certain Lagrangian foliation.

The Hamiltonian vector fields with generating functions in $\mathscr{F}(P)$ commute pairwise, are tangent to the leaves and are constant with respect to the canonical affine structure on the leaves. For a pseudo-Kähler polarization P the algebra $\mathscr{F}(P)$ consists of the holomorphic functions with respect to a certain complex structure. In the general case an explicit description of $\mathscr{F}(P)$ is for the present not known.

Let us denote by $G(P)$ the group of automorphisms of (M, ω) which preserve the polarization P. The Lie algebra $\mathfrak{g}(P)$ of this infinite-dimensional group admits of a simple description. The preimage of $\mathfrak{g}(P)$ in $\mathscr{P}(M, \omega)$ coincides with the normalizer of the commutative subalgebra $\mathscr{F}(P)$. For a real polarization P the algebra $\mathfrak{g}(P)$ consists of the vector fields whose generating functions are affine on the leaves of the polarization P.

3.2. Polarizations on Homogeneous Manifolds. Let us suppose that the symplectic manifold (M, ω) is homogeneous, that is, it admits a transitive action of a Lie group G which preserves the form ω. Let us briefly call to mind the classification of these manifolds which was formulated above in theorem 1.1.

Let us note first of all that without loss of generality one may consider the group G to be simply connected.

Let $\mathfrak{g}$ be the Lie algebra of the group G, and $\mathfrak{g}^*$ the dual space to $\mathfrak{g}$. To each element $X \in \mathfrak{g}$ corresponds a Hamiltonian vector field ξ_X on M. Passing if necessary to a covering $\tilde{M}$ of the manifold M, we may assume that the field ξ_X possesses a generating function f_X. One may also assume that the mapping $X \mapsto f_X$ from $\mathfrak{g}$ to $C^\infty(\tilde{M})$ is linear. The Poisson bracket $\{f_X, f_Y\}$ can differ from the function $f_{[X,Y]}$ only by a constant, which we denote by $c(X, Y)$. It is easily verified that $c(X, Y)$ is a 2-cocycle on the Lie algebra $\mathfrak{g}$, that is, it has the properties

$$c(X, Y) = -c(Y, X),$$

$$c([X, Y], Z) + c([Y, Z], X) + c([Z, X], Y) = 0. \tag{3.3}$$

This cocycle defines a one-dimensional central extension $\tilde{\mathfrak{g}}$ of the Lie algebra $\mathfrak{g}$. As a linear space $\tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}$, but the multiplication law in $\tilde{\mathfrak{g}}$ has the form:

$$[X \oplus \alpha, Y \oplus \beta] = [X, Y] \oplus c(X, Y). \tag{3.4}$$

The mapping

$$X \oplus \alpha \mapsto f_{X \oplus \alpha} = f_X + \alpha \tag{3.5}$$

is a homomorphism of the Lie algebra $\tilde{\mathfrak{g}}$ into the Poisson algebra $\mathscr{P}(\tilde{M}, \tilde{\omega})$. Let $\tilde{G}$ be the simply connected Lie group with Lie algebra $\tilde{\mathfrak{g}}$. It acts transitively on $\tilde{M}$ and this *action* is *Poisson* (see [3], [46]).

If the cocycle c is trivial, that is,

$$c(X, Y) = b([X, Y]) \tag{3.6}$$

for some $b \in \mathfrak{g}^*$, then the algebra $\tilde{\mathfrak{g}}$ is the direct sum of $\mathfrak{g}$ and $\mathbb{R}$. The group $\tilde{G}$ will in this case be the direct product of G and $\mathbb{R}$, where the second factor acts on $\tilde{M}$ trivially.

Thus, the transition from M, G to $\tilde{M}, \tilde{G}$ allows one to consider the group action to be Poisson. In this case the *momentum mapping* is defined, which takes a point $x \in M$ over into the functional $F_x \in \mathfrak{g}^*$ defined by the identity

$$F_x(X) = f_X(x). \tag{3.7}$$

The momentum mapping is equivariant (commutes with the action of the group G) and is a local homeomorphism. Its image is one of the G-orbits in $\mathfrak{g}^*$ with respect to the coadjoint representation.

Example 3.1. Let $M = \mathbb{R}^{2n}$ with the standard symplectic structure ω, and let $G = \mathbb{R}^{2n}$ be the group of parallel translations on M. In this case $\tilde{G}$ is a nontrivial central extension of the group G, called the *Heisenberg group* (W. Heisenberg).

The algebra $\tilde{\mathfrak{g}}$ may be realized in the form of the space of affine functions on $\mathbb{R}^{2n}$ (that is, polynomials of degree ≤ 1) with the operation of the Poisson bracket. One can identify the space $\tilde{\mathfrak{g}}^*$ with $\tilde{\mathfrak{g}}$ with the aid of a nondegenerate bilinear form on $\tilde{\mathfrak{g}}$:

$$(P, Q) = P\left(\frac{\partial}{\partial x}\right) Q(x) \bigg|_{x=0}.$$

The momentum mapping has the form

$$x \mapsto \omega(x, \cdot) + 1$$

and takes M over into the orbit consisting of all affine functions equal to 1 at the point 0.

Example 3.2. Let $M = \mathbb{R}^{2n} \setminus \{0\}$ with the standard symplectic structure ω, and let $G = \mathrm{Sp}(2n, \mathbb{R})$ be the group of linear symplectic transformations. The Lie algebra $\mathfrak{g}$ can be realized in the form of the space $\mathrm{Sym}(2n, \mathbb{R})$ of symmetric matrices with the bracket

$$[A, B] = A\omega B - B\omega A,$$

but also as the space of quadratic forms on $\mathbb{R}^{2n}$ with the operation of the Poisson bracket. The space $\mathfrak{g}^*$ may be identified with $\mathfrak{g}$ just as above. The momentum mapping has the form:

$$x \mapsto xx'$$

(where x is a column vector, $'$ means transposition) and it takes $\mathbb{R}^{2n}$ over into the cone of non-negative matrices of rank 1 in $\mathrm{Sym}(2n, \mathbb{R})$.

For a G-homogeneous symplectic manifold M it is natural to ask about the existence of G-invariant polarizations on M. Since G acts transitively, a G-invariant polarization P is determined by the space $P(x_0) \subset T^{\mathbb{C}}_{x_0}(M)$ for any point $x_0 \in M$. Let H be the isotropy group of the point x_0, and $\mathfrak{h}$ its Lie algebra. The space $T_{x_0}(M)$ can be identified with $\mathfrak{g}/\mathfrak{h}$ and $T^{\mathbb{C}}_{x_0}(M)$ with $\mathfrak{g}^{\mathbb{C}}/\mathfrak{h}^{\mathbb{C}}$. Let us denote by $\mathfrak{p}$ the subspace of $\mathfrak{g}^{\mathbb{C}}$ containing $\mathfrak{h}^{\mathbb{C}}$ and such that $\mathfrak{p}/\mathfrak{h}^{\mathbb{C}} = P(x_0)$. Let us recall that a *subalgebra* $\mathfrak{a} \subset \mathfrak{g}$ is called *subordinate* to a functional $F \in \mathfrak{g}^*$ if F vanishes on the derived algebra $[\mathfrak{a}, \mathfrak{a}]$ (see [20]).

Theorem 3.1. *For a G-invariant distribution P on M to be a polarization, it is necessary and sufficient that the space $\mathfrak{p}$ be a subalgebra of $\mathfrak{g}^{\mathbb{C}}$ subordinate to the functional $F \in \mathfrak{g}^*$ which is the image of $x_0 \in M$ under the momentum mapping.*

It is known [23] that for solvable Lie groups invariant polarizations exist for all orbits in the coadjoint representation. This is true also for generic orbits in an arbitrary complex Lie algebra.

A convenient way of constructing polarizations has been proposed by Michèle Vergne (see the reference in [23]). This construction is based on the following simple assertion.

Theorem 3.2. *Let s: $0 = V_0 \subset V_1 \subset \ldots \subset V_n = V$ be a chain of linear spaces, $\dim V_k = k$. Let us suppose that on V a bilinear skew-symmetric form B is given and that B_k is the restriction of B to V_k.*

Let us set $W(s, B) = \sum_{k=1}^{n} \ker B_k$. Then:

a) *$W(s, B)$ is a maximal isotropic subspace of V with respect to B.*

b) *If V is a Lie algebra, the V_k are ideals in V, and the form B has the shape $B(X, Y) = \langle F, [X, Y] \rangle$ for some $F \in V^*$, then $W(s, B)$ is a subalgebra of V.*

Examples are also known of orbits which do not admit invariant polarizations. In particular, the orbits of minimal dimension for the symplectic groups of rank ≥ 2 are of this kind (see example 3.2 above).

Recently interest in invariant polarizations on homogeneous manifolds has grown in connection with the so-called group approach in the theory of completely integrable Hamiltonian systems (see [1], [12], [14], [15], [34], [36] and also volume 16 of this edition).

§4. Quantization

4.1. The Space of a Quantization. As was already noted above, the space of a prequantization is too large for the completeness condition (condition 4 of sect. 2.1) to be fulfilled. In ordinary quantum mechanics the wave functions depend on only half of the coordinates of the classical phase space: in the coordinate representation the wave functions do not depend on the momenta, in the momentum representation they do not depend on the coordinates. In the language of symplectic geometry one may say that the space of the quantization in both cases consists of the functions which are constant along the leaves of a certain real polarization.

Therefore in the general case it is natural to try to construct the space of the quantization out of those elements of the prequantization space (i.e. the sections of the bundle L over M) which are covariantly constant along some polarization P, that is, have the property that

$$\nabla_\xi s = 0 \text{ for any } P\text{-admissible field } \xi. \tag{4.1}$$

We shall denote the space of such sections by $\Gamma(L, M, P)$.

There are at least three obstructions to the construction of the quantization space out of the elements of $\Gamma(L, M, P)$.

The first obstacle arises in the case when the polarization is real and its leaves are not simply connected. Parallel translation of a section along a closed path Γ lying in a leaf results in multiplication of the section by a number $Q(\Gamma)$. If Γ can not be contracted to a point, the number $Q(\Gamma)$ is generally different from 1. It is clear that in this case any solution of equation (4.1) vanishes on the leaf under consideration.

The mapping $\Gamma \mapsto Q(\Gamma)$ defines a character of the fundamental group of the leaf (compare sect. 2.2 above). Let us denote by M_0 the union of those leaves for which this character is trivial. The subset M_0 is called the *Bohr–Sommerfeld subvariety* (N. Bohr–A. Sommerfeld) in M. In place of $\Gamma(L, M, P)$ one may consider the set of generalized solutions of equation (4.1). They all are concentrated on M_0 and they form a space which we shall denote by $\Gamma_{\text{gen}}(L, M, P)$.

Example 4.1. Let $M = \mathbb{R}^2 \setminus \{(0, 0)\}$, $\omega = dx \wedge dy$, and let the polarization P have as its leaves the circles $x^2 + y^2 = \text{const}$. Let us choose $\frac{1}{2}(xdy - ydx)$ as the form θ. In polar coordinates r, ϕ the form θ has the form $\frac{1}{2}r^2 d\phi$, and for the path Γ which goes once around the circle $r = \text{const}$ we get $Q(\Gamma) = \exp(2\pi^2 r^2/ih)$. Thus, the Bohr–Sommerfeld subvariety consists in this case of the circles which bound an area $\pi r^2 = nh$, $n \in \mathbb{Z}$.

Thus, one way of circumventing the obstacle which arises lies in the passage from ordinary sections to generalized ones.

Another method consists in considering the sheaf $\mathscr{C}_P$ of germs of sections in $\Gamma(L, M, P)$ and its higher cohomology groups $H^k(M, \mathscr{C}_P)$. In [39] it is shown that in the case of example 4.1 the transition from $H^0(M, \mathscr{C}_P) = \Gamma(L, M, P)$ to $H^1(M, \mathscr{C}_P)$ leads to the same quantization space as is obtained by the method of generalized sections.

This corroborates L.D. Faddeev's well-known thesis: "cohomology means the same functions, but with singularities".

The second obstacle is that the scalar square of a section $s \in \Gamma(L, M, P)$ is a function which is constant on the leaves of the polarization P. If the leaves are not compact, then the integral of this function over the manifold M diverges and one must replace it by the integral over the set of leaves, on which there is no measure *a priori*.

A way out of this difficulty is given by the concept, introduced by Blattner and Kostant, of L-valued half-forms on M normal to a given polarization P.

(The origins of this concept may be seen in the construction, proposed by G.W. Mackey about 30 years ago, of the "intrinsic Hilbert space" $L_2(M)$, which could be applied to an arbitrary smooth manifold M. In modern language the elements of this space may be described as *half-densities* (or densities of weight 1/2) on M, that is, sections of a bundle with a one-dimensional fibre over M whose transition functions are square roots of the absolute values of Jacobian determinants.)

Before giving the exact definition of L-valued half-forms and of the scalar product for them, it will be helpful to consider first the model situation of a flat space M with a constant polarization P. This will be done in sect. 4.2.

Finally, a third obstacle is that the prequantization operators $\check{F}$ generally do not preserve the space $\Gamma(L, M, P)$. More precisely, we have

Theorem 4.1. *If the field $c(F)$ is complete, then the operator* $\exp(t\check{F})$ *takes* $\Gamma(L, M, P)$ *over into* $\Gamma(L, M, P_t)$, *where P_t is the polarization into which P goes over under the diffeomorphism* $\phi_t = \exp(tc(F))$.

The proof follows immediately from the commutation relation between $\check{F}$ and ∇_ξ:

$$[\check{F}, \nabla_\xi] = \frac{h}{2\pi i} \nabla_{[c(F), \xi]}, \tag{4.2}$$

which in turn follows from (2.5), (2.7) and theorem 2.1.

The way out of this difficulty is to construct a canonical isomorphism between the spaces $\Gamma(L, M, P)$ for various P. It turns out this can also be achieved (at least in the flat case) by the introduction of L-valued half-forms.

Returning to the first obstacle, let us observe that the transition from sections of L to L-valued half-forms leads to a modification of the Bohr–Sommerfeld conditions. Namely, in place of the condition $Q(\Gamma) = 1$ the condition $Q(\Gamma) \cdot \chi(\Gamma) = 1$ must be fulfilled, where χ is the character of the fundamental group of the leaf corresponding to the natural flat connection on the bundle of half-forms over this leaf. In the case of example 4.1 $\chi(\Gamma) = -1$ for a generating cycle Γ. Therefore the *modified Bohr–Sommerfeld condition* takes the form $\pi r^2 = (n + \frac{1}{2})h$, $n \in \mathbb{Z}$. This result (in contrast to the one obtained above) agrees with the well-known structure of the spectrum of the energy operator for the harmonic oscillator.

4.2. Quantization of a Flat Space. Let $M = \mathbb{R}^{2n}$ with the standard symplectic form $\omega = \sum_{k=1}^{n} dp_k \wedge dq_k$. We shall consider only constant polarizations on M, for which $P(x) \equiv P \subset \mathbb{C}^{2n}$. The set of such polarizations forms the complex

Lagrangian Grassmann manifold $\Lambda(\mathbb{C}^{2n})$. This is a compact complex manifold of complex dimension $\frac{1}{2}n(n+1)$. In it, let us single out the subset $\Lambda_+(\mathbb{C}^{2n})$ of positive polarizations P, for which the restriction of the form (3.1) to P is non-negative. The group $\mathrm{Sp}(2n, \mathbb{R})$ of linear symplectic transformations of M acts on $\Lambda(\mathbb{C}^{2n})$ and preserves $\Lambda_+(\mathbb{C}^{2n})$. Under the action of $\mathrm{Sp}(2n, \mathbb{R})$ the set $\Lambda_+(\mathbb{C}^{2n})$ splits into $n+1$ orbits, numbered by the rank of the form (3.1) on P:

$$\Lambda_+(\mathbb{C}^{2n}) = \bigcup_{k=0}^{n} \Lambda_+^k(\mathbb{C}^{2n}).$$

The subset $\Lambda_+^n(\mathbb{C}^{2n})$ is open and consists of the Kähler polarizations; the subset $\Lambda_+^0(\mathbb{C}^{2n})$ is closed and consists of the real polarizations. One can show that $\Lambda_+^0(\mathbb{C}^{2n})$ is a skeleton of the boundary of $\Lambda_+(\mathbb{C}^{2n})$. For $n=1$ $\Lambda(\mathbb{C}^2)$ is the Riemann sphere, $\Lambda_+(\mathbb{C}^2)$ is a closed hemisphere, $\Lambda_+^1(\mathbb{C}^2)$ is the interior of the hemisphere and $\Lambda_+^0(\mathbb{C}^2)$ is the equator.

One can introduce suitable coordinates on the set $\Lambda_+(\mathbb{C}^{2n})$ which map it onto a bounded region in $\mathbb{C}^{n(n+1)/2}$. Namely, let $z_j = p_j + iq_j$, $1 \le j \le n$, be complex coordinates on M and let $\tau = \|\tau_{jk}\|$ be a complex matrix of order $n \times n$. Let us consider the subspace P_τ of $\mathbb{C}^{2n}$ generated by the vector fields

$$\xi_j = \frac{\partial}{\partial \bar{z}_j} + \sum_{k=1}^{n} \tau_{jk} \frac{\partial}{\partial z_k}, \quad 1 \le j \le n. \tag{4.3}$$

Theorem 4.2. a) *The subspace P_τ is a polarization if and only if the matrix τ is symmetric.*

b) *The polarization P_τ is positive if and only if the matrix $1-\tau^*\tau$ is non-negative definite (an equivalent condition: $\|\tau\| \le 1$ with respect to the standard Hilbert structure on $\mathbb{C}^n$).*

c) *The polarization P_τ is real if and only if the matrix τ is unitary (an equivalent condition: τ has n linearly independent real eigenvectors).*

The group $G = \mathrm{Sp}(2n, \mathbb{R})$ of linear symplectic transformations can be written in the coordinates z_j, $\bar{z}_j$ as a block matrix of the form[1]

$$\begin{pmatrix} A & B \\ \bar{B} & \bar{A} \end{pmatrix}, \quad AB' = BA', \quad AA^* - BB^* = 1. \tag{4.4}$$

In the τ coordinates the action of this group on $\Lambda_+(\mathbb{C}^{2n})$ has the form

$$\tau \mapsto (A\tau + B)(\bar{B}\tau + \bar{A})^{-1}. \tag{4.5}$$

For $n=1$ this action turns into the well-known fractional linear action of the group $\mathrm{Sp}(2, \mathbb{R}) \cong \mathrm{SU}(1, 1)$ on the unit disk.

Let us construct the quantization space $\mathscr{H}_P$ corresponding to a polarization $P \in \Lambda_+(\mathbb{C}^{2n})$. The prequantization bundle over $M = \mathbb{R}^{2n}$ is trivial. One can identify its sections with functions on M if one fixes a non-vanishing section s_0. By virtue of (2.4) the choice of s_0 up to a constant factor is equivalent to the choice of a 1-form θ giving the connection on L. It is convenient to set

[1] Here $'$ denotes transposition.

$$\theta = \frac{1}{2} \sum_{k=1}^{n} (p_k dq_k - q_k dp_k). \tag{4.6}$$

(This is the only 1-form on M having the property $d\theta = \omega$ and invariant with respect to the linear symplectic group $\mathrm{Sp}(2n, \mathbb{R})$.) Since θ is a real form, by (2.13) the section s_0 (times the proper constant factor) has unit norm at all points of M.

Let us define the *action function* S_P on M by the condition

$$dS_P = \theta \quad \text{on} \quad P. \tag{4.7}$$

This definition makes sense, since $d\theta = \omega$ vanishes on P. Therefore θ is a closed, and hence also an exact form on P. Condition (4.7) defines the function S_P up to a summand in $\mathscr{F}(P)$ (that is, a function which is annihilated by all P-admissible vector fields — see sect. 3.1). With the aid of the action function it is easy to describe the space $\Gamma(L, M, P)$ which interests us. Namely,

$$\Gamma(L, M, P) = \mathscr{F}(P) \cdot \exp\left(\frac{2\pi i}{h} S_P\right) \cdot s_0. \tag{4.8}$$

In the flat case we may take as S_P a suitable quadratic form on M. The choice of this form becomes uniquely determined by the additional condition of linearity along P. If $F_1, \ldots, F_n$ are complex linear functions on M for which the fields $c(F_1), \ldots, c(F_n)$ give a basis of P, and if the linear functions $G_1, \ldots, G_n$ are defined by the conditions $\{F_j, G_k\} = \delta_{jk}$, $\{G_j, G_k\} = 0$, then

$$S_P = -\frac{1}{2} \sum_{k=1}^{n} F_k G_k. \tag{4.9}$$

Let $P \in \Lambda_+^k(\mathbb{C}^{2n})$. In the notation of sect. 3.1 the space $\mathscr{F}(P)$ consists of the functions $f(x_1, \ldots, x_{n-k}, z_1, \ldots, z_k)$ which are holomorphic in $z_1, \ldots, z_k$. With a fixed action function S_P the space $\Gamma(L, M, P)$ can be identified with $\mathscr{F}(P)$. For the construction of the quantization space $\mathscr{H}_P$ a suitable scalar product remains to be introduced on $\mathscr{F}(P)$. The choice of the scalar product is dictated (up to a factor) by the following circumstance. Let us denote by $\mathscr{P}_1(M)$ the set of all real polynomials of degree ≤ 1 on M. Clearly $\mathscr{P}_1(M)$ is a Lie subalgebra of the Poisson algebra $\mathscr{P}(M, \omega)$. It is called the *Heisenberg algebra* and is a nilpotent algebra with a one-dimensional centre $\mathscr{P}_0(M)$ consisting of the constants. The prequantization operators $\check{F}$ for $F \in \mathscr{P}_1(M)$ preserve each of the spaces $\Gamma(L, M, P)$ and hence give a representation of the Heisenberg algebra on each $\mathscr{F}(P)$. It is natural to require the corresponding representations on $\mathscr{H}_P$ to be unitary, irreducible, and equivalent to one another. It turns out this can be achieved if one defines the scalar product in $\mathscr{F}(P)$ for $P \in \Lambda_+^k(\mathbb{C}^{2n})$ by the formula:

$$(f_1, f_2)_P = c_P \int_{\mathbb{R}^{n-k}} \int_{\mathbb{C}^k} f_1(x, z) \overline{f_2(x, z)} \exp\left(-\frac{4\pi \operatorname{Im} S_P}{h}\right) d^{n-k}x\, d^k u\, d^k v. \tag{4.10}$$

Let us note that the function under the integral sign can be written in the form $\langle s_1, s_2 \rangle$, where $s_i \in \Gamma(L, M, P)$ and $\langle \cdot, \cdot \rangle$ is the Hermitian structure on the fibres of L.

Furthermore, the condition of equivalence of the representations defines (up to a factor) a *pairing* between $\mathscr{H}_{P_1}$ and $\mathscr{H}_{P_2}$ for an arbitrary pair of polarizations P_1, $P_2 \in \Lambda_+(\mathbb{C}^{2n})$. To describe this pairing, we need

Lemma 4.1. *Let P_1 and P_2 be positive polarizations. Then*
a) $P_1 + \bar{P}_2 = E_{12}^{\mathbb{C}}$ *for subspace* $E_{12} \subset \mathbb{R}^{2n}$;
b) $P_1 \cap \bar{P}_2 = D_{12}^{\mathbb{C}}$ *for some subspace* $D_{12} \subset \mathbb{R}^{2n}$;
c) *there obtains a self-dual (with respect to passage to the orthogonal complement in $(\mathbb{R}^{2n}, \omega)$) diagram of inclusions*

$$0 \subset D_{12} \begin{matrix} \subset D_1 \subset E_1 \subset \\ \subset D_2 \subset E_2 \subset \end{matrix} E_{12} \subset \mathbb{R}^{2n},$$

that is, $D_i^{\perp} = E_i$, $i = 1, 2, 12$.

Just as in sect. 3.1, we may choose particular coordinates

$$x_1, \ldots, x_{n-m};\ y_1, \ldots, y_{n-m};\ u_1, \ldots, u_m;\ v_1, \ldots, v_m$$

on $\mathbb{R}^{2n}$ such that D_{12} is generated by the fields $\partial/\partial y_j$, $1 \le j \le n-m$, and E_{12} is generated by the fields $\partial/\partial y_j$, $1 \le j \le n-m$, and by $\partial/\partial u_k$ and $\partial/\partial v_k$, $1 \le k \le m$. In these coordinates the pairing between $\mathscr{H}_{P_1}$ and $\mathscr{H}_{P_2}$ has the form:

$$(f_1, f_2)_{P_1, P_2} = c_{P_1, P_2} \int\limits_{\mathbb{R}^{n-m}} \int\limits_{\mathbb{C}^m} f_1(x, u, v)\overline{f_2(x, u, v)} \times \exp\left(\frac{2\pi i}{h}(S_{P_1} - \bar{S}_{P_2})\right) d^{n-m}x\, d^m u\, d^m v. \qquad (4.11)$$

When $P_1 = P_2 = P$ this formula goes over into (4.10). Let us note that the function under the integral sign in (4.11) has the form $\langle s_1, s_2 \rangle$, where $s_k = f_k \cdot \exp[(2\pi i/h)S_{P_k}] \cdot s_0 \in \Gamma(L, M, P_k)$. Let us investigate the "geometric meaning" of the expression $c_{P_1, P_2} d^{n-m}x d^m u\, d^m v$.

Let us denote by $\wedge^n P$ the vector bundle with one-dimensional fibre over $\Lambda(\mathbb{C}^{2n})$ which is the n-th exterior power of the "tautological" bundle over $\Lambda(\mathbb{C}^{2n})$ (the fibre of this bundle over a point $P \in \Lambda(\mathbb{C}^{2n})$ is the space $P \subset \mathbb{C}^{2n}$ itself). The group $\mathrm{Sp}(2n, \mathbb{R})$ acts in a natural way on $\wedge^n P$.

A *metaplectic structure* on $M = (\mathbb{R}^{2n}, \omega)$ is a bundle with one-dimensional fibre over $\Lambda(\mathbb{C}^{2n})$, denoted by $\sqrt{\wedge^n P}$, together with an action on it of the *metaplectic group* $\mathrm{Mp}(2n, \mathbb{R})$ (the connected double covering of $\mathrm{Sp}(2n, \mathbb{R})$, see [16], [29], [30]), having the property

$$\sqrt{\wedge^n P} \otimes \sqrt{\wedge^n P} \cong \wedge^n P, \qquad (4.12)$$

where $\cong$ denotes isomorphism of $\mathrm{Mp}(2n, \mathbb{R})$-spaces.

The sections of the bundle $\sqrt{\wedge^n P}$ may be written formally in the form $\sqrt{s}$, where s is a section of $\wedge^n P$. For a section $\sqrt{s}$ it makes sense, just as for s, to speak of its being "constant along P". An example of such a section is given by

$$\lambda = \sqrt{dF_1 \wedge \ldots \wedge dF_n}, \quad \text{where } F_i \in \mathscr{F}(P). \qquad (4.13)$$

(With the aid of ω we identify the 1-form dF_i with the vector field $c(F_i)$; for $F \in \mathscr{F}(P)$ the field $c(F)$ will be P-admissible.) In other words, the introduction of a metaplectic structure on M is equivalent to a consistent choice of the sign of the square root in the expressions (4.13).

Now let P_1 and P_2 be two positive polarizations. From the exact sequence of linear spaces

$$\begin{matrix} 0 \to & P_1 \cap \bar{P}_2 & \to P_1 \oplus \bar{P}_2 \to & P_1 + \bar{P}_2 & \to 0 \\ & \| & & \| & \\ & D_{12}^{\mathbb{C}} & & E_{12}^{\mathbb{C}} & \end{matrix}$$

it follows that there is an isomorphism

$$\wedge^n P_1 \otimes \wedge^n \bar{P}_2 \cong \wedge^{n-m} D_{12}^{\mathbb{C}} \otimes \wedge^{n+m} E_{12}^{\mathbb{C}}. \qquad (4.14)$$

Furthermore, the form ω restricted to E_{12} has D_{12} as its kernel and defines a symplectic structure on E_{12}/D_{12}. From this it follows that there is an isomorphism

$$\wedge^{n-m} D_{12} \cong \wedge^{n+m} E_{12}. \qquad (4.15)$$

Combining (4.14) and (4.15), we see that

$$\wedge^n P_1 \otimes \wedge^n \bar{P}_2 \cong (\wedge^{n+m} E_{12}^{\mathbb{C}})^2 \cong (\wedge^{n+m} (\mathbb{R}^{2n}/D_{12})^{\mathbb{C}})^2.$$

It turns out one can "take the square root" of the latter equality. Namely,

Lemma 4.2. *There is a natural isomorphism*

$$\sqrt{\wedge^n P_1} \otimes \sqrt{\wedge^n \bar{P}_2} \cong \wedge^{n+m} (\mathbb{R}^{2n}/D_{12})^{\mathbb{C}}. \qquad (4.16)$$

Thus, to a pair of sections λ_1, λ_2 of the bundles $\sqrt{\wedge^n P_1}$ and $\sqrt{\wedge^n P_2}$ there corresponds a differential form of highest degree on M/D_{12}, which we shall denote by $\mu(\lambda_1 \otimes \bar{\lambda}_2)$. It turns out that just this form must appear in (4.11) in place of the expression $c_{P_1, P_2} d^{n-m}x d^m u d^m v$, if one takes a system of units in which the Planck constant $h = 1$. (In the general case, one must use not the form ω, but the form $\Omega = h^{-1}\omega$ in the isomorphism (4.15).)

Now we can sum up and define the quantization space in purely geometric terms.

Namely, the *quantization space* $\mathscr{H}_P$ consists of the sections of the bundle $L \otimes \sqrt{\wedge^n P}$ which are constant along P. These sections are called *L-valued half-forms* on M with respect to P. It is convenient to write them as $s \otimes \lambda$, where $s \in \Gamma(L, M, P)$ and λ is a section of the bundle $\sqrt{\wedge^n P}$ of the form (4.13).

The pairing between $\mathscr{H}_{P_1}$ and $\mathscr{H}_{P_2}$ has the form:

$$(s_1 \otimes \lambda_1, s_2 \otimes \lambda_2)_{P_1, P_2} = \int_{M/D_{12}} \langle s_1, s_2 \rangle \mu(\lambda_1 \otimes \bar{\lambda}_2). \tag{4.17}$$

The expression under the integral sign in (4.17) has been given the name *Blattner–Kostant–Sternberg kernel* (R. Blattner–B. Kostant–S. Sternberg). Its generalization to the non-linear situation will be considered in sect. 4.4.

The quantization of a flat space constructed in this section gives simple explicit formulas for the quantization operators $\hat{F}$ in the case where F is a polynomial of degree no greater than two. Let us note that the space $\mathscr{P}_2(M)$ of such polynomials forms a Lie subalgebra of $\mathscr{P}(M, \omega)$ in which $\mathscr{P}_1(M)$ is a nilpotent ideal. The quotient algebra $\mathscr{P}_2(M)/\mathscr{P}_1(M)$ is isomorphic to the Lie algebra of the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$. The corresponding operators generate the *Weil representation* (A. Weil) (see sect. 4.3).

The quantization operators look the simplest for $F \in \mathscr{F}(P)$. Namely, if $F \in \mathscr{F}(P)$, then $\hat{F}$ coincides with the operator of multiplication by F. This fact together with the property $\hat{\bar{F}} = \hat{F}^*$ allows one to compute $\hat{F}$ for all $F \in \mathscr{P}_1(M)$ in the case of a Kähler polarization.

Let P_+ be a fixed Kähler polarization and $a_1, \ldots, a_n$ an orthonormal basis of P_+ with respect to the form (3.1). The operators $\hat{a}_1, \ldots, \hat{a}_n$ are called creation operators, and the adjoint operators $\hat{a}_1^*, \ldots, \hat{a}_n^*$ are called annihilation operators. They are tied together by the commutation relations

$$[\hat{a}_i, \hat{a}_k] = [\hat{a}_i^*, \hat{a}_k^*] = 0, \quad [\hat{a}_i^*, \hat{a}_k] = \frac{h}{2\pi} \delta_{ik}. \tag{4.18}$$

In the quantization space there exists a unique vector (up to a factor) which is annihilated by all annihilation operators. It is called the *vacuum vector*.

Example 4.2. We shall cite explicit quantization formulas for all constant polarizations in the case $n = 1$, where $M = \mathbb{R}^2$ is the ordinary plane with coordinates p, q and symplectic form $\omega = dp \wedge dq$.

A constant positive polarization P_τ is given by a complex number τ with the condition $|\tau| \leq 1$. It is generated by the field $\xi = \partial/\partial\bar{z} + \tau\,\partial/\partial z$, where $z = p + iq$, $\partial/\partial z = 1/2(\partial/\partial p - i\,\partial/\partial q)$.

If $|\tau| < 1$ then P_τ is a Kähler polarization. Let us introduce a complex coordinate $z_\tau = (z - \tau\bar{z})/\sqrt{1 - |\tau|^2}$. Then P_τ is generated by the field $\partial/\partial\bar{z}_\tau$. We have: $\omega = (i/2) dz_\tau \wedge d\bar{z}_\tau$, $\theta = (i/4)(z_\tau d\bar{z}_\tau - \bar{z}_\tau dz_\tau)$, $S_\tau = (i/4)|z_\tau|^2$. The quantization space $\mathscr{H}_\tau$ consists of the expressions of the form

$$\psi = f(z_\tau) \cdot e^{-(\pi/2h)|z_\tau|^2} \cdot s_0 \otimes \sqrt{dz_\tau},$$

where f is a holomorphic function. Henceforth we shall identify ψ with f and $\mathscr{H}_\tau$

with the space of holomorphic functions with the norm

$$\|f\|^2 = \int_{\mathbb{C}} |f(z_\tau)|^2 e^{-(\pi/h)|z_\tau|^2} \frac{du \wedge dv}{h},$$

where $u = \operatorname{Re} z_\tau$, $v = \operatorname{Im} z_\tau$.

The coordinate and momentum operators have the form:

$$\hat{q} = \frac{1 - \bar{\tau}}{2\sqrt{1 - |\tau|^2}} z_\tau - \frac{(1 - \tau)h}{2\pi i \sqrt{1 - |\tau|^2}} \frac{d}{dz_\tau},$$

$$\hat{p} = \frac{1 + \bar{\tau}}{2\sqrt{1 - |\tau|^2}} z_\tau + \frac{(1 + \tau)h}{2\pi i \sqrt{1 - |\tau|^2}} \frac{d}{dz_\tau}.$$

The vacuum vector is proportional to $\exp(-\bar{\tau} z_\tau^2/2h)$.

If on the other hand $|\tau| = 1$, then the polarization P_τ is real. Let $\tau = e^{2i\alpha}$; let us introduce the coordinates $p_\alpha = p \cos\alpha + q \sin\alpha$ and $q_\alpha = -p \sin\alpha + q \cos\alpha$. Then P_τ is generated by the field $\partial/\partial p_\alpha$. We have: $\omega = dp_\alpha \wedge dq_\alpha$, $\theta = \frac{1}{2}(p_\alpha dq_\alpha - q_\alpha dp_\alpha)$, $S_\tau = -\frac{1}{2} p_\alpha q_\alpha$. The quantization space $\mathscr{H}_\tau$ consists of the expressions of the form

$$\psi = f(q_\alpha) e^{-(\pi i/h) p_\alpha q_\alpha} s_0 \otimes \sqrt{dq_\alpha}.$$

Henceforth ψ will be identified with f and $\mathscr{H}_\tau$ with the space $L_2(\mathbb{R}, dq_\alpha)$. In this realization the coordinate and momentum operators have the form:

$$\hat{q} = q_\alpha \cos\alpha + \frac{h}{2\pi i} \sin\alpha \frac{\partial}{\partial q_\alpha}, \quad \hat{p} = -q_\alpha \sin\alpha + \frac{h}{2\pi i} \cos\alpha \frac{\partial}{\partial q_\alpha}.$$

The vacuum vector is proportional to $\exp(-\pi q_\alpha^2/h)$.

The reader will find it a useful exercise to check that the isomorphism between $\mathscr{H}_{\tau_1}$ and $\mathscr{H}_{\tau_2}$ established by the pairing (4.17) turns into the ordinary Fourier transform in the case $|\tau_1| = |\tau_2| = 1$, $\tau_1 = -\tau_2$. If on the other hand $\tau_1 = 1$ and $\tau_2 = 0$, then this pairing gives the well-known *Bargmann isomorphism* (V. Bargmann) between $L_2(\mathbb{R}, dq)$ and the *Fock space* $L_2^{\mathrm{hol}}(\mathbb{C}, e^{-\pi(p^2+q^2)} dp dq)$.

4.3. The Connection with the Maslov Index and with the Weil Representation. The pairing (4.17) generates a set of unitary operators $U_{P_2, P_1}\colon \mathscr{H}_{P_1} \to \mathscr{H}_{P_2}$ for which

$$(\psi_1, \psi_2)_{P_1, P_2} = (U_{P_2, P_1}\psi_1, \psi_2)_{P_2}. \tag{4.19}$$

These operators have the properties

$$U_{P_1, P_2} = U_{P_2, P_1}^{-1},$$

$$U_{P_1, P_2} \cdot U_{P_2, P_3} \cdot U_{P_3, P_1} = c(P_1, P_2, P_3) \cdot 1, \tag{4.20}$$

where $c(P_1, P_2, P_3)$ is a complex number equal to 1 in absolute value. If the polarizations P_1, P_2, P_3 are Kählerian, then $c(P_1, P_2, P_3) \equiv 1$. On the other hand, if these polarizations are real, then, as is shown in [29] and [30],

$$c(P_1, P_2, P_3) = \exp\frac{\pi i}{4} m(P_1, P_2, P_3), \qquad (4.21)$$

where $m(P_1, P_2, P_3)$ is a skew-symmetric integral function, called the *Maslov index* (V.P. Maslov) in the Leray–Kashiwara form (J. Leray–M. Kashiwara). This index is defined as the signature of the quadratic form Q on $P_1^{\mathbb{R}} \oplus P_2^{\mathbb{R}} \oplus P_3^{\mathbb{R}}$ (where $P_i^{\mathbb{R}} = P_i \cap \mathbb{R}^{2n}$) given by the formula

$$Q(x_1 \oplus x_2 \oplus x_3) = \omega(x_1, x_2) + \omega(x_2, x_3) + \omega(x_3, x_1). \qquad (4.22)$$

Let $\Lambda(\mathbb{R}^{2n}) = \Lambda_+^0(\mathbb{C}^{2n})$ be the real Lagrangian Grassmann manifold (= the set of real polarizations of $\mathbb{R}^{2n}$). It is well known that the fundamental group of $\Lambda(\mathbb{R}^{2n})$ is isomorphic to $\mathbb{Z}$ and therefore for each natural number q a q-fold covering $\Lambda_q(\mathbb{R}^{2n})$ is defined. By $\Lambda_\infty(\mathbb{R}^{2n})$ we shall denote the universal covering. If one lifts the Leray–Kashiwara index $m(P_1, P_2, P_3)$ to $\Lambda_\infty(\mathbb{R}^{2n})$ then it admits a decomposition

$$m(\tilde{P}_1, \tilde{P}_2, \tilde{P}_3) = \mu(\tilde{P}_1, \tilde{P}_2) + \mu(\tilde{P}_2, \tilde{P}_3) + \mu(\tilde{P}_3, \tilde{P}_1), \qquad (4.23)$$

where $\mu(\tilde{P}_1, \tilde{P}_2)$ is a skew-symmetric function of the pair of points on $\Lambda_\infty(\mathbb{R}^{2n})$. Precisely this function was originally introduced by V.P. Maslov in [31] as the index of a path leading from P_1 to P_2 on $\Lambda(\mathbb{R}^{2n})$.

The operators U_{P_1, P_2} for real polarizations are closely connected with the Weil representation of the metaplectic group $\mathrm{Mp}(2n, \mathbb{R})$ (see [16], [29], [30]). Namely, to each element $g \in \mathrm{Mp}(2n, \mathbb{R})$ there corresponds an automorphism of the bundle $L \otimes \sqrt{\wedge^n P}$, taking P over to $g(P)$, and hence a unitary operator $T(g, P)$: $\mathscr{H}_P \to \mathscr{H}_{g(P)}$ (compare theorem 4.1). The composition $W_P(g) = T(g, P)^{-1} \circ U_{g(P), P}$ acts on the space $\mathscr{H}_P$.

Theorem 4.3. *The correspondence $g \mapsto W_P(g)$ is a unitary representation of the group $\mathrm{Mp}(2n, \mathbb{R})$ on $\mathscr{H}_P$. The equivalence class of the representation W_P does not depend on the choice of $P \in \Lambda_+^0(\mathbb{C}^{2n})$.*

This representation is called the *Weil representation* (other names are: the oscillator representation, the spinor representation, the Shale–Weil representation (D. Shale–A. Weil), the metaplectic representation). It corresponds to an orbit of the minimal dimension $2n$ in the coadjoint representation of the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$. One may regard the Weil representation as a projective representation of the symplectic group $\mathrm{Sp}(2n, \mathbb{R})$. The corresponding 2-cocycle on $\mathrm{Sp}(2n, \mathbb{R})$ is connected with the Maslov index by the equality ([29], [30]):

$$c(g_1, g_2) = \exp\frac{\pi i}{4} m(P, g_1(P), g_1 g_2(P)). \qquad (4.24)$$

The group $\mathrm{Sp}(2n, \mathbb{R})$ acts not only on the real Lagrangian Grassmann manifold $\Lambda(\mathbb{R}^{2n})$, but also on its double covering $\Lambda_2(\mathbb{R}^{2n})$ (the elements of $\Lambda_2(\mathbb{R}^{2n})$ are the oriented Lagrangian subspaces in $\mathbb{R}^{2n}$).

One can check that the Maslov index $\mu(\tilde{P}_1, \tilde{P}_2) \bmod 2q$ is well defined on $\Lambda_q(\mathbb{R}^{2n})$. Therefore the cocycle (4.24) becomes trivial by virtue of (4.23) on the group $\mathrm{Mp}(2n, \mathbb{R})$, acting on $\Lambda_4(\mathbb{R}^{2n})$.

4.4. The General Scheme of Geometric Quantization. The starting material for quantization consists of 1) a symplectic manifold (M, ω), 2) a prequantization bundle L over M with a connection ∇ and a Hermitian structure on the fibres, 3) an admissible positive polarization P on M, and finally 4) a metaplectic structure on M. The first three items were discussed in §§1–3 respectively. A *metaplectic structure* on a curvilinear *manifold M* consists of a consistent assignment of metaplectic structures on the tangent spaces $T_x(M)$ (see sect. 4.2) for all $x \in M$. This is equivalent to a lifting of the structure group of the bundle TM from $\mathrm{Sp}(2n, \mathbb{R})$ to $\mathrm{Mp}(2n, \mathbb{R})$.

Theorem 4.4 ([26]). *In order that there exist a metaplectic structure on M, it is necessary and sufficient that the first Chern class of the bundle TM (more precisely, of the principal $U(n)$-bundle associated with it; recall that a maximal compact subgroup of* $\mathrm{Sp}(2n, \mathbb{R})$ *is isomorphic to $U(n)$) be divisible by 2 in $H^2(M, \mathbb{Z})$. If this condition is fulfilled, then the set of metaplectic structures on M is a principal homogeneous space for $H^1(M, \mathbb{Z}_2)$.*

If all the structures enumerated are on hand, the construction of the quantization space $\mathscr{H}_P$ out of the sections of the bundle $L \otimes \sqrt{\wedge^n P}$ which are covariantly constant along P can be carried out just as in the flat case (see sect. 4.2).

The attempt to define the pairing between the spaces $\mathscr{H}_{P_1}$ and $\mathscr{H}_{P_2}$ in the "curvilinear" case runs into the following obstruction, discovered by Blattner. The value of the Hermitian form $\langle s_1, s_2 \rangle$ need not be constant along D_{12}.

Theorem 4.5 ([9]). *Let P_i be two positive polarizations on the symplectic manifold (M, ω) and let L be a prequantization bundle with a connection ∇. The equality*

$$\xi \langle s_1, s_2 \rangle = 0 \qquad (4.25)$$

holds for all $s_i \in \Gamma(L, M, P_i)$ and $\xi(x) \in D_{12}(x)$ if and only if the covector field $\chi_{P_1, P_2} \in T^*M/(P_1 + \bar{P}_2)$ defined by the following formula vanishes:

$$\chi_{P_1, P_2} = \sum_{i=1}^{k} \omega([v_i, w_i], \cdot), \qquad (4.26)$$

where $k = \dim P_1 \cap \bar{P}_2$ and $v_1, \ldots, v_k, w_1, \ldots, w_k$ are $(P_1 + \bar{P}_2)$-admissible fields normalized by the conditions

$$\omega(v_i, v_j) = \omega(w_i, w_j) = 0, \quad \omega(v_i, w_j) = \delta_{ij}.$$

The simplest example of two polarizations for which $\chi_{P_1, P_2} \neq 0$ can be constructed for $M = \mathbb{R}^4$, $\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$:

$$P_1 = \left\{ a_1 \frac{\partial}{\partial p_1} + a_2 \frac{\partial}{\partial p_2} \right\},$$
$$P_2 = \left\{ a\left(p_1 \frac{\partial}{\partial p_1} + p_2 \frac{\partial}{\partial p_2} \right) + b\left(p_2 \frac{\partial}{\partial q_1} - p_1 \frac{\partial}{\partial q_2} \right) \right\}. \tag{4.27}$$

Let us note that χ_{P_1, P_2} vanishes if $P_1 + \bar{P}_2$ is an integrable distribution (in particular, this is true for all constant polarizations).

If $\chi_{P_1, P_2} = 0$, then it is possible, just as in the flat case, to define the pairing (4.17) and the Blattner–Kostant–Sternberg integral operator U_{P_1, P_2}. General theorems which guarantee that this operator is unitary are still not known at present. The investigation of the possibilities which arise here presents itself as being of extraordinary interest.

4.5. The Quantization Operators. If a Hamiltonian field $c(F)$ preserves some positive polarization P (that is, $c(F) \in \mathfrak{g}(P)$ in the terminology of sect. 3.1), then the desired operator $\hat{F}$ can be defined on the space $\mathscr{H}_P$ as follows

$$\hat{F}(s \otimes \lambda) = \check{F}(s) \otimes \lambda + s \otimes L_{c(F)} \lambda, \tag{4.28}$$

where by $\check{F}$ is denoted the prequantization operator (2.5). In particular, if the function F is constant along P, then the field $c(F)$ is P-admissible, the operator $\check{F}$ reduces to multiplication by F, and $L_{c(F)}\lambda = 0$, since λ is a covariantly constant section of $\sqrt{\wedge^n P}$ with respect to the affine structure on P. The spectrum of the operator $\hat{F}$ is defined in this case by the modified Bohr–Sommerfeld variety (see sect. 4.1).

Now suppose the field $c(F)$ does not preserve the polarization P. If $c(F)$ is complete, that is, generates a one-parameter group $\phi_t = \exp tc(F)$ of transformations of M, then one can define a corresponding group $\tilde{\phi}_t = \exp(2\pi i/h)t\check{F}$ of transformations of $\Gamma(L, M)$. The transformation ϕ_t takes the polarization P over into a polarization P_t:

$$P_t(x) = (\phi_t)_* P(\phi_t^{-1}(x)). \tag{4.29}$$

By theorem 4.1 the mapping $\exp(t\check{F})$ takes $\Gamma(L, M, P)$ over into $\Gamma(L, M, P_t)$. In addition, the mapping $(\phi_t)_*$ takes sections of the bundle $\sqrt{\wedge^n P}$ constant along P over into sections constant along P_t. Thus, there arises a family of operators

$$\hat{\phi}_t\colon \mathscr{H}_P \to \mathscr{H}_{P_t}\colon s \otimes \lambda \mapsto \exp\left(\frac{2\pi i}{h} t\check{F} \right) s \otimes (\phi_t)_* \lambda, \tag{4.30}$$

where $s \in \Gamma(L, M, P)$, $\lambda \in \sqrt{\wedge^n P}$.

In the case when $\chi_{P, P_t} = 0$ (for example, when P is an admissible positive polarization) the operators $\hat{\phi}_t$ are unitary with respect to the Hilbert-space structures on $\mathscr{H}_P$ and $\mathscr{H}_{P_t}$ defined by the pairing (4.17). If, besides, the operator $U_{P_t, P}$ is defined and unitary, then one can define the quantization operator by the equality

$$\hat{F} = \frac{h}{2\pi i} \frac{d}{dt} (U_{P_t, P} \circ \hat{\phi}_t)|_{t=0}. \tag{4.31}$$

This is the most general definition being used at present in the theory of geometric quantization. In the homogeneous situation it turns out to be adequate in order to construct a complete set of irreducible unitary representations for a broad class of Lie groups (see [4], [10], [11], [23]).

In the non-homogeneous situation general theorems are lacking for the time being, but a number of concrete examples have already been studied. Thus in [40] the case is analyzed of $M = T^*N$ where N is a Riemannian manifold, $\omega = dp \wedge dq$, $H = p^2 + V(q)$. For the quantum energy operator the expression

$$\hat{H} = -\Delta + V(q) + \tfrac{1}{6}R \tag{4.32}$$

is obtained, where Δ is the Laplace–Beltrami operator on N and R is the scalar curvature of the manifold N.

The example of a relativistic system representing a charged particle in an external electromagnetic field is analyzed in the same work. The manifold M in this case is a "twisted" cotangent bundle over Minkowski space:

$$(M, \omega) = \left(T^*N, \sum_{k=1}^{4} dp_k \wedge dq_k + e\pi^* F(q) \right),$$

where F is the electromagnetic field tensor on N and π is the projection of T^*N onto N.

The quantum energy operator is given by the previous formula (4.32), where Δ denotes the "twisted" Laplace–Beltrami operator, in which the usual covariant derivatives ∇_k are replaced by "long" derivatives $\nabla_k + (2\pi ie/h)A_k$ which include the electromagnetic potential A.

The case of the harmonic oscillator has been analyzed in many works (see, for example, [16], [38], [41]) and the standard result on the structure of the energy levels has been obtained.

Also popular is the quantum Kepler problem, which describes the motion of a particle in a Coulomb field ([35], [38]). In a number of works the n-dimensional generalization of this problem is considered. The most natural realization of the phase space of this system is an orbit of minimal dimension (equal to $2n$) in the coadjoint representation of the group $G = SO(n+1, 2)$. As a symplectic G-space this orbit is isomorphic to T^*Q, where Q is the quadric in $\mathbb{RP}(n+1)$ associated with the cone of isotropic vectors in $\mathbb{R}^{n+1, 1}$. The various affine forms of this quadric correspond to the three different cases of the Kepler problem (see [35]).

The connection of geometric quantization with the twistor approach of Penrose is discussed in the book of N. Woodhouse [47].

An interesting direction which connects the ideas of geometric quantization with the Selberg trace formula has been under development recently by N.E. Hurt [18].

A geometrical approach to thermodynamical problems is being advocated by one of the founders of geometric quantization, J.M. Souriau [43]. In the case of an infinite number of degrees of freedom the method of geometric quantization cannot be applied literally and demands essentially new ideas. One such idea is the concept, introduced by J.M. Souriau, of a diffeology. In place of a smooth manifold structure, which does not exist in many concrete examples (because of the presence of singularities, non-separability, or infinite dimension), he suggests axiomatizing the class of smooth mappings from M to the space which interests us [44].

Another idea, which has proved very useful in the theory of representations of infinite-dimensional groups, is the examination by G.I. Ol'shanskij of the semigroup of contraction operators lying in the closure of the image of the group under a unitary irreducible representation of it.

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Integrable Systems. I

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Contents

Introduction

Integrable systems which do not have an "obvious" group symmetry, beginning with the results of Poincaré and Bruns at the end of the last century, have been perceived as something exotic. The very insignificant list of such examples practically did not change until the 1960's. Although a number of fundamental methods of mathematical physics were based essentially on the perturbation-theory analysis of the simplest integrable examples, ideas about the structure of nontrivial integrable systems did not exert any real influence on the development of physics.

The situation changed radically with the discovery of the *inverse scattering method*. The ever-growing interest in this method is connected with the fact that it has proved to be applicable to a number of nonlinear equations of mathematical physics which, as became clear in the mid-sixties, possess a remarkable universality property. They arise in the description (in the simplest approximation after the linear one) of the most diverse phenomena in plasma physics, the theory of elementary particles, the theory of superconductivity, in nonlinear optics and in a number of other problems which are reducible to spatially one-dimensional ones. Among the equations referred to are the Korteweg–de Vries equation, the nonlinear Schrödinger equation, the sine-Gordon equation and many others.

The inverse scattering method allowed people for the first time to discover and to understand a number of principally new effects which had not become apparent in any way in the theory of perturbations. The most striking and important of them are connected with the concept of solitons and their periodic analogues (which will be the topic of discussion to a significant degree later on). The concept of solitons has become one of the fundamental ones in contemporary nonlinear physics.

Although after the papers [54], [59] it subsequently became clear that the equations to which the inverse scattering method is applicable are Hamiltonian and, what is more, are the field analogues of completely integrable Hamiltonian systems, the integration of these equations within the framework of the inverse scattering method does not make use of the Hamiltonian theory. The Hamiltonicity of these equations, the construction for them of variables of the action-angle type turn out to be essential during the following stages—in the construction of a theory of perturbations and of diverse versions of the averaging methods, in the construction of the quantum analogue of the inverse scattering method. These sections remain outside the scope of the present article.

The goal of the present survey is the presentation of the modern theory of integrable systems as a constituent part of the inverse scattering method. Just as in classical analytical mechanics, special emphasis is laid on finite-dimensional systems.

The finite-dimensional dynamical systems to which the inverse scattering method is applicable and to which, basically, this article is devoted (and among them are contained all the known classical completely integrable systems) are finite-dimensional in their original physical formulation or they arise during the construction of particular classes of exact solutions of the field-theoretic equations as restrictions of the latter to finite-dimensional invariant submanifolds.

One should especially stress the significantly greater effectiveness of the inverse scattering method as compared with the classical methods of integrating Hamiltonian systems. For completely integrable systems, in contrast to the ineffective integration procedure given by Liouville's theorem, the inverse scattering method allows one to explicitly produce solutions of the equations of motion, as well as canonical action-angle variables, in terms of special classes of functions.

In the first chapter of the survey the modern views of the Hamiltonian formalism of both finite-dimensional and field-theoretic systems are presented. Also set forth are the methods, going back to the classical ones, of integrating Hamiltonian systems which have an explicit symmetry or which admit of separation of the variables.

The second chapter is the nucleus of the present survey. In it the paramount concept of the *commutation representation* of evolution systems is introduced, which is the starting point of all the integration schemes which are unified by the ideas of the inverse scattering method. A scheme based on the application of the methods of classical algebraic geometry has proven to be the most fruitful one in the theory of integrable finite-dimensional systems. This scheme, and also its numerous applications, are presented in the second chapter.

It needs to be noted that naturally abutting on the present survey there will be a survey "Integrable systems II" by A.M. Perelomov, M.A. Ol'shanetskij, and M.A. Semenov-Tyan-Shanskij, which will be published in one of the following volumes of the present series. Its first chapter is devoted to group-theoretic methods of integration of some special finite-dimensional systems. The second chapter is devoted to geometric quantization of the open Toda lattice and its generalizations.

Chapter 1

Hamiltonian Systems. Classical Methods of Integration

§ 1. The General Concept of the Poisson Bracket. The Principal Examples

From the modern point of view, the concept of the *Poisson bracket* (S.D. Poisson) lies at the basis of the Hamiltonian formalism. Let y^i, $i = 1, \ldots, N$ be local coordinates on a manifold Y—the *phase space*. The Poisson bracket of two functions $f(y)$ and $g(y)$ is given by a tensor field $h^{ij}(y)$,

$$\{f, g\} = h^{ij}(y) \frac{\partial f}{\partial y^i} \frac{\partial g}{\partial y^j} \tag{1.1}$$

(here and further on the summation over repeated indices is implied). Here it is required that the following properties be fulfilled:

a) bilinearity

$$\{\lambda f + \mu g, h\} = \lambda\{f, h\} + \mu\{g, h\}, \quad \lambda, \mu = \text{const}, \tag{1.2}$$

and skew-symmetry

$$\{g, f\} = -\{f, g\}; \tag{1.3}$$

b) the *Leibniz identity* (G.W. Leibniz)

$$\{fg, h\} = g\{f, h\} + f\{g, h\}; \tag{1.4}$$

c) the *Jacobi identity* (C.G.J. Jacobi)

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0. \tag{1.5}$$

Let us note that

$$h^{ij}(y) = \{y^i, y^j\}, \tag{1.6}$$

and the definition (1.1) can be written in the form

$$\{f, g\} = \frac{\partial f}{\partial y^i} \frac{\partial g}{\partial y^j} \{y^i, y^j\} \tag{1.7}$$

Hamiltonian systems by definition have the form:

$$\dot{y}^i = \{y^i, H\}, \quad i = 1, \ldots, N, \tag{1.8}$$

where $H = H(y)$ is an arbitrary function, called the *Hamiltonian*. The vector field $\xi_H = (\xi_H^i)$ corresponding to the Hamiltonian system (1.8) has the form:

$$\xi_H^i(y) = h^{ij}(y) \frac{\partial H(y)}{\partial y^j} = \{y^i, H(y)\}, \quad i = 1, \ldots, N. \tag{1.9}$$

Such *vector fields* are called *Hamiltonian*. The commutator of two Hamiltonian fields is connected with the Poisson bracket by the relation

$$[\xi_H, \xi_F] = -\xi_{\{H, F\}}. \tag{1.10}$$

It is clear that the derivative of an arbitrary function $f = f(y)$ by means of the Hamiltonian system (1.8) has the form

$$\dot{f} = \{f, H\} = \xi_H^i \frac{\partial f}{\partial y^i}. \tag{1.11}$$

The flow of (1.8) preserves the Poisson bracket:

$$\{y^i(t), y^j(t)\} = \{y^i(0), y^j(0)\}. \tag{1.12}$$

(*Transformations* which preserve the Poisson bracket are called *canonical*. Any one-parameter group of canonical transformations for non-degenerate brackets $\det(h^{ij}) \neq 0$ has the form (1.8), where the Hamiltonian is possibly defined locally [42] (see also the second chapter of the article by V.I. Arnol'd and A.B. Givental').

It is possible that there are nontrivial functions $f_q(y)$ (maybe given locally on the manifold) such that

$$\{f_q, g\} = 0 \tag{1.13}$$

for any function $g(y)$. In this case the Poisson bracket is called degenerate: the matrix $h^{ij}(y)$ is degenerate. (For a degenerate matrix $h^{ij}(y)$ of constant rank the functions $f_q(y)$ of (1.13) locally always exist.) If all such quantities $f_l(y)$ have been found, then on their common level surface

$$f_l(y) = \text{const} \quad (l = 1, 2, \ldots) \tag{1.14}$$

the Poisson bracket no longer stays degenerate.

Let z^q be coordinates on the level surface (1.14). The restriction of the tensor h^{qr} to this surface is no longer degenerate, and there is an inverse matrix

$$h_{qp} h^{pr} = \delta_q^r. \tag{1.15}$$

The inverse matrix defines a 2-form

$$\Omega = h_{qp}(z)\, dz^q \wedge dz^p. \tag{1.16}$$

From property (1.5) it follows that the form Ω is closed,

$$d\Omega = 0, \quad \text{i.e.} \quad \frac{\partial h_{qp}}{\partial z^r} + \frac{\partial h_{rq}}{\partial z^p} + \frac{\partial h_{pr}}{\partial z^q} = 0. \tag{1.17}$$

If the Poisson bracket was non-degenerate right from the start, then the closedness condition (1.17) turns out to be equivalent to the Jacobi identity (1.5) [42]. Thus phase spaces with a non-degenerate Poisson bracket are *symplectic manifolds*.

Let us examine the basic types of phase spaces.

Type I. Constant brackets and Lagrangian variational problems. Let the matrix h^{ij} be constant and skew-symmetric. The Jacobi identity is automatically fulfilled in this case: on a plane where the matrix h^{qr} becomes non-degenerate the corresponding 2-form $\Omega = h_{qr} dy^q \wedge dy^r$ has constant coefficients and is therefore closed. By a linear transformation constant brackets can be reduced to the form

$$(h^{ij}) = \begin{pmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & & \\ & & \ddots & & & 0 & \\ & & & 0 & 1 & & \\ & & & -1 & 0 & & \\ & & & & & 0 & \\ & 0 & & & & & \ddots \\ & & & & & & & 0 \end{pmatrix}. \tag{1.18}$$

Locally any Poisson brackets of constant rank can be brought into this form (the Darboux theorem (G. Darboux)). In the non-degenerate case coordinates $(y) = (x^1, \ldots, x^n, p_1, \ldots, p_n)$ may be introduced such that

$$h^{ij} = -h_{ij} = \begin{pmatrix} & & & 1 & & \\ & 0 & & & \ddots & 0 \\ & & & 0 & & \ddots \\ & & & & & & 1 \\ -1 & & & & & & \\ & \ddots & & 0 & & & \\ & & \ddots & & & 0 & \\ & 0 & & \ddots & & & \\ & & & -1 & & & \end{pmatrix}. \tag{1.19}$$

The coordinates (x, p) are called canonical ones.

The equations (1.8) take on the form:

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x^i}, \quad i = 1, \ldots, n. \tag{1.20}$$

These are the canonical Hamilton equations (W.R. Hamilton). They can be obtained from the variational principle $\delta S = 0$, where

$$S = \int L(x, \dot{x}) dt, \tag{1.21}$$

(x) belongs to the configuration space and $L(x, \dot{x})$ is the Lagrangian, if one performs a *Legendre transformation* (A.M. Legendre)

$$p_i = \frac{\partial L}{\partial \dot{x}^i}, \quad i = 1, \ldots, n$$
$$H(x, p) = p_i \dot{x}^i - L(x, \dot{x}) \tag{1.22}$$

(it is assumed that the equations $p_i = \partial L / \partial \dot{x}^i$ can be solved in the form $\dot{x}^i = \dot{x}^i(x, p)$).

Conversely, if $H(x, p)$ is a Hamiltonian, then we have a Lagrangian $L(x, \dot{x})$ defined from the equations

$$\dot{x}^i = \frac{\partial H(x, p)}{\partial p_i}, \quad L(x, \dot{x}) = p_i \dot{x}^i - H(x, p). \tag{1.23}$$

It is assumed that one can solve the equations $\dot{x}^i = \partial H / \partial p_i$ for the variables p_i.

Similar variational problems with higher derivatives [42]

$$\delta S = 0, \quad S = \int L(x, \dot{x}, \ldots, x^{(k)}) dt \tag{1.24}$$

can be brought into the form (1.20) with the aid of *M.V. Ostrogradskij's transformation*

$$q^j = x^{(j-1)}, \quad p_j = \sum_{s=0}^{k-j} (-1)^s \frac{d^s}{dt^s} \frac{\partial L}{\partial x^{(j+s)}}, \quad j = 1, \ldots, k; \qquad (1.25)^1$$

$$-H(q, p) = L - p_1 q^2 - \ldots - p_{k-1} q^k - p_k x^{(k)}, \qquad (1.26)^1$$

$$\dot{q}^j = \partial H / \partial p_j, \quad \dot{p}_j = -\partial H / \partial q^j, \quad j = 1, \ldots, k,$$

if the equations (1.25) can be solved uniquely in the form

$$x = x(q, p), \quad \dot{x} = \dot{x}(q, p), \ldots, x^{(2k-1)} = x^{(2k-1)}(q, p). \tag{1.27}$$

Type II. *Lie–Poisson brackets.* Let us now consider the second case in order of complexity, when the tensor h^{ij} is not constant, but depends linearly[2] on the coordinates (y)

$$h^{ij} = c_k^{ij} y^k, \quad c_k^{ij} = \text{const}. \tag{1.28}$$

Let us consider the set L of all linear functions on the phase space, which we shall denote by L^*. For the basis linear forms—the coordinates y^i—the bracket defines a "commutation" operation

$$[y^i, y^j] = c_k^{ij} y^k = \{y^i, y^j\}. \tag{1.29}$$

The requirements (1.3), (1.5) imply that the operation (1.29) turns the linear space L into a Lie algebra (S. Lie) whose dual space L^* is the phase space for the Poisson bracket (1.28). The bracket of this form was first examined by Lie [93]. It was rediscovered by F.A. Berezin [14] and used by A.A. Kirillov and B. Kostant [67] (in the less convenient language of symplectic manifolds) in the theory of

[1] Note that in these equations the q^j and p_j are *vectors*, and j essentially indexes the derivatives of different *orders* of the vector x, not the coordinates of x as in (1.20)–(1.23).

[2] The third case in order of complexity, when the tensor $h^{ij}(y)$ depends quadratically on y, is also very interesting and has recently begun to be studied [129].

infinite-dimensional representations of Lie groups. The bracket (1.28) is in general degenerate.

Example 1. The fundamental example of the Hamiltonian formalism of type 1 is the phase space T^*M—the space of covectors (with lowered indices) on a manifold M (the configuration space). On T^*M there are local coordinates x^i (on M) and conjugate momenta p_j (on the fibre) with Poisson brackets

$$\{x^i, x^j\} = \{p_i, p_j\} = 0, \quad \{x^i, p_j\} = \delta^i_j \tag{1.30}$$

and with the form

$$\Omega = dp_i \wedge dx^i. \tag{1.31}$$

Example 2. It is useful to consider also a Poisson bracket of the form (1.30) which in addition is distorted by an "external field" $F_{ij} = -F_{ji}(x)$:

$$\{x^i, x^j\} = 0, \quad \{x^i, p_j\} = \delta^i_j, \quad \{p_i, p_j\} = F_{ij}(x), \tag{1.32}$$

where the 2-form $F = F_{ij} dx^i \wedge dx^j$ is closed, $dF = 0$. The corresponding 2-form Ω has the form:

$$\Omega = dp_i \wedge dx^i + F_{ij} dx^i \wedge dx^j. \tag{1.33}$$

The equations of motion with a Hamiltonian $H(x, p)$ and the Poisson bracket (1.32) represent (for $n = 2, 3$) the equations of motion of a charged particle in the external magnetic field ($n = 2, 3$) F_{ij} (or electromagnetic field for $n = 4$). In a region where $F = dA$ the bracket (1.32) can be reduced to the standard form (1.30). As a rule one can reduce to the form (1.32) (globally) non-degenerate Poisson brackets on the space T^*M which satisfy the following requirement: any two functions f, g on the base space M (not depending on the variables on the fibre, which consists of all the covectors) have a vanishing Poisson bracket: $\{f, g\} = 0$ (see [112]).

Let us turn now to examples connected with the Lie–Poisson brackets.

Example 3. Let L be the Lie algebra of the rotation group SO(3). The Killing metric (W. Killing) on L is Euclidean and it allows us not to distinguish between L and L^* (all indices will be considered to be lower ones). The Poisson bracket of the basis functions M_i on L^* has the form:

$$\{M_i, M_j\} = \varepsilon_{ijk} M_k, \tag{1.34}$$

where

$$\varepsilon_{ijk} = c^{ij}_k = \begin{cases} \text{the signum of the permutation } (i, j, k), \text{ if} \\ \quad i, j, k \text{ are all different} \\ 0, \text{ if there is a pair of coinciding} \\ \quad \text{indices } i, j, k. \end{cases} \tag{1.35}$$

The function $M^2 = \sum M_i^2$ is such that

$$\{M^2, M_i\} = 0, \quad i = 1, 2, 3. \tag{1.36}$$

On the level surfaces $M^2 = \text{const}$ (spheres) the bracket (1.34) becomes non-degenerate. The Hamiltonian systems on L^* have the form:

$$\dot{M}_i = \{M_i, H(M)\}. \tag{1.37}$$

Let $\omega^i = \partial H/\partial M_i$; the Killing metric allows us not to distinguish between upper and lower indices. The equations (1.37) reduce to the form of the "*Euler equations*" (L. Euler)

$$\dot{M} = [M, \omega], \tag{1.38}$$

where the square brackets denote the commutator in L. (When $H = \frac{1}{2}(a_1 M_1^2 + a_2 M_2^2 + a_3 M_3^2)$ the equations (1.37) coincide with the equations of motion of a rigid body fixed at its centre of gravity). The derivation of the equations (1.38) is valid for all compact (and semisimple) Lie groups on which there is a Killing metric—a Euclidean (pseudo-Euclidean) metric on the Lie algebra which is invariant with respect to inner automorphisms

$$L \to gLg^{-1}, \tag{1.39}$$

where g is an element of the Lie group, and L is the Lie algebra. Such systems on the groups SO(N) are called the "many-dimensional analogue of a rigid body", in accordance with V.I. Arnol'd, if the Hamiltonian has the aspect of a quadratic form on the space of skew-symmetric matrices $M = (M_{ij})$, where

$$H(M) = \sum_{i<j} d_{ij} M_{ij}^2, \quad d_{ij} = q_i + q_j, \quad q_i > 0. \tag{1.40}$$

Example 4. With the Lie algebra L of the group $E(3)$ of motions of three-dimensional Euclidean space some important systems arising in hydrodynamics are connected. This algebra is no longer semisimple. On the phase space L^* there are 6 coordinates $\{M_1, M_2, M_3, p_1, p_2, p_3\}$ and the Poisson brackets

$$\{M_i, M_j\} = \varepsilon_{ijk} M_k, \quad \{M_i, p_j\} = \varepsilon_{ijk} p_k, \quad \{p_i, p_j\} = 0. \tag{1.41}$$

The bracket (1.41) possesses two independent functions $f_1 = \sum p_i^2$, $f_2 = \sum p_i M_i$ such that

$$\{f_q, M_i\} = \{f_q, p_i\} = 0, \quad q = 1, 2, \quad i = 1, 2, 3. \tag{1.42}$$

On the level surfaces $f_1 = p^2$, $f_2 = ps$ the bracket (1.41) is non-degenerate. The substitution $\sigma_i = M_i - (s/p) p_i$ sets up an isomorphism of these level surfaces with the tangent bundle T^*S^2 of the sphere, $\sum \sigma_i p_i = 0$. (We identify the tangent bundle with the cotangent bundle by means of the standard Riemannian metric on the sphere.) On these level surfaces the restriction of the Poisson bracket (1.41) is no longer degenerate (when $p \neq 0$). It turns out that the brackets which arise on T^*S^2 can be reduced globally to the form (1.32). The appropriate substitution (see [112]) has the form:

$$p_1 = p\cos\theta\cos\psi, \quad p_2 = p\cos\theta\sin\psi, \quad p_3 = p\sin\theta,$$
$$\sigma_1 = p_\psi\tan\theta\cos\psi - p_\theta\sin\psi, \quad \sigma_2 = p_\psi\tan\theta\sin\psi - p_\theta\cos\psi, \tag{1.43}$$
$$\sigma_3 = -p_\psi,$$

where $-\pi/2 \le \theta \le \pi/2, 0 \le \psi \le 2\pi, \sigma_i = M_i - sp^{-1}p_i$. It is easy to deduce from formula (1.43) that

$$\{\theta,\psi\} = \{p_\theta,\psi\} = \{p_\psi,\theta\} = 0, \quad \{\theta,p_\theta\} = \{\psi,p_\psi\} = 1, \quad \{p_\theta,p_\psi\} = s\cos\theta. \tag{1.44}$$

The corresponding 2-form Ω takes on the form (1.33),

$$\Omega = dp_\theta \wedge d\theta + dp_\psi \wedge d\psi + s\cos\theta\, d\theta \wedge d\psi = d\xi_i \wedge dy^i + F, \tag{1.45}$$

where $y^1 = \theta$, $y^2 = \psi$, $\xi^1 = p_\theta$, $\xi^2 = p_\psi$, $F = s\cos\theta\, d\theta \wedge d\psi$. The integral of the form F (and Ω) over the basis cycle $[S^2] \in H_2(T^*S^2) = \mathbb{Z}$ has the form

$$\iint_{S^2} F = \iint_{[S^2]} \Omega = 4\pi s = 4\pi f_2 f_1^{-1/2}. \tag{1.46}$$

Thus we obtain the standard Poisson bracket on T^*S^2, supplementarily distorted by an effective magnetic field F. When $s \ne 0$ the effective magnetic field is always different from zero and represents a "Dirac monopole" (non-quantized).

Let $H(M, p)$ be a Hamiltonian. Let us introduce the notation $u^i = \partial H/\partial p_i$, $\omega^i = \partial H/\partial M_i$. The Hamilton equations will assume the form of "*Kirchhoff's equations*" (P. Kirchhoff)

$$\dot{p} = [p, \omega], \quad \dot{M} = [M, \omega] + [p, u] \tag{1.47}$$

(the square brackets denote the vector product). The equations (1.47) coincide (for quadratic Hamiltonians $H(M, p)$) with Kirchhoff's equations for the motion of a rigid body in a fluid—in a fluid which is perfect, incompressible, and at rest at infinity [107]. The motion of the fluid itself is considered to be potential. In this case H is the energy, M and p are the total angular momentum and the momentum of the body-fluid system in a moving coordinate system rigidly connected with the body. The energy $H(M, p)$, quadratic in M, p and positive definite, can be given in the form

$$2H = \sum a_i M_i^2 + \sum b_{ij}(p_i M_j + M_i p_j) + \sum c_{ij} p_i p_j. \tag{1.48}$$

One can reduce to the form (1.47) the equations of motion of a rigid body with a fixed point in an axially symmetric force field with a potential $W(z)$. The corresponding Hamiltonian has the form:

$$H = \frac{1}{2}\sum a_i M_i^2 + W(l^i p_i), \tag{1.49}$$

where l^i is the constant vector giving the position of the centre of mass relative to the principal axes and the point of attachment. The quantities p_i here are dimensionless and do not have the physical meaning of momenta. They are the direction cosines of a unit vector, i.e. one always has $f_1 = p^2 = 1$. The equations of the dynamics of the spin in the A-phase of superfluid ^{3}He can also be reduced to the form (1.47) (see [112]).

On the surface $f_1 = p^2, f_2 = ps$ the Hamiltonians H of the form (1.48) or (1.49) can be written as follows in the variables (y, ξ):

$$H = \frac{1}{2} g^{ab}(y)\xi_a\xi_b + A^a(y)\xi_a + V(y). \tag{1.50}$$

Here for the Hamiltonian (1.48) we shall have

$$\sum a_i\sigma_i^2 = g^{ab}\xi_a\xi_b, \quad \sigma_i = M_i - sp^{-1}p_i, \tag{1.51}$$
$$A^a\xi_a = s\left(\sum a_i p_i p^{-1}\sigma_i\right) + p\sum b_{ij}(\sigma_i p_j p^{-1} + \sigma_j p_i p^{-1}), \tag{1.52}$$
$$2V = s^2\sum a_i p_i^2 p^{-2} + 2ps\sum b_{ij}p_i p_j p^{-2} + p^2\sum c_{ij}p_i p_j p^{-2}. \tag{1.53}$$

In view of homogeneity, the Hamiltonian H depends only on sp^{-1}. For the top (1.49), the Hamiltonian can also be written on the level surface $f_1 = 1, f_2 = s$ in the form (1.50), where the metric g^{ab} again has the form (1.51), but

$$A^a\xi_a = s\sum a_i\sigma_i p_i, \tag{1.54}$$
$$2V = s^2\sum a_i p_i^2 + 2W(l^i p_i). \tag{1.55}$$

Deduction ([112]). *The equations of the Kirchhoff type reduce to a system which is mathematically equivalent to a classical charged particle moving on the sphere S^2 with the Riemannian metric $g_{ab}(y)$, $g^{ab}g_{bc} = \delta^a_c$, in a potential field $U(y)$,*

$$U(y) = V(y) - \frac{1}{2} g_{ab}A^a A^b, \tag{1.56}$$

and also in an effective magnetic field $\tilde{F}_{ab}(y)$,

$$\tilde{F}_{12} = s\cos\theta - \partial_1 A_2 + \partial_2 A_1, \quad A_a = g_{ab}A^b, \quad s = f_2 f_1^{-1/2}. \tag{1.57}$$

The form $A_a dy^a$ is globally defined on the sphere S^2, therefore

$$\iint_{S^2} \tilde{F}_{12}\, d\theta \wedge d\psi = \iint_{S^2} F = 4\pi s \tag{1.58}$$

by virtue of (1.46).

Remark. It has recently become clear that on SO(4) there arise systems which in certain cases describe the motion of a rigid body with cavities filled with a fluid. The integrable cases here were found by V.A. Steklov [133] and have been rediscovered in a number of modern papers (see, for example, [19], [138]).

A number of other applications of Euler equations on Lie algebras in problems of mathematical physics have been found just lately by O.I. Bogoyavlenskij together with the integrable cases in these problems (see [18]).

Now let us consider infinite-dimensional examples of phase spaces—spaces of fields $u=(u^1(x), \ldots, u^n(x))$ of some type, where $x=(x^1, \ldots, x^m)$ is one of the indices in the formulas. The Poisson bracket is given by a matrix

$$\{u^i(x), u^j(y)\} = h^{ij}(x, y) \tag{1.59}$$

of functions $h^{ij}(x, y)$ (generalized functions), which in general depend on the fields. For two "functions" (functionals) $F[u]$, $G[u]$ the Poisson bracket can be computed by the formula

$$\{F, G\} = \iint \frac{\delta F}{\delta u^i(x)} h^{ij}(x, y) \frac{\delta G}{\delta u^j(y)} d^m x d^m y. \tag{1.60}$$

Here $\delta F/\delta u^i(x)$ are variational derivatives, defined by the equalities

$$\delta F = \int \frac{\delta F}{\delta u^i(x)} \delta u^i(x) d^m x. \tag{1.61}$$

Example 1. The *local field-theoretic brackets* of Lagrangian variational problems. There are two sets of fields $u=(q^1(x), \ldots, q^n(x), p_1(x), \ldots, p_n(x))$ with pairwise Poisson brackets of the form

$$\begin{gathered} \{q^i(x), q^j(y)\} = \{p_i(x), p_j(y)\} = 0, \\ \{q^i(x), p_j(y)\} = \delta^i_j \delta(x-y), \quad i, j = 1, \ldots, n. \end{gathered} \tag{1.62}$$

The Poisson bracket of two functionals F and G has the form:

$$\{F, G\} = \int \left[\frac{\delta F}{\delta q^i(x)} \frac{\delta G}{\delta p_i(x)} - \frac{\delta F}{\delta p_i(x)} \frac{\delta G}{\delta q^i(x)} \right] d^m x. \tag{1.63}$$

Hamilton's equations can be written in the form

$$\begin{gathered} \dot{q}^i(x) = \{q^i(x), \mathscr{H}\} = \frac{\delta \mathscr{H}}{\delta p_i(x)}, \\ \dot{p}_i(x) = \{p_i(x), \mathscr{H}\} = -\frac{\delta \mathscr{H}}{\delta q^i(x)}, \end{gathered} \tag{1.64}$$

where $\mathscr{H} = \mathscr{H}[p, q]$ is the Hamiltonian. They arise, in particular, from the field variational principle

$$\frac{\delta S}{\delta q^i} \equiv \frac{\partial \Lambda}{\partial q^i} - \partial_x \frac{\partial \Lambda}{\partial q^i_x} - \partial_t \frac{\partial \Lambda}{\partial q^i_t} = 0, \quad i = 1, \ldots, n, \tag{1.65}$$

$$S = \int dt \int d^m x \Lambda(q, q_x, q_t), \tag{1.66}$$

where $\Lambda(q, q_x, q_t)$ is the density of the Lagrangian, with the aid of the field-

theoretic version of the Legendre transformation

$$p_i = \frac{\partial \Lambda}{\partial q^i_t}, \quad \mathscr{H} = \int d^m x (p_i(x) q^i_t(x) - \Lambda) \tag{1.67}$$

(it is assumed, just as above in the finite-dimensional case, that the equations $p_i = \partial \Lambda(q, q_x, q_t)/\partial q^i_t$ can be solved for q^i_t).

One can also consider, by analogy with the finite-dimensional case, the distortion of the brackets (1.62) by a "magnetic field"—a closed 2-form on the space of fields $q(x)$. Let us analyze an example connected with the inclusion of "external fields" in the theory of *chiral fields.* As is well-known (see, for example, [115]), the definition of a nonlinear chiral field is as follows: one has arbitrary Riemannian manifolds N^q and M^n; let there be defined a functional $S_0(f)$ on the mappings $f: N^q \to M^n$. The functional $S_0(f)$ has the form of a Dirichlet functional, quadratic in the derivatives of the mapping f, possibly with some additional terms. Thus the standard "chiral Lagrangian" for a principal chiral field, where $M^n = G$ is a Lie group with a two-sided invariant metric, has the form:

$$S_0(f) = \frac{1}{2} \int_{N^q} \operatorname{tr}(g^{\mu\nu} A_\mu A_\nu) \sqrt{g} d^q y, \tag{1.68}$$

where $g_{\mu\nu}$ is the metric of N^q, $A_\mu = f^{-1}(y) \partial f(y)/\partial y^\mu$.

If the manifold N^q is presented in the form of a product $N^q = P^{q-1} \times \mathbb{R}$, where $\mathbb{R}$ is the axis of the time t and $y=(x, t)$ (for example, N^q is the Minkowski space $N^q = \mathbb{R}^{q-1,1}$), then the Euler–Lagrange equations for the action (1.68) can be brought into the Hamiltonian form by means of the Legendre transformation (1.67), $q(x) = f(x)$.

Now let us define the procedure for including an external field. Let us note beforehand that an arbitrary differential form ω of degree $q+r$ on the manifold M^n defines a differential r-form Ω_r on the space of mappings $\{N^q \xrightarrow{f} M^n\}$ via the formula

$$\Omega_r(\delta_1 f, \ldots, \delta_r f)|_f = \int_{N^q} f^*(i_{\xi_1} \ldots i_{\xi_r} \omega), \tag{1.69}$$

where

$$\xi_k(y) = \delta_k f(y), \quad k = 1, \ldots, r \tag{1.70}$$

are "tangent vectors" to the space of mappings (vector fields on M^n at the points $f(y)$), $i_\xi \omega$ is the inner product of the form $\omega = (\omega_{i_1 \ldots i_{q+r}})$ with the vector $\xi = (\xi^i)$,

$$(i_\xi \omega)_{i_2 \ldots i_{q+r}} = \xi^i \omega_{i i_2 \ldots i_{q+r}}. \tag{1.71}$$

If the form ω is closed, $d\omega = 0$, then the form Ω_r on the infinite-dimensional space of mappings is also closed [112].

On M^n let us fix a closed $(q+1)$-form ω (the "external field"). Then it defines a closed 1-form Ω_1 on the space of mappings $N^q \xrightarrow{f} M^n$ in the way cited above. The closed 1-form

$$\delta S = \delta S_0 + \Omega_1, \tag{1.72}$$

where the functional S_0 is of the type (1.68), defines a so-called "*multi-valued functional*" S of the chiral field f in the external field ω [112]. The extremals of this functional can be determined, as usual, from the Euler–Lagrange equations (L. Euler–J.L. Lagrange)

$$\delta S = 0. \tag{1.73}$$

It turns out that for $N_y^q = P_x^{q-1} \times \mathbb{R}_t$ the inclusion of an external field is equivalent to distortion of the Poisson brackets by a "magnetic field" F—a closed 2-form on the space of fields $\{P^{q-1} \xrightarrow{f} M^n\}$ —without changing the Hamiltonian. This 2-form $F = \Omega_2$ can be defined via (1.69) with N^q replaced by P^{q-1}.

Example 2. More generally, the field-theoretic brackets (1.59) are called local if the generalized functions $h^{ij}(x, y)$ present themselves as finite sums of the delta function $\delta(x-y)$ and its derivatives with coefficients which depend on the values of the field variables and their derivatives at the points x, y. For these brackets and for local Hamiltonians of the form

$$\mathscr{H} = \int h(u, u_x, \ldots, u^{(s)}) d^m x \tag{1.74}$$

the Hamilton equations $\dot{u} = \{u, \mathscr{H}\}$ can be written in the form of partial differential equations.

Important Example. The case $m=1$, $n=1$. Here one has a bracket (the C. Gardner–V.E. Zakharov–L.D. Faddeev bracket) which arises in the theory of the *Korteweg–de Vries equation* (D.J. Korteweg–G. de Vries) (KdV)

$$\{u(x), u(y)\} = \delta'(x-y). \tag{1.75}$$

The Poisson bracket of two functionals has the form:

$$\{F, G\} = \int \frac{\delta F}{\delta u(x)} \frac{\partial}{\partial x} \frac{\delta G}{\delta u(x)} dx. \tag{1.76}$$

The skew-symmetry of the brackets (1.75), (1.76) is obvious; the correctness of the Jacobi identity follows from the fact that the "tensor" h^{ij} is constant here (it does not depend on the field variables). The bracket (1.75) is degenerate; the functional $I_{-1} = \int u dx$ has vanishing bracket with any other functional F:

$$\{F, I_{-1}\} = 0. \tag{1.77}$$

On a subspace $I_{-1} = \int u dx = c$ (for example, $c=0$) the bracket (1.75), (1.76) is no longer degenerate. The KdV equation itself is given by the Hamiltonian

$$I_1 = \mathscr{H} = \int \left(\frac{u_x^2}{2} + u^3 \right) dx, \tag{1.78}$$

$$u_t = \frac{\partial}{\partial x} \frac{\delta \mathscr{H}}{\delta u(x)} = 6uu_x - u_{xxx}. \tag{1.79}$$

The quantity

$$I_0 = \int \frac{u^2}{2} dx, \quad \{u(x), I_0\} = u_x(x), \tag{1.80}$$

plays the rôle of the momentum (the generator of the translations in x). It is curious that one of the manifestations of the integrability of the KdV equation (by the inverse scattering method, see chap. 2 below) is the presence of another local bracket [94] of the form

$$\{F, G\} = \int \frac{\delta F}{\delta u(x)} A \frac{\delta G}{\delta u(x)} dx, \qquad A = -\frac{d^3}{dx^3} + 2\left(u\frac{d}{dx} + \frac{d}{dx} u \right). \tag{1.81}$$

There is even a family of brackets: one can replace the operator A by $A + \lambda(d/dx)$ (λ is an arbitrary constant). In the new Hamiltonian structure the KdV itself has the form

$$u_t = A \frac{\delta I_0}{\delta u(x)}. \tag{1.82}$$

Example 3. Now let us consider continuous examples of the Lie–Poisson brackets, connected with infinite-dimensional Lie algebras L. The starting point for the subsequent constructions will be the Lie algebra L of vector fields on an m-dimensional space. The commutator of two fields $v^i(x)$, $w^i(x)$ has the form:

$$[v, w]^i(x) = v^j(x) \frac{\partial w^i(x)}{\partial x^j} - w^j(x) \frac{\partial v^i(x)}{\partial x^j}. \tag{1.83}$$

The rôle of the index is played here by pairs (x, i)—a point x and an index i. The operation (1.83) should be written in terms of "structure constants" in the form

$$[v, w]^i(x) = \int d^m y\, d^m z\, c^i_{jk}(x, y, z) v^j(y) w^k(z). \tag{1.84}$$

By comparing (1.83) with (1.84) we obtain

$$c^i_{jk}(x, y, z) = \delta^i_j \delta(z-x) \partial^{(y)}_k \delta(y-z) - \delta^i_k \delta(y-x) \partial^{(z)}_j \delta(z-y), \tag{1.85}$$

$$\partial^{(x)}_j = \frac{\partial}{\partial x^j}, \quad \int f(z) \partial^{(z)}_j \delta(z-x) d^m z = -\frac{\partial f(x)}{\partial x^j}. \tag{1.86}$$

The variables $p_i(x)$ conjugate to the velocity components, on the dual space L^* to the vector fields $v^i(x)$, must be such that the quantity

$$\int p_i(x) v^i(x) d^m x \tag{1.87}$$

is scalar with respect to change of variables. This means that the variables $p_i(x)$ are covector densities, which under changes of variables are additionally multiplied by the Jacobian determinant (we shall call them momentum densities).

By (1.85) the Poisson bracket has the form:

$$\{p_j(y), p_k(z)\} = \int c^i_{jk}(x, y, z)\, p_i(x) d^m x$$
$$= p_k(y) \partial^{(y)}_j \delta(y-z) - p_j(z)\, \partial^{(z)}_k \delta(z-y). \quad (1.88)$$

In the important special case $m=1$ we get

$$\{p(y), p(z)\} = p(y)\delta'(y-z) - p(z)\delta'(z-y). \quad (1.89)$$

The substitution $p=u^2$ reduces this bracket to the bracket (1.75).

In the algebra L of vector fields on Euclidean space (where there is a distinguished Euclidean metric and the volume element is the mass density, which is considered to be constant) a subalgebra L_0 of divergence-free fields

$$\partial_i v^i = 0 \quad (1.90)$$

is given. The dual space L_0^* can be obtained by factoring by the gradients

$$L_0^* = L^*/(\partial_i \phi). \quad (1.91)$$

In other words, momentum densities $p_i(x)$ give trivial linear forms on L_0 if $p_i(x) = \partial_i \phi(x)$:

$$\int p_i v^i d^m x = \int v^i \partial_i \phi d^m x = -\int \phi \partial_i v^i d^m x = 0. \quad (1.92)$$

The Euler equations for the hydrodynamics of a perfect incompressible fluid can be written as a Hamiltonian system [7], [112] on the space L_0^* with the Hamiltonian

$$H = \int \frac{\rho v^2}{2} d^m x, \quad \rho = \text{const}, \quad \partial_i v^i = 0, \quad p_i = \rho v^i \quad (1.93)$$

and the Poisson brackets (1.88). One always writes these equations on the full space L^*, which is equivalent to the space of velocities in the given case

$$\begin{cases} \rho v^i_t = \{p_i, H\} + \partial_i p, \\ \partial_i v^i = 0. \end{cases} \quad (1.94)$$

The terms $\partial_i p$ have arisen because of the transition from L_0^* to the space L^*, where quantities of the form $\partial_i p$ are equivalent to zero. The pressure p is only defined up to a constant here. The Poisson bracket on the space L_0^* may be written in the form

$$\{v_i(x), v_j(y)\} = \frac{1}{\rho}(\partial_i v_j - \partial_j v_i)\delta(x-y),$$
$$p_i = \rho v_i, \quad \rho = \text{const}. \quad (1.95)$$

The Hamiltonian formalism for a perfect compressible fluid cannot be realized on the algebra L; it represents a special case of the Hamiltonian formalism for fluids with internal degrees of freedom. Even the ordinary compressible fluid has such internal degrees of freedom—the mass density ρ and the entropy density s, whose inclusion requires the extension of the Lie algebra L of vector fields. Besides the vector fields v^i, we shall add another pair of fields v^ρ and v^s with commutators of the form

$$[(v, v^\rho, v^s), (w, w^\rho, w^s)] = ([v, w], v^i \partial_i w^\rho - w^i \partial_i v^\rho, v^i \partial_i w^s - w^i \partial_i v^s). \quad (1.96)$$

We shall denote the algebra (1.96) by $L_{\rho,s}$. The corresponding variables in the dual space $L^*_{\rho,s}$ we shall denote by ρ (the mass density) and s (the entropy density). The Poisson brackets in $L^*_{\rho,s}$ have the form:

$$\{p_i(x), \rho(y)\} = \rho(x)\partial_i \delta(y-x),$$
$$\{p_i(x), s(y)\} = s(x)\partial_i \delta(y-x),$$
$$\{\rho(x), \rho(y)\} = \{s(x), s(y)\} = \{\rho(x), s(y)\} = 0, \quad (1.97)$$
$$\{v_i(x), v_j(y)\} = \frac{1}{\rho}(\partial_i v_j - \partial_j v_i)\delta(x-y),$$

(the velocities are here the covectors $v_i = p_i \rho^{-1}$). The Hamiltonian $H = \int [p^2/2\rho + \varepsilon(\rho, s)] d^m x$ is just the energy. The quantities $M = \int \rho d^m x$ and $S = \int s d^m x$ have vanishing Poisson brackets (the trivial conservation laws). Essentially the Poisson brackets (1.97) were appropriately chosen so that mass and entropy would be transported together with the particles, in contrast to the energy, which is conserved only as a whole. Other examples of Lie–Poisson brackets which arise in hydrodynamics can be found in [112].

Example 4. General *brackets of hydrodynamic type*. The Poisson brackets and Hamiltonians considered in the previous example have the following properties:

1) The Hamiltonians have the form:

$$H = \int h(u) d^m x, \quad (1.98)$$

where the densities $h(u)$ depend only on the fields $u = (u^1, \ldots, u^n)$ and not on their derivatives.

2) The Hamilton equations

$$u^i_t(x) = \{u^i(x), H\} \quad (1.99)$$

are first-order quasilinear equations

$$u^i_t = v^i_j(u) u^j_x, \quad i = 1, \ldots, n. \quad (1.100)$$

The most general form for Poisson brackets which lead to the equations (1.100) for Hamiltonians (1.98) is as follows:

$$\{u^i(x), u^j(y)\} = g^{ij\alpha}(u(x)) \frac{\partial \delta(x-y)}{\partial y^\alpha}$$
$$+ b^{ij\alpha}_k(u(x)) \frac{\partial u^k(x)}{\partial x^\alpha} \delta(x-y). \quad (1.101)$$

The form of equations (1.100), the Hamiltonians (1.98) and the brackets (1.101) is invariant with respect to local changes of field variables

$$u=u(w). \tag{1.102}$$

Let us consider here the one-dimensional case $m=1$:

$$\{u^i(x), u^j(y)\}=g^{ij}(u(x))\delta_y(x-y)+b^{ij}_k(u(x))u^k_x(x)\delta(x-y). \tag{1.103}$$

In this case the symmetric matrix $g^{ij}(u)=g^{ji}(u)$ behaves under the changes (1.102) like a metric (with upper indices) on the space of fields u. If it is non-degenerate, then the quantities Γ^i_{jk}, defined by the equalities

$$b^{ij}_k=-g^{is}\Gamma^j_{sk}, \tag{1.104}$$

transform under the changes (1.102) like Christoffel symbols (see [48]). It turns out [48] that the expression (1.103) gives a Poisson bracket if and only if the connection Γ^i_{jk} is symmetric, compatible with the metric g^{ij} and has zero curvature. This means that by local coordinate changes (1.102), the metric g^{ij} can be reduced to the Euclidean (or a pseudo-Euclidean) one, the connection to zero, and the bracket (1.103), by the same token, to a constant one:

$$\{w^i(x), w^j(y)\}=\pm\delta^{ij}\delta'(x-y). \tag{1.105}$$

It should be noted that the natural "physical" variables u^i in which the equations (1.100) and the brackets (1.103) arise are essentially "curvilinear", i.e. the metric $g^{ij}(u)$ is nontrivial in the coordinates u^i.

In the multidimensional case $m>1$ a family of metrics $g^{ij\alpha}$, $\alpha=1, \ldots, m$ actually arises. If they are non-degenerate, then the connections $\Gamma^{i\alpha}_{jk}$, where $b^{ij\alpha}_k=-g^{is\alpha}\Gamma^{j\alpha}_{sk}$, are compatible with these metrics, symmetric, and have zero curvature. However all the metrics $g^{ij\alpha}$ cannot as a rule be reduced to a constant form by a single transformation. The obstruction to such a reduction are the tensors $T^{ijk\alpha\beta}=b^{ij\alpha}_l g^{lk\beta}-b^{kj\beta}_l g^{li\alpha}$. For example, for the brackets (1.88) when $m>1$ such a reduction is impossible. Let us also note that for $m>2$ the metrics $g^{ij\alpha}=p_s(\delta^{is}\delta^{j\alpha}+\delta^{js}\delta^{i\alpha})$ corresponding to the brackets (1.88) are always degenerate. For non-degenerate metrics $g^{ij\alpha}$ the conditions under which the expression (1.101) gives a Poisson bracket can be written as a set of relations on the tensors $T^{ijk\alpha\beta}$ that we shall not discuss here (see [49]).

§2. Integrals and Reduction of the Order of Hamiltonian Systems. Systems with Symmetry

A function $F(y)$ is called an *integral of the Hamiltonian system* (1.8) if its bracket with the Hamiltonian $H(y)$ is equal to zero:

$$\{F, H\}=0. \tag{1.106}$$

Taking (1.11) into account, we get: the quantity F is conserved along the trajectories of the Hamiltonian system (1.8). In particular, the Hamiltonian H itself (if it does not depend on time) is always a conserved quantity. The trajectories of the system (1.8) lie entirely on a level surface $F=\text{const}$. If the Poisson bracket is degenerate, then there are always "trivial" integrals (1.13), which commute with any Hamiltonian. We have looked at an example of the reduction of the Hamiltonian formalism with the help of trivial integrals in § 1 in connection with equations of the Kirchhoff type. Hamiltonian systems with one degree of freedom ($N=2$) with a time-independent Hamiltonian can always be integrated by quadratures. The presence of non-trivial integrals when $N>2$ which do not depend on the "energy" H[3] allows one to reduce the order of the Hamiltonian system (1.8) by two all at once. Let us give the appropriate construction (see also vol. 3 of the current publication and chap. 3, § 3 of the article by V.I. Arnol'd and A.B. Givental'). Let $F(y)$ be an integral of a Hamiltonian system with Hamiltonian H, where the vector $(\xi^i_F(y))=(h^{ij}(y)(\partial F(y)/\partial y^j)$ is independent of ξ_H. Let us consider a level surface M_c:

$$F(y)=c, \tag{1.107}$$

and on it the Hamiltonian flow defined by the Hamiltonian $F(y)$,

$$y^i_\tau=\{y^i, F(y)\}, \quad i=1, \ldots, N. \tag{1.108}$$

The flow (1.8) with Hamiltonian H permutes the trajectories of the flow (1.108) by virtue of the commutation (1.106), and therefore defines a dynamical system on the set of trajectories of the flow (1.108). The trajectories of the flow (1.108) lying on the level M_c are "indexed" by the points of a surface M^0_c (in general defined locally) transversal to these trajectories. Let us define the "reduction operation" of the original Poisson bracket $h^{ij}(y)$ onto M^0_c. Let us consider the subalgebra of all functions $z(y)$ which commute with $F(y)$,

$$\{F(y), z(y)\}=0. \tag{1.109}$$

Let the independent functions $z^1(y), \ldots, z^{N-2}(y)$ satisfy (1.109) and not functionally depend on $F(y)$. They are constant along the trajectories of the flow (1.108) and together with $F(y)$ and τ they define local coordinates in a neighbourhood of the transverse surface M^0_c. By the same token, the quantities $z^1, \ldots, z^{N-2}$ give local coordinates on the transverse surface M^0_c. We obviously have

$$\{\tau, F\}=1, \quad \{\tau, z^q\}=f^q(z, F), \quad \{z^p, z^q\}=\tilde{h}^{pq}(z, F). \tag{1.110}$$

Therefore we may impose on the choice of coordinates $z^1, \ldots, z^{N-2}$ the useful

[3] With the aid of the energy integral H the order of the system can also be reduced by two, but the Hamiltonian of the reduced system will depend explicitly on time [7].

additional conditions

$$\{\tau, z^q\}=0, \quad q=1, \ldots, N-2. \tag{1.111}$$

The reduced Poisson bracket on M_c^0 has by definition the form

$$\{z^p, z^q\}_{\text{red}}=\{z^p(y), z^q(y)\}=\tilde{h}^{pq}(z, c), \quad p, q=1, \ldots, N-2. \tag{1.112}$$

Obviously, the right-hand side depends only on the coordinates on M_c^0 (and on c), and does not depend on the choice of the surface M_c^0. By virtue of (1.106) the Hamiltonian has the form

$$H(y)=\tilde{H}(z^1, \ldots, z^{N-2}, F), \tag{1.113}$$

and therefore the original Hamiltonian system has a well-defined restriction to M_c^0:

$$\dot{z}^q=\{z^q, \tilde{H}(z, c)\}_{\text{red}}, \quad q=1, \ldots, N-2. \tag{1.114}$$

Thus, integration of the original system (1.8) is reduced to the integration of the Hamiltonian system (1.114), whose order has been lowered by two. After this the dependence of the coordinate τ on time can be determined from the equation (taking (1.110), (1.111) into account)

$$\dot{\tau}=\{\tau, \tilde{H}(z, F)\}=\frac{\partial \tilde{H}(z, F)}{\partial F} \tag{1.115}$$

(by one quadrature).

Whether it is possible to carry out the reduction procedure globally requires a supplementary investigation. It is sufficient, for example, to suppose that c is a regular value of the function $F(y)$ and the one-parameter group G_τ of translations along the trajectories of the system (1.108) is compact and has no fixed points. In the practical realization of the procedure described above the main difficulty lies in the construction of the "transversal" coordinates $z^1, \ldots, z^{N-2}$.

Example 1. Let $H=H(x, p)$ be a Hamiltonian on the phase space $\mathbb{R}^{2n}$ with canonical coordinates $(x^1, \ldots, x^n, p_1, \ldots, p_n)$ of the form (1.19). Let us suppose that $H(x, p)$ is invariant with respect to "spatial translations"

$$x^i \mapsto x^i+a, \quad p_i \mapsto p_i, \tag{1.116}$$

i.e.

$$\sum_{i=1}^n \frac{\partial H}{\partial x^i}=0. \tag{1.117}$$

For this it is sufficient that the Hamiltonian have the form

$$H(x^1, \ldots, x^n, p_1, \ldots, p_n)=\hat{H}(x^1-x^n, \ldots, x^{n-1}-x^n, p_1, \ldots, p_n). \tag{1.118}$$

Then, obviously, the quantity ("total momentum")

$$F=\sum_{i=1}^n p_i \tag{1.119}$$

commutes with H, $\{H, F\}=0$. The coordinates $z=(z^1, \ldots, z^{2n-2})=(\tilde{x}^q, \tilde{p}_q)$ on the reduced phase space have the form

$$\begin{aligned} \tilde{x}^q&=x^q-x^n, \quad q=1, \ldots, n-1, \\ \tilde{p}_q&=p_q, \quad q=1, \ldots, n-1. \end{aligned} \tag{1.120}$$

The reduced Hamiltonian $\tilde{H}(\tilde{x}, \tilde{p}; c)$ on the surface

$$p_1+\ldots+p_n=c \tag{1.121}$$

has the form

$$\tilde{H}(\tilde{x}, \tilde{p}; c)=\hat{H}\left(\tilde{x}^1, \ldots, \tilde{x}^{n-1}, \tilde{p}_1, \ldots, \tilde{p}_{n-1}, c-\sum_{q=1}^{n-1} \tilde{p}_q\right). \tag{1.122}$$

The reduced brackets are the canonical ones:

$$\begin{gathered} \{\tilde{x}^q, \tilde{x}^r\}_{\text{red}}=\{\tilde{p}_q, \tilde{p}_r\}_{\text{red}}=0, \quad \{\tilde{x}^q, \tilde{p}_r\}=\delta_r^q, \\ q, r=1, \ldots, n-1, \end{gathered} \tag{1.123}$$

and the original Hamiltonian system reduces to the system on $\mathbb{R}^{2n-2}$

$$\dot{\tilde{x}}^q=\frac{\partial \tilde{H}}{\partial \tilde{p}_q}, \quad \dot{\tilde{p}}_q=-\frac{\partial \tilde{H}}{\partial \tilde{x}^q}, \quad q=1, \ldots, n-1. \tag{1.124}$$

The dependence of the quantity $\tau=x^n$ on the time t can be found from the equation

$$\dot{\tau}=\tilde{H}_c. \tag{1.125}$$

Now let us suppose that the Hamiltonian system (1.8) with the Hamiltonian H possesses several integrals. Let us note, first of all, a simple but important assertion: the integrals of the system (1.8) form a subalgebra with respect to the Poisson bracket. The proof is obvious from the Jacobi identity (1.5).

The presence of nontrivial pairwise commuting integrals $F_1(y), \ldots, F_k(y)$,

$$\{F_i, H\}=0, \quad \{F_i, F_j\}=0, \quad i, j=1, \ldots, k, \tag{1.126}$$

allows one, according to the scheme described above, to reduce the order of the Hamiltonian system by $2k$. In particular, if the initial Poisson bracket was non-degenerate, $N=2n$, then the presence of n pairwise commuting integrals for the Hamiltonian system allows one, in principle, to integrate this system by quadratures. We shall discuss the properties and examples of such systems in § 3.

A set of non-commuting integrals also allows one to reduce the order of the original Hamiltonian system; however, here the reduction algorithm is more complicated. Let us first analyze a simple example.

Example 2. Let $H(x, p)=(|p|^2/2m)+U(|x|)$ be a spherically symmetric Hamiltonian on $\mathbb{R}^6$, $x=(x_1, x_2, x_3)$, $p=(p_1, p_2, p_3)$. Here one has the three "angular momentum integrals"

$$M_1 = x_2 p_3 - x_3 p_2, \quad M_2 = x_3 p_1 - x_1 p_3, \quad M_3 = x_1 p_2 - x_2 p_1 \tag{1.127}$$

with pairwise brackets

$$\{M_i, M_j\} = \varepsilon_{ijk} M_k. \tag{1.128}$$

(In fact, the whole angular momentum vector

$$M = [x, p] \tag{1.129}$$

is conserved.) Here there are three integrals, but because of their non-commutativity the reduced phase space will have dimension 2. Let us fix a value of the angular momentum

$$M = m \neq 0. \tag{1.130}$$

Without loss of generality we may assume that $m = (\mu, 0, 0)$, and the conditions (1.130) can be written in the form

$$x_2 p_3 - x_3 p_2 = \mu, \quad x_3 p_1 - x_1 p_3 = 0, \quad x_1 p_2 - x_2 p_1 = 0. \tag{1.131}$$

From the last two equations it follows when $\mu \neq 0$ that $x_1 = p_1 = 0$. i.e. the motion takes place in the (x_2, x_3) plane. The flow with the Hamiltonian M_1, which represents a rotation in the (x_2, x_3) and (p_2, p_3) planes by the same angle, thus acts on the three-dimensional surface $x_1 = p_1 = 0$, $x_2 p_3 - x_3 p_2 = \mu$. If we factor by this flow, we obtain the desired reduced phase space. For the factorization it is most convenient to use polar coordinates r, ϕ in the (x_2, x_3) plane, putting

$$x_2 = r\cos\phi, \qquad x_3 = r\sin\phi \tag{1.132}$$

and introducing the conjugate momenta

$$\begin{aligned} p_2 &= p_r \cos\phi - \frac{p_\phi}{r}\sin\phi, \\ p_3 &= p_r \sin\phi + \frac{p_\phi}{r}\cos\phi. \end{aligned} \tag{1.133}$$

Then r, p_r serve as canonical coordinates on the reduced phase space; the reduced Hamiltonian has the form

$$\tilde{H}(r, p_r) = \frac{p_r^2}{2m} + \frac{\mu^2}{2mr^2} + U(r). \tag{1.134}$$

The dependence of ϕ on time is obtained separately from the equation $p_\phi = \mu$, from which we get

$$\dot{\phi} = \frac{\mu}{mr^2} = \frac{\partial H}{\partial p_\phi}. \tag{1.135}$$

The construction of example 2 admits of obvious generalizations, which go back to Jacobi and H. Poincaré and have been formulated in the language of symplectic manifolds in a number of works in the last decades (see also vol. 3 of the present publication and chap. 3, §3 of the article by V.I. Arnol'd and A.B. Givental').

Suppose the Hamiltonian system (1.8) possesses integrals $F_1, \ldots, F_r$ whose pairwise brackets can be expressed as linear combinations of these same functions,

$$\{F_i, F_j\} = c_{ij}^k F_k, \tag{1.136}$$

the coefficients c_{ij}^k being constants (this is the next case in order of complexity after commuting integrals; compare the formulas (1.128) of example 2). Thus the space of linear combinations

$$L = \{a^i F_i(y)\}, \quad a^1, \ldots, a^r \text{ are constants,} \tag{1.137}$$

is closed with respect to the Poisson bracket and for this reason forms a finite-dimensional Lie algebra. The functions $F_1(y), \ldots, F_r(y)$ form a basis of L, and the c_{ij}^k are the structure constants. Let G be the corresponding Lie group. Then G acts locally on the phase space by canonical transformations (ones which preserve Poisson brackets): the one-parameter subgroups of G which correspond to the basis vectors F_i of the Lie algebra L are the Hamiltonian flows

$$y_\tau^j = \{y^j, F_i\}, \quad j = 1, \ldots, N. \tag{1.138}$$

For a fixed y the collection of numbers $(F_1(y), \ldots, F_r(y)) = F(y)$ may be considered as the coordinates of a linear form on the Lie algebra L: if $(a^1, \ldots, a^r)$ is a vector in L, then

$$F(y)(a) = a^i F_i(y). \tag{1.139}$$

Thereby we have defined the *momentum mapping*

$$y \mapsto F(y) \in L^*. \tag{1.140}$$

Let us fix some element $c = (c_1, \ldots, c_r) \in L^*$ and let us consider the momentum level surface (the simultaneous level surface of the integrals $F_1, \ldots F_r$)

$$F(y) = c \leftrightarrow (F_1(y) = c_1, \ldots, F_r(y) = c_r). \tag{1.141}$$

Let us suppose that this surface M_c is a manifold. The Hamiltonian flow corresponding to the Hamiltonian $f_a(y) = a^i F_i(y)$ preserves the level surface M_c if the vector $a = (a^1, \ldots, a^r)$ satisfies the linear relations

$$\{f_a, F_j\}|_{M_c} = a^i c_{ij}^k c_k = 0, \quad j = 1, \ldots, r. \tag{1.142}$$

Such vectors a form a Lie subalgebra $L_c \subset L$. Let l be the dimension of this subalgebra; a basis of it is constituted by the functions

$$f_s(y) = a_s^i F_i(y), \quad s = 1, \ldots, l, \tag{1.143}$$

where (a_s^i) is a fundamental system of solutions of the equations (1.142). (The subgroup $G_c \subset G$ with the Lie algebra $L_c \subset L$ is just the isotropy subgroup of the element $c \in L^*$ in the coadjoint representation Ad*. In example 2 (above) the subgroup G_c coincided with the rotations about the $c=(\mu, 0, 0)$ axis.)

If we factor M_c by the action of the flows with the Hamiltonians f_a out of the subalgebra L_c, we obtain the reduced phase space M_c^0 (of codimension $r+l$). (All M_c is fibred (locally) over M_c^0 with fibre G_c). As coordinates on M_c^0 one may take functions $z^q = z^q(y)$ such that

$$\{z^q, F_j\}|_{M_c} = 0, \quad j=1, \ldots, r, \tag{1.144}$$

which do not depend on the functions $f_1, \ldots, f_l$ (1.143) (the gradients of the functions z^q and f_s must generate the whole tangent space to M_c). As above, we define the reduced brackets on M_c^0 by the equality

$$\{z^q, z^p\}_{\text{red}} = \{z^q(y), z^p(y)\}. \tag{1.145}$$

The Hamiltonian also restricts to M_c^0 in a well-defined way. We obtain the reduced Hamiltonian system

$$\dot{z}^q = \{z^q, \tilde{H}(z, c)\}_{\text{red}}, \quad q=1, \ldots, N-l-r. \tag{1.146}$$

It is clear that in the commutative case $c_{ij}^k = 0$ the subalgebra L_c coincides with the whole Lie algebra L, i.e. $l=r$ and the order of the system is reduced by $2r$.

We have not yet discussed the mechanisms by which integrals of Hamiltonian systems arise. The best known of these mechanisms is a *symmetry of the Hamiltonian system*, i.e. the presence of a continuous group G of canonical transformations of the phase space, preserving the Hamiltonian:

$$H(gy) = H(y), \quad g \in G. \tag{1.147}$$

Let L be the Lie algebra of the group G, let $e_1, \ldots, e_r$ be a basis of L, and let the commutators in L have the form:

$$[e_i, e_j] = c_{ij}^k e_k. \tag{1.148}$$

Each one-parameter subgroup $\exp(te_i)$ of transformations of the phase space has a Hamiltonian $F_i(y)$ (defined perhaps locally), i.e. the transformations $y \mapsto \exp(\tau e_i)y$ are translations along the trajectories of the system

$$y_\tau^j = \{y^j, F_i(y)\}, \quad j=1, \ldots, N. \tag{1.149}$$

The functions $F_i(y)$ are integrals of the Hamiltonian system (1.147). Indeed,

$$\{H(y), F_i(y)\} = \frac{d}{d\tau} H(\exp(\tau e_i)y)_{\tau=0} = 0.$$

The functions $F_i(y)$ are called *generators of the canonical action of G*.

A canonical *Lie group action* is called *Poisson* if the functions $F_i(y)$ are defined globally and their Poisson brackets have the form (1.136), where the c_{ij}^k are the structure constants (1.148) of the Lie algebra L.

Example. Let the phase space have the form of the cotangent bundle T^*M of a smooth n-dimensional manifold M with the standard brackets (1.30) and let G act on M as a group of diffeomorphisms. The corresponding action of G on T^*M is canonical. Let us construct the functions $F_i(y)$, where $y=(x, p)$ are the canonical coordinates on T^*M (locally). Let

$$X_i^k(x) = \frac{d}{d\tau}(\exp(\tau e_i)x)_{\tau=0}^k, \quad k=1, \ldots, n. \tag{1.150}$$

Let us put

$$F_i(x, p) = p_k X_i^k(x), \quad i=1, \ldots, r. \tag{1.151}$$

The functions $F_i(x, p)$ are defined globally on T^*M and are generators of the action of G. Their Poisson brackets, as is easy to see, have the form (1.136). Thus the action of G on T^*M is Poisson.

For an arbitrary phase space a canonical action of G might not be Poisson. In the first place, even if the Poisson bracket is non-degenerate, the integrals $F_i(y)$ might not be globally defined (and single-valued); only their differentials dF_i are well-defined. Let us suppose further that the functions F_i are defined globally (up to a constant). It is not difficult to show that then their Poisson brackets have the form:

$$\{F_i, F_j\} = c_{ij}^k F_k + b_{ij}, \tag{1.152}$$

where the c_{ij}^k are the structure constants of the Lie algebra L, and the $b_{ij} = -b_{ji}$ are certain constants. The skew-symmetric matrix b_{ij} defines a bilinear form on the Lie algebra L, $B(\xi, \eta) = b_{ij}\xi^i\eta^j$, which is a (two-dimensional) cocycle:

$$B([\xi, \eta], \zeta) + B([\zeta, \xi], \eta) + B([\eta, \zeta], \xi) = 0 \tag{1.153}$$

(a consequence of the Jacobi identity (1.5)). For the action of the group G to be Poisson it is necessary that the matrix b_{ij} should have the form (β_k are certain constants):

$$b_{ij} = \beta_k c_{ij}^k \tag{1.153'}$$

(the cocycle b_{ij} is cohomologous to zero). In this case, by substituting $F_j \mapsto F_j + \beta_j$ we obtain a Poisson action.

If the action of the group G on the phase space is Poisson, then the reduction procedure described above for the Hamiltonian formalism can be carried out globally under certain additional restrictions. It is sufficient, for example, to suppose that c is a regular value for the momentum mapping (1.140) (i.e., M_c is a manifold), the isotropy subgroup G_c of the element $c \in L^*$ with respect to the coadjoint representation Ad* is compact, and its elements act on M_c without fixed points. Thus, for the case T^*M, where the group G acts on M, the reduced phase space has the form $T^*(M/G)$, if of course the quotient manifold M/G is defined.

Example 3 ([112]). *A.J. Leggett's equations* for the dynamics of the "order parameters" in the B-phase of superfluid ^{3}He. In the state of hydrodynamical rest and with non-zero spin a state in the B-phase is defined by a pair—a rotation matrix $R=(R_{ij})\in SO(3)$ and $s=(s_i)$, $i=1, 2, 3$,—the "magnetic moment".

The variables s_i represent coordinates on the dual space to the Lie algebra of the group SO(3), analogously to the angular momenta M_i. In the variables (s_i, R_{jk}) the standard Poisson brackets on $T^*SO(3)$ are written thus:

$$\{s_i, s_j\}=\varepsilon_{ijk}s_k, \quad \{R_{ij}, R_{kl}\}=0, \quad \{s_i, R_{jl}\}=\varepsilon_{ijk}R_{kl}. \tag{1.154}$$

The Hamiltonian of the Leggett system in the B-phase and in an external magnetic field has the form:

$$H=\tfrac{1}{2}as^2+b\sum s_iF_i+V(\cos\theta), \tag{1.155}$$

where a, b are constants, $F=(F_i)$ is the external field,

$$V(\cos\theta)=\mathrm{const}(\tfrac{1}{2}+2\cos\theta)^2; \tag{1.156}$$

here R_{ij} is the rotation by the angle θ about the axis of the vector (n_i), $\sum n_i^2=1$:

$$R_{ij}=\cos\theta\,\delta_{ij}+(1-\cos\theta)n_in_j+\sin\theta\,\varepsilon_{ijk}n_k, \tag{1.157}$$

$$1+2\cos\theta=R_{ii}=\mathrm{tr}\,R. \tag{1.158}$$

After the substitution

$$as_i=\omega_i, \quad \Omega_{jk}=\varepsilon_{jki}\omega_i=(\dot{R}R^{-1})_{jk} \tag{1.159}$$

we will obtain a Lagrangian system in the variables $(R_{ij}, \dot{R}_{ij})$ on $T^*SO(3)$, where the kinetic energy is defined by a two-sided invariant Killing metric, and the potential $V(\cos\theta)$ is invariant with respect to inner automorphisms

$$R\mapsto gRg^{-1}, \quad s\mapsto gs, \quad g\in SO(3). \tag{1.160}$$

If the field $F=(F_i)$ is constant, then the whole Hamiltonian is invariant with respect to the one-parameter group of transformations (1.160), where g belongs to the group of rotations around the axis of the field F. Suppose, further, $F=(F, 0, 0)$.

With zero flux $F=0$ the system admits the group SO(3) of transformations (1.160) and is completely integrated in [95]. The transformations (1.160) generate the vector, conserved when $F=0$:

$$A=(A_j)=(1-\cos\theta)\left[n, \cot\frac{\theta}{2}S+[n, S]\right], \tag{1.161}$$

where the Poisson brackets are the same as for the ordinary angular momentum

$$\{A_i, A_j\}=\varepsilon_{ijk}A_k.$$
$$\{A_i, \tfrac{1}{2}as^2+V(\cos\theta)\}=0. \tag{1.162}$$

The variables s^2 and θ which enter into the Hamiltonian when $F=0$ generate the

closed Poisson bracket algebra $\{s^2, s_\|, \theta\}$ [112], where

$$s_\|=\sum s_in_i, \tag{1.163}$$

$$\{s^2, \theta\}=2s_\|, \quad \{s_\|, \theta\}=1, \quad \{s^2, s_\|\}=\frac{1+\cos\theta}{\sin\theta}(s^2-s_\|^2). \tag{1.164}$$

The quantity $A^2=\sum A_i^2=(1-\cos\theta)(s^2-s_\|^2)$ has zero Poisson bracket with everything in this subalgebra

$$\{A^2, s^2\}=\{A^2, s_\|\}=\{A^2, \theta\}=0. \tag{1.165}$$

In a non-zero magnetic field $(F, 0, 0)$ there remains only one integral (besides the energy)

$$\{A_1, H\}=0. \tag{1.166}$$

In the present case it proves to be possible to carry through the reduction procedure for the Hamiltonian formalism to the end (globally) and to reduce the system to two degrees of freedom.

The integral A_1 generates a group (1.160), where g is a rotation around the first axis, that of $n=(1, 0, 0)$. Variables which are invariant with respect to this subgroup are the following:

$$s^2, s_\|, \theta, n_1, s_1, \tau=s_2n_3-n_2s_3 \tag{1.167}$$

with a constraint of purely geometric origin

$$s^2\tau^2=(s^2-s_1^2)(s^2-s_\|^2)-(s^2n_1-s_1s_\|)^2. \tag{1.168}$$

It is not hard to work out that the variables (1.167) form a closed Poisson bracket algebra which contains the Hamiltonian H (1.155) and has functional dimension 5. The quantity A_1, which lies in this algebra, has vanishing bracket with all the variables

$$0=\{A_1, s^2\}=\{A_1, s_\|\}=\{A_1, \theta\}=\{A_1, n_1\}=\{A_1, s_1\}=\{A_1, \tau\}. \tag{1.169}$$

Therefore, if we impose the condition $A_1=\mathrm{const}$ it is possible as before to make use of the formulas for the Poisson brackets of the quantities (1.167) which follow from (1.164). With the condition $A_1=\mathrm{const}$, we shall choose the following as basis variables:

$$A^2, s_\|, \theta, n_1=n. \tag{1.170}$$

Their brackets have the form:

$$\begin{gathered}\{s_\|, \theta\}=1, \quad \{\theta, n\}=0,\\ \{A^2, s^2\}=\{A^2, \theta\}=\{A^2, s_\|\}=0,\\ \{A^2, n\}=\sqrt{\tfrac{1}{2}(1-n^2)A^2-\tfrac{1}{4}A_1^2}.\end{gathered} \tag{1.171}$$

Thus, the canonical variables can be chosen in the form

$$x^1=\theta, \quad p_1=p_\theta=s_\parallel, \qquad x^2=n, \quad p_2=p_n=\sqrt{\frac{2A^2}{1-n^2}-\frac{A_1^2}{(1-n^2)^2}}. \tag{1.172}$$

The Hamiltonian takes on the form:

$$H=\frac{1}{2}a\left[p_\theta^2+\frac{1-n^2}{2(1-\cos\theta)}\left(p_n^2+\frac{A_1^2}{(1-n^2)^2}\right)\right] +bF\left(np_\theta+\frac{1-n^2}{2}\sin\theta p_n+A_1^2\frac{2-\sin^2\theta}{2(1-\cos\theta)}\right)+V(\cos\theta). \tag{1.173}$$

Now let us introduce spherical coordinates

$$\theta=2\chi, \quad n=n_1=\sin\phi \tag{1.174}$$

and let us pass over to the Lagrangian formalism. We will obtain

$$L=2a(\dot\chi^2+\sin^2\chi\dot\phi^2)-\tilde A_1\dot y^1-\tilde A_2\dot y^2-U(y), \tag{1.175}$$

where $y^1=\chi$, $y^2=\phi$,

$$\tilde A_1=2b\sin\phi, \quad \tilde A_2=8bF\cos\phi\sin^3\chi\cos\chi,$$

$$U=V(\cos\theta)+\frac{aA_1^2}{4\sin^2\chi\cos^2\phi}+bF\frac{A_1(1-\sin^2\chi\cos^2\chi)}{2\sin^2\chi} -\frac{1}{2}b^2F^2(\sin^2\phi+4\cos^2\phi\sin^3\chi\cos\chi).$$

Thus we have obtained a system on a region of the sphere S^2 with the usual metric, where there is an effective magnetic field and a scalar potential. When $A_1\neq 0$ this system cannot be extended onto the whole sphere, since it has a singularity for $\phi=0, \pi$.

Now let us consider examples of continuous systems with spare integrals. Let us first consider the case of Lagrangian field-theoretic systems (1.65), described in the Hamiltonian form with the aid of the transformation (1.67). In the case when the density of the Lagrangian $\Lambda(q, q_x, q_t)$ does not depend explicitly on the space-time coordinates x^α, t, $\alpha=1, \ldots, m$, there hold laws of conservation of the total energy

$$\dot E=0, \quad E=\mathscr{H}=\int d^mx(p_iq_t^i-\Lambda) \tag{1.176}$$

and of the total momentum vector

$$\dot P_\alpha=0, \quad P_\alpha=\int p_i\frac{\partial q^i}{\partial x^\alpha}d^mx. \tag{1.177}$$

The functionals P_α are generators of the translations in the spatial variables, i.e.

$$\{p_i(x), P_\alpha\}=\frac{\partial p_i(x)}{\partial x^\alpha}, \quad \{q^i(x), P_\alpha\}=\frac{\partial q^i(x)}{\partial x^\alpha}, \tag{1.178}$$

$$\{P_\alpha, P_\beta\}=0, \quad \alpha, \beta=1, \ldots, m. \tag{1.179}$$

Let us stress that the generators of the spatial translations are local field integrals. The conservation laws (1.176), (1.177) are often written in infinitesimal form, by introducing the *energy–momentum tensor*

$$T_b^a=q_{x^b}^i\frac{\partial\Lambda}{\partial q_{x^a}^i}-\delta_b^a\Lambda, \tag{1.180}$$

where $a, b=0, 1, \ldots, m$, $x^0=t$. We have [142]

$$E=\int T_0^0d^mx, \quad P_\alpha=\int T_\alpha^0d^mx, \tag{1.181}$$

$$\frac{\partial T_b^a}{\partial x^a}=0, \quad b=0, 1, \ldots, m. \tag{1.182}$$

More generally, one can consider variational problems of the form

$$\delta S=0, \quad S=\int\Lambda(x, q, q_x)d^{m+1}x \tag{1.183}$$

(here, as above, we put $x=(x^a)$, $a=0, 1, \ldots, m$, $x^0=t$) which are invariant with respect to more general one-parameter groups of transformations $G_\tau(x, q)$ of the form

$$x_\tau^a=X^a(x), \quad a=0, 1, \ldots, m, \qquad q_\tau^i=Q^i(x, q), \quad i=1, \ldots, n. \tag{1.184}$$

To each such group there corresponds a "conserved current"

$$J^a=\Lambda X^a+\frac{\partial\Lambda}{\partial q_a^i}(Q^i-q_{x^b}^iX^b), \tag{1.185}$$

$$\frac{\partial J^a}{\partial x^a}=0 \tag{1.186}$$

(*Noether's theorem* (E. Noether) [17]). The quantity

$$\int_{x^0=\text{const}} J^0d^mx \tag{1.187}$$

is conserved. If the transformations (1.184) do not affect time, i.e. $X^0=\text{const}$, then they define a family of canonical transformations on the space of fields $(p(x), q(x))$, whose generator is a local field integral with the density

$$J^0=p_i(Q^i-q_{x^\alpha}^iX^\alpha). \tag{1.188}$$

The second Noether theorem concerns variational problems which admit symmetries with functional parameters (as, for example, in the theory of gauge fields [53]). The equations for the extremals (1.65) are not independent in this case, but satisfy some system of differential relations. We shall not discuss this theorem here.

§ 3. Liouville's Theorem. Action-Angle Variables

In this section we shall restrict ourselves to the consideration of phase spaces with a non-degenerate Poisson bracket. The important *theorem of Liouville* (J. Liouville) studies the case of Hamiltonian systems with n degrees of freedom (i.e. on a $2n$-dimensional phase space) where there are exactly n functionally independent integrals $F_1=H, F_2, \ldots, F_n$ whose pairwise Poisson brackets are equal to zero, $\{F_i, F_j\}=0, j=1, \ldots, n$. People often call such systems *completely integrable*. In this case the level surfaces of the integrals

$$F_1=c_1, \ldots, F_n=c_n \tag{1.189}$$

are quotient groups of $\mathbb{R}^n$ by lattices of finite rank $\leq n$; in particular, compact non-singular level surfaces are n-dimensional tori. If the level surface (1.189) is compact, then in a neighbourhood of it one can introduce coordinates $s_1, \ldots, s_n, \phi_1, \ldots, \phi_n$ $(0\leq\phi_i<2\pi)$ ("*action-angle*" *variables*) such that:

a)

$$\{s_i, s_j\}=\{\phi_i, \phi_j\}=0, \quad \{\phi_i, s_j\}=\delta_{ij}; \tag{1.190}$$

b) $s_i=s_i(F_1, \ldots, F_n)$, ϕ_j are coordinates on the level surfaces (1.189); c) in the coordinates (s_i, ϕ_j) the initial Hamiltonian system has the form:

$$\left.\begin{aligned} \dot{s}_i&=0,\\ \dot{\phi}_i&=\omega_i(s_1, \ldots, s_n),\end{aligned}\right\} \quad i=1, \ldots, n. \tag{1.191}$$

Let us give the idea of the proof of this theorem (see, for example, [42]). A level surface M_c of the form (1.189) is a smooth manifold by virtue of the independence of the integrals $F_1, \ldots, F_n$ (i.e. the independence of their "gradients" $(\xi_i)^j = h^{jk}(\partial F_i/\partial y^k)$. The group $\mathbb{R}^n$ of the flows with the Hamiltonians $F_1, \ldots, F_n$ acts on this manifold. Let us choose an initial point $x_0=x_0(c)\in M_c$ and let us pick out a lattice in $\mathbb{R}^n$: a vector $d\in\mathbb{R}^n$ belongs to the lattice if d, acting on x_0, yields x_0 again. A subgroup $\{d\}\subset\mathbb{R}^n$ arises. This subgroup is discrete and is therefore isomorphic to a lattice spanned by k vectors of $\mathbb{R}^n$, where $k\leq n$. Obviously, only for $k=n$ will we obtain a compact manifold (a torus T^n).

Now let us construct the action-angle variables. On the given level surface M_c one may put together linear combinations of the fields ξ_i:

$$\eta_i=b_i^j\xi_j, \quad i=1, \ldots, n, \tag{1.192}$$

such that the coordinates introduced with their help on the group $\mathbb{R}^n$ acting on

the torus $T^n=M_c$ coincide with the angles $0\leq\phi_j<2\pi$ ($\phi_i=0$ is just the point x_0). The coefficients b_i^j will depend on the collection $c_1, \ldots, c_n$ in a neighbourhood of the chosen level surface. Thus we have

$$\eta_i=b_i^j(F_1, \ldots, F_n)\xi_j. \tag{1.193}$$

This introduces coordinates $\tilde{\phi}_1, \ldots, \tilde{\phi}_n$ on a whole region around the given M_c. In this region we have coordinates $(F_1, \ldots, F_n, \tilde{\phi}_1, \ldots, \tilde{\phi}_n)$ and a non-degenerate matrix of Poisson brackets

$$\begin{pmatrix} \{F_i, F_j\}=0 & \{F_i, \tilde{\phi}_j\} \\ \{\tilde{\phi}_i, F_j\} & \{\tilde{\phi}_i, \tilde{\phi}_j\}\end{pmatrix}, \tag{1.194}$$

where $\det\{F_i, \tilde{\phi}_j\}\neq 0$. Now let us introduce the action variables. For the phase space $\mathbb{R}^{2n}$ with the canonical coordinates $(x^1, \ldots, x^n, p_1, \ldots, p_n)$ the action variables have the form:

$$s_i=\frac{1}{2\pi}\oint_{\gamma_i} p_k\, dx^k, \quad i=1, \ldots, n. \tag{1.195}$$

Here γ_i is the i-th basis cycle of the torus T^n,

$$\gamma_i\colon 0\leq\tilde{\phi}_i\leq 2\pi, \quad \tilde{\phi}_j=\text{const} \quad \text{for} \quad j\neq i. \tag{1.196}$$

We get:

$$\{\tilde{\phi}_j, s_i\}=\delta_{ij}, \quad i, j=1, \ldots, n. \tag{1.197}$$

On an arbitrary phase space with a form $\Omega=h_{ij}dy^i\wedge dy^j$ of the type (1.17) one must do the following: the form Ω vanishes on the tori $T^n=M_c$. Therefore on some neighbourhood of the given torus $T^n=M_c$ this form is exact:

$$\Omega=d\omega.$$

The action variables have the form, analogous to (1.195):

$$s_i=\frac{1}{2\pi}\oint_{\gamma_i}\omega, \quad i=1, \ldots, n. \tag{1.198}$$

Now let us set

$$\phi_i=\tilde{\phi}_i+b_i(s_1, \ldots, s_n), \quad i=1, \ldots, n. \tag{1.199}$$

Let us select the b_i according to the condition $\{\phi_i, \phi_j\}=0$. This can always be done by virtue of (1.197). On each level surface M_c the coordinates $\phi_1, \ldots, \phi_n$ coincide up to a translation with the angles $\tilde{\phi}_1, \ldots, \tilde{\phi}_n$ chosen earlier. The matrix of Poisson brackets takes on the form (1.190), and the Hamiltonian $H=F_1$ can be written in the form

$$H=\tilde{H}(s_1, \ldots, s_n). \tag{1.200}$$

The equations of motion will have the form (1.191). This is conditionally periodic motion along an n-dimensional torus with the frequencies

$$\omega_i(s_1, \ldots, s_n) = \frac{\partial \tilde{H}(s_1, \ldots, s_n)}{\partial s_i}. \tag{1.201}$$

Example 1. Let the level surface $H(x, p) = E$ of a system with one degree of freedom be compact. Then we have the canonical action-angle coordinates

$$s(E) = \oint_{H=E} p\,dx, \quad \{s, \phi\} = 1. \tag{1.202}$$

Now let us consider some examples of completely integrable systems with two degrees of freedom. Here, according to Liouville's theorem, it is sufficient for "complete integrability" to know one integral not dependent on the energy H.

Example 2. The equations of the rotation of a heavy rigid body with a fixed point can be represented, in accordance with §1, in the form of a Hamiltonian system on $E(3)$ with the Hamiltonian

$$H(M, p) = \frac{M_1^2}{2I_1} + \frac{M_2^2}{2I_2} + \frac{M_3^2}{2I_3} + \gamma_1 p_1 + \gamma_2 p_2 + \gamma_3 p_3. \tag{1.203}$$

Here the axes of the coordinate system coincide with the principal axes of the body, the origin is at the point of attachment, I_1, I_2, I_3 are the principal moments of inertia of the body, $\gamma_1, \gamma_2, \gamma_3$ are the coordinates of the centre of mass. The Poisson brackets have the form (1.41). The phase space is six-dimensional here, but the rank of the matrix of Poisson brackets is equal to 4. Therefore for integrability according to Liouville it is enough to know one integral (besides the energy integral). Well known are the following cases of integrability.

a) *The Euler case*: $\gamma_1 = \gamma_2 = \gamma_3 = 0$. The extra integral is the square of the total angular momentum $M^2 = M_1^2 + M_2^2 + M_3^2$.

b) *The Lagrange case*: $I_1 = I_2$, $\gamma_1 = \gamma_2 = 0$. Here there is an axial symmetry (with respect to the third axis). This gives the extra integral $M_3 = \text{const}$.

c) *S. V. Kovalevskaya's case* $I_1 = I_2 = I_3/2$, $\gamma_3 = 0$. Here the appearance of the spare integral

$$F = |I_1(M_1 + iM_2)^2 - 2(\gamma_1 + i\gamma_2)(p_1 + ip_2)|^2 \tag{1.204}$$

is not connected with a symmetry of the system (see chap. 2 below).

Example 3. The problem of the motion of a rigid body in a perfect fluid (see above §1) is far richer in integrable cases. The simplest of them is the *Kirchhoff case*, where the Hamiltonian has the form (1.48), with $a_1 = a_2$, $b_{11} = b_{22}$, $b_{ij} = 0$ for $i \neq j$, $c_{11} = c_{22}$, $c_{ij} = 0$ for $i \neq j$. Here, just as in the Lagrange case, there is an axial symmetry, and the extra integral is M_3. More complicated integrable cases (with a "hidden symmetry") have the following form.

a) *The Clebsch case* (R. Clebsch). Here the coefficients of the Hamiltonian (1.48) are like this:

$$b_{ij} = 0, \quad c_{ij} = c_i \delta_{ij}, \tag{1.205}$$

where the coefficients a_i and c_j satisfy the relation

$$\frac{c_2 - c_3}{a_1} + \frac{c_3 - c_1}{a_2} + \frac{c_1 - c_2}{a_3} = 0. \tag{1.206}$$

The supplementary integral has the form

$$M_1^2 + M_2^2 + M_3^2 - (a_1 p_1^2 + a_2 p_2^2 + a_3 p_3^2). \tag{1.207}$$

b) *The Lyapunov–Steklov–Kolosov case*: $b_{ij} = b_i \delta_{ij}$, $c_{ij} = c_i \delta_{ij}$, where

$$b_j = \mu(a_1 a_2 a_3) a_j^{-1} + \nu, \quad c_1 = \mu^2 a_1 (a_2 - a_3)^2 + \nu', \ldots \tag{1.208}$$

(μ, ν, $\nu' = \text{const}$). The supplementary integral is

$$\sum_j [M_j^2 - 2\mu(a_j + \nu) M_j p_j] + \mu^2[(a_2 - a_3)^2 + \nu''] p_1^2 + \ldots \tag{1.209}$$

(the parameters ν, ν', ν'' are unessential).

§4. The Hamilton–Jacobi Equation. The Method of Separation of Variables—The Classical Method of Integration and of Finding Action-Angle Variables

The theory of completely integrable Hamiltonian systems set forth in the preceding section arose as a generalization of the *Hamilton–Jacobi method* of integrating the canonical equations.

We shall consider here the phase space $\mathbb{R}^{2n}$ with the canonical coordinates $(x^1, \ldots, x^n, p_1, \ldots, p_n)$ (see (1.19)). The Hamiltonian system with the Hamiltonian $H = H(x, p)$ has the form:

$$\left.\begin{aligned} \dot{x}^i &= \frac{\partial H}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H}{\partial x^i} \end{aligned}\right\} \quad i = 1, \ldots, n. \tag{1.210}$$

Let us consider a canonical transformation (i.e. one preserving the Poisson brackets (1.19)) of the coordinates (x, p) to coordinates (X, P) of the form

$$p_i = \frac{\partial S}{\partial x^i}, \quad X^i = \frac{\partial S}{\partial P_i}, \quad S = S(x, P), \tag{1.211}$$

$$dS = p_i dx^i + X^i dP_i, \quad dp_i \wedge dx^i = dP_i \wedge dX^i. \tag{1.212}$$

In the new coordinates the system (1.210) can be written in the form

$$\left.\begin{aligned} \dot{X}^i &= \frac{\partial K}{\partial P_i} \\ \dot{P}_i &= -\frac{\partial K}{\partial X^i} \end{aligned}\right\} \quad i=1, \ldots, n, \tag{1.213}$$

where the Hamiltonian $K=K(X, P)$ has the form

$$K(X, P)=H(x(X, P), p(X, P)). \tag{1.214}$$

The idea of the Hamilton–Jacobi method consists in choosing the transformation (1.211), (1.214) appropriately so that in the new coordinates the Hamiltonian K does not depend on X: $K=K(P)$. In this case the variables $P_1, \ldots, P_n$ will obviously be variables of action type, and the conjugate variables $X^1, \ldots, X^n$ will be the corresponding angles, i.e. the system (1.213) can be written in the form (1.191)

$$\left.\begin{aligned} \dot{X}^i &= \frac{\partial K(P)}{\partial P_i} \\ \dot{P}_i &= 0 \end{aligned}\right\} \quad i=1, \ldots, n. \tag{1.215}$$

Thus, the problem of integrating the canonical equations (1.210) reduces to finding a function $S(x, P)$ satisfying the *Hamilton–Jacobi equation*

$$H\left(x, \frac{\partial S}{\partial x}\right)=K, \tag{1.216}$$

depending on n parameters $P_1, \ldots, P_n$ (it is necessary that the function $S=S(x^1, \ldots, x^n, P_1, \ldots, P_n)$ be a general integral of equation (1.216), i.e. that $\det(\partial^2 S/\partial x^i \partial P_j) \neq 0$ [7]).

The only method of integrating the Hamilton–Jacobi equation employed with success in classical analytical mechanics is *the method of separation of variables.* Namely, suppose the Hamilton–Jacobi equation (1.216) can be written in the form

$$h\left(f_1\left(x^1, \frac{\partial S}{\partial x^1}\right), \ldots, f_n\left(x^n, \frac{\partial S}{\partial x^n}\right), K\right)=0, \tag{1.217}$$

where the $f_i(x^i, p_i)$ are certain functions. In this case its general integral may be sought in the form

$$S=S_1(x^1; c_1)+S_2(x^2; c_2)+ \ldots +S_n(x^n; c_n), \tag{1.218}$$

where the equations for the functions $S_1, \ldots, S_n$ will be written in the form

$$f_i\left(x^i, \frac{\partial S_i}{\partial x^i}\right)=c_i, \quad i=1, \ldots, n, \tag{1.219}$$

and in an obvious manner can be integrated by quadratures. The dependence of the Hamiltonian on the new variables $H(x, p)=K(c_1, \ldots, c_n)$ is determined by the equation

$$h(c_1, \ldots, c_n, K)=0. \tag{1.220}$$

The variables

$$c_i=\phi_i(x^i, p_i), \quad i=1, \ldots, n, \tag{1.221}$$

if they are globally defined, will be variables of action type. The corresponding variables of angle type can be computed by formulas (1.211).

Example 1. *Geodesics on an ellipsoid* (Jacobi, 1839). Let the ellipsoid have the form:

$$\frac{x_1^2}{a_1}+\frac{x_2^2}{a_2}+\frac{x_3^2}{a_3}=1, \quad a_1>a_2>a_3>0. \tag{1.222}$$

The elliptical coordinates λ_1, λ_2, λ_3 in space are defined as the roots of the equation

$$\frac{x_1^2}{a_1-\lambda}+\frac{x_2^2}{a_2-\lambda}+\frac{x_3^2}{a_3-\lambda}=1, \tag{1.223}$$

where $\lambda_3<a_3<\lambda_2<a_2<\lambda_1<a_1$. The ellipsoid (1.222) is obtained for $\lambda_3=0$. The Hamiltonian of the free motion of a unit point mass on the surface of the ellipsoid coincides with the kinetic energy (the metric) and has the form

$$H=\frac{2}{\lambda_1-\lambda_2}\left[\frac{(a_1-\lambda_1)(\lambda_1-a_2)(\lambda_1-a_3)}{\lambda_1}p_1^2 + \frac{(a_1-\lambda_2)(a_2-\lambda_2)(\lambda_2-a_3)}{\lambda_2}p_2^2\right], \tag{1.224}$$

where

$$p_j=(-1)^{j+1}(\lambda_1-\lambda_2)\frac{\lambda_j\dot{\lambda}_j}{4(a_1-\lambda_j)(a_2-\lambda_j)(a_3-\lambda_j)}, \quad j=1, 2. \tag{1.225}$$

The variables have been separated. It is not hard to show [120] that integration of the equations of motion reduces to the hyperelliptic quadratures (of genus 2)

$$\frac{d\lambda_1}{\sqrt{R(\lambda_1)}}=\frac{dt}{\lambda_1-\lambda_2}, \quad \frac{d\lambda_2}{\sqrt{R(\lambda_2)}}=\frac{dt}{\lambda_2-\lambda_1}, \tag{1.226}$$

where

$$R(\lambda)=-\frac{(\lambda-\alpha)(\lambda-a_1)(\lambda-a_2)(\lambda-a_3)}{\lambda}, \tag{1.227}$$

and $a_3<\alpha<a_1$ is an arbitrary constant. These equations were integrated in 1861 by K. Weierstrass in theta functions of two variables. The solution of the

Hamilton–Jacobi equation has the form:

$$S(\lambda_1, \lambda_2; \alpha, E) = \frac{\sqrt{E}}{\sqrt{2}} \int \frac{\lambda_1 - \alpha}{\sqrt{R(\lambda_1)}} d\lambda_1 + \frac{\sqrt{E}}{\sqrt{2}} \int \frac{\lambda_2 - \alpha}{\sqrt{R(\lambda_2)}} d\lambda_2, \tag{1.228}$$

where $E = H$ is the energy. From this the variables of angle type are found by the formulas (1.211)

$$\phi_\alpha = \frac{\partial S}{\partial \alpha}, \quad \phi_E = \frac{\partial S}{\partial E}. \tag{1.229}$$

The change of variables (1.229) $(\lambda_1, \lambda_2) \mapsto (\phi_\alpha, \phi_E)$ is an Abel map, corresponding to the hyperelliptic Riemann surface of the root $\sqrt{\lambda(\lambda-\alpha)(\lambda-a_1)(\lambda-a_2)(\lambda-a_3)}$ (of genus 2; see chap. 2 below). Therefore the invariant tori here can be extended to the complex domain and are abelian.

The question of separation of variables for Hamiltonian systems was studied intensively in the second half of the last century (see the bibliography in [92]). The following criterion was established (by T. Levi-Civita, [91]): the system with Hamiltonian $H(x, p)$ is integrable by the method of separation of variables in a given coordinate system if and only if the function H satisfies the following system of equations

$$\frac{\partial H}{\partial p_j}\frac{\partial H}{\partial p_k}\frac{\partial^2 H}{\partial x^j \partial x^k} - \frac{\partial H}{\partial p_j}\frac{\partial H}{\partial x^k}\frac{\partial^2 H}{\partial x^j \partial p_k} - \frac{\partial H}{\partial x^j}\frac{\partial H}{\partial p_k}\frac{\partial^2 H}{\partial p_j \partial x^k} + \frac{\partial H}{\partial x^j}\frac{\partial H}{\partial x^k}\frac{\partial^2 H}{\partial p_j \partial p_k} = 0, \tag{1.230}$$

$1 \le j < k \le n$ (there is no summation over repeated indices). The application of this criterion to the investigation of the integrability (via Hamilton–Jacobi) of Hamiltonian systems is a non-trivial problem; advances in certain special classes of Hamiltonians were obtained in [27], [32].

To conclude this section let us note that the system of S.V. Kovalevskaya mentioned above cannot be integrated by the method of separation of variables, and action-angle variables for it were not found until very recently (see below chap. 2).

Chapter 2

Modern Ideas on the Integrability of Evolution Systems

§ 1. Commutational Representations of Evolution Systems

The algebraic mechanism lying at the foundation of the procedure for integrating the KdV equation with initial conditions rapidly decreasing in x

which was proposed in the famous paper of C.S. Gardner, J. Green, M. Kruskal and R.M. Miura [60] was cleared up in P.D. Lax's paper [89]. It was observed that this equation

$$4u_t = 6uu_x + u_{xxx} \tag{2.1}$$

is equivalent to the commutation condition

$$\left[L, \frac{\partial}{\partial t} - A\right] = 0 \Leftrightarrow \frac{\partial L}{\partial t} = [A, L] \tag{2.2}$$

for the auxiliary linear differential operators

$$L = \frac{\partial^2}{\partial x^2} + u(x,t); \quad A = \frac{\partial^3}{\partial x^3} + \frac{3}{2}u\frac{\partial}{\partial x} + \frac{3}{4}u_x. \tag{2.3}$$

Beginning with this paper, all schemes for producing new equations to which the inverse scattering method is applicable were based on various generalizations of the *commutation equation* (2.2).

The first and most natural step is the generalization of equation (2.2) to the case when L and A are arbitrary differential operators

$$L = \sum_{i=0}^{n} u_i(x,t)\frac{\partial^i}{\partial x^i}; \quad A = \sum_{i=0}^{m} v_i(x,t)\frac{\partial^i}{\partial x^i} \tag{2.4}$$

with matrix $(l \times l)$ or scalar coefficients.

New physically important equations to which the inverse scattering method is applicable were discovered along just this path in the papers of V.E. Zakharov–A.B. Shabat [125], [126] and G.L. Lamb [88].

Let u_n and v_m be constant non-degenerate diagonal matrices with distinct entries on the diagonals. By conjugating with a suitable matrix function $g(x, t)$: $L \to gLg^{-1}$ and $A \to gAg^{-1}$ one can always achieve that $u_{n-1}^{\alpha\alpha} = 0$, $v_{m-1}^{\alpha\alpha} = 0$, $\alpha = 1, \dots, l$. The equations (2.2) form a system of $n+m$ matrix equations in the coefficients of the operators L and A. It turns out that from the first m equations, obtained by equating to zero the coefficients of the $\partial^k/\partial x^k$, $k = n, \dots, n+m-1$, one can successively find the $v_j(x, t)$, the matrix entries of which are differential polynomials in the matrix entries $u_i^{\alpha\beta}(x, t)$ and certain constants h_j^α; $\alpha, \beta = 1, \dots, l$. If we substitute the expressions obtained into the remaining n equations, we obtain a system of evolution equations only in the coefficients of the operator L, and these are called equations of Lax type. There exist a great number of schemes (see, for example, [1], [34], [75], [106], [115], [126], [127]) which by one method or another realize a reduction of the general equation (2.2) to equations in the coefficients of the operator L.

The system (2.2) represents a family of *Lax equations*, parametrized by the constants h_j^α. For example, if $(l = 1)$

$$L = \partial^2 + u, \quad A = \partial^3 + v_1\partial + v_2, \quad \partial = \partial/\partial x, \tag{2.5}$$

then

$$v_1 = \tfrac{3}{2}u + h_1, \quad v_2 = \tfrac{3}{4}u_x + h_2 \tag{2.6}$$

and equation (2.2) is equivalent to the family of equations

$$4u_t = u_{xxx} + 6uu_x + 4h_1 u_x. \tag{2.7}$$

Let us give a few simplest examples of equations of Lax type.

Example 1 ([142]). If $(l=1)$

$$L = \partial^3 + \tfrac{3}{2}u(x,t)\partial + w(x,t), \quad A = \partial^2 + u(x,t), \tag{2.8}$$

then equation (2.2) leads to the system

$$\tfrac{3}{4}u_t = w_x - \tfrac{3}{4}u_{xx}, \tag{2.9}$$

$$w_t = w_{xx} - u_{xxx} - \tfrac{3}{2}uu_x. \tag{2.10}$$

If we eliminate w from these equations, we arrive for $u(x, t)$ at the Boussinesq equation (J.V. Boussinesq)

$$3u_{tt} + (u_{xxx} + 6uu_x)_x = 0. \tag{2.11}$$

Two-dimensional systems which admit a representation of the Lax type (2.2) were first discovered in [37], [126]. An important example of such systems is the "two-dimensionalized" KdV equation—the Kadomtsev–Petviashvili (KP) equation

$$\begin{cases} \tfrac{3}{2}u_y + \tfrac{3}{2}u_{xx} - 2w_x = 0, \\ w_x - u_t + u_{xxx} + \tfrac{3}{2}uu_x - w_{xx} = 0. \end{cases} \tag{2.12}$$

In this case

$$L = -\partial_y + \partial_x^2 + u(x,y,t), \quad A = \partial_x^3 + \tfrac{3}{2}u(x,y,t)\partial_x + w(x,y,t). \tag{2.13}$$

Remark. A number of authors have shown that the usual Lax representation for two-dimensional systems is possible only for operators which involve differentiation of no higher than first order with respect to one of the variables. For example, for the two-dimensional Schrödinger operator $L = \partial_x^2 + \partial_y^2 + u(x, y)$ there is not a single non-trivial equation of Lax type. The correct (non-trivial) analogue of the equations of Lax type for $2+1$-systems was found later in the works [98], [43], [117].

Example 2 ([125]). The non-linear Schrödinger equation ($\text{NLS}_\pm$)

$$ir_t = r_{xx} \pm |r|^2 r. \tag{2.14}$$

The operators L and A in this case are matricial and are equal to

$$L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\partial}{\partial x} + \begin{pmatrix} 0 & r \\ q & 0 \end{pmatrix}, \quad q = \pm\bar{r}, \tag{2.15}$$

$$A = i\left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{\partial^2}{\partial x^2} + \frac{1}{2}\left[\begin{pmatrix} 0 & r \\ q & 0 \end{pmatrix}\frac{\partial}{\partial x} + \frac{\partial}{\partial x}\begin{pmatrix} 0 & r \\ q & 0 \end{pmatrix}\right] + \begin{pmatrix} rq & 0 \\ 0 & -rq \end{pmatrix}\right]. \tag{2.16}$$

Example 3. The equations of three-wave interaction

$$u_{13t} + v_{13}u_{13x} = i\varepsilon u_{12}u_{23}; \quad u_{12t} + v_{12}u_{12x} = i\varepsilon u_{13}\bar{u}_{23},$$
$$u_{23t} + v_{23}u_{23x} = i\varepsilon u_{13}\bar{u}_{12}, \quad u_{ij} = u_{ij}(x,t), \quad v_{ij} = \text{const}. \tag{2.17}$$

The operators L and A are matricial (3×3) and are equal to

$$L = I\frac{\partial}{\partial x} + [I, Q], \quad A = J\frac{\partial}{\partial x} + [J, Q],$$
$$I_{ij} = a_i\delta_{ij}, \quad a_{i+1} > a_i, \quad J_{ij} = b_i\delta_{ij}, \tag{2.18}$$
$$Q_{ij} = \bar{Q}_{ji} = -iu_{ij}\sqrt{a_j - a_i}, \quad j > i.$$
$$v_{ij} = \frac{a_i b_j - b_i a_j}{a_i - a_j}. \tag{2.19}$$

Example 4. The Toda lattice ((M. Toda) [96], [55], [56]) and the difference analogue of the KdV equation [96]. The inverse scattering method is also applicable to certain differential-difference systems. If L and A are difference operators of the form

$$L\psi_n = c_n\psi_{n+1} + v_n\psi_n + c_{n-1}\psi_{n-1}, \tag{2.20}$$

$$A\psi_n = \frac{c_n}{2}\psi_{n+1} - \frac{c_{n-1}}{2}\psi_{n-1}, \tag{2.21}$$

then equation (2.2) leads to the equations

$$2\dot{c}_n = c_n(v_{n+1} - v_n), \tag{2.22}$$

$$\dot{v}_n = c_n^2 - c_{n-1}^2, \tag{2.23}$$

which, if one sets $c_n = \exp(\frac{1}{2}(x_{n+1} - x_n))$, $v_n = \dot{x}_n$, coincide with the equations of motion of the so-called Toda lattice—the Hamiltonian system of particles on the line with the Hamiltonian

$$H = \tfrac{1}{2}\sum_n p_n^2 + \sum_n \exp(x_{n+1} - x_n). \tag{2.24}$$

If in (2.20) one sets $v_n \equiv 0$ and for A one chooses

$$A\psi_n = \tfrac{1}{2}[c_n c_{n+1}\psi_{n+1} - c_{n-1}c_{n-2}\psi_{n-2}], \tag{2.25}$$

then (2.20) leads to the difference analogue of the KdV equation

$$\frac{d}{dt}\tilde{c}_n = \tilde{c}_n(\tilde{c}_{n+1} - \tilde{c}_{n-1}), \quad \tilde{c}_n = c_n^2. \tag{2.26}$$

With each operator L there is connected a whole *hierarchy of Lax-type equations*, which are the reduction to equations in the coefficients of L of the equations (2.2) with operators A of different orders. One of the most important facts in the theory of integrable equations is the commutativity of all the equations which enter into the general hierarchy.

For the KdV equation the corresponding equations are called the "higher-order KdV equations". They have the form

$$u_\tau = \sum_{k=1}^{n} h_k Q_k(u, \ldots, u^{(2k+1)})$$

and are the commutativity condition for the Sturm–Liouville operator with the operators $\partial/\partial t - A$, where A has order $2n+1$.

In the paper [111] a representation of a different type than (2.2) for the higher-order KdV equations was used for the first time—a representation of Lax type in matrix functions depending on an additional spectral parameter.

For the general equation (2.2) such a λ-representation can be constructed in the following manner.

The equation

$$Ly = \lambda y \tag{2.27}$$

is equivalent to the first-order matrix equation

$$\left[\frac{\partial}{\partial x} - U_L(x, t, \lambda)\right] Y(x, t, \lambda) = 0, \tag{2.28}$$

where U_L is an $nl \times nl$ matrix (n is the order of L, and the matricial coefficients of L are $l \times l$). The vector Y is the column composed of the vectors $(\partial^i/\partial x^i) y(x, t, \lambda)$, $i = 0, \ldots, n-1$.

The matrix U_L has the following block structure

$$U_L = u_n^{-1} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & & \cdots & 0 & 1 \\ \lambda - u_0 & -u_1 & & \cdots & -u_{n-2} & -u_{n-1} \end{pmatrix}. \tag{2.29}$$

If we act with the operator A on the coordinates of the vector Y and use (2.27) to express the $(\partial^n/\partial x^n)y$ in terms of lower-order derivatives and the parameter λ, we obtain that on the space of solutions of (2.27) the equation

$$\left(\frac{\partial}{\partial t} - A\right) y(x, t, \lambda) = 0 \tag{2.30}$$

is equivalent to an equation

$$\left(\frac{\partial}{\partial t} - V_A(x, t, \lambda)\right) Y(x, t, \lambda) = 0. \tag{2.31}$$

The matrix entries of V_A depend polynomially on λ, the matrix entries of the $u_i(x, t)$, and their derivatives.

The compatibility of equations (2.27) and (2.30) implies the compatibility of (2.28) and (2.31). Hence

$$\left[\frac{\partial}{\partial x} - U_L, \frac{\partial}{\partial t} - V_A\right] = 0. \tag{2.32}$$

For the KdV equation the matrices U_L, V_A have the form:

$$U_L = \begin{pmatrix} 0 & 1 \\ \lambda - \mu & 0 \end{pmatrix}, \tag{2.33}$$

$$V_A = \begin{pmatrix} -\frac{u_x}{4} & \lambda + \frac{u}{2} \\ \lambda^2 - \frac{u\lambda}{2} - \frac{u^2}{2} - \frac{u_{xx}}{4} & \frac{u_x}{4} \end{pmatrix}. \tag{2.34}$$

Subsequently, equation (2.32), where U and V were now arbitrary rational functions of the parameter λ, was proposed in [127] as a general scheme for the production of one-dimensional integrable equations. The beginning of this program goes back to the paper [1], in which for the integration of the sine-Gordon equation an example of a *rational family* was first introduced.

Let $U(x, t, \lambda)$ and $V(x, t, \lambda)$ be arbitrary matrix functions depending rationally on the parameter λ:

$$\begin{aligned} U(x, t, \lambda) &= u_0(x, t) + \sum_{k=1}^{n} \sum_{s=1}^{h_k} u_{ks}(x, t)(\lambda - \lambda_k)^{-s}, \\ V(x, t, \lambda) &= v_0(x, t) + \sum_{r=1}^{m} \sum_{s=1}^{d_r} v_{rs}(x, t)(\lambda - \mu_r)^{-s}. \end{aligned} \tag{2.35}$$

The condition of compatibility of the linear problems

$$\begin{aligned} &\left(\frac{\partial}{\partial x} - U(x, t, \lambda)\right) \Psi(x, t, \lambda) = 0, \\ &\left(\frac{\partial}{\partial t} - V(x, t, \lambda)\right) \Psi(x, t, \lambda) = 0 \end{aligned} \tag{2.36}$$

is represented by the *equation of zero curvature*

$$U_t - V_x + [U, V] = 0, \tag{2.37}$$

which must be fulfilled for all λ. This equation is equivalent to a system of $1 + \sum_k h_k + \sum_r d_r$ matrix equations in the unknown functions $u_{ks}(x, t)$, $v_{rs}(x, t)$, $u_0(x, t)$, $v_0(x, t)$. These equations arise when one equates to zero all the singular terms on the left-hand side of (2.37) at the points $\lambda=\lambda_k$, $\lambda=\mu_r$, and also the absolute term, equal to $u_{0t}-v_{0x}+[u_0, v_0]$.

The number of equations is one matrix equation fewer than the number of unknown matrix functions. This underdeterminacy is connected with a "gauge symmetry" of the equations (2.37). If $g(x, t)$ is an arbitrary non-degenerate matrix function, then the transformation

$$U\to g_x g^{-1}+gUg^{-1},\quad V\to g_t g^{-1}+gVg^{-1}, \tag{2.38}$$

called a "gauge transformation", takes the solutions of (2.37) over into solutions of the same equation.

A choice of conditions on the matrices $U(x, t, \lambda)$, $V(x, t, \lambda)$ compatible with the equations (2.37) and destroying the gauge symmetry is called a setting of the gauge. The simplest gauge is the pair of conditions $u_0(x, t)=v_0(x, t)=0$.

Just as in the above-considered case of commutation equations for differential operators, the equations (2.37) are essentially generating equations for a whole family of integrable systems. If the poles of U and V coincide, then these equations can be reduced to a family of equations, which are parametrized by arbitrary constants, in the coefficients only of $U(x, t, \lambda)$. Here, by changing the multiplicity of the poles of V, we will obtain a hierarchy of commuting flows associated with $U(x, t, \lambda)$.

In singling out some particular equations from (2.37), an important issue is singling out the invariant submanifolds for the equation (2.37). This problem reduces to describing the orbits of the coadjoint representation of the current algebra [2], in the framework of which the Hamiltonian theory of equations of zero curvature can naturally be introduced (see [30], [36], [52], [124]).

Leaving aside the further analysis of the questions of reduction and gauge equivalence of systems, which may be found in the papers [2], [36], [52], [106], [126], let us cite the two simplest examples.

If U and V have one pole each, which do not coincide,

$$U=\frac{u(\xi,\eta)}{1-\lambda},\quad V=\frac{v(\xi,\eta)}{\lambda+1}, \tag{2.39}$$

then the equations (2.37) (after the substitution $x\to\xi=x'-t'$; $t\to\eta=x'+t'$) lead to the equations of a principal *chiral field* ([106], [122])[4]

$$u_\eta+\tfrac{1}{2}[u, v]=0,\quad v_\xi=\tfrac{1}{2}[u, v]. \tag{2.40}$$

Here $u(\xi,\eta)$, $v(\xi,\eta)$ are the currents of the chiral field

$$u=G_\xi G^{-1};\quad v=G_\eta G^{-1}. \tag{2.41}$$

[4] It is interesting that this example was first brought to the open together with the notion of zero curvature in a remarkable (but forgotten) classical paper by René Garnier [61].

Equation (2.40) yields

$$2G_{\xi\eta}=G_\xi G^{-1}G_\eta+G_\eta G^{-1}G_\xi. \tag{2.42}$$

The last equations are Lagrangian with Lagrangian (1.68) (the currents A_μ in the notation of (1.68) correspond, as is evident from (2.41), to u and v).

As was remarked in [135], the representation (2.37), where U and V are given by the formulas (2.39), simultaneously gives the solutions of the equations of motion of a principal chiral field with a "multi-valued additional term". If in the definition of the corresponding Lagrangian (1.72) one introduces a coupling constant κ, i.e.

$$\delta S=\delta S_0+\kappa\Omega_1,$$

then the equations of motion can be written in terms of the currents in the form

$$\partial_\xi v=\frac{1+\kappa}{2}[v, u],\quad \partial_\eta u=\frac{1-\kappa}{2}[v, u]. \tag{2.43}$$

In the inverse scattering method, as will be stressed repeatedly in the following, the road to constructing solutions of the equations (2.37) goes via the construction with the aid of various schemes ("dressing", algebraic geometric schemes, etc.) of functions $\Psi(\xi,\eta,\lambda)$ which by their construction satisfy the equations (2.36) with some U, V (which will automatically satisfy (2.37)). In the case under consideration, after the function $\Psi(\xi,\eta,\lambda)$ has been constructed by one means or another, the desired solutions of the equations of motion are defined by the formula

$$G(\xi,\eta)=\Psi(\xi,\eta,\kappa).$$

As the second example let us cite the sine-Gordon equation [134], [1]

$$u_{\xi\eta}=4\sin u, \tag{2.44}$$

which arose in the theory of surfaces of constant negative curvature. By the number of applications (in the theory of superconductivity, in the theories of quasi-one-dimensional conductors, of film-formation processes on crystalline substrates, in the theory of fields) leading to equation (2.44), this equation belongs to the most important in contemporary mathematical physics.

As in the general case of a principal chiral field, the matrices U and V have one pole each

$$U(\xi,\eta,\lambda)=\begin{pmatrix}\frac{iu_\xi}{2} & 1\\ \lambda^{-1} & -\frac{iu_\xi}{2}\end{pmatrix}, \tag{2.45}$$

$$V(\xi,\eta,\lambda)=\begin{pmatrix}0 & \lambda e^{-iu}\\ e^{iu} & 0\end{pmatrix}. \tag{2.46}$$

It should be noted that an automatic generalization of equations (2.37) to the case of matrices U and V whose spectral parameter is defined on an algebraic curve Γ of genus greater than zero (the case of rational families corresponds to $g=0$) is obstructed by the Riemann–Roch theorem (G.F.B. Riemann–G. Roch).

Indeed, let $U(x, t, \lambda)$ and $V(x, t, \lambda)$ be meromorphic functions on Γ having poles of total multiplicity N and M. By the Riemann–Roch theorem [131] the number of independent variables (i.e. the dimension of the space of matrix functions with the same poles as U and V) equals $l^2(N-g+1)$ for U and $l^2(M-g+1)$ for V. The commutator $[U, V]$ has poles of total multiplicity $N+M$. Hence the equations (2.37) are equivalent to $l^2(N+M-g+1)$ equations. With gauge equivalence taken into account the number of equations is always greater than the number of variables.

There are two ways of circumventing the obstacle mentioned. One of them was proposed in the papers [84], [86], where the matrices U and V were allowed to have, besides poles stationary with respect to x and t, gl poles depending on x and t in a definite fashion. It was shown that here the number of equations with gauge equivalence taken into account is equal to the number of the independent variables, which (just as in the rational case) are the singular parts of U and V at the stationary poles.

An example of such an equation is

$$c_t=\frac{1}{4}c_{xxx}+\frac{3}{8c_x}(1-c_{xx}^2)-\frac{1}{2}Q(c)c_x^2. \tag{2.47}$$

Here the quantity $Q=\partial\Phi/\partial c+\Phi^2$ is defined through

$$\Phi(c, y)=\zeta(-2c)+\zeta(c-y)+\zeta(c+y)$$

and does not depend, as is easy to verify, on y (ζ is Weierstrass's ζ-function [11]).

This equation together with the following pair with an elliptic spectral parameter (*an elliptic family*) was obtained in [84]. The matrix U equals

$$U=A_1\zeta(\lambda-\gamma_1)+B_1\zeta(\lambda-\gamma_2)+\begin{pmatrix}0&0\\1&0\end{pmatrix}\zeta(\lambda)+A_1\zeta(\gamma_1)+B_1\zeta(\gamma_2)+\begin{pmatrix}0&1\\-u&0\end{pmatrix},$$

where

$$A_1=\frac{1}{\alpha_2-\alpha_1}\begin{pmatrix}0&0\\\alpha_1&1\end{pmatrix},\quad B_1=\frac{1}{\alpha_1-\alpha_2}\begin{pmatrix}0&0\\\alpha_2&1\end{pmatrix}.$$

For V we have the formulas

$$V=A_2\zeta(\lambda-\gamma_1)+B_2\zeta(\lambda-\gamma_2)+\begin{pmatrix}0&0\\1&0\end{pmatrix}\wp(\lambda)+\begin{pmatrix}0&1\\-\frac{u}{2}&0\end{pmatrix}\zeta(\lambda)+D,$$

where

$$A_2=\begin{pmatrix}\dfrac{\alpha_1\alpha_2}{\alpha_1-\alpha_2}&\dfrac{\alpha_2}{\alpha_1-\alpha_2}\\[2ex]\dfrac{\alpha_1 u}{2(\alpha_1-\alpha_2)}&\dfrac{u}{2(\alpha_1-\alpha_2)}\end{pmatrix},\quad B_2=\begin{pmatrix}\dfrac{\alpha_1\alpha_2}{\alpha_2-\alpha_1}&\dfrac{\alpha_1}{\alpha_2-\alpha_1}\\[2ex]\dfrac{\alpha_2 u}{2(\alpha_2-\alpha_1)}&\dfrac{u}{2(\alpha_2-\alpha_1)}\end{pmatrix}$$

and

$$D=A_2\zeta(\gamma_1)+B_2\zeta(\gamma_2)+\begin{pmatrix}w_1&\dfrac{u}{2}\\w_2&-w_1\end{pmatrix}.$$

The equations (2.37) are equivalent to a system of equations in the functions γ_i, α_i, w_i, u. By successively eliminating from this system the variables w_i, which equal

$$w_1=\frac{1}{2(\alpha_1-\alpha_2)}(\wp(\gamma_1)-\wp(\gamma_2))-\frac{u_x}{4},$$

$$w_2=w_{1x}-\frac{u^2}{2}+\wp(\gamma_1)+\wp(\gamma_2),$$

and afterwards the α_i (the formulas for which we shall omit here) and u (see (2.48)), we shall arrive finally at equation (2.47), where

$$\gamma_1=c(x, t)+y,\quad \gamma_2=y-c(x, t)+c_0.$$

Each solution of equation (2.47) defines by the formula

$$8u(x, y, t)=(c_{xx}^2-1)c_x^{-2}+8\Phi c_{xx}+4c_x^2\left(\frac{\partial\Phi}{\partial c}-\Phi\right)-2c_{xxx}c_x^{-1} \tag{2.48}$$

a solution of the Kadomtsev–Petviashvili equation (KP).

Equation (2.47) also describes the deformation of commuting linear differential operators of orders 4 and 6. Such an operator of order 4 has the form:

$$\mathscr{L}=(\partial^2+u)^2+c_x(\wp(\gamma_2)-\wp(\gamma_1))\frac{d}{dx}+\frac{d}{dx}(c_x(\wp(\gamma_2)-\wp(\gamma_1)))$$
$$-\wp(\gamma_2)-\wp(\gamma_1). \tag{2.49}$$

Equation (2.47), as is shown in [130], is the only one of the equations of the form

$$c_t=\text{const}\cdot c_{xxx}+f(c, c_x, c_{xx})$$

possessing a "hidden symmetry" which cannot be reduced to the ordinary KdV by transformations of "Miura type" $w=w(c, c_x, \ldots)$.

By the substitution $v=\zeta(c)$ equation (2.47) reduces to the algebraic form

$$v_t=\frac{1}{4}v_{xxx}+\frac{3}{8v_x}(v_{xx}^2-P_3(v)),\qquad P_3(v)=4v^3-g_2v-g_3. \tag{2.50}$$

The second way of introducing a non-rational spectral parameter is based on the choice of a special form for the matrices U and V and has been successfully realized only in certain examples on elliptic curves Γ $(g=1)$. The physically most interesting example of such equations is the Landau–Lifshitz equation

$$S_t=S\times S_{xx}+S\times JS, \tag{2.51}$$

where S is a three-dimensional vector of unit length, $|S|=1$, and $J_{\alpha\beta}=J_\alpha\delta_{\alpha\beta}$ is a diagonal matrix. As has been shown in the papers [22], [128], equation (2.51) is the compatibility condition for the linear equations (2.35), (2.36), where the (2×2) matrices U and V are

$$U=-i\sum_{\alpha=1}^{3}w_\alpha(\lambda)S_\alpha(x,t)\sigma_\alpha,\qquad V=-i\sum_{\alpha,\beta,\gamma}w_\alpha(\lambda)\sigma_\alpha S_\beta S_{\gamma x}\varepsilon^{\alpha\beta\gamma}-2i\sum_\alpha a_\alpha(\lambda)S_\alpha\sigma_\alpha, \tag{2.52}$$

where the σ_α are the Pauli matrices (W. Pauli), and

$$w_1=\frac{\rho}{\mathrm{sn}(\lambda,k)},\quad w_2=\rho\frac{\mathrm{dn}(\lambda,k)}{\mathrm{sn}(\lambda,k)},\quad w_3=\rho\frac{\mathrm{cn}(\lambda,k)}{\mathrm{sn}(\lambda,k)},$$

$$a_1=-w_2w_3,\quad a_2=-w_3w_1,\quad a_3=-w_1w_2,$$

(where $\mathrm{sn}(\lambda,k)$, $\mathrm{cn}(\lambda,k)$, $\mathrm{dn}(\lambda,k)$ are Jacobi's elliptic functions [11]).

The parameters J_α are given by the relations

$$k=\sqrt{\frac{J_2-J_1}{J_3-J_1}},\quad \rho=\frac{1}{2}\sqrt{J_3-J_1},\quad 0<k<1. \tag{2.53}$$

Another interesting example are the equations of an anisotropic O(3)-field [29]

$$u_\xi=[u,Jv],\quad v_\eta=[v,Ju], \tag{2.54}$$

where u and v are currents: $u=g_\eta g^{-1}$, $v=g_\xi g^{-1}$.

The Lagrangian of this model is equal to

$$\mathscr{L}=-\tfrac{1}{2}\int\mathrm{tr}(g_\xi g^{-1}Jg_\eta g^{-1})\,dx. \tag{2.55}$$

The equations (2.54) are equivalent to (2.37), where U and V are closely connected with the pair for the Landau–Lifshitz equation and, just as this latter pair, contain an elliptic dependence on the spectral parameter.

For a long time the commutational representation

$$\dot{L}=[M,L] \tag{2.56}$$

of Lax type for the equations of motion of the *Moser–Calogero system* (J. Moser–F. Calogero) appeared to be a special case.

This is a system of particles x_n on the line with the Hamiltonian

$$H=\frac{1}{2}\sum_{n=1}^{N}p_n^2+2\sum_{i<j}\wp(x_i-x_j), \tag{2.57}$$

where $\wp$ is the Weierstrass function. For this system L and M are matrices generally not depending on a spectral parameter, and have the form:

$$L_{ij}=p_i\delta_{ij}+(1-\delta_{ij})\phi(x_i-x_j), \tag{2.58}$$

$$M_{ij}=\left(\sum_{l\neq i}z(x_i-x_l)\right)\delta_{ij}-(1-\delta_{ij})\,y(x_i-x_j), \tag{2.59}$$

$$y(\xi)=\phi'(\xi),\quad z(\xi)=-\frac{\phi''}{2\phi},\quad \phi(\xi)=\frac{1}{\mathrm{sn}(\xi,k)}.$$

The representations (2.56) are sufficient for the construction of integrals of the system (2.57), equal to $J_k=(1/k)\,\mathrm{tr}\,L^k$, but they are insufficient for the explicit construction of angle-type variables and the integration of the equations of motion in terms of theta functions.

As will be shown in §5, the matrices L and M admit the introduction of a spectral parameter on an elliptic curve, but here the matrix entries turn out not to be meromorphic functions, but to have an exponential essential singularity.

The examples which have been brought by far do not exhaust all systems to which the inverse scattering method is applicable. A series of important examples will be cited and analyzed in detail in the following sections. Meanwhile, to conclude this section let us underline once more those basic features which are characteristic for systems to which the inverse method is applicable.

First, all such $(1+1)$ systems are equivalent to the compatibility condition (2.37) for the pair of linear problems (2.36), where $U(x,t,\lambda)$ and $V(x,t,\lambda)$ are meromorphic functions of the "spectral parameter λ" (defined in the basic examples on a rational or elliptic curve). Second, each such equation is included in a whole hierarchy of flows which commute with it. The commutativity of the flows allows one to restrict the original system to the set of stationary points of any other flow which enters into the same hierarchy.

Restriction of the KdV equation to the stationary points of the "n-th higher-order analogue" of the KdV equation served as the starting point in the construction of the theory of *finite gap Sturm–Liouville operators* and in the further development of algebraic-geometric integration methods which are applicable to all systems admitting commutational representations.

Surveys of the various stages in the development of the theory of *finite gap* or "algebraic-geometric" *integration* can be found in [40], [41], [76], [83], [86], [113], [115].

The stationary points of the "n-th higher-order analogue of the KdV equation" are described by the ordinary differential equation

$$\sum_{k=1}^{n} h_k Q_{2k+1}(u, \ldots, u^{(2k+1)}) = 0, \tag{2.60}$$

which is equivalent to the condition of commutation of the Sturm–Liouville operator L and an operator L_1 of order $2n+1$

$$[L, L_1] = 0. \tag{2.61}$$

These equations, as was shown in [111], are a completely integrable Hamiltonian system. In the following section a procedure will be set forth for the integration in *Riemann theta functions* of both equation (2.61) and the restriction to the space of its solutions of the KdV equation, and also for the integration of all generalizations of these equations.

For general Lax equations (2.2) the condition which picks out the algebraic-geometric solutions of these equations also has the form (2.61), where L is the operator (2.4) which enters into the original Lax pair and L_1 is an auxiliary operator. Equation (2.61) describes an invariant submanifold of the initial equation (2.2). When we increase the order of L_1 we obtain an ever-ascending family of such submanifolds, which in a number of cases (for example, for the KdV equation) are everywhere dense in the space of periodic solutions.

The problem of classifying commuting linear differential operators with scalar coefficients was considered from a purely algebraic point of view in the papers of J. Burchnall and T. Chaundy [26]. It was shown by them that for such operators there can be found a polynomial $Q(\lambda, \mu)$ in two variables such that

$$Q(L, L_1) = 0. \tag{2.62}$$

In the case of operators of relatively prime orders, to each point of the curve Γ given by the equation $Q(\lambda, \mu) = 0$ there corresponds a joint eigenfunction $\psi(x, P)$, $P = (\lambda, \mu)$, unique up to proportionality, of the operators L and L_1, i.e.

$$L\psi(x, P) = \lambda\psi(x, P), \quad L_1\psi(x, P) = \mu\psi(x, P). \tag{2.63}$$

The logarithmic derivative $\psi_x\psi^{-1}$ is a meromorphic function on Γ which generically has g poles $\gamma_1(x), \ldots, \gamma_g(x)$, where g is the genus of the curve Γ (the remaining poles do not depend on x).

In the paper [26] it was shown that commuting operators of relatively prime orders are uniquely determined by the polynomial Q and the assignment of the generic points $\gamma_1(x_0), \ldots, \gamma_g(x_0)$, although finite formulas were not obtained. A program for putting these results to effective use was proposed by H. Baker [9], who pointed out that the analytic properties of $\psi(x, P)$ on Γ coincide with those

which at the end of the nineteenth century were placed by R. Clebsch and P.A. Gordan at the foundation of the definition of the analogue of the "exponential function" on algebraic curves (see [8]). Unfortunately, Baker's program was never realized and in the course of a long time these papers were undeservedly forgotten.

As has already been said, the equations (2.61) describe the invariant submanifolds of equations of Lax type. These equations were considered from this point of view in the papers [74], [75], in which the results of the twenties were made significantly more effective and were generalized to the case of operators with matrix coefficients. For the coefficients of commuting scalar operators of relatively prime order, explicit expressions in terms of Riemann theta functions were found in these papers which showed that the general solutions of the equations (2.61) in this case are quasi-periodic functions. This permitted connecting the local theory of commuting operators with the Floquet spectral theory of operators with periodic coefficients, where an eigenfunction is defined non-locally—via the operator of translation by a period.

The first progress in the problem of classifying commuting operators of arbitrary orders was obtained in [35] on the basis of an algebraization of the scheme of the papers [74], [75]. This problem was completely solved in [78].

In the general case the solutions of (2.63) form an r-dimensional linear space, where r is a divisor of the orders of the operators L and L_1. It has been shown that a ring of commuting operators $\mathscr{A}$ is determined by a curve Γ and a matrix divisor of rank r. The reconstruction of the coefficients of the operators from these data comes down to a linear Riemann problem.

In individual cases, as was shown in [85], it is possible to eliminate the necessity of solving a Riemann problem and to obtain explicit formulas for the operators L and L_1 (the formula for an operator L of order 4 which commutes with an operator of 6th order was quoted above (2.49)).

Let us give a general definition of finite gap solutions for the equations of zero curvature (2.37), which generalizes condition (2.61) in a natural way.

We shall speak of "*finite gap*" or algebraic-geometric *solutions* of equations admitting a commutational representation to mean solutions for which a matrix-valued function $W(x, t, \lambda)$ can be found, depending meromorphically on the parameter λ (which is defined on the same curve as the parameter in U and V), such that

$$\left[\frac{\partial}{\partial x} - U(x, t, \lambda),\ W(x, t, \lambda)\right] = 0, \tag{2.64}$$

$$\left[\frac{\partial}{\partial t} - V(x, t, \lambda),\ W(x, t, \lambda)\right] = 0. \tag{2.65}$$

The equations (2.64), (2.65), which in a definition like this one play only an auxiliary rôle in the process of integrating the original equation (2.37), are also of independent interest, as will be evident in the sequel. To them one can reduce practically all interesting examples of finite-dimensional Hamiltonian systems which are integrable by the inverse scattering method.

§2. Algebraic-Geometric Integrability of Finite-Dimensional λ-Families

The basic goal of this section is the presentation of a procedure for integrating equations (2.64) and (2.65).

Let us denote by $\Psi(x, t, \lambda)$ the fundamental solution matrix of the equation

$$\left(\frac{\partial}{\partial x} - U(x, t, \lambda)\right)\Psi(x, t, \lambda) = 0,$$
$$\left(\frac{\partial}{\partial t} - V(x, t, \lambda)\right)\Psi(x, t, \lambda) = 0, \tag{2.66}$$

normalized by the condition

$$\Psi(0, 0, \lambda) = 1. \tag{2.67}$$

It follows from (2.64) that

$$W(x, t, \lambda)\Psi(x, t, \lambda) \tag{2.68}$$

is also a solution of equation (2.66). Any solution of (2.66) is uniquely determined by its initial conditions and has the form $\Psi(x, t, \lambda)G(\lambda)$, where G does not depend on (x, t). Taking the normalization condition (2.67) into account, we get that

$$W(x, t, \lambda)\Psi(x, t, \lambda) = \Psi(x, t, \lambda)\, W(0, 0, \lambda). \tag{2.69}$$

Consequently, the coefficients of the characteristic equation

$$Q(\lambda, \mu) = \det(W(x, t, \lambda) - \mu \cdot 1) = 0 \tag{2.70}$$

are integrals of the equations (2.64), (2.65). They are polynomials in the matrix entries of W, or, as was explained in the preceding section, they are differential polynomials of the basic phase variables — the matrix entries of $U(x, t, \lambda)$.

Generically equation (2.70) defines a nonsingular *algebraic curve* Γ (i.e. a compact Riemann surface), to each point of which (i.e. pair (λ, μ)) there corresponds an eigenvector h, unique up to proportionality, of the matrix W

$$W(x, t, \lambda)h(x, t, \gamma) = \mu h(x, t, \gamma); \quad \gamma = (\lambda, \mu) \in \Gamma. \tag{2.71}$$

Remark. The notion of genericity in the question at hand is not completely trivial. Although the levels of the integrals of equation (2.64) for which the eigenvalues of W are r-fold degenerate identically in λ (r is obliged to be a divisor of l, where W is an $l \times l$ matrix) unconditionally have nonzero codimension, in the procedure for reconstructing the matrices U and V from these data additional functional parameters appear, as is shown by the results of the papers [36], [78]. The elaboration of this direction, which has led to the construction of finite gap solutions of rank $r > 1$ for two-dimensional equations of the Kadomtsev–Petviashvili type depending on functional parameters (see [84], [85], [86]), remains beyond the scope of this article.

Let us normalize the vector h by requiring, for example, that $h_1(x, t, \gamma) = 1$ or $\sum h_i(x, t, \gamma) = 1$. In both cases all of the coordinates $h_i(x, t, \gamma)$ are rational functions of λ and μ, i.e. meromorphic functions on the curve Γ.

If $\Psi^j(x, t, \lambda)$ are the columns of the matrix $\Psi(x, t, \lambda)$, then it follows from (2.69) and (2.71) that the vector function

$$\psi(x, t, \gamma) = \sum_j h_j(0, 0, \gamma)\,\Psi^j(x, t, \lambda); \quad \gamma = (\lambda, \mu), \tag{2.72}$$

simultaneously satisfies the equations

$$\left(\frac{\partial}{\partial x} - U(x, t, \lambda)\right)\psi(x, t, \gamma) = \left(\frac{\partial}{\partial t} - V(x, t, \lambda)\right)\psi(x, t, \gamma) = 0, \tag{2.73}$$

$$W(x, t, \lambda)\psi(x, t, \gamma) = \mu\psi(x, t, \gamma). \tag{2.74}$$

The curve Γ l-foldly covers the curve $\hat{\Gamma}$ on which the parameter λ is defined. Outside the poles λ_i of the matrices U and V the matrix $\Psi(x, t, \lambda)$ is a holomorphic function of the parameter λ. Consequently, the vector function $\psi(x, t, \gamma)$ is meromorphic on Γ outside the points P_α — the preimages of the poles λ_i of the matrices U and V. (We observe that above the points λ the curve Γ may branch). The poles of $\psi(x, t, \gamma)$ coincide with the poles of $h(0, 0, \gamma)$ and therefore do not depend on x, t.

Let us further restrict ourselves to the case of a *rational family* (i.e. λ is defined on the ordinary complex plane). Let us consider the matrix $H(x, t, \lambda)$ whose columns are the vectors $h(x, t, \gamma_i)$, where the $\gamma_i = (\lambda, \mu_i)$ are the preimages of the point λ on Γ under the natural projection $\Gamma \to \mathbb{C}$, $(\lambda, \mu) \to \lambda$.

The function

$$r(x, t, \lambda) = (\det H(x, t, \lambda))^2 \tag{2.75}$$

does not depend on the numbering of the γ_i and is therefore a well-defined function of λ. Since the $h_i(x, t, \gamma)$ are meromorphic on Γ, $r(x, t, \lambda)$ is a rational function of λ. It has double poles at the images of the poles of $h(x, t, \gamma)$ and zeroes at the points above which Γ branches. The number of poles of r is equal to the number of zeroes. Hence

$$2N = \nu, \tag{2.76}$$

where ν is the number of branch points with multiplicities counted. A formula is well-known which connects the genus g of a smooth curve Γ l-foldly covering $\mathbb{C}$ with the number of branch points [131]

$$2g - 2 = \nu - 2l. \tag{2.77}$$

Consequently, the number of poles of $h(x, t, \gamma)$ and hence also of ψ is equal to

$$N = g + l - 1. \tag{2.78}$$

Now let us find the behaviour of $\psi(x, t, \gamma)$ in the neighbourhood of the points P_α — the preimages of the poles of $U(x, t, \lambda)$, $V(x, t, \lambda)$.

It follows from (2.71) and (2.74) that the vectors h and ψ are proportional

$$\psi(x, t, \gamma) = f(x, t, \gamma) h(x, t, \gamma). \tag{2.79}$$

Let us denote by $\tilde{\Psi}(x, t, \lambda)$ the matrix whose columns are the vectors $\psi(x, t, \gamma_i)$, $\gamma_i = (\lambda, \mu_i)$, and by $F(x, t, \lambda)$ the diagonal matrix $F_{ij}(x, t, \lambda) = f(x, t, \gamma_i)\delta_{ij}$. Then one can write (2.79) in the form

$$\tilde{\Psi}(x, t, \lambda) = H(x, t, \lambda) F(x, t, \lambda). \tag{2.80}$$

We have

$$\begin{aligned} U(x, t, \lambda) &= \tilde{\Psi}_x \tilde{\Psi}^{-1} = H_x H^{-1} + H F_x F^{-1} H^{-1}, \\ V(x, t, \lambda) &= \tilde{\Psi}_t \tilde{\Psi}^{-1} = H_t H^{-1} + H F_t F^{-1} H^{-1}. \end{aligned} \tag{2.81}$$

Without loss of generality we may suppose that H is regular and nondegenerate at the points λ_i. From this we get that $F_x F^{-1}$ coincides modulo $O(1)$ with the eigenvalues of the singular part of $U(x, t, \lambda)$ at λ_i; and similarly for $F_t F^{-1}$.

Thus, in a neighbourhood of P_α

$$f(x, t, \gamma) = \exp(q_\alpha(x, t, k_\alpha)) f_\alpha(x, t, \gamma). \tag{2.82}$$

Here $k_\alpha^{-1}(P)$ is a local parameter in the neighbourhood of P_α, $k_\alpha^{-1}(P_\alpha) = 0$, $q_\alpha(x, t, k)$ is polynomial in k, and f_α is a regular function in the neighbourhood of P_α.

Summing up, we come to the following assertion.

Theorem 2.1. *The vector function $\psi(x, t, \gamma)$*

1°. *is meromorphic on Γ outside the points P_α. Its divisor of poles does not depend on x, t. If W is nondegenerate, then generically the curve Γ is nonsingular. The number of poles of ψ (counting multiplicity) is equal to $g + l - 1$, where g is the genus of the curve Γ.*

2°. *in a neighbourhood of the points P_α the function $\psi(x, t, \gamma)$ has the form:*

$$\psi(x, t, \gamma) = \left(\sum_{s=0}^{\infty} \xi_{s\alpha}(x, t) k_\alpha^{-s} \right) \exp(q_\alpha(x, t, k_\alpha)), \tag{2.83}$$

where the first factor is the expansion with respect to the local parameter $k_\alpha^{-1} = k_\alpha^{-1}(\gamma)$ of a holomorphic vector, and $q_\alpha(x, t, k)$ is a polynomial in k.

The basic idea of the algebraic-geometric version of the inverse problem consists in reconstructing the vector $\psi(x, t, \gamma)$ from the enumerated analytic properties. The specific nature of these properties guarantees the existence of $U(x, t, \lambda)$ and $V(x, t, \lambda)$, $W(x, t, \lambda)$ such that (2.66) and (2.69) hold. A consequence of the compatibility of these systems are the equations (2.37), (2.64), (2.65).

As has already been said, the development of the fundamental stages of the theory of *finite gap integration* is mirrored in detail in [115] and in the surveys [40], [41], [44], [45], [76], [83], [86], [113].

Before going over to the procedure for reconstructing ψ, let us quote what we need to know from the classical algebraic geometry of Riemann surfaces and the theory of theta functions.

An arbitrary compact *Riemann surface* can be given by an equation

$$R(\lambda, \mu) = \sum a_{ij} \lambda^i \mu^j, \tag{2.84}$$

where i, j run through some finite set of integers. Generically this curve will be nonsingular. The genus of this curve can be found conveniently with the aid of the so-called Newton polygon, which is what one calls the convex hull of the integer points with the coordinates i, j for which $a_{ij} \neq 0$ in (2.84). The genus of the curve is equal to the number of integer points lying within the Newton polygon.

A basis of the holomorphic differentials (of the first kind) on a nonsingular curve has the form

$$\eta_{ij} = \frac{\lambda^i \mu^j}{R_\mu(\lambda, \mu)} d\lambda, \tag{2.85}$$

where the i, j belong to the interior of the Newton polygon.

On the curve Γ one can choose a basis of cycles $a_1, \ldots, a_g, b_1, \ldots, b_g$ with the following intersection numbers

$$a_i \circ a_j = b_i \circ b_j = 0, \qquad a_i \circ b_j = \delta_{ij}. \tag{2.86}$$

By taking suitable linear combinations, we obtain a canonical basis of the holomorphic differentials

$$\omega_1, \ldots, \omega_g \tag{2.87}$$

normalized by the conditions

$$\oint_{a_k} \omega_i = \delta_{ik}, \quad i, k = 1, \ldots, g. \tag{2.88}$$

The matrix

$$B_{ik} = \oint_{b_k} \omega_i \tag{2.89}$$

is called the period matrix of the Riemann surface Γ. It is symmetric and has a positive definite imaginary part. The unit basis vectors in $\mathbb{C}^g$ and the vectors B_i with the coordinates B_{ik} generate a lattice in $\mathbb{C}^g$, the quotient by which is a $2g$-dimensional torus $T^{2g} = J(\Gamma)$, called the *Jacobi variety* (or Jacobian) of the curve Γ.

The Riemann theta function of the surface Γ is constructed in terms of the matrix B

$$\theta(z)=\sum_{N\in\mathbb{Z}^g}\exp(\pi i\langle BN, N\rangle+2\pi i\langle N, z\rangle),$$
$$z=(z_1,\ldots,z_g),\qquad N=(N_1,\ldots,N_g), \tag{2.90}$$
$$\langle N, z\rangle=N_1z_1+\ldots+N_gz_g,$$
$$\langle BN, N\rangle=\sum B_{ij}N_iN_j.$$

This function is entire. Under translation of the argument by a vector of the lattice it is transformed according to the law

$$\theta(z+N+BM)=\exp(-\pi i(\langle BM, M\rangle+2\langle z, M\rangle))\theta(z),\quad N, M\in\mathbb{Z}^g. \tag{2.91}$$

Also often used are theta functions with characteristics

$$\theta[\alpha,\beta](z)=\exp(\pi i(\langle B\alpha,\alpha\rangle+2\langle z+\beta,\alpha\rangle))\theta(z+\beta+B\alpha),\quad \alpha,\beta\in\mathbb{R}^g. \tag{2.92}$$

Characteristics $[\alpha,\beta]$ for which all the coordinates of α,β equal 0 or 1/2 are called half-periods. A half-period $[\alpha,\beta]$ is even if $4\langle\alpha,\beta\rangle=0$ (mod 2), and odd otherwise.

The Abel map of a Riemann surface Γ into its Jacobi variety $A(P)=(A_1(P),\ldots,A_g(P))$ is given in the following way

$$A_k(P)=\int_Q^P\omega_k, \tag{2.93}$$

where Q is a fixed point on Γ.

A *divisor* on Γ is a formal integer combination of points on Γ,

$$D=\sum n_iP_i,\quad n_i\in\mathbb{Z}. \tag{2.94}$$

For any function f meromorphic on Γ there is defined the divisor (f) of its zeroes $P_1,\ldots,P_n$ and poles $Q_1,\ldots,Q_m$ (with multiplicities $p_1,\ldots,p_n$, $q_1,\ldots,q_m$ respectively)

$$(f)=p_1P_1+\ldots+p_nP_n-q_1Q_1-\ldots-q_mQ_m \tag{2.95}$$

(such divisors are called principal).

The divisors form an abelian group. The degree of a divisor is the number

$$\deg D=\sum n_i. \tag{2.96}$$

The Abel map (N.H. Abel) (2.93) can be extended linearly to the group of all divisors.

A divisor for which all $n_i\geq 0$ is called a positive divisor $D\geq 0$ (or an effective divisor).

For any divisor D the linear space $l(D)$ associated with it is the space of meromorphic functions f on Γ such that

$$(f)+D\geq 0.$$

The dimension of this space is given by the Riemann–Roch theorem [131]. For a divisor of degree greater than or equal to g,

$$\dim l(D)\geq\deg D-g+1. \tag{2.97}$$

For generic divisors (2.97) is an equality. The corresponding divisors are called non-special.

Let us consider the *Abel map*[5] of unordered sets $P_1,\ldots,P_g$ of points of Γ, i.e. of the g-th symmetric power of Γ

$$A\colon S^g\Gamma\to J(\Gamma),\quad A(P_1,\ldots,P_g)=\sum_i A(P_i). \tag{2.98}$$

The problem of inverting this map is known as the Jacobi inversion problem. Its solution (Riemann) can be given in the language of theta functions. Namely, if for the vector $\zeta=(\zeta_1,\ldots,\zeta_g)$ the function $\theta(A(P)-\zeta)$ is not identically equal to zero on Γ, then it has on Γ exactly g zeroes $P_1,\ldots,P_g$, giving the solution of the inversion problem

$$A(P_1)+\ldots+A(P_g)=\zeta-\mathcal{K}, \tag{2.99}$$

where $\mathcal{K}=(\mathcal{K}_1,\ldots,\mathcal{K}_g)$ is the vector of Riemann constants [40], which depend only on the Riemann surface, the choice of the basis of cycles on it, and the initial point of the Abel map.

Now we are ready to pass over to the solution of the inverse problem of reconstructing the "eigen"-vector $\psi(x,t,\gamma)$ of the operators of (2.64), (2.65) from its analytic properties.

The fundamental algebraic-geometric tool in the theory of finite gap linear operators and in the algebraic-geometric version of the inverse scattering method are the Clebsch–Gordan–Baker–Akhiezer functions. The general definition of these functions, including the multi-point ones, was given in [75] on the basis of a generalization of the analytic properties of Bloch eigenfunctions of operators with periodic and almost periodic coefficients [38], [45], [64]. Multi-point functions are functions which have essential singularities of exponential type at several points. Single-point functions of this kind (of the "exponential type" $\exp(\lambda x)$, where λ is on an algebraic curve) were introduced in the nineteenth century by Gordan and Clebsch (see [8]). Their connection with a joint eigenfunction of a pair of scalar commuting operators of relatively prime orders was first noted by Baker in the paper [9]; N.I. Akhiezer [5] stated examples of the interpretation of such functions in the spectral theory of operators on the half-line. The connection with periodic problems was not known until the beginning of the seventies (see § 6).

Definition. Let $P_1,\ldots,P_n$ be points on a Riemann surface Γ of genus g; let $k_\alpha^{-1}(P)$ be local parameters in the neighbourhood of these points, $k_\alpha^{-1}(P_\alpha)=0$,

[5] Also often called the *Jacobi map* (translator's note).

$\alpha=1, \ldots, n$; let $q_1(k), \ldots, q_n(k)$ be a set of polynomials; and let D be a divisor on Γ. An *n-point Baker–Akhiezer function* given by these data is a function: a) meromorphic on Γ outside the points P_α, with the divisor of its poles and zeroes (ψ) satisfying the condition $(\psi)+D\geq 0$; and such that b) for $P\to P_\alpha$ the product $\psi(P)\exp(-q_\alpha(k_\alpha(P))$ is analytic.

Theorem 2.2. *For a non-special divisor D of degree N the dimension of the linear space of functions with the enumerated properties is equal to $N-g+1$. In particular, if D is a generic set of g points, then ψ is uniquely determined up to a factor. It has the form:*

$$\psi(P)=c\exp\left(\sum_{\alpha=1}^{n}\int_Q^P\Omega_{q_\alpha}\right)\frac{\theta\left(A(P)+\sum_\alpha U^{(q_\alpha)}-\zeta\right)}{\theta(A(P)-\zeta)}. \tag{2.100}$$

Here Ω_{q_α} is a normalized abelian differential of the second kind with a principal part at the point P_α of the form $dq_\alpha(k_\alpha(P))$ (normalization means

$$\oint_{a_i}\Omega_{q_\alpha}=0; \tag{2.101}$$

with this condition Ω_{q_α} exists and is unique); the vector $2\pi iU^{(q_\alpha)}$ is the vector of b-periods of the differential Ω_{q_α}; $\zeta=A(D)+\mathcal{K}$.

The proof of formula (2.100) amounts to checking that it correctly defines a function on Γ. Changing the path of integration from Q to P leads to a translation of the arguments of the theta functions by a vector of the period lattice, $N+BM$. The exponent of the exponential is translated by $2\pi i\left\langle\sum_\alpha U^{(q_\alpha)}, M\right\rangle$.

From (2.91) it follows that the value of $\psi(P)$ does not depend on the choice of the path of integration. From (2.100) it follows that the function possesses all the necessary analytic properties. □

By virtue of theorem 2.1, to each finite gap solution of rank 1 of the equations (2.37), i.e. solution of the system (2.64), (2.65), there is associated a Riemann surface Γ, which generically can be considered to be nonsingular, a set of polynomials $q_\alpha(x, t, k)$ and a non-special divisor of degree $g+l-1$, where g is the genus of the curve Γ. Let us make use of theorem 2.2 for the construction of the inverse mapping.

So, let there be given the set of data enumerated above. In the linear space of Baker–Akhiezer functions corresponding to these data let us choose an arbitrary basis $\psi_i(x, t, P)$ (the polynomials $q_\alpha(x, t, k)$ depend on x and t as parameters; the ψ_i also will obviously depend on the same parameters).

By theorem 2.2 one may choose for the ψ_i the functions given by formula (2.100), in which ζ has been set equal to

$$\zeta_i=\sum_{s=1}^{g-1}A(P_s)+A(P_{g-1+i})+\mathcal{K}. \tag{2.102}$$

Theorem 2.3. *Let $\psi(x, t, P)$ be the vector function whose coordinates are the $\psi_i(x, t, P)$ constructed above. There exist unique matrix functions $U(x, t, \lambda)$, $V(x, t, \lambda)$, $W(x, t, \lambda)$, rational in λ, such that*

$$\partial_x\psi=U\psi,\quad \partial_t\psi=V\psi,\quad W\psi=\mu\psi,\quad P=(\lambda,\mu). \tag{2.103}$$

For the proof of the theorem it is enough to consider the matrix $\tilde\Psi(x, t, \lambda)$ whose columns are the vectors $\psi(x, t, P_j)$, $P_j=(\lambda, \mu_j)$. This matrix depends on the numbering of the columns (i.e. of the points P_j); however, the matrices

$$(\partial_x\tilde\Psi)\tilde\Psi^{-1},\quad (\partial_t\tilde\Psi)\tilde\Psi^{-1},\quad \tilde\Psi\hat\mu\tilde\Psi^{-1} \tag{2.104}$$

are already well-defined (i.e. do not depend on this numbering) and by virtue of the analytic properties of ψ they are rational functions of λ. These matrices are designated by U, V, W respectively. Here $\hat\mu$ is the diagonal matrix equal to $\hat\mu_{ij}=\mu_i\delta_{ij}$.

By using the path of the proof of equation (2.78) in the opposite direction, we get that $\det\tilde\Psi\neq 0$ if λ is not a branch point. From this it follows that U and V have poles only at the projections of the distinguished points P_α, and W only at the projections of the points on Γ where μ has poles.

Corollary. *The matrices U, V, W constructed by the formulas* (2.104) *satisfy the equations* (2.37), (2.64), (2.65).

Remark. The formulas (2.104) give the most economical way of proving the theorem in general, relating to arbitrary rational families. However in a majority of cases, especially those corresponding to reductions of equations, the explicit computation of the matrices U, V, W can be carried out from the requirement that in the neighbourhoods of the P_α there should hold the congruences:

$$\partial_x\psi(x, t, P)\equiv U(x, t, \lambda)\psi(x, t, P)(\mathrm{mod}\ O(1)\exp q_\alpha(x, t, k_\alpha))$$

(and the analogous congruences for V and W). Here the matrix entries of U, V, W turn out to be differential polynomials in the $\xi_{s\alpha}(x, t)$ of the expansion (2.83) of the regular part of ψ at the point P_α. This path will be traced in detail later on in examples of the construction of finite gap solutions of equations of the Lax type (see [40], [74], [75], [76]).

In the construction of the vector $\psi(x, t, P)$ from the set of data given before theorem 2.3 there is an arbitrariness connected with the possibility of choosing different bases ψ_i in the linear space of Baker–Akhiezer functions corresponding to the polar divisor D.

To this arbitrariness, under which $\psi(x, t, P)$ goes over into $g(x, t)\psi(x, t, P)$, where g is a nondegenerate matrix, there corresponds a gauge symmetry (2.38) of the equations (2.37), (2.64), (2.65) (the matrix W goes over under such a transformation into

$$W\to gWg^{-1}). \tag{2.105}$$

Let us consider two vector functions $\psi(x, t, P)$, $\tilde\psi(x, t, P)$, corresponding to two equivalent divisors D and $\tilde D$. The equivalence of these divisors means that there exists a meromorphic function $f(P)$ such that its poles coincide with D and its zeroes with $\tilde{D}$. From the definition it follows that the components of $f\tilde{\psi}$ possess the same analytic properties as the components of the vector function $\psi(x, t, P)$. Hence

$$\psi(x, t, P) = g(x, t) f(P) \tilde{\psi}(x, t, P), \tag{2.106}$$

and the functions ψ and $\tilde{\psi}$ define gauge-equivalent solutions.

We shall consider both the equations (2.37), (2.64), (2.65) and their solutions up to gauge transformations (2.38), (2.105). From (2.105) and the definition of Γ (2.70) it follows that the gauge transformations leave the curves Γ invariant.

Theorem 2.4. *The set of finite gap solutions (considered up to gauge equivalence) corresponding to a nonsingular curve Γ is isomorphic to a torus—$J(\Gamma)$—the Jacobi variety of this curve.*

The assertion of the theorem follows from the fact that by virtue of the well-known theorem of Abel two divisors are equivalent if and only if

$$\deg \mathrm{D} = \deg \tilde{\mathrm{D}}, \quad A(D) \equiv A(\tilde{D}).$$

The congruence sign means congruence modulo periods of the Jacobian of the curve Γ.

The coefficients of the polynomial $Q(\lambda, \mu)$ are integrals of the equations (2.37), (2.64). The theorem just formulated means that the level set of these integrals is generically a torus.

For special values of the integrals, for which the surface Γ has singularities, the level manifold of these integrals is isomorphic to the generalized Jacobian of the curve, which is the product of a torus with a linear space.

To multisoliton and rational solutions of the equations (2.37) correspond rational curves with singularities. To the different singularity types there also correspond different solution types. For example, in the case of singularities of the self-intersection type multisoliton solutions are obtained (see, for example, for the KdV equation § 3, [115]), and in the case of singularities of the "cusp" type rational solutions are gotten.

Let us consider at greater length a number of examples connected with *hyperelliptic curves*. As has already been said, finite gap solutions of the KdV equation are the restriction of this equation to the stationary points of one of the higher-order analogues of the KdV equation. They satisfy an ordinary differential equation equivalent to the operator equation

$$[L, A_n] = 0, \tag{2.107}$$

where A_n is a differential operator of order $2n+1$. As was shown above, this equation admits the λ-representation (2.32), where U_L has the form (2.33), and

$$W_A = \begin{pmatrix} 0 & \lambda^n \\ \lambda^{n+1} & 0 \end{pmatrix} + O(\lambda^{n-1}). \tag{2.108}$$

Hence the characteristic equation (2.70),

$$\det(\mu \cdot 1 - W_A(x, t, \lambda)) = \mu^2 - R_{2n+1}(\lambda) = 0, \tag{2.109}$$

gives a hyperelliptic curve Γ. The coefficients of the polynomial

$$R_{2n+1} = \lambda^{2n+1} + \sum_{i=1}^{2n+1} r_i \lambda^{2n+1-i}$$

are polynomials in $u, u', \ldots, u^{(2n+1)}$ and, in the proved fashion, integrals of the equation (2.107).

The curve Γ may be represented as being glued together out of two copies of the λ plane along cuts joining the E_i—the zeroes of the polynomial R_{2n+1}—and the point at infinity $E = \infty$.

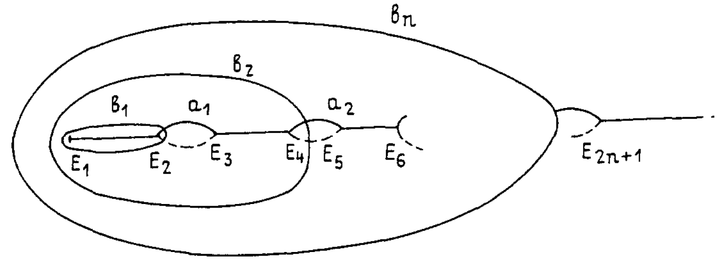

Fig. 1

For real $u(x)$ the E_i are the simple points of the spectrum of the periodic and antiperiodic problems for the *Sturm–Liouville operator* (J.C.F. Sturm–J. Liouville)

$$L = -\frac{d^2}{dx^2} + u(x), \tag{2.110}$$

the segments $[E_{2i-1}, E_{2i}]$ are the allowed bands of the spectrum of L on the entire axis, and the $[E_{2i}, E_{2i+1}]$, $i = 1, \ldots, n$ are the forbidden bands[6]. (For more details about the spectral theory of finite gap operators see § 6.)

As the a-cycles it is convenient to choose the cycles situated above the forbidden bands, and as the b-cycles, cycles encompassing the segment $[E_1, E_{2i}]$ of the real axis.

Let $\psi(x, t, P)$ be the Baker–Akhiezer function having n poles (n is the genus of Γ) $\gamma_1, \ldots, \gamma_n$ and in a neighbourhood of $P_0 = \infty$ the form

$$\psi(x, t, P) = \exp(kx + k^3 t)\left(1 + \sum_{s=1}^{\infty} \xi_s(x, t) k^{-s}\right), \tag{2.111}$$

$k = \sqrt{\lambda}$.

[6] In English, the allowed and forbidden bands are often called the *stable* and *unstable bands* respectively (translator's note).

By theorem 2.2 it exists and is unique. Let $u(x,t)=2\xi_1'$. Then a straight substitution of (2.111) gives

$$(-\partial_x^2+u(x,t)+\lambda)\psi(x,t,P)=e^{kx+k^3t}O(k^{-1}). \tag{2.112}$$

The function $\tilde{\psi}(x,t,P)$, equal to the left-hand side of (2.112), satisfies all the requirements defining ψ except one. Its expansion (2.111) in the neighbourhood of P_0 begins with $\tilde{\xi}_1 k^{-1}+\ldots$. From the uniqueness of ψ it follows that $\tilde{\psi}=0$.

Analogously, there exist unique functions $v_1(x,t)$ and $v_2(x,t)$ such that

$$A\psi-\partial_t\psi=O(k^{-1})e^{kx+k^3t}, \tag{2.113}$$

where

$$A=\partial_x^3+v_1\partial_x+v_2. \tag{2.114}$$

From (2.113) we have

$$v_1=-3\xi_1'=-\tfrac{3}{2}u.$$

The compatibility of (2.113) and (2.112) is equivalent to the KdV equation.

The normalized *abelian differentials of the second kind* having poles at P_0 of second and fourth order are representable in the form:

$$\Omega^{(2)}=\frac{\lambda^n+\sum_{i=1}^{n} r_i\lambda^{n-i}}{\sqrt{R_{2n+1}(\lambda)}}\,d\lambda, \tag{2.115}$$

$$\Omega^{(4)}=\frac{\lambda^{n+1}-s_1\lambda^n+\sum_{i=1}^{n}\tilde{r}_i\lambda^{n-i}}{\sqrt{R_{2n+1}(\lambda)}}\,d\lambda, \tag{2.116}$$

where $R_{2n+1}(\lambda)=\prod_{i=1}^{2n+1}(\lambda-E_i)$, $s_1=\sum_{i=1}^{2n+1}E_i$.

The coefficients r_i, $\tilde{r}_i$ are determined by the normalization conditions

$$\int_{E_{2i}}^{E_{2i+1}}\Omega^{(2)}=\int_{E_{2i}}^{E_{2i+1}}\Omega^{(4)}=0,\quad i=1,\ldots,n. \tag{2.117}$$

By theorem 2.2

$$\psi(x,t,P)=c\exp\left(x\int_{E_1}^{P}\Omega^{(2)}+t\int_{E_1}^{P}\Omega^{(4)}\right)\frac{\theta(A(P)+Ux+Vt-\zeta)}{\theta(A(P)-\zeta)},\quad c=c(x,t), \tag{2.118}$$

where

$$\pi i U_k=\int_{E_1}^{E_{2k}}\Omega^{(2)}, \tag{2.119}$$

$$\pi i V_k=\int_{E_1}^{E_{2k}}\Omega^{(4)}. \tag{2.120}$$

If we choose E_1 as the initial point for the Abel mapping, then after an explicit calculation of the vector of Riemann constants we get

$$\zeta_k=\sum_{i=1}^{n}\int_{E_{2i}}^{\gamma_i}\omega_k. \tag{2.121}$$

In the neighbourhood of P_0 we have

$$A(P)=-Uk^{-1}+O(k^{-2}) \tag{2.122}$$

and by expanding (2.118) we arrive finally at the Matveev-Its formula [64] for the finite gap solutions of the equation

$$u(x,t)=-2\partial_x^2\ln\theta(Ux+Vt-\zeta)+\text{const}. \tag{2.123}$$

Remark 1. The above construction of finite gap solutions of the KdV equation was carried out with the aid of the original commutational representation without an explicit transition to the λ-representation (2.32) in (2×2) matrices. For a comparison with theorems 2.1, 2.2 let us indicate only that the components of the vector ψ which figures in their formulation are given by $\psi(x,t,\lambda)$ and $\psi_x(x,t,\lambda)$. The divisor of poles of ψ coincides with the divisor of poles of $\psi(x,t,\lambda)$ together with the point P_0.

Let us also indicate that theorem 2.3 associates with the algebraic-geometric spectral data $\Gamma,\gamma_1,\ldots,\gamma_g$, besides the operators L and A, a matrix W which is W_{A_n} = the λ-representation of the operator A_n of (2.107). The operator A_n itself can be recovered from these data in an analogous way to the construction of the operator L.

Its coefficients can be determined uniquely from the congruence

$$A_n\psi\equiv\mu\psi\ (\mathrm{mod}\ e^{kx+k^3t}O(k^{-1})),\quad \mu=k^{2n+1}+a_1k^{2n-1}+\ldots.$$

From the congruence there follows the exact equality

$$A_n\psi(x,t,P)=\mu(P)\psi(x,t,P);\quad (\mu,\lambda)=P.$$

Remark 2. In a number of applications the equations of motion of the zeroes $\gamma_i(x,t)$ of the *Bloch function* $\psi(x,t,P)$, which were first obtained in [38], turn out to be useful.

For this let us consider the functions $\psi_x\psi^{-1}$ and $\psi_t\psi^{-1}$. The function $\psi_x\psi^{-1}$ has poles at the points $\gamma_i(x,t)$ and the form $k+O(k^{-1})$ in the neighbourhood of P_0. Hence it is representable unambiguously in the form

$$\frac{\psi_x}{\psi}=\frac{\sqrt{R(\lambda)}+P(x,t,\lambda)}{\prod_{i=1}^{n}(\lambda-\gamma_i(x,t))},$$

where $P(x,t,\lambda)$ is a polynomial of degree $(n-1)$. It is uniquely determined by the fact that $\psi_x\psi^{-1}$ has a pole over $\lambda=\gamma_i$ only for one sign of $\sqrt{R}$ (for example, for the sign plus). Hence

$$P(x, t, \gamma_i(x, t)) = \sqrt{R(\gamma_i(x, t))}.$$

In the neighbourhood of a pole of $\psi_x\psi^{-1}$ we have

$$\frac{\psi_x}{\psi} = \frac{\gamma_i'(x, t)}{\lambda - \gamma_i(x, t)} + O(1), \qquad \gamma_i' = \frac{\partial}{\partial x}\gamma_i.$$

Comparing the preceding equalities, we finally find that

$$\gamma_i'(x, t) = \frac{2\sqrt{R(\gamma_i(x, t))}}{\prod_{j \neq i}(\gamma_i(x, t) - \gamma_j(x, t))}.$$

Analogously

$$\frac{\psi_t}{\psi} = \frac{\lambda\sqrt{R(\lambda)} + P_1(x, t, \lambda)}{\prod_i (\lambda - \gamma_i(x, t))}$$

and, repeating the derivation of the equations for the γ_i', we get

$$\dot{\gamma}_i = \frac{2\gamma_i\sqrt{R(\gamma_i)}}{\prod_{j \neq i}(\gamma_i - \gamma_j)}, \quad \dot{\gamma}_i = \frac{\partial}{\partial t}\gamma_i.$$

The Abel isomorphism (2.98) linearizes these equations on the Jacobian $J(\Gamma)$.

As a second example let us consider the construction of finite gap solutions of the sine-Gordon equation (2.44), which were first obtained in [68].

It follows from (2.64), (2.65) that $W(\xi, \eta, 0)$ commutes with the singular part of U at $\lambda = 0$; $W(\xi, \eta, \infty)$ commutes with the singular part of V at the point $\lambda = \infty$. Hence the hyperelliptic curve Γ corresponding to a finite gap solution of the sine-Gordon equation has branching at the points $\lambda = 0$, $\lambda = \infty$.

Without retracing word for word the course of the proof of theorem 2.1, let us give the form of the Baker–Akhiezer vector functions for this equation. The components $\psi_i(\xi, \eta, P)$ have n poles γ_i outside the branch points P_+ and P_-, situated above $\lambda = 0$, $\lambda = \infty$. In a neighbourhood of these points

$$\psi_1^{\pm} = e^{k(x \pm t)}\left(\sum_{s=0}^{\infty} \chi_{s1}^{\pm}(\xi, \eta)k_{\pm}^{-s}\right), \tag{2.124}$$

$$\psi_2^{\pm} = e^{k(x \pm t)}k^{\pm 1}\left(\sum_{s=0}^{\infty} \chi_{s2}^{\pm}(\xi, \eta)k_{\pm}^{-s}\right), \tag{2.125}$$

$$\xi = x + t, \quad \eta = x - t, \quad k_{\pm} = \lambda^{\mp 1/2}.$$

The functions ψ_i are determined uniquely by the normalization $\chi_{0i}^{+} \equiv 1$. (The divisor D of degree $n + l - 1 = n + 1$ is equal to $\gamma_1 + \ldots + \gamma_n + P_+$.)

It follows from the definitions of ψ_1 and ψ_2 that $\partial_\eta\psi_1$ and $\lambda\psi_2$ have the same analytic properties. So they are proportional. For the computation of the constant of proportionality one must compare the coefficients of the term $\lambda^{1/2}$ in the expansions of these functions at P_+. We have

$$\partial_\eta\psi_1 = e^{-iu}\lambda\psi_2, \qquad e^{-iu} = \frac{\chi_{01}^{-}}{\chi_{02}^{-}}. \tag{2.126}$$

Analogously,

$$\partial_\eta\psi_2 = e^{iu}\psi_1. \tag{2.127}$$

In the same way it can be shown that

$$\partial_\xi\psi_1 = \frac{iu_\xi}{2}\psi_1 + \psi_2, \tag{2.128}$$

$$\partial_\xi\psi_2 = \lambda^{-1}\psi_1 - \frac{iu_\xi}{2}\psi_2. \tag{2.129}$$

Corollary. *The function $u(\xi, t)$ defined out of (2.126) is a solution of the sine-Gordon equation.*

Let us find its explicit appearance. It can be shown analogously to theorem 2.2 that

$$\psi_n(\xi, \eta, P) = r_n(\xi, \eta)\cdot\exp\left(\xi\int_Q^P \Omega_+^{(2)} + \eta\int_Q^P \Omega_-^{(2)} + n\int_Q^P \Omega_{+-}\right) \times \frac{\theta(A(P) - \zeta + U^+\xi + U^-\eta + Vn)}{\theta(A(P) - \zeta)}. \tag{2.130}$$

Here the $\Omega_{\pm}^{(2)}$ are normalized abelian differentials with poles of second order at the points $P_\pm$, Ω_{+-} is a differential of the third kind with the residues ± 1 at $P_\pm$, $2\pi i U^{\pm}$, $2\pi i V$ are the vectors of b-periods of these differentials.

The factor $r_n(\xi, \eta)$ is chosen via the condition that the multiplier in front of the exponential be equal to one at the point P_+. Then χ_{0n}^{-} equals the value of this multiplier at P_-.

After simple computations we finally arrive at the following expression for the finite gap solutions

$$e^{iu} = \text{const}\,\frac{\theta^2(U^+\xi + U^-\eta - \zeta)}{\theta(U^+\xi + U^-\eta - \zeta + V)\theta(U^+\xi + U^-\eta - \zeta - V)}. \tag{2.131}$$

Let us note that the vector V is equal to a half-period, since by virtue of the Riemann relations and Abel's theorem

$$2V = 2(A(P_+) - A(P_-)) \equiv 0$$

(the last congruence holds inasmuch as the divisors $2P_+$ and $2P_-$ are equivalent, being the zeroes and the poles of λ on Γ).

Until now we have been talking about the construction of complex solutions of nonlinear equations which admit a commutational representation of one of the enumerated forms. Picking out the real nonsingular solutions among them turns out to be comparatively easy in those cases in which the auxiliary linear problem

$$L\psi=\lambda\psi \tag{2.132}$$

for the Lax representation, or

$$\left(\frac{\partial}{\partial x}+U(x,t,\lambda)\right)\psi=0, \tag{2.133}$$

in the case of the general representation (2.37), is self-adjoint. However for almost all nonlinear equations (the nonlinear Schrödinger equation, the sine-Gordon equation, the equations of the nonlinear interaction of wave packets etc.) the corresponding linear problems are not self-adjoint.

The typical conditions which select physically interesting real solutions have one of the following types

$$U(x,t,\lambda)=JU^{+}(x,t,\sigma(\lambda))J^{-1}, \tag{2.134}$$

$$U(x,t,\lambda)=J\bar{U}(x,t,\sigma(\lambda))J^{-1}, \tag{2.135}$$

where the cross denotes the Hermitian adjoint, $\sigma(\lambda)$ is an antiholomorphic involution of the λ plane (e.g., $\lambda\to\bar{\lambda}$, $\lambda\to\bar{\lambda}^{-1}$) and J is a diagonal matrix with entries $\varepsilon_k=\pm 1$.

Since one can also subject the matrix $W(x,t,\lambda)$ to the same realness conditions, the curves Γ, given by equation (2.70), which arise in the construction of real finite gap solutions are real, i.e. there is an antiholomorphic involution defined on them

$$\tau\colon\ \Gamma\to\Gamma,$$

which leaves the distinguished points P_α fixed, or permutes them in a well-defined fashion.

The description of real curve types and, what is the most difficult and interesting part, the distribution on them of the poles γ_i of Baker–Akhiezer functions leading to real solutions, pose problems of real algebraic geometry which until comparatively recently had not been worked out at all. (The first serious progress in the solution of these problems in connection with the nonlinear Schrödinger equation and the sine-Gordon equation was made in [68] and [28], although the results obtained in these papers are far from being effective).

A detailed exposition of recent achievements in real finite gap integration is given in [13], [41], [44], [46]. Here let us describe on the basis of the two examples analyzed above the two basic types of involutions on the set of divisors, whose various combinations give all the realness conditions known at present.

Let Γ be a *real hyperelliptic curve*, i.e. a curve given by equation (2.109) with a real polynomial R_{2n+1}. If the set of points $\gamma_1,\ldots,\gamma_n$ is invariant with respect to the antiholomorphic involution

$$\tau\colon\ (\lambda,\mu)\to(\bar{\lambda},\bar{\mu})$$

and $\psi(x,t,P)$ is the Baker–Akhiezer function corresponding to them, then

$$\bar{\psi}(x,t,\tau(P))=\psi(x,t,P), \tag{2.136}$$

since both the right and left-hand sides have the same analytic properties and are equal to each other by virtue of the uniqueness of ψ. From this it follows at once that the corresponding finite gap solution $u(x,t)$ of the KdV equation is real.

Now let us suppose that τ has $n+1$ fixed ovals $a_1,\ldots,a_{n+1}$ on Γ, on one of which the point $P_0=\infty$ lies. (In real algebraic geometry the curves of genus g with $g+1$ real ovals are called M-curves[7].) In the case under consideration this means that all the branch points E_i are real.

If the points γ_i are distributed one on each oval, $\gamma_i\in a_i$, then $u(x,t)$ has no singularities. Indeed, as is evident from the construction of ψ, a pole of $u(x,t)$ arises only when one of the n zeroes of ψ hits P_0 (here $\theta(Ux+Vt-\zeta)=0$). But by virtue of (2.136) ψ is real on real ovals. Since ψ has a pole on each a_i it also has at least one zero. Since there are n zeroes in all, they are separate from P_0.

For the finite gap solutions of the sine-Gordon equation to be real it is necessary that the hyperelliptic curve Γ be real [68]. On it let us consider the anti-involution

$$\tau\colon\ (\lambda,\mu)\to(\bar{\lambda},-\bar{\mu}).$$

The action of this anti-involution on the local parameters $k_\pm^{-1}$ is such that

$$\tau^*(k_\pm)=-\bar{k}_\pm.$$

Let the polar divisor of $\psi_n(\xi,\eta,P)$ satisfy the condition

$$D+\tau(D)\equiv K+P_++P_-, \tag{2.137}$$

where K is the canonical class, i.e. the zero divisor of a holomorphic differential on Γ.

Condition (2.137) means that $D,\tau(D)$ are the zeroes of a differential of the third kind

$$\omega=\frac{d\lambda}{\lambda}\,\frac{\lambda^{n+1}+\alpha_1\lambda^n+\ldots+\alpha_{n+1}}{\sqrt{R_{2n+1}(\lambda)}} \tag{2.138}$$

with poles at the points $P_\pm$.

[7] For some authors, M-curves are those with $g+1$ *components*, which need not all be ovals (translator's note).

Let us consider the differential

$$\psi_1(\xi, \eta, P)\bar{\psi}_1(\xi, \eta, \tau(P))\omega. \tag{2.139}$$

From (2.124) and (2.137) it follows that this is a meromorphic differential with its only poles at the points $P_\pm$. The residue of this differential at P_+ is equal to 1. Since the sum of the residues of any meromorphic differential is equal to zero, then

$$\chi_{01}^- \cdot \bar{\chi}_{01}^- = 1. \tag{2.140}$$

Analogously, if we consider the differential

$$\lambda\psi_2(\xi, \eta, P)\bar{\psi}_2(\xi, \eta, \tau(P))\omega,$$

we get

$$\chi_{02}^- \cdot \bar{\chi}_{02}^- = 1.$$

Hence, by (2.126),

$$|e^{-iu}| = |\chi_{01}^-| / |\chi_{02}^-| = 1$$

and $u(\xi, \eta)$ is real.

§ 3. The Hamiltonian Theory of Hyperelliptic λ-Families

In this section, following [116], [118], we shall present the Hamiltonian theory of systems which are connected with *hyperelliptic curves* (see the examples of the preceding and the following sections). These systems usually come from systems of the form (2.64), (2.65), where the matrices are (2×2).

The equations (2.64) are ordinary differential equations.

The initial "physical" coordinates on the finite-dimensional space of their solutions are the values of the matrix entries of U and W at some initial point $x = x_0$ (or rather the values of the matrix entries of the singular parts of U and W at their poles).

For example, for the "higher-order KdV equations" the space of solutions of the commutativity equations for the Sturm–Liouville operator L and an operator A_n of order $2n+1$ has dimension $3n+1$. Coordinates on it are given by

$$u(x_0), \ldots, u^{(2n+1)}(x_0), h_2, \ldots, h_n,$$

where the constants h_j arise in expressing the coefficients of the operator A_n in terms of u and its derivatives.

In the preceding section an isomorphism was set up between this space and the space

$$(\Gamma, P_1, \ldots, P_k) = N^{n+k},$$

where Γ is a hyperelliptic curve, given in the form

$$w^2 = \prod_{i=1}^{2g+1} (\lambda - \lambda_i)$$

(as for the KdV equation, the sine-Gordon equation, where $\lambda_1 = 0$), or in the form

$$w^2 = \prod_{i=1}^{2g+2} (\lambda - \lambda_i)$$

(the nonlinear Schrödinger equation, the Toda lattice etc.). Coordinates in the neighbourhood of P_j are given by $\lambda(P)$—the projections of the points onto the λ plane. In the following (when this does not give rise to misunderstanding) the points P_j will be denoted as $\gamma_j = \lambda(P_j)$ without indicating $\varepsilon_j = \pm$, the number of the sheet of the surface Γ.

The space N^{n+k} is fibred over M^n, the manifold of hyperelliptic curves. Coordinates on M^n are given by the λ_i. The fibre of this fibration

$$N^{n+k} \to M^n$$

is $S^k\Gamma$—the kth symmetric power of the curve Γ.

In the fundamental examples $k = g$ (KdV, sine-Gordon) or $k = g+1$ (NLS). In the first case the fibre, by virtue of Abel's theorem, is birationally isomorphic to a complex torus—the Jacobian of the curve, $J(\Gamma)$.

Let us define *analytic Poisson brackets* on the phase space N^{n+k} of our systems.

a) Let A be some set of functions on N^{n+k} which depend only on the point of the base space M^n, i.e. on the hyperelliptic curve. (In the sequel A will play the rôle of the annihilator of the Poisson bracket, which becomes nondegenerate on the manifolds N_A given by the equations $f = \text{const}$ for all $f \in A$; $N_A \to M_A$, $M_A \subset M^n$.)

b) Let a meromorphic 1-form $Q(\Gamma)$ be given on the Riemann surface Γ or on a covering of it $\hat{\Gamma} \to \Gamma$. In local notation

$$Q(\Gamma) = Q(\Gamma, \lambda) d\lambda. \tag{2.141}$$

The derivatives of $Q(\Gamma)$ in all directions of the base space tangent to the manifolds M_A are required to be globally defined meromorphic differential forms on the Riemann surface Γ itself (and not on the covering).

c) In all of the major examples the form Q has turned out either to be meromorphic on Γ right from the start, or to be meromorphic on a regular covering $\hat{\Gamma}$ with an abelian monodromy group, where the image of $\pi_1(\hat{\Gamma}) \to \pi_1(\Gamma)$ is generated by a set of cycles with vanishing pairwise intersection numbers.

Definition. If the closed 2-form

$$\Omega_Q = \sum dQ(\Gamma, \gamma_j) \wedge d\gamma_j \tag{2.142}$$

is nondegenerate at a "general" point of a region of the N_A, where the pair (A, Q) has the properties a), b), c), then it will be said that an analytic Poisson bracket with annihilator A is given on an open region of N^{n+k}. The dimension of N_A in this case must be equal to $2k$.

By definition, the Poisson bracket of (2.142) is given by the properties

$$\{\gamma_i, \gamma_j\}=0, \qquad \{Q(\gamma_i), Q(\gamma_j)\}=0,$$
$$\{Q(\gamma_i), \gamma_j\}=\delta_{ij}, \tag{2.143}$$
$$\{f, \gamma_j\}=\{f, Q(\gamma_k)\}=0, \qquad f\in A.$$

If λ_i are any coordinates on the manifold M_A then it follows from (2.142) that Ω_Q contains only terms of the form $d\lambda_i \wedge d\gamma_j$ in its expansion. This at once implies the proposition:

Any two functions g, h which depend only on $\Gamma\in M^n$ are in involution

$$\{g(\Gamma), h(\Gamma)\}=0. \tag{2.144}$$

Let $\tau_1, \ldots, \tau_k$ be the tangent directions to M_A at a "point in general position". By the definition of an analytic Poisson bracket, $\nabla_{\tau_i}Q$ is a meromorphic differential. Like any other differential, it can be decomposed uniquely (if a basis of a-cycles is fixed (2.86)) as a sum of a holomorphic differential ω_i and normalized (see (2.101)) differentials $\tilde{\omega}_i$ and $\tilde{\tilde{\omega}}_{it}$ of the second and third kinds respectively:

$$\nabla_{\tau_i}Q=\omega_i+\tilde{\omega}_i+\sum \tilde{\tilde{\omega}}_{it}, \tag{2.145}$$

where $\tilde{\tilde{\omega}}_{it}$ has a pair of first-order poles at points (P_t', P_t'').

Without loss of generality, when $k\geq g$ one may assume that locally the coordinates τ_i have been chosen so that the ω_i for $i\leq g$ form a normalized

$$\oint_{a_i} \omega_j=\delta_{ij}$$

basis of the holomorphic differentials, and for $j>g$ $\omega_j=0$.

Let us consider the flows on N^{n+k} generated by Hamiltonians of the form $H(\Gamma)$, which by (2.144) commute with each other.

Theorem 2.5. *Let the coordinates τ_i be as indicated above. Then at a point*

$$(\Gamma_0, \gamma_1, \ldots, \gamma_k)$$

in general position the complex variables

$$\psi_j=\sum_{i=1}^{k} \int_{P_0}^{\gamma_i} \nabla_{\tau_j}Q \tag{2.146}$$

are independent and have dynamics linear with respect to time.

The proof can be obtained entirely analogously to the standard Liouville procedure.

The definition of the ψ_j (2.146) depends on the choice of the paths between P_0 and the γ_i. Therefore these quantities are determined up to the lattice in $\mathbb{C}^k$ generated by the $2g+l$ periods e_q, e_q', η_s of the gradient ∇Q:

$$e_q^i=\oint_{a_q} \nabla_{\tau_i}Q=\delta_q^i, \quad e_q'^i=\oint_{b_q} \nabla_{\tau_i}Q, \quad \eta_s^i=\operatorname{res}_{P_s}\nabla_{\tau_i}Q. \tag{2.147}$$

The transformation (2.146) allows one on the basis of an analytic Poisson bracket (A, Q) satisfying the requirements enumerated above to construct a fibration

$$N_{(Q,A)} \xrightarrow{J_Q(\Gamma)} M_A$$

whose fibre is the quotient of $\mathbb{C}^k$ by the lattice generated by the vectors (2.147).

Let κ be the number of functionally independent (modulo A) residues of the form Q, $\kappa\leq l$.

In general,

$$2g+\kappa\leq 2k. \tag{2.148}$$

The variables ψ_j form a compact torus T^{2k} only in the case when $2g+\kappa=2k$. By no means will one always get an abelian torus. For this it is necessary and sufficient that $k=g$ and that all the forms $\nabla_{\tau_i}Q$ be holomorphic.

Comparing (2.146) with the definition of the Abel map (2.93), we get the following theorem.

Theorem 2.6. *The Abel transformation $S^g\Gamma\to J(\Gamma)$ linearizes the dynamics of all Hamiltonians of the form $H(\Gamma)$ for Poisson brackets given by generic pairs (A, Q), if and only if the derivatives $\nabla_{\tau_i}Q$ in all directions tangent to M_A give a basis of the holomorphic differentials on Γ.*

Only for the real theory does it make sense to discuss a special choice of the vectors τ_i, corresponding to the differentiation of Q with respect to so-called "action variables" canonically conjugate to the "angles" on tori, varying from 0 to 2π.

Let us consider *real hyperelliptic curves* Γ. These are curves with an antiholomorphic involution σ_Γ: $\Gamma\to\Gamma$, which is induced by an antiholomorphic involution σ on the space M^n of all hyperelliptic curves.

The form Q and the annihilator A must also be compatible with σ, σ_Γ in a natural way:

$$\text{a)}\quad \sigma_\Gamma^* Q=\bar{Q}, \qquad \text{b)}\quad \sigma^* A=\bar{A}. \tag{2.149}$$

The simplest example of real structures, which may be called the "elementary" ones for the Hamiltonian systems which interest us, actually already appeared earlier in the description of the real nonsingular finite gap solutions of the KdV equation.

Let σ_Γ have $g+1$ or g fixed ovals on Γ. (Such curves are called M-curves or $M-1$-curves). In the first case $k=g$ or $k=g+1$. In the second case $k=g$.

Lemma. *If the Poisson bracket (A, Q) has the properties (2.149) then sets of points γ_i $(i=1, \ldots, g+1$ or $i=i_1, \ldots, i_g)$ lying on pairwise distinct fixed ovals a_i of the anti-involution σ_Γ are invariant with respect to the dynamics generated by the real Hamiltonians $H(\Gamma)$, $H(\sigma(\Gamma))=\bar{H}(\Gamma)$.*

For M-curves and $k=g$ the admissible sets of $\gamma_i \in a_i$ form $g+1$ connected components isomorphic to the real torus T^g. For M-curves and $k=g+1$ or for $(M-1)$-curves and $k=g$ there is only one connected component—a real torus T^{g+1} or T^g.

An example of a non-elementary real structure arose in the description of the real solutions of the sine-Gordon equation (§ 2).

The effective assignment of such a structure is possible only in terms of coordinates on N connected with γ by the transformation (2.146).

By a non-elementary real structure will be meant an anti-involution

$$\tau\colon N_{(Q,A)} \to N_{(Q,A)}$$

which is compatible with the fibration

$$N_{(Q,A)} \xrightarrow{J_Q(\Gamma)} M_A.$$

On the fibres of $J_Q(\Gamma)$ there must be a superposition of a translation and an automorphism of $J_Q(\Gamma)$ as a real commutative group.

The real submanifolds in the *phase space* are picked out by the conditions

$$\tau(\eta)=\bar{\eta} \tag{2.150}$$

or

$$\tau(\eta)=-\bar{\eta}+\eta_0^\alpha. \tag{2.151}$$

The case (2.151) is realized for the sine-Gordon and the NLS (2.14). For a given curve the vector η_0^α may take on a finite number of values. Their computation for the sine-Gordon equation was done for the first time in [46].

Theorem 2.7. *For analytic Poisson brackets satisfying the elementary and non-elementary realness conditions, the action variables J_j, canonically conjugate to the coordinates on the tori T^k varying from 0 to 2π, are given by the formula*

$$J_j=\frac{1}{2\pi}\oint_{a_j} Q(\Gamma,\lambda)\,d\lambda. \tag{2.152}$$

The proof of the theorem follows in essence from the course of the proof of Liouville's theorem and from the fact that (2.152) represents the quantity

$$J_j=\frac{1}{2\pi}\oint_{\tilde{a}_j} p\,dq, \tag{2.153}$$

where the $\tilde{a}_j$ are the basis 1-cycles on the tori T^k.

For the class of Poisson brackets under study the action variables J_j (2.153) acquire an important interpretation as integrals over the elements a_j of the group $H_1(\Gamma\backslash P,\mathbb{Z})$, where P is the set of poles of the form Q. This results in a significantly greater effectiveness in the construction of action variables than in Liouville's theorem. In particular, for example, until [116] no explicit construction of action variables for the Kovalevskaya case (see below) was known, since the only method for constructing them known to the classical workers was the method of separation of the variables in the Hamilton–Jacobi equation.

Theorem 2.6 gives necessary and sufficient conditions on the bracket (Q,A) which guarantee the linearization by the Abel mapping of the Hamiltonian flows generated by Hamiltonians $H(\Gamma)$.

As is well known, the Abel mapping linearizes all the higher-order KdVs. Let us express the Hamiltonians corresponding to these flows in terms of the form Q.

Theorem 2.8. *The coefficients of the expansion*

$$Q(\Gamma,\lambda)=\sum_{k=0}^{\infty}\left(\frac{z}{2}\right)^k q_k(\Gamma), \qquad z=\lambda^{-1/2}, \tag{2.154}$$

are such that the $h_l(\Gamma)=q_{2l+3}(\Gamma)$ are the Hamiltonians of the higher-order KdVs with the number $l\ge 0$. The remaining coefficients q_k belong to the annihilator A.

In conclusion let us enumerate a number of major examples.

Example 1. The *Gardner–Zakharov–Faddeev bracket.* From [56] one can extract

$$Q=2ip(\lambda)d\lambda, \quad A=\left\{T_1,\ldots,T_g,\bar{u}=\lim_{T\to\infty}\frac{1}{T}\int_0^T u\,dx\right\}, \tag{2.155}$$

$p(\lambda)$ is the quasimomentum, where $dp(\lambda)$ is a differential of the second kind with a unique pole at $\lambda=0$,

$$\oint_{a_j} dp(\lambda)=0, \qquad j=1,\ldots,g. \tag{2.156}$$

The periods T_j of the quasiperiodic potential $u(x)$ are defined as

$$\frac{1}{2\pi i}\oint_{b_j} dp=T_j. \tag{2.157}$$

Example 2. The *Magri bracket* (F. Magri) [94]. In this bracket the higher-order KdV equations have the form

$$\dot{u}=\left(al+b\frac{\partial}{\partial x}\right)\frac{\delta H}{\delta u}, \qquad l=\frac{1}{2}\frac{\partial^3}{\partial x^3}+u\frac{\partial}{\partial x}+\frac{\partial}{\partial x}u. \tag{2.158}$$

Here we have

$$Q=2ip(\lambda)(a\lambda+b)^{-1}\,d\lambda. \tag{2.159}$$

For $b=0$ the annihilator is

$$A=\{T_1,\ldots,T_g,J\},$$

where

$$J=\sum_{k=0}^{g+1} c_k I_{g-k+1} \tag{2.160}$$

is a linear combination of Kruskal integrals, and its extremals are given by the finite gap solutions constructed over Γ.

Example 3. The Hamiltonian formalism of the stationary problem for the higher-order KdV.

The commutativity equation (2.61) may be presented in the form

$$\delta J = 0, \tag{2.161}$$

where J is the same as in (2.160). This representation naturally gives rise to the Hamiltonian formalism of the system (2.61) (see [18], [20]). From [6] one can extract

$$Q = \sqrt{-R(\lambda)d\lambda}, \qquad R(\lambda) = \prod(\lambda - \lambda_i).$$

The annihilator of the bracket is generated by the first $(g+1)$ symmetric polynomials in the λ_i.

Example 4. The Hamiltonian structure generated by the "hidden isomorphism of Moser and Trubowitz" [136], [110] (for more details on which see the examples of the next section) between the dynamics of the KdV on the space of finite gap potentials and the Neumann systems (2.183), (2.184):

$$\begin{aligned} &Q = \sqrt{-R(\lambda)} \prod_j (\lambda - \lambda_{2j})^{-1} d\lambda, \\ &A = \{\lambda_0, \lambda_2, \ldots, \lambda_{2g}\}. \end{aligned} \tag{2.162}$$

Example 5. The integrable case of Goryachev–Chaplygin in the dynamics of a rigid body with a fixed point [71].

Here

$$Q(\Gamma, \lambda) = \arcsin \frac{1}{\mu}\left(\frac{\lambda^2}{2} - \frac{1}{2}H - \frac{2G}{\lambda}\right),$$

where H is the energy of the top, G is the Goryachev–Chaplygin integral, μ is a parameter. The curve Γ is given by the equation

$$y^2 = 4\mu^2\lambda^2 - (\lambda^3 - H\lambda - 4G)^2.$$

Example 6. In the well-known *Kovalevskaya case* the action variables formerly could not be calculated. In the notation of [71] (and of the next section, see (2.174), (2.181)) we have:

$$Q(\Gamma, \lambda) = \frac{1}{2\sqrt{-\lambda}} \ln(\sqrt{-\lambda}(\lambda - 6h)^2 - k^4) + \frac{v^2}{2\sqrt{-\lambda}}(\lambda - 8l^2) + \sqrt{-R_5(\lambda)},$$

where R_5 is given by (2.181).

The curve Γ is given by the equation $y^2 = R_5(\lambda)$. By integrating Q over the "real" cycles a_j, on which the $\gamma_j = s_j$ lie—the Kovalevskaya variables, we obtain the action variables J_j.

§4. The Most Important Examples of Systems Integrable by Two-Dimensional Theta Functions

By the example analyzed in §2, the hyperelliptic curve Γ of genus 2

$$y^2 = R_5(\lambda) = \prod_{i=1}^{5} (\lambda - \lambda_i) \tag{2.163}$$

generates a pair of commuting operators

$$L = \frac{\partial^2}{\partial x^2} + u(x) \tag{2.164}$$

$$\begin{aligned} A_5 = 16\frac{\partial^5}{\partial x^5} &+ 20\left(u\frac{\partial^3}{\partial x^3} + \frac{\partial^3}{\partial x^3}u\right) + 30u\frac{\partial}{\partial x}u - 5\left(u''\frac{\partial}{\partial x} + \frac{\partial}{\partial x}u''\right) \\ &+ h_1\left[4\frac{\partial^3}{\partial x^3} + 3\left(u\frac{\partial}{\partial x} + \frac{\partial}{\partial x}u\right)\right] + h_2\frac{\partial}{\partial x}. \end{aligned} \tag{2.165}$$

The commutativity equation (2.107) for the operators L and A_5 on the function u may be written in the Lagrangian form

$$\delta \int \Lambda\, dx = 0 \tag{2.166}$$

with the Lagrangian

$$\Lambda = \frac{u''^2}{2} - \frac{5}{2}u''u^2 + \frac{5}{2}u'' + h_1\left(\frac{u'^2}{2} + u^3\right) + h_2u^2 + h_3u. \tag{2.167}$$

According to [115], the equation (2.166) is equivalent to a Hamiltonian system with two degrees of freedom and with the Hamiltonian

$$H = p_1p_2 + V(q_1, q_2), \tag{2.168}$$

$$\begin{aligned} &q_1 = u, \quad q_2 = u'' - 5u^2, \quad p_1 = q_2', \quad p_2 = u', \\ &V = -\frac{q_2^2}{2} - \frac{5}{2}q_2q_1^2 - \frac{5}{8}q_1^4 + \frac{h_2}{2}q_1^2 + h_3q_1 \end{aligned} \tag{2.169}$$

(by a substitution $u \to u + \text{const}$ the constant h_1 has been made zero in equations (2.168), (2.169)).

The integrals of the system (2.168) in involution have the form $J_1 = H$,

$$\begin{aligned} &J_2 = p_1^2 + 2q_1p_1p_2 + (2q_2 - h_2)p_2^2 + D(q_1, q_2), \\ &D = q_1^5 + h_2q_1^3 - 4q_1q_2^2 + 2h_2q_1q_2 + 2h_3q_2. \end{aligned}$$

The integrals J_i define a curve (2.163). The corresponding polynomial R_5 is equal to

$$R_5 = \lambda^5 + \frac{h_2}{2}\lambda^3 + \frac{h_3}{16}\lambda^2 + \left(\frac{J_1}{32} + \frac{h_2^2}{16}\right)\lambda + \frac{J_2 - h_2h_3}{256}. \tag{2.170}$$

The results of § 2 indicate that coordinates on the level manifold $J_1=\text{const}$, $J_2=\text{const}$ are given by γ_1, γ_2—the locations of the poles of the corresponding Baker–Akhiezer function. Their connection with the initial variables is given by means of the so-called trace formulas

$$\gamma_1+\gamma_2=\frac{\dot{u}}{2}, \qquad \gamma_1\gamma_2=\frac{1}{8}(3u^2+u'')+\frac{1}{2}\sum_{i\neq j}\lambda_i\lambda_j, \tag{2.171}$$

where the λ_i are the zeroes of the polynomial (2.170).

The equations on the γ_i which are equivalent to the original system have, in the given case, the form (see § 2)

$$\gamma_1'=\frac{2i\sqrt{R_5(\gamma_1)}}{\gamma_1-\gamma_2}, \qquad \gamma_2'=\frac{2i\sqrt{R_5(\gamma_2)}}{\gamma_2-\gamma_1}. \tag{2.172}$$

These equations, as was already noted in § 2, are linearized by the *Abel transformation.* The two-gap potential $u(x)$ equals (2.123)

$$u(x)=2\frac{\partial^2}{\partial x^2}\ln\theta(Ux-\zeta)+\text{const}. \tag{2.173}$$

Later a number of examples will be cited of systems leading to two-gap potentials and integrable, as a consequence, by two-dimensional theta functions.

S.V. Kovalevskaya's Problem. The equations of motion of a heavy rigid body with a fixed point in Kovalevskaya's case have the form:

$$\begin{cases} 2\dot{p}=qr, \\ 2\dot{q}=-pr-\mu\gamma_3, \\ \dot{r}=\mu\gamma_2, \end{cases} \qquad \begin{cases} \dot{\gamma}_1=r\gamma_2-q\gamma_3, \\ \dot{\gamma}_2=p\gamma_3-r\gamma_1, \\ \dot{\gamma}_3=q\gamma_1-p\gamma_2, \quad \mu=\text{const}. \end{cases} \tag{2.174}$$

(A representation of the Lax type for this system was found in [121].)

The equations (2.170) have the following integrals

$$H=2(p^2+q^2)+r^2-2\mu\gamma_1 \quad \text{(the energy)},$$
$$L=2(p\gamma_1+q\gamma_2)+r\gamma_3 \quad \text{(the angular momentum)}, \tag{2.175}$$
$$K=(p^2-q^2+\mu\gamma_1)^2+(2pq+\mu\gamma_2)^2 \quad \text{(Kovalevskaya's integral)}.$$

In addition there is fulfilled the constraint condition

$$\gamma_1^2+\gamma_2^2+\gamma_3^2=1. \tag{2.176}$$

Let us consider the combined level surface of these integrals

$$H=6h^2, \qquad L=2l, \qquad K=k^2. \tag{2.177}$$

Under fulfillment of the constraint (2.176) these equations give a two-dimensional invariant submanifold of the original system (2.174).

The Kovalevskaya variables—coordinates on this surface—are defined in the following manner

$$s_{1,2}=3h+\frac{R(x_1,x_2)\mp\sqrt{R(x_1)R(x_2)}}{(x_1-x_2)^2}, \tag{2.178}$$

where

$$x_{1,2}=p\pm iq, \qquad R(z)=-z^4+6hz^2+4\mu lz+\mu^2-k^2,$$
$$R(x_1,x_2)=-x_1^2x_2^2+6x_1x_2h+2\mu l(x_1+x_2)+\mu^2-k^2. \tag{2.179}$$

An easy computation shows that in the variables s_i the equations (2.174) have the form:

$$\dot{s}_1=\frac{i}{2}\frac{\sqrt{R_5(s_1)}}{s_1-s_2}, \qquad \dot{s}_2=\frac{i}{2}\frac{\sqrt{R_5(s_2)}}{s_2-s_1}, \tag{2.180}$$

$$R_5=(\lambda[(\lambda-3h)^2+\mu^2-k^2]-2\mu^2l^2)((\lambda-3h)^2-k^2). \tag{2.181}$$

These equations coincide up to a factor with the equations (2.172). Consequently, they will be linearized by the Abel substitution.

The expressions for the original variables $p, q, r, \gamma_1, \gamma_2, \gamma_3$ in terms of the Kovalevskaya variables are cited in [63]. As for the variables s_i, they, by the results of § 2, can be defined as solutions of the equations

$$\theta(A(s_i)+Ut-\zeta)=0. \tag{2.182}$$

Here $A\colon \Gamma\to J(\Gamma)$ and Γ is given by equation (2.163) with R_5 equal to (2.181).

The Neumann and Jacobi Problems. The General Garnier System. The equations of motion of a particle on the $(n-1)$-dimensional sphere

$$x^2=\sum_{i=1}^{n}x_i^2=1 \tag{2.183}$$

under the action of a quadratic potential

$$U(x)=\frac{1}{2}\sum_{i=1}^{n}a_ix_i^2, \qquad a_i=\text{const}, \tag{2.184}$$

have the form:

$$\ddot{x}_i=-a_ix_i+u(t)x_i, \tag{2.185}$$

where $u(t)$ is a Lagrange multiplier arising because of the imposition of the constraints (2.183). When $n=3$ this system bears the name Neumann system.

The *Neumann system* may be obtained from the Hamiltonian flow on $\mathbb{R}^6$ with the Hamiltonian

$$H=\frac{1}{2}\sum_{i=1}^{3}a_ix_i^2+\frac{1}{2}(x^2y^2-(xy)^2) \tag{2.186}$$

by restriction to the surface $x^2=1$ (here $xy=\sum_i x_i y_i$). The functions

$$F_k(x, y)=x_k^2+\sum_{i\neq k}\frac{x_k y_i - y_k x_i}{a_i-a_k}, \quad k=1, 2, 3, \tag{2.187}$$

are a system of integrals in involution for (2.186). The Hamiltonian H itself has the form:

$$H=\frac{1}{2}\sum_{i=1}^{3} a_i F_i. \tag{2.188}$$

The transformation

$$\tilde{x}=y, \qquad \tilde{y}=-x, \qquad \tilde{H}=\frac{1}{2}\sum_{i=1}^{3}\frac{F_i}{a_i} \tag{2.189}$$

takes the constructed Hamiltonian flow over into a geodesic flow on the triaxial ellipsoid (when $a_i>0$)

$$\sum_{i=1}^{3}\frac{x_i^2}{a_i}=1.$$

The problem of geodesics on a triaxial ellipsoid is called the Jacobi problem.

In the work [110] a trajectory isomorphism was established between the equations in x for the periodic n-gap potentials of the Sturm–Liouville operator and the equations in $t\to x$ for the periodic trajectories of the system (2.183), (2.184). The full phase isomorphism of the systems for n-gap potentials and the system (2.183), (2.184) was proved in [136]. By the same token the general solutions of the latter system can be expressed in n-dimensional theta functions, and those of the Neumann system (and the solutions of the Jacobi problem) in two-dimensional theta functions.

Let us remark that although these systems are trajectorially isomorphic, the corresponding Hamiltonian structures (as was shown in the preceding section) are different.

Let us consider a Baker–Akhiezer function $\psi(x, P)$ associated with a hyperelliptic curve Γ with real branch points $\lambda_1<\lambda_2<\dots<\lambda_{2n+1}$. In §2 it was shown that it satisfies the equation

$$\psi''(x, P)=-\lambda\psi(x, P)+u(x)\psi(x, P), \tag{2.190}$$

where $P=(\lambda, \sqrt{R})$ is a point of Γ. Let us denote $\psi^+(x, P)=\psi(x, \sigma(P))$, where σ is the involution which exchanges the sheets of Γ. Its operation on the local parameter is $\sigma^*(k)=-k$. Hence the function $\psi(x, P)\psi^+(x, P)$ is regular at the point at inifinity P_0. Besides, it does not depend on the choice of the sheet of Γ, and hence is a rational function of λ

$$\psi(x, P)\psi^+(x, P)=\frac{\prod_{i=1}^{n}(\lambda-\gamma_i(x))}{\prod_{i=1}^{n}(\lambda-\gamma_i)}. \tag{2.191}$$

For an arbitrary polynomial $P(\lambda)$ of degree n (whose coefficients may depend on parameters) and for arbitrary points μ_i, $i=1, \dots, n+1$, there holds the simple identity

$$\sum_{i=1}^{n+1}\frac{P(\mu_i)}{\prod_{j\neq i}(\mu_i-\mu_j)}\equiv 1. \tag{2.192}$$

Since at the branch points $\psi(x, \lambda_i)=\psi^+(x, \lambda_i)$, it follows from (2.191) and (2.192) that the functions

$$\phi_i(x)=\psi(x, \lambda_{2i+1})\prod_j\left(\frac{\lambda_{2i+1}-\gamma_j}{\lambda_{2i+1}-\lambda_{2j+1}}\right)^{1/2} \tag{2.193}$$

satisfy the identity

$$\sum_{i=1}^{n}\phi_i^2(x)\equiv 1. \tag{2.194}$$

The equalities (2.190) and (2.194) coincide (after renaming x to t) with the equations (2.185) and (2.183). The expressions in terms of theta functions for $\psi(x, P)$ which were obtained in §2 thereby give the solutions of the system (2.183), (2.184).

For the Neumann system we get, in particular,

$$x_i(t)=\alpha_i\frac{\theta[\nu_i](tU+\zeta)}{\theta[\nu_0](tU+\zeta)}, \tag{2.195}$$

where

$$\alpha_1=-\frac{\theta[\nu_1](0)}{\theta[\nu_0](0)}, \quad \alpha_2=-i\frac{\theta[\nu_2](0)}{\theta[\nu_0](0)}, \quad \alpha_3=\frac{\theta[\nu_3](0)}{\theta[\nu_0](0)}, \tag{2.196}$$

and the characteristics $[\nu_i]$ of the theta functions equal

$$\begin{aligned} \nu_0&=[(1/2, 1/2), (1/2, 1/2)], & \nu_1&=[(1/2, 0), (0, 1/2)],\\ \nu_2&=[(0, 0), (0, 1/2)], & \nu_3&=[(0, 0), (1/2, 1/2)]. \end{aligned} \tag{2.197}$$

The system (2.183), (2.184) can be obtained from a more general system, discovered by Garnier [61]

$$\begin{aligned} x_i''&=x_i\left(\sum x_i y_i+a_i\right),\\ y_i''&=y_i\left(\sum x_i y_i+a_i\right). \end{aligned} \tag{2.198}$$

On the invariant plane $x_i=a_i y_i$ we exactly get the Neumann system on the sphere. Another interesting case is the system of anharmonic oscillators, which is obtained from (2.198) by restricting to the plane $x_i=y_i$, [62].

The *Garnier system* is equivalent (under a suitable choice of the parameter τ) to the commutation conditions

$$\frac{dA(\lambda)}{d\tau}=[A(\alpha), A(\lambda)]/(\lambda-\alpha), \tag{2.199}$$

where the matrix $A=(A_{ij})$ has the form:

$$A_{11}=\lambda^2-\sum x_i y_i, \tag{2.200}$$
$$A_{1i}=x_{i-1}\lambda+x'_{i-1}, \quad A_{i1}=y_{i-1}\lambda-y'_{i-1},$$
$$A_{ij}=x_{i-1}y_{i-1}-a_{i-1}\delta_{ij}, \quad i, j\geq 2.$$

The Motion of a Body in an Ideal Fluid. Integration of the Clebsch Case. As was already said in chap. 1, the equations of motion of a rigid body in an ideal fluid have the form:

$$\dot{p}=p\times\frac{\partial H}{\partial M}, \quad \dot{M}=p\times\frac{\partial H}{\partial p}+M\times\frac{\partial H}{\partial M}, \tag{2.201}$$

where $H=H(M, p)$ is the Hamiltonian (1.48),

$$M=\{M_1, M_2, M_3\}; \qquad p=\{p_1, p_2, p_3\};$$
$$\frac{\partial H}{\partial M}=\left\{\frac{\partial H}{\partial M_i}\right\}; \quad \frac{\partial H}{\partial p}=\left\{\frac{\partial H}{\partial p_i}\right\}. \tag{2.202}$$

Below we shall give the commutational representation of equations (2.201) in the integrable cases of Clebsch and of Lyapunov–Steklov–Kolosov (see (1.205)–(1.209)).

The commutational representation for the Clebsch system was found in [119]. The matrix L has the form:

$$L=\lambda A+L_0-\lambda^{-1}P, \tag{2.203}$$

$$L_0=\begin{pmatrix} 0 & M_3 & -M_2 \\ -M_3 & 0 & M_1 \\ M_2 & -M_1 & 0 \end{pmatrix}, \quad P_{ij}=p_i p_j; \quad A_{ij}=a_i\delta_{ij}. \tag{2.204}$$

The matrix M equals

$$M=\lambda C+\begin{pmatrix} 0 & a_3 M_3 & -a_2 M_2 \\ -a_3 M_3 & 0 & a_1 M_1 \\ a_2 M_2 & -a_1 M_1 & 0 \end{pmatrix}, \tag{2.205}$$

$C_{ij}=c_i\delta_{ij}$.

The Clebsch case is the limit under contraction of the group SO(4) to E(3) of the integrable tops obtained in [97] and considered in the next topic. If one fixes

a basis of the algebra so(4) with the commutation relations

$$\{M_i, M_j\}=\varepsilon_{ijk}M_k; \quad \{M_i, p_j\}=\varepsilon_{ijk}p_k; \quad \{p_i, p_j\}=\varepsilon_{ijk}M_k, \tag{2.206}$$

then this contraction corresponds to a passage to a limit, under which

$$M_i=M'_i, \quad p_i\to Np'_i, \quad N\to\infty.$$

The Lax pair (2.212) for tops on so(4) diverges under the contraction, although its integrals hold out under this passage and coincide after it with the integrals of the Clebsch case. On the other hand, the pair (2.203)–(2.205) will not endure the deformation of so(4) to e(3). It is of interest to remark that not only the tie indicated above exists between these systems. As was found in [10], [15], the Kirchhoff equations for the Clebsch case go over into the Manakov equations for the algebra so(4) (see below) after a suitable linear change of variables. An analogous linear change takes the integrable case of Lyapunov–Steklov–Kolosov (see below) over into the integrable case of Steklov [133] for the rotation of a rigid body with an ellipsoidal cavity filled with a fluid [15].

The Clebsch case was integrated in [69], [132], [140].

The Lyapunov–Steklov–Kolosov Case. In this case the Hamiltonian, with (1.208) taken into account, has the form:

$$2H=\sum_{\alpha=1}^{3} b_\alpha(M_\alpha-(b_1+b_2+b_3-b_\alpha)\sigma p_\alpha)^2+A\sum_{\alpha=1}^{3} p_\alpha^2+B\sum_{\alpha=1}^{3} p_\alpha M_\alpha. \tag{2.207}$$

Let us set

$$2z_\alpha=M_\alpha-\sigma(b_1+b_2+b_3-b_\alpha)p_\alpha. \tag{2.208}$$

The fourth integral has the form

$$2I=\sum_{\alpha=1}^{3} b_\alpha p_\alpha^2. \tag{2.209}$$

The *Lyapunov–Steklov–Kolosov case* was integrated in the paper [70]. It is curious that in this paper for this system there was practically used, between the lines, a commutational representation with an elliptic spectral parameter.

F. Kötter's representation for the equations of motion had the form:

$$\frac{d}{dt}(z_1+\sigma s p_1)=2(s-b_2)(z_2+s\sigma p_2)\frac{b_3 z_3}{s}-2(s-b_3)(z_3+s\sigma p_3)\frac{b_2 z_2}{s}$$

(the remaining equations have the same appearance up to a cyclic permutation of the indices).

Here s is the "spectral parameter". These equations, as is simple to verify, are equivalent to the Lax equation $\dot{L}=[L, M]$, where

$$L=\begin{pmatrix} 0 & \sqrt{c_3}v_3 & -\sqrt{c_2}v_2 \\ -\sqrt{c_3}v_3 & 0 & \sqrt{c_1}v_1 \\ \sqrt{c_2}v_2 & -\sqrt{c_1}v_1 & 0 \end{pmatrix},$$

$$M=\frac{2}{s}\begin{pmatrix} 0 & \sqrt{c_1c_2}b_3z_3 & -\sqrt{c_1c_3}b_2z_2 \\ -\sqrt{c_1c_2}b_3z_3 & 0 & \sqrt{c_2c_3}b_1z_1 \\ \sqrt{c_1c_3}b_2z_2 & -\sqrt{c_2c_3}b_1z_1 & 0 \end{pmatrix}. \quad (2.210)$$

Here $v_i=z_i+\sigma sp_i$, $c_i=s-b_i$.

Let us set $e_i=b_i-\frac{1}{3}(b_1+b_2+b_3)$, $s=\wp(\lambda)+(b_1+b_2+b_3)/3$, where $\wp$ is the Weierstrass function corresponding to the elliptic curve Γ with the branch points e_i. Then L, M are elliptic functions of λ, defined on Γ.

The Clebsch and Lyapunov–Steklov–Kolosov cases exhaust all the possibilities when the system (2.201) with the Hamiltonian (1.48) has a fourth integral quadratic in (M, p) [119]. Let us note that for general diagonal metrics, as was shown in [73], for the equations of motion (with the exception of the Clebsch case) a splitting of the separatrices occurs, i.e. they are non-integrable.

A Multidimensional Free Rigid Body. The equations of a multidimensional rigid body have the form [7]:

$$\dot{M}=[\Omega, M], \quad M=J\Omega+\Omega J \quad (2.211)$$

and $J_{ij}=J_i\delta_{ij}$ is the inertia operator[8] of the rigid body. The complete integrability of this system for all n was proved in [97][9]. As was remarked in this paper, the system (2.211) is equivalent to the system

$$[A, \dot{V}]=[[A, V], [B, V]], \quad (2.212)$$

$$[B, V]=\Omega, \quad A=J^2, \quad B=J. \quad (2.213)$$

The commutational representation for (2.211) has the form:

$$\left[\frac{d}{dt}-[B, V]+\lambda B, \lambda A-[A, V]\right]=0.$$

By the results of § 2, the general solutions can be expressed via theta functions of the Riemann surface Γ of the form

$$\det(\lambda A-[A, V]-\mu\cdot 1)=0. \quad (2.214)$$

[8] Often called the inertia *tensor* (translator's note).

[9] For $n=4$ in [108].

The rigorous and abstract exposition of the Manakov theory and the direct verification of the independence of the Manakov integrals constructed (the coefficients of the characteristic polynomial (2.214)), without relying on the spectral theory of operators, were realized in the paper [57]. A series of subsequent papers, of which a survey is given in [58], were devoted to the transfer of this technique to some other Lie algebras. An investigation of the dynamics of these systems not only on the Lie algebra, but also on the whole Lie group was given in the paper [105].

The equations (2.212) were integrated (for arbitrary A and B) in the paper [39]. As was remarked in [97], for general diagonal matrices A and B the equations (2.212) coincide with the equations of a motion on SO(N) with the diagonal metric

$$\omega_{ij}=\frac{a_i-a_j}{b_i-b_j},$$

which for $N=4$ go over under the contraction of SO(4) to E(3) into the integrable case of Clebsch.

The solutions of the general equations (2.212):

$$v_{ij}=\frac{\lambda_i}{\lambda_j}\frac{\theta(A(P_i)-A(P_j)+tU+\zeta)}{\theta(tU+\zeta)\varepsilon(P_i, P_j)}, \quad i\neq j, \quad (2.215)$$

$$\varepsilon(P, Q)^{-1}=\frac{\sqrt{\partial_{U(P)}\theta[v](0)\partial_{U(Q)}\theta[v](0)}}{\theta[v](A(P)-A(Q))},$$

$$\lambda_i=\lambda_i^0\exp\left(t\sum_{k\neq i}c_i^k b_k\right),$$

$$c_i^k=-\frac{\partial}{\partial P}\ln\varepsilon(P, P_i)|_{P=P_i}.$$

Here the λ_i^0 are arbitrary nonzero constants, the θ-function is constructed with respect to a curve of the form (2.214); P_i are the points at infinity of this curve, where $\mu/\lambda\to a_i$ when $P\to P_i$; the vector U has the form:

$$U=\sum_j b_jU(P_j),$$

$U(P)$ is the period vector of the differential $\Omega_P^{(2)}$ with a double pole at P, $[v]$ is an arbitrary nondegenerate (grad $\theta[v](0)\neq 0$) odd half-period.

Waves in the Landau–Lifshitz Equation. Following [137], let us look at solutions of the travelling-wave type

$$S(x, t)=q(x-at)$$

for the Landau–Lifshitz equation (2.51).

We have

$$-a\dot{q}=q\times(\ddot{q}+Jq). \tag{2.216}$$

Taking the vector product of this equality with q and using the condition $q^2=1$, we get

$$\ddot{q}+Jq=\lambda q+a\dot{q}\times q,\quad \lambda=(q,Jq)-\dot{q}^2.$$

Let us introduce the variable

$$M=\dot{q}\times q+aq.$$

Then equation (2.216) turns out to be equivalent to the already analyzed Clebsch system (2.204)

$$\dot{M}=M\times Jq,$$
$$\dot{q}=q\times M.$$

In [16] finite gap solutions are constructed in terms of Prym theta functions, starting from the Lax pair (2.52) for the Landau–Lifshitz equations (2.51). As a special case they also contain solutions of the travelling-wave type.

A generalization of the Landau–Lifshitz equations is given by the equations

$$\begin{aligned} u_t&=u\times(u_{xx}+Jv),\\ v_t&=v\times(v_{xx}+Ju), \end{aligned} \tag{2.217}$$

which were considered in [139] and which describe a two-sublattice system. In [139] it is shown that the equations describing the travelling wave

$$u=u(x-at),\quad v=v(x-at),$$

can be integrated. They correspond to the Hamiltonian system on E(3)+E(3) with the Hamiltonian

$$H=\tfrac{1}{2}(M^2+N^2+2(Jp,q)). \tag{2.218}$$

Here $p(\xi)=u(\xi)$, $q(\xi)=v(\xi)$ and $M=u\times u_\xi+au$, $N=v\times v_\xi+av$. The pairs p, M and q, N satisfy the commutation relations of E(3).

The matrices L and A which enter into the commutational representation for the equations of motion of the system of (2.218) have the block form

$$L=\begin{pmatrix}\lambda\hat{M} & \lambda^2 p_iq_j+J_i\delta_{ij}\\ \lambda^2 q_ip_j+J_i\delta_{ij} & \lambda\hat{N}\end{pmatrix},$$

$$A=\begin{pmatrix}0 & \lambda p_iq_j\\ \lambda q_ip_j & 0\end{pmatrix}.$$

Here $\hat{M}$ and $\hat{N}$ are skew-symmetric 3×3 matrices corresponding in a canonical way to the vectors $\{M_i\}$, $\{N_i\}$.

A Top in a Gravitational Field. Let us consider, following [19], the problem of the rotation of a top, fixed at its centre of gravity, in the gravitational field being created by an arbitrary body V. Let $\rho(x)$ be the mass density of the body V at its point x; let $R(x)$ be the distance from the point x to the fixation point of the top. Let us write the equations in a coordinate system S rigidly connected with the top, where we shall orient the axes of this system along the principal axes of the inertia operator, i.e. the inertia operator I of the top is diagonal, $I=(I_i\delta_{ij})$. We shall suppose the dimensions of the top to be small in comparison with the distances $R(x)$ to the body V. In this approximation the equations of the rotation of the top can be written in the form

$$\dot{M}=M\times\omega+\int_V 3G\rho(x)R^{-3}(x)\gamma(x)\times I\gamma(x)\,d^3x, \tag{2.219}$$

where $\gamma(x)$ is the unit vector of the direction going from the point x of the body to the top (written in the system S!), M and ω are the angular momentum and angular velocity vectors, G is the gravitational constant. Let us supplement the equations (2.219) with the obvious relation

$$\dot{\gamma}(x)=\gamma(x)\times\omega. \tag{2.220}$$

We shall show that the equations (2.219), (2.220) are integrable, and the integration procedure does not depend on the body V.

Let us associate to the vectors $\gamma(x)=(\gamma_i(x))$, $M=(M_i)$, $\omega=(\omega_i)$ skew-symmetric matrices $\hat{\gamma}(x)=(\hat{\gamma}_{ij}(x))$, $\hat{M}=(\hat{M}_{ij})$, $\hat{\omega}=(\hat{\omega}_{ij})$, setting $\hat{\gamma}_{ij}(x)=\varepsilon_{ijk}\gamma_k(x)$, and so on. Let us introduce, further, the matrix

$$u(x)=\int_V 3G\rho(x)R^{-3}(x)\hat{\gamma}^2(x)\,d^3x.$$

The equations (2.219), (2.220) can be written in the form of the system

$$\begin{cases}\dot{\hat{M}}=[\hat{M},\hat{\omega}]+[u,C],\\ \dot{u}=[u,\hat{\omega}],\end{cases}\quad \hat{M}=I\hat{\omega}+\hat{\omega}I. \tag{2.221}$$

Here $C=\operatorname{diag}(C_1,C_2,C_3)$. The system (2.221) is Hamiltonian on the Lie algebra whose elements are pairs of 3×3 matrices (ω,u), where ω is a skew-symmetric matrix and u is a symmetric matrix, and the commutators have the form:

$$[\omega_1,\omega_2]=\omega_1\omega_2-\omega_2\omega_1,\quad [\omega,u]=\omega u-u\omega,\quad [u_1,u_2]=0.$$

The Hamiltonian has the form $H=\operatorname{tr}(\frac{1}{2}\hat{M}\hat{\omega}+uC)$. The Lax representation for the system (2.221), obtained in [19], has the form $\dot{L}=[L,A]$, where

$$\begin{aligned} &L(\lambda)=\hat{M}+\lambda B+\lambda^{-1}u,\quad A(\lambda)=\hat{\omega}+\lambda C,\\ &B=\operatorname{diag}(B_1,B_2,B_3),\quad B_i=I_1I_2I_3I_i^{-1}, \end{aligned} \tag{2.222}$$

where to simplify the formulas we assume that $I_1+I_2+I_3=0$. From this it follows that the system (2.221) can be integrated in theta functions of the

Riemann surface Γ given by the equation $\det(L(\lambda)-\mu\cdot 1)=0$. On the surface Γ of genus 4 an obvious involution of the form $(\lambda,\mu)\mapsto(-\lambda,-\mu)$ acts with six fixed points, corresponding to $\lambda=0$ and $\lambda=\infty$. Therefore this surface doubly covers an elliptic curve, and the phase variables of the system (2.221) can be expressed via the Prym theta functions (of three variables) of this covering.

Another application of systems of the form (2.221) is the proof of the integrability of the problem of the rotation of a rigid body about a fixed point in a Newtonian field with an arbitrary quadratic potential $U=2^{-1}a_{ij}x^i x^j$ [19] (the possibility of applying L-A pairs of type (2.222) to a top in the field of a quadratic potential was noted in [123]). Here the equations of motion can be written in the form (2.221), where the matrix u is constructed as follows. Let Q be the transition matrix from the S-system to the fixed system. Then $u=Q^TaQ$, where $a=(a_{ij})$.

§ 5. Pole Systems

The program for research on the dynamics of poles of solutions of equations to which the inverse scattering method is applicable goes back to the article [87]. In two-dimensional hydrodynamics the poles of the solutions correspond to the dynamics of vortices. In the case of a finite number of vortices the corresponding system turns out to be a finite-dimensional Hamiltonian system.

The connection of the dynamics of poles of rational and elliptic solutions of the KdV equation to the equations of motion of the system (2.57) was first discovered in the paper [4].

Let us remark that elliptic solutions of the KdV equation of the form

$$u(x,t)=2\wp(x-x_1(t))+2\wp(x-x_2(t))+2\wp(x-x_3(t))$$

were first constructed without any connection to finite-dimensional systems in the paper [47].

Originally the theory of the *Moser–Calogero systems* (2.57), integrable by the method of L, A pairs, and of their generalizations, which will be discussed in detail in the second part of this work, was developed without the use of a direct connection with solutions of partial-differential wave equations of the KdV type to which the inverse scattering method is applicable. The construction of solutions of the equations of motion of these systems was based on the theory of Lie algebras. For the system (2.57) in degenerate cases of the Weierstrass $\wp$-function—potentials x^{-2} or $\sinh^{-2}x$—it was shown that the coordinates of the particles $x_j(t)$ are the eigenvalues of a matrix depending linearly on t, i.e.

$$\text{const}\times\prod_j(x-x_j(t))=\det(At+B-x\cdot 1) \tag{2.223}$$

(the matrix entries of A and B can be expressed explicitly in terms of the initial coordinates and momenta of the particles).

But in the elliptic case only the involutivity and independence of the integrals

$$J_k=\frac{1}{k}\operatorname{tr}L^k \tag{2.224}$$

was known, where L is given in (2.58), $J_2=H$.

In the paper [4] already mentioned above it was shown that the dynamics of the poles $x_j(t)$ of solutions of the KdV equation rational in x, which are obliged to have the form

$$u(x,t)=2\sum_{j=1}^N(x-x_j(t))^{-2}, \tag{2.225}$$

coincides with the Hamiltonian flow generated by the integral J_3, restricted to the fixed points of the original system, grad $H=0$. The necessity of restricting the flow to the stationary points grad $H=0$ leads also to a restriction on the number of particles, which may have only the form $N=d(d+1)/2$.

The connection between the rational Moser–Calogero system and rational solutions of nonlinear equations proves to be more natural in the case of two-dimensional systems.

As was remarked in [77], all solutions rational in x of the KP equation

$$\frac{3}{4}u_{yy}=\frac{\partial}{\partial x}\left(u_t+\frac{1}{4}(u_{xxx}-6uu_x)\right) \tag{2.226}$$

which subside as $|x|\to\infty$ have the form $u=2\sum_{j=1}^N(x-x_j(y,t))^{-2}$. Here the dynamics of the poles of $x_j(y,t)$ in y and t correspond to the two commuting flows $J_2=H$, J_3 (2.224). The number N is arbitrary. Using this connection in [77], it was shown that the construction of [77] gives *all* rational solutions of the KP equation.

Rational multisoliton solutions for the KP equation were constructed within the framework of the inverse scattering method in [99].

In the paper [31] the isomorphism of the two problems indicated in [77] was carried over to the elliptic case as well. However till [80] both problems—the construction of angle-type variables for the system (2.57) and the integration of its equations of motion in terms of theta functions, but also the problem of constructing elliptic solutions of the KP equation—remained completely unsolved (except for the simplest, two-particle case).

At the basis of the paper [80], where these problems were solved, lay the commutational representation found for the equations of motion

$$\ddot{x}_i=4\sum_{j\neq i}\wp(x_i-x_j) \tag{2.227}$$

of the system (2.57). This commutational representation (in contrast to (2.58), (2.59)) involves a spectral parameter defined on the elliptic curve Γ. Moreover with respect to this parameter the matrix entries of U and V are Baker–Akhiezer functions.

Let us define matrices

$$U_{ij} = \dot{x}_i \delta_{ij} + 2(1-\delta_{ij})\Phi(x_{ij}, \lambda), \tag{2.228}$$

$$V_{ij} = \delta_{ij}\left(2\sum_{k\neq i} \wp(x_{ik}) - \wp(\lambda)\right) + 2(1-\delta_{ij})\Phi'(x_{ij}, \lambda), \tag{2.229}$$

where

$$\Phi(z, \lambda) = \frac{\sigma(z-\lambda)}{\sigma(\lambda)\sigma(z)} e^{\zeta(\lambda)z}, \tag{2.230}$$

$$\Phi'(z, \lambda) = \frac{\partial}{\partial z}\Phi(z, \lambda) \tag{2.231}$$

and $x_{ij}(t) = x_i(t) - x_j(t)$.

Proposition. *The equations* (2.227) *are equivalent to the commutational equation*

$$U_t = [V, U]. \tag{2.232}$$

It follows from (2.232) that the function

$$R(k, \lambda) = \det(2k + U(\lambda, t)) \tag{2.233}$$

does not depend on t. The matrix U, which has essential singularities for $\lambda = 0$, can be represented in the form

$$U(\lambda, t) = g(\lambda, t)\tilde{U}(\lambda, t)g^{-1}(\lambda, t), \tag{2.234}$$

where $\tilde{U}$ does not have an essential singularity at $\lambda = 0$, and $g_{ij} = \delta_{ij}\exp(\zeta(\lambda)x_i)$. Consequently, $r_i(\lambda)$—the coefficients of the expression

$$R(k, \lambda) = \sum_{i=0}^{n} r_i(\lambda) k^i, \tag{2.235}$$

are elliptic functions with poles at the point $\lambda = 0$. The functions $r_i(\lambda)$ are representable as a linear combination of the $\wp$-function and its derivatives. The coefficients of such an expansion are integrals of the system (2.57). Each set of fixed values for these integrals gives by means of the equation $R(k, \lambda) = 0$ an algebraic curve Γ_n which n-foldly covers the original elliptic curve Γ.

Generically the genus of the curve which arises is equal to n. The Jacobian of the curve Γ_n is isomorphic to the level manifold of the integrals r_n, and the variables on it are variables of the angle type.

A further putting to good effect of the solution of equations (2.227) uses the connection of equation (2.232) with the existence of solutions of a special form for the non-stationary Schrödinger equation with an elliptic potential.

Theorem 2.9. *The equation*

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + 2\sum_{i=1}^{n} \wp(x - x_i(t))\right)\psi = 0 \tag{2.236}$$

has a solution ψ of the form

$$\psi = \sum_{i=1}^{n} a_i(t, k, \lambda)\Phi(x - x_i, \lambda)\, e^{kx + k^2 t} \tag{2.237}$$

if and only if the $x_i(t)$ satisfy equations (2.227).

Here $\Phi(z, \lambda)$ is given by formula (2.230).

A function ψ of the form (2.237), as a function of the variable x, has simple poles at the points $x_i(t)$. Substituting it into (2.236) and equating the coefficients of $(x - x_i)^{-2}$ and $(x - x_i)^{-1}$ to zero, we get that ψ satisfies (2.236) if and only if the vector $a = (a_1, \ldots, a_n)$ satisfies the equations

$$U(\lambda, t)a = -2ka, \tag{2.238}$$

$$\left(\frac{\partial}{\partial t} + V(\lambda, t)\right)a = 0, \tag{2.239}$$

where U and V are the same as in (2.228), (2.229).

The analytic properties of a on the Riemann surface Γ_n can be clarified analogously to §2. Let us formulate the final assertion.

Theorem 2.10. *The eigenfunction $\psi(x, t, \gamma)$ of the non-stationary Schrödinger equation* (2.236) *is defined on the n-fold covering Γ_n of the original elliptic curve. The function ψ is a Baker–Akhiezer function with a unique essential singularity of the form*

$$\exp(n\lambda^{-1}(x - x_1(0)) + n^2\lambda^{-2}t)$$

at an isolated preimage P_0 on Γ_n of the point $\lambda = 0$.

The explicit expressions for $\psi(x, t, \gamma)$ which were obtained in §2 give that the poles of ψ in x coincide with the zeroes of the function $\theta(U^{(2)}x + U^{(3)}t + \zeta)$. Comparing with theorem 2.9, we finally get

Theorem 2.11. *Let Γ_n be given by the equation $R(k, \lambda) = 0$, where R is defined in* (2.233). *Then the equation in x*

$$\theta(U^{(2)}x + U^{(3)}t + \zeta) = 0 \tag{2.240}$$

has n roots in the fundamental cell with periods 2ω, $2\omega'$—the $x_i(t)$ which satisfy equation (2.227).

Here θ is a Riemann theta function corresponding to the surface Γ_n, and $U^{(2)}$, $U^{(3)}$ are the periods of differentials of the second kind with poles of the second and third order at the distinguished point P_0. These quantities can be expressed in terms *of the integrals r_i of the equations* (2.227). *The vector ζ in* (2.240) *is arbitrary and corresponds to variables of the angle type.*

All of the parameters in (2.240) can be expressed by quadratures in terms of $x_i(0)$ and $\dot{x}_i(0)$.

§6. Integrable Systems and the Algebraic-Geometric Spectral Theory of Linear Periodic Operators

The original approach to the construction of finite gap solutions of the KdV equations, the nonlinear Schrödinger equation and a number of others was based on the spectral theory of linear operators with periodic coefficients (see [38], [45], [64], [90], [100], [102], [111], [115]). The term "finite gap solutions" is connected with just this approach. Let us briefly point out the interconnection between this approach and the algebraic-geometric one which was set forth in §2.

Let $U(x, t, \lambda)$ and $V(x, t, \lambda)$ be solutions of the equations of zero curvature depending periodically on x. Let us consider the matrix

$$W(x, t, \lambda)=\Psi(x+T, t, \lambda)\Psi^{-1}(x, t, \lambda), \tag{2.241}$$

where T is the period and Ψ is a solution of the equations (2.36). This matrix is called the monodromy matrix (describing the translation by a period of the solutions of the linear equations (2.36)).

From the fact that $\Psi(x+T, t, \lambda)$ is also a solution of equations (2.36) it follows that

$$[\partial_x-U, W]=[\partial_t-V, W]=0,$$

and we arrive at equations (2.64), (2.65).

The matrix $W(x, t, \lambda)$ is analytic outside the poles of U and V, where it generically has essential singularities.

The vector function $\psi(x, t, \gamma)$ defined by the equations (2.66)–(2.72) is an eigenfunction of the period-translation operator: $\psi(x+T, t, \gamma)=\mu(\gamma)\psi(x, t, \gamma)$. In the theory of operators with periodic coefficients such functions are called *Bloch functions.* The Riemann surface Γ on which a Bloch function becomes single-valued has infinite genus in the general case (its branch points accumulate at the poles of U and V).

The finite gap periodic solutions are singled out by the condition that the genus of the surface Γ is finite, which is equivalent to the existence of a solution $W(x, t, \lambda)$ which is rational in λ for equations (2.64), (2.65).

Thus, the periodic solutions of the equations (2.37), (2.64), (2.65) have the property that their corresponding Bloch function is defined on a Riemann surface of finite genus and coincides with the *Clebsch–Gordan–Baker–Akhiezer function.*

It is clear that the finite gap notion can be carried over verbatim to an arbitrary linear operator $\partial_x - U(x, \lambda)$ irrespective of nonlinear equations. The corresponding matrices U are called finite gap potentials.

The spectral properties of a Sturm–Liouville operator with finite gap potentials (properties obtained in the work presented in [45], [115]) were briefly cited in §2.

Below we shall describe these properties in greater detail and shall give sketches of the proofs of the fundamental assertions using the example of the spectral theory of the Schrödinger difference operator (2.20)

$$L\psi_n=c_n\psi_{n+1}+v_n\psi_n+c_{n-1}\psi_{n-1} \tag{2.242}$$

($c_n=c_{n+N}\neq 0,\ v_n=v_{n+N}$), which enters into the Lax representation for the equations of the Toda lattice and for the KdV difference equation (when $v_n\equiv 0$).

Remark. In recent years there have been discovered new remarkable applications of the algebraic-geometric spectral theory to Peierls–Fröhlich problems, which are among the most fundamental ones in the theory of quasi-one-dimensional conductors. In the continuous limit this model was investigated in the papers [12], [24], where indeed a connection between the Peierls model and the theory of *finite gap* Sturm–Liouville *operators* was discovered for the first time. This theory (the formulas for the variational derivatives of Kruskal integrals, variation with respect to the period group) was first applied to a full extent in [12]. The latter papers served as a starting point for the subsequent investigations [23], [50], [51], [81], in which these results were carried over to the discrete Peierls model and considerably developed.

The *Peierls–Fröhlich model* (R.E. Peierls, H. Fröhlich) describes the self-consistent behaviour of a lattice of atoms with coordinates $x_n<x_{n+1}$ and electrons. There are two models. In the first the atom at each lattice site also possesses an internal degree of freedom: v_n. In the second model, $v_n\equiv 0$.

The electronic energy levels are defined as the points $E_1<E_2\leq\ldots\leq E_N$ of the spectrum of the periodic problem for an operator L which has the form (2.242), where $c_n=\exp(x_n-x_{n+1})$, $c_n=c_{n+N}$, $v_n=v_{n+N}$. The energy of the system consists of the energy of the electrons, which at absolute zero occupy the m lowest levels, and the elastic energy of the lattice:

$$H=\frac{1}{N}\left(\sum_{i=1}^{m}E_i+\sum_{n=0}^{N-1}\Phi(c_n, v_n)\right).$$

Here m is the number of electrons and $\Phi(c_n, v_n)$ is the elastic energy potential.

In [23] the case was considered of

$$\sum_n\Phi(c_n, v_n)=\sum_n[\kappa(v_n^2+2c_n^2)-P\ln c_n]$$

and the more general one of

$$\sum_n \Phi(c_n, v_n) = \sum_{k=1}^{l} \kappa_k I_k,$$

where the I_k are integrals of the Toda lattice or the Langmuir lattice (J. Langmuir) ($v_n \equiv 0$).

In the first case it was shown that $H(c_n, v_n)$ has a unique extremal, corresponding to a one-gap operator L.

In the continuous limit this extremal goes over into the extremals obtained in [13], [24], which proves that in these papers the ground state was found.

For the more general models the stability of the extremals was investigated and the ground state was found. In addition, the speed of sound and of a charge density wave were found.

The systematic construction of an algebraic-geometric Bloch–Floquet spectral theory on $\mathscr{L}_2(\mathbb{Z})$ for the Schrödinger difference operator (2.242) was first begun by S.P. Novikov [45, chap. 3, § 1] and by S. Tanaka–E. Date [33]. With the aid of the trace formulas for the function $\chi_n = \psi_{n+1}/\psi_n$ formulas were obtained for v_n. In [45] the symmetric case $v_n = 0$ was also studied. This theory was carried through to the finish in [45], but only in the elliptic case. In the paper [33] the expressions for v_n were written in the form

$$v_n = \frac{\partial}{\partial t} \ln \frac{\theta(Un + Vt + Z)}{\theta(U(n+1) + Vt + Z)} + \text{const.} \tag{2.243}$$

(In [45] an insignificant error was committed, which was rectified in the book [115].)

In the case of the Toda lattice, by virtue of the condition $\dot{x}_n = v_n$ formula (2.243) determines $x_n(t)$ up to a choice of the numbers $x_n(0)$, $-\infty < n < \infty$. The difference KdV was not considered in [33].

These investigations received their completion in [79], in which explicit expressions were obtained for the x_n and the solutions of the difference KdV. The idea of [79] consists in using explicit expressions for the ψ_n in terms of theta functions and analogues of the "trace identities" for the ψ_n, in contrast to [45], [33], where, as has already been said, trace formulas for the χ_n were used, analogously to the continuous case. In the later paper [82] "local trace identities" $c_n = c_n(\gamma_1, \ldots, \gamma_n)$ were explicitly obtained whose existence had been ineffectively proved in [45].

The basic contemporary approach to spectral problems for periodic operators is the analysis of the analytic properties of the solutions of the equation

$$L\psi_n = E\psi_n \tag{2.244}$$

(here L is the operator (2.242) with periodic coefficients) for all values, among them also complex ones, of the parameter E.

For any E the space of solutions of equation (2.244) is two-dimensional. Having given arbitrary values to ψ_0 and ψ_1, one can find the remaining values ψ_n in a recursive manner. The standard basis $\phi_n(E)$ and $\theta_n(E)$ is given by the conditions $\phi_0 = 1$, $\phi_1 = 0$, $\theta_0 = 0$, $\theta_1 = 1$. From the recursive procedure for computing $\phi_n(E)$ and $\theta_n(E)$ it follows that (for $n > 0$) they are polynomials in E

$$\begin{aligned} \phi_n(E) &= \frac{c_0}{c_1 \ldots c_{n-1}} \left(E^{n-2} - \left(\sum_{k=2}^{n-1} v_k \right) E^{n-3} + \ldots \right), \\ \theta_n(E) &= \frac{1}{c_1 \ldots c_{n-1}} \left(E^{n-1} - \left(\sum_{k=1}^{n-1} v_k \right) E^{n-2} \right. \\ &\quad \left. + \left(\sum_{0<i<j}^{n-1} v_i v_j - \sum_{k=1}^{n-3} c_k^2 \right) E^{n-3} + \ldots \right). \end{aligned} \tag{2.245}$$

The matrix $W(E)$ of the monodromy operator $\hat{T}: y_n \to y_{n+N}$ in the basis ϕ_n and θ_n has the form:

$$W(E) = \begin{pmatrix} \phi_N(E) & \theta_N(E) \\ \phi_{N+1}(E) & \theta_{N+1}(E) \end{pmatrix}. \tag{2.246}$$

It easily follows from (2.244) that for any two solutions of this equation, in particular for ϕ and θ, the expression (analogue of the Wronskian)

$$c_n(\phi_n \theta_{n+1} - \phi_{n+1} \theta_n) \tag{2.247}$$

does not depend on n. Since $c_0 = c_N$, we have

$$\det W = \phi_N \theta_{N+1} - \phi_{N+1} \theta_N = \phi_0 \theta_1 - \theta_0 \phi_1 = 1. \tag{2.248}$$

The eigenvalues w of the monodromy operator can be determined from the characteristic equation

$$w^2 - 2Q(E)w + 1 = 0, \quad 2Q(E) = \phi_N(E) + \theta_{N+1}(E). \tag{2.249}$$

The polynomial Q has degree N and its highest-order terms have the form

$$\begin{aligned} 2Q(E) &= \frac{1}{c_0 \ldots c_{N-1}} \left(E^N - \left(\sum_{k=0}^{N-1} v_k \right) E^{N-1} \right. \\ &\quad \left. + \left(\sum_{i<j} v_i v_j - \sum_{k=0}^{N-1} c_k^2 \right) E^{N-2} + \ldots \right). \end{aligned} \tag{2.250}$$

The spectra $E_i^{\pm}$ of the periodic and antiperiodic problems for L can be determined from the equations $Q(E_i^{\pm}) = \pm 1$, since when this holds $w = \pm 1$.

Let us denote by E_i, $i = 1, \ldots, 2q+2$, $q \le N-1$, the simple points of the spectrum of the periodic and antiperiodic problems for L, i.e. the simple roots of the equation

$$Q^2(E) = 1. \tag{2.251}$$

For a point E in general position the equation (2.249) has two roots w and w^{-1}. To each root there corresponds a unique eigenvector normalized by the condition $\psi_0=1$

$$L\psi_n=E\psi_n, \qquad \psi_{n+N}=w\psi_n. \tag{2.252}$$

This solution is called a Bloch solution.

Theorem 2.12. *The two-valued function $\psi_n^\pm(E)$ is a single-valued meromorphic function $\psi_n(P)$ on the hyperelliptic curve Γ, $P\in\Gamma$, corresponding to the Riemann surface of the function $\sqrt{R(E)}$*

$$R(E)=\prod_{i=1}^{2q+2}(E-E_i). \tag{2.253}$$

Outside the points at infinity it has q poles $\gamma_1,\ldots,\gamma_q$. In the neighbourhood of the points at infinity

$$\psi_n^\pm(E)=e^{\pm x_n}E^{\pm n}\left(1+\sum_{s=1}^{\infty}\xi_s^\pm(n)E^{-s}\right). \tag{2.254}$$

Here the $\pm$ signs correspond to the upper and lower sheets of the surface Γ (by the upper sheet will be meant the one on which at infinity $\sqrt{R}\sim E^{q+1}$).

A Bloch solution, like any other solution of equation (2.244), has the form $\psi_n=\psi_0\phi_n+\psi_1\theta_n$. The vector (ψ_0,ψ_1) is an eigenvector for the matrix W. Hence $\psi_0=1$, $\psi_1=\dfrac{w-\phi_N}{\theta_N}$ or

$$\psi_n=\phi_n(E)+\frac{w-\phi_N(E)}{\theta_N(E)}\theta_n(E). \tag{2.255}$$

Let e_j, $j=1,\ldots,N-q-1$ be the double roots of the equation

$$Q^2(E)=1, \quad \text{i.e.}$$

$$Q^2(E)-1=C^2r^2(E)R(E), \quad r(E)=\prod_{j=1}^{N-q-1}(E-e_j), \tag{2.256}$$

$$C^{-1}=c_0\ldots c_{N-1}.$$

At the points e_j the matrix of the operator $\hat{T}$ with respect to a Bloch basis is equal to ±1. So it is equal to ±1 in any other basis. Hence

$$\theta_N(E)=r(E)\tilde{\theta}_N(E), \quad \phi_{N+1}(E)=r(E)\tilde{\phi}_{N+1}(E),$$
$$\phi_N(e_j)=\theta_{N+1}(e_j)=w(e_j)=\pm1. \tag{2.257}$$

From (2.257) it follows that $Q(E)-\phi_N(E)=r(E)\tilde{Q}(E)$.

Here $\tilde{\theta}_N$, $\tilde{\phi}_{N+1}$, $\tilde{Q}$ are polynomials in E. Substituting $w=Q+Cr\sqrt{R}$ in (2.255) and using the preceding equalities, we get

$$\psi_n^\pm=\phi_n(E)+\frac{\tilde{Q}(E)\pm C\sqrt{R(E)}}{\tilde{\theta}_N(E)}\theta_n(E). \tag{2.258}$$

And this equality means in fact that the double-valued function $\psi_n^\pm$ is a single-valued meromorphic function of the point of Γ. The poles of ψ lie at points $\gamma_1,\ldots,\gamma_q$ disposed one above each of the roots of the polynomial $\tilde{\theta}_N(E)$. Indeed, if $\tilde{\theta}_N(E)=0$ then the two roots $w_{1,2}$ are equal to $\phi_N(E)$ and $\theta_{N+1}(E)$. In addition $\phi_N(E)\neq\theta_{N+1}(E)$. Consequently, for one of the roots w (i.e. on one of the sheets of Γ over the root of $\tilde{\theta}_N(E)=0$) the numerator of the fraction in (2.258) vanishes. The pole of ψ_n lies on the second sheet.

To complete the proof of the theorem it remains to consider the behaviour of $\psi_n^\pm(E)$ when $E\to\infty$. From (2.258) it follows that ψ_1 has a simple pole at P^+. We immediately get from (2.244) that ψ_n has a pole at P^+ of n-th order for all $n>0$. Similarly, ψ_{-n} has a pole of n-th order at P^-. This, together with the fact that w has a pole of N-th order at P^+ and a zero of multiplicity N at P^-, implies equation (2.254), where the x_n are such that $x_0=0$, $c_n=\exp(x_n-x_{n+1})$. □

The parameters γ_i, or rather their projections onto the E plane (which, as earlier, we shall for brevity denote the same) have a natural spectral meaning.

Lemma 2.2. *The set of points e_j (the double points of the spectrum of the periodic and antiperiodic problems for L) and $\{\gamma_i\}$ are the spectrum for the problem* (2.244) *with zero boundary conditions.*

Proof. The surface Γ has two sheets above the points e_j, on each of which w takes on the same value 1 or -1.

As $\tilde{\psi}_n$ one may take

$$\tilde{\psi}_n(e_j)=\psi_n^+(e_j)-\psi_n^-(e_j)=\frac{2C\sqrt{R(e_j)}}{\tilde{\theta}_N(e_j)}\theta_n(e_j). \tag{2.259}$$

The points γ_i are zeroes of $\theta_N(E)$. As was already said above, when $E=\gamma_i$ then for one of the signs in front of $\sqrt{R}$ in (2.258) the numerator of the second term vanishes. Hence for the second it is different from zero. Let this, for example, be the plus sign. Then

$$\tilde{\psi}_n(\gamma_j)=(\tilde{Q}(\gamma_j)+C\sqrt{R(\gamma_j)})\theta_n(\gamma_j) \tag{2.260}$$

is a non-trivial solution of equation (2.244), $E=\gamma_j$, with zero boundary conditions. □

Let us consider the inverse problem. Let arbitrary distinct points E_i be given, $i=1,\ldots,2q+2$, and points $\gamma_1,\ldots,\gamma_q$ on the Riemann surface Γ of the function $\sqrt{R(E)}$, whose projections to the E plane are all different. In difference problems the analogue of theorem 2.2 is the Riemann–Roch theorem [131]. In the given case it states that there exists a meromorphic function $\psi_n(P)$ on Γ, unique up to proportionality, having poles at the points $\gamma_1,\ldots,\gamma_q$, an n-th order pole at P^+ and an *n-th* order zero at P^-. The function $\psi_n(P)$ can be normalized up to sign by requiring that the coefficients of $E^{\pm n}$ on the upper and lower sheets at infinity be reciprocal. Having fixed the signs arbitrarily, we shall denote the corresponding coefficients by $e^{\pm x_n}$. With this ψ_n will have the form (2.254) in a neighbourhood of infinity.

Lemma 2.3. *The constructed functions $\psi_n(P)$ satisfy equation* (2.244), *where the coefficients of the operator L equal*

$$c_n = e^{x_n - x_{n+1}}, \quad v_n = \xi_1^+(n) - \xi_1^+(n+1). \tag{2.261}$$

Proof. Let us consider the function $\tilde{\psi}_n = L\psi_n(P) - E\psi_n(P)$. It has poles at the points $\gamma_1, \ldots, \gamma_q$. From (2.254), (2.261) it follows that $\tilde{\psi}_n$ has an $(n-1)$-st order pole at P^+ and an n-th order zero at the point P^-. By the Riemann–Roch theorem $\tilde{\psi}_n = 0$.

The method of obtaining explicit formulas for the ψ_n and the coefficients of L is completely analogous to the continuous case. As before, let us fix a canonical set of cycles on Γ. Let us denote by idp the normalized abelian differential of the third kind with its only singularities at infinity

$$idp = \frac{E^q + \sum_{i=0}^{q-1} \alpha_i E^{q-i-1}}{\sqrt{R(E)}} dE = \frac{h(E)\,dE}{\sqrt{R(E)}}. \tag{2.262}$$

The coefficients α_i are determined from the normalization conditions

$$\oint_{a_i} dp = 0, \quad i = 1, \ldots, q. \tag{2.263}$$

Lemma 2.4. *The function $\psi_n(P)$ has the form*

$$\psi_n(P) = r_n \exp\left(in \int_{e_1}^{P} dp\right) \frac{\theta(A(P) + nU + \zeta)}{\theta(A(P) + \zeta)}, \tag{2.264}$$

where $U_k = (1/2\pi) \oint_{b_k} dp$, r_n is a constant.

In a neighbourhood of the point at infinity on the upper sheet we have

$$\exp\left(i \int_{e_1}^{P} dp\right) = Ee^{-I_0}(1 - I_1 E^{-1} + \ldots), \quad P = (E, \sqrt{R}) \in \Gamma.$$

It follows from (2.254) that $e^{2x_n + 2I_0 n}$ equals the ratio of the values of the multipliers attached to the exponential in (2.264) taken at the images $A(P^\pm) = \pm z_0$. From (2.264) and the fact that by the Riemann bilinear relations $2z_0 = -U$, we get

$$c_n^2 = e^{2I_0} \frac{\theta((n-1)U + \tilde{\zeta})\theta((n+1)U + \tilde{\zeta})}{\theta^2(nU + \tilde{\zeta})}, \tag{2.265}$$

where $\tilde{\zeta} = \zeta - z_0$.

In a neighbourhood of P^+ we have

$$A(P) = z_0 + VE^{-1} + \ldots,$$

where the coordinates V_k of the vector V are defined by the equality

$$\omega_k = (V_k + O(E^{-1}))dE^{-1}.$$

By expanding (2.264) in a series in E^{-1}, we get from (2.261)

$$v_n = \frac{d}{dt} \ln \frac{\theta((n-1)U + \tilde{\zeta} + Vt)}{\theta(nU + \tilde{\zeta} + Vt)}\bigg|_{t=0} + I_1. \tag{2.266}$$

Theorem 2.13. *The formulas* (2.265), (2.266) *recover the coefficients of L from the parameters E_i and γ_j.*

It is important to note that in general formulas (2.265), (2.266) define quasiperiodic functions c_n and v_n. For c_n and v_n to be periodic it is necessary and sufficient that for the corresponding differential dp the conditions be fulfilled:

$$U_k = \frac{1}{2\pi} \oint_{b_k} dp = \frac{m_k}{N}, \quad \text{the } m_k \text{ being integers.} \tag{2.267}$$

As follows from the definition of the ψ_n, the parameters E_i, γ_j determine them up to sign.

Changes of the signs of the ψ_n result in a change of the signs of the c_n. Operators differing only in the signs of the c_n need not be distinguished, since their eigenfunctions can be trivially transferred into one another.

So far we have been talking about operators L with arbitrary complex coefficients. Now let c_n and v_n be real; then all of the polynomials $\theta_n(E)$, $\phi_n(E)$, $Q(E)$ introduced above will be real. In addition, the periodic and the antiperiodic problems for L are self-adjoint. Hence there are N real points in the spectrum for each of these problems, i.e. the polynomial $Q^2 - 1$ has $2N$ real roots. Hence at the extrema of the polynomial $Q(E)$, $dQ/dE = 0$, one has that $|Q(E)| \geq 1$. The graph of the polynomial Q has the form:

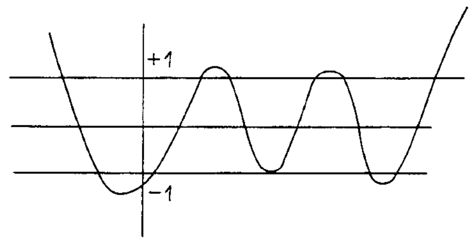

Fig. 2

The intervals $[E_{2i-1}, E_{2i}]$, on which $|Q(E)| \leq 1$, are called the allowed bands[10] In these intervals $|w| = 1$ and the many-valued function $p(E)$ defined from the equality $w = e^{ipN}$ is real. It is called the quasimomentum. Its differential coincides

[10] Or the stable bands (translator's note).

with (2.262), where in (2.263) the a_i are the cycles situated above the forbidden bands[11] $[E_{2i}, E_{2i+1}]$.

Lemma 2.5. *The poles γ_i of the Bloch function $\psi_n(P)$ of a real operator L are distributed one in each of the finite forbidden bands, $E_{2i} \le \gamma_i \le E_{2i+1}$.*

Proof. The poles γ_i are the zeroes of the polynomial $\theta_N(E)$. At these points

$$1 = \det W = \phi_N(\gamma_i)\theta_{N+1}(\gamma_i).$$

Since ϕ_N and θ_{N+1} are real, we have

$$|Q(\gamma_i)| = \tfrac{1}{2}|\phi_N(\gamma_i) + \theta_{N+1}(\gamma_i)| \ge 1$$

and γ_i lies either in a forbidden band or in one of the collapsed bands—the points e_j. At the latter $\psi_n(P)$, as was shown above, has no singularities. □

Theorem 2.14. *It the points $E_1, \ldots, E_{2q+2}$ are real and the points $\gamma_1, \ldots, \gamma_q$ of the corresponding Riemann surface lie one above each of the forbidden bands $[E_{2i}, E_{2i+1}]$, then the coefficients v_n and c_n of the operator L determined by them by virtue of theorem 2.13 are real.*

Proof. The necessity of the conditions of the theorem within the class of periodic operators is given by lemma 2.5.

Let the E_i be real. Complex conjugation induces an anti-involution τ on the curve Γ, τ: $P = (E, \sqrt{R}) \to \tau(P) = (\bar{E}, \sqrt{R(E)})$. The fixed ovals of this anti-involution are the cycles disposed above the intervals $[E_{2i}, E_{2i+1}]$ and above the infinite band which joins through infinity the points E_{2q+2}, E_1.

Let us consider $\bar{\psi}_n(\tau(P))$. This function possesses all the analytic properties of ψ_n. Since ψ_n is determined by these properties up to sign, we get

$$\bar{\psi}_n(\tau(P)) = \pm\psi_n(P). \qquad (2.268)$$

From (2.262) it follows that the v_n are real, and the c_n are either real or pure imaginary (i.e. c_n^2 is real).

Let us prove that under the assumptions of the theorem $c_n \ne 0$, $c_n \ne \infty$. The negation of this assertion is equivalent to one or several of the zeroes $\gamma_i(n)$ of the function $\psi_n(P)$ finding themselves at infinity on the upper or the lower sheet of Γ. From (2.268) it follows that on the cycles disposed above $[E_{2i}, E_{2i+1}]$ ψ_n is either real or pure imaginary. On each cycle there is one pole γ_i; therefore there is also at least one zero. Since there are q zeroes in all, the $\gamma_i(n)$ are distributed, like the γ_i, one above each $[E_{2i}, E_{2i+1}]$, and hence are separate from infinity.

By virtue of what has been proved, the sign of c_n^2 does not change under continuous deformations of E_i and γ_i for which the conditions of the theorem are fulfilled. Let us deform them so that all the forbidden bands close up. Here it is

[11] Or unstable bands (translator's note).

easy to check that the operator L is deformed into an operator L_0 which has $v_n = 0$ and $c_n^2 = \text{const} > 0$. The theorem is proved. □

To conclude the section let us examine the conditions which pick out operators L for which $v_n = 0$, i.e.

$$L\psi_n = c_n\psi_{n+1} + c_{n-1}\psi_{n-1}. \qquad (2.269)$$

Theorem 2.15 ([45], chap. 3, § 1). *Necessary and sufficient conditions for the operator L reconstructed by virtue of theorem 2.13 from the data E_i and γ_j to have the form (2.269), i.e. to have $v_n = 0$, are symmetry of the points E_i relative to zero and invariance of the set $\{\gamma_j\}$ with respect to the involution on Γ*

$$(E, \sqrt{R(E)}) \to (-E, \sqrt{R(E)}), \quad R(E) = \prod_{i=1}^{q+1} (E^2 - E_i^2).$$

The necessity of the conditions follows from the fact that if $\psi_n(P)$ is a Bloch solution for the operator (2.269), then for $\tilde{\psi}_n(E) = (-1)^n\psi_n(E)$ we have

$$L\tilde{\psi}_n = -E\tilde{\psi}_n, \quad \tilde{\psi}_{n+N} = (-1)^N w\tilde{\psi}_n.$$

The sufficiency of the conditions can be proved analogously to the proof of theoreem 2.14. □

Let us define a function $\psi_n(t, P)$ which is meromorphic on Γ outside $P^{\pm}$, has poles $\gamma_1, \ldots, \gamma_q$, and in a neighbourhood of $P^{\pm}$ has the form:

$$\psi_n^{\pm}(t, E) = e^{\pm x_n} E^{\pm n}\left(1 + \sum_{s=1}^{\infty} \xi_s^{\pm}(n, t)E^{-s}\right)e^{\mp t/2}. \qquad (2.270)$$

It can be proved in the standard way that such a function satisfies the linear equations

$$L\psi_n = E\psi_n, \quad \frac{d}{dt}\psi_n = A\psi_n, \qquad (2.271)$$

where L and A have the form (2.20), (2.21). Consequently, the $x_n = x_n(t)$ satisfy the equations of the periodic Toda lattice.

Analogously to lemma 2.4, it is possible to write out an explicit formula for the $\psi_n(t, P)$ and to find explicit expressions for the $x_n(t)$.

Theorem 2.16. *The functions*

$$x_n(t) = \ln\frac{\theta(Un + Vt + \zeta)}{\theta(U(n+1) + Vt + \zeta)} + I_1 t - nI_0 \qquad (2.272)$$

satisfy the equations of the Toda lattice.

(Here I_1 is the average momentum, $-I_0$ is the mean distance between particles.)

The parameters of the theta function, the vectors U, V, ζ can be expressed by quadratures in terms of the initial data $x_n(0)$, $\dot{x}_n(0)$.

To conclude the chapter, let us cite on the basis of this example one more aspect of the theory of finite gap integration — its connections with variational principles for functionals of the Kruskal type (M. Kruskal).

Let us define the functionals $I_k = I_k\{c_n, v_n\}$ by the formula

$$ip(E) = \ln E - \sum_{k=0}^{\infty} I_k E^{-k}, \tag{2.273}$$

where $p(E)$ is the quasimomentum. These functionals have the form:

$$I_k = \frac{1}{N} \sum_{n=1}^{N} r_k(c_{n+i}, v_{n+i} \,|\, |i| < k),$$

where the local densities r_k are polynomials.

From (2.250) we have

$$I_0 = \frac{1}{N} \sum_{n=1}^{N} \ln c_n, \quad I_1 = \frac{1}{N} \sum_{n=1}^{N} v_n, \quad I_2 = \frac{1}{N} \sum_{n=1}^{N} \left(c_n^2 + \frac{v_n^2}{2} \right)$$

etc.

Theorem 2.17. *The operator L is q-gap if and only if its coefficients are extremals of the functional H,*

$$H = I_{q+2} + \sum_{k=0}^{q+1} \alpha_k I_k. \tag{2.274}$$

This assertion follows from the formula

$$i\delta p = \frac{l_0 E^{q+1} + \ldots + l_{q+1}}{\sqrt{R(E)}}, \quad l_i = l_i(\delta c_n, \delta v_n),$$
$$\delta = \sum_n \left(\frac{\partial}{\partial c_n} \delta c_n + \frac{\partial}{\partial v_n} \delta v_n \right). \tag{2.275}$$

In fact, by expanding (2.275) in the neighbourhood of P^+ and comparing with the coefficients of (2.273), we get

$$l_0 = -\delta I_0, \quad l_1 = -\delta I_1 + \frac{s_1}{2} \delta I_0, \quad s_1 = \sum_i E_i,$$

$$l_2 = -\delta I_2 + \frac{s_1}{2} \delta I_1 + \left(\frac{s_1^2}{8} - \frac{s_2}{2} \right) \delta I_0, \quad s_2 = \sum_{i<j} E_i E_j,$$

$$l_k = -\delta I_k + \sum_{i=0}^{k-1} \beta_{ik} \delta I_i. \tag{2.276}$$

From the first $q+1$ equalities the coefficients l_k will be expressed via the δI_k, $k \le q+1$. Equating the coefficients of E^{-q-2} in the expansion (2.273) and in (2.275), we get that

$$\delta H = 0, \tag{2.277}$$

where the α_k are the symmetric polynomials in the E_i.

The proof of formula (2.274) can be obtained in an entirely analogous way to the proof of its continuous version [38].

References*

1. The basic concepts of the Hamiltonian formalism go back to the classical work in analytical mechanics, to Poisson, Hamilton, Jacobi, Lie. Different versions of the presentation of these classical concepts are to be found in quite a number of textbooks (see, for example, [7], [42]) and surveys (see [112]). Infinite-dimensional analogues of the Hamiltonian formalism until recently were considered only for Lagrangian field systems in connection with the needs of quantum field theory (see, for example, [17]). More complicated examples arose in the hydrodynamics of a perfect incompressible fluid (see [7], appendix 2), and also in the theory of the Korteweg–de Vries equation [54], [59]. The modern formalism of Poisson brackets in application to infinite-dimensional (field theoretic) systems was systematically developed in [112]. The general concept of a Poisson bracket of hydrodynamic type was introduced and studied in [48].

2. The use of the symmetry of Hamiltonian systems to construct their integrals and to reduce their order also goes back to the classical works of Jacobi, Poincaré and others; for a modern presentation see [101] (see also [7], appendix 5). The construction of integrals for field-theoretic Lagrangian systems with symmetry was given by E. Noether.

3. The concept of a completely integrable Hamiltonian system arose in the works of Bour and Liouville (see the textbooks [7], [42]). We do not discuss here degenerate completely integrable systems with a larger number of integrals than the number of degrees of freedom (see, for example, [57] and V.V. Kozlov's survey article [72]).

4. The fundamental material of section 4 (chap. 1) is contained in the classical works of Hamilton and Jacobi (see, for example, the text [7]).

5. Beginning with the paper [89], in which the mechanism for integrating the KdV equation which had been proposed in the pioneering paper [60] was cleared up, all schemes for producing integrable evolution equations have been based on representing them in the form of a compatibility condition for the auxiliary linear problems.

The scheme based on the equations of "zero curvature for rational families of operators", proposed in [127], included in a natural way all examples known up till then, in particular such key stages in the development of the method as [54], [88], [96], [56], [59], [1]. (These examples and a number of others are presented together with the history of the development of the first stages of the inverse scattering method in the books [115], [25]). A representation of the KdV equation in the form of the zero curvature equation for polynomial families of operators was first proposed in [111], and an example of a rational family was met with in the paper [1].

For the anisotropic Landau–Lifshitz equation, the papers [22], [128] first used a zero-curvature representation for families of operators with a spectral parameter on an elliptic curve. This line received a further development in the papers [30], [29]. In the papers [84], [86] another way was

* For the convenience of the reader, reference to reviews in Zentralblatt für Mathematik (Zbl.), compiled using the MATH database, and Jahrbuch über die Fortschritte der Mathematik (FdM.), have, as far as possible, been included in this bibliography.

proposed of generalizing the zero-curvature equations for rational families to the case where the spectral parameter is defined on an algebraic curve of non-zero genus. In the article [80] (for greater detail see [83]) a representation for the Moser–Calogero system was proposed in which the dependence of the matrix entries on a parameter defined on an elliptic curve contained essential singularities of a special form.

6. The program for integration of the periodic problem for the KdV equation was initiated by the paper [111] (somewhat later and in a less effective form it was considered in [90]). The employment of the methods of algebraic geometry for the construction of periodic and quasiperiodic solutions of the KdV and nonlinear Schrödinger equations was begun in the articles [38], [39], [45], [47], [64]. (Later the papers [102], [103] appeared.) For the sine-Gordon equation finite gap solutions were constructed in [68]. The question of whether one can approximate an arbitrary periodic potential by finite gap potentials of a Sturm–Liouville operator with conservation of the period was settled positively in [100], [104].

The first stage of the theory of finite gap integration was presented in [45], [115].

A general scheme for integrating two-dimensional equations of the type of the Kadomtsev–Petviashvili equation with the aid of the methods of algebraic geometry was proposed in [74], [75]. It also included in a natural way the constructions of solutions of one-dimensional evolution equations which were proposed in the works cited above. The concept of the Clebsch–Gordan–Baker–Akhiezer function became the central concept of this scheme. The definition of such functions, including the multi-point ones, was given in [75] on the basis of a generalization of the analytic properties of Bloch functions of finite gap periodic and quasiperiodic operators. "Single-point" functions of this kind were introduced as a formal generalization of the concept of exponentials in the 19th century by Gordan and Clebsch (see [8]). Their connection with a joint eigenfunction of a pair of commuting operators of relatively prime orders was first noted in [9] by Baker. N.I. Akhiezer indicated examples of the interpretation of such functions in the spectral theory of operators on the half-line.

The isolation of the real non-singular solutions within the framework of the general scheme, for equations for which the auxiliary linear problem is not self-adjoint, was begun in [28] and was earnestly pushed forward in [13], [41], [44], [46].

7. A general Hamiltonian theory of systems whose integration is connected with hyperelliptic curves was proposed in [116], [118]. This theory made it possible to examine from a single point of view and to unify not only the Hamiltonian structure itself of diverse systems, but also to give a unified construction of variables of the action-angle type. For Kovalevskaya's system a construction of variables of the action type was obtained for the first time in just these articles. A construction of action-angle variables for the Hamiltonian systems connected with finite gap Sturm–Liouville operators was first obtained in [6], [56]. The relation of the stationary and non-stationary Hamiltonian formalisms for these systems was obtained in [18], [20], [34].

8. References to works devoted to the algebraic-geometric integration of a number of classical systems of mechanics and hydrodynamics are cited in sections 3 and 4 of Chapter 2 in the course of the analysis of a series of prime examples.

9. The program of the research on the dynamics of the poles of solutions of equations to which the inverse scattering method is applicable goes back to the paper [87]. The connection of the dynamics of the poles of rational and elliptic solutions of the KdV equation with the rational and elliptic Moser–Calogero systems was first discovered in [4]. Without any connection with finite-dimensional systems, elliptic solutions of the KdV equation with three poles were constructed in [47]. The isomorphism of the rational Moser–Calogero system and the polar system of rational solutions of the KP equation was established in [77]. In [31] this result was carried over to the elliptic case. The construction of variables of the angle type for the elliptic Moser–Calogero system and the construction of all elliptic solutions of the KP equation were obtained in [80].

10. The algebraic-geometric Floquet spectral theory of linear operators with periodic coefficients was developed in the publications [38], [45], [115], [64], [111], [100], [90]. The starting point for these works was the problem of constructing periodic solutions of equations of the KdV type.

The possibilities for applying the algebraic-geometric spectral theory to the continuous Peierls–Fröhlich model were discovered in [12], [24].

The construction of the algebraic-geometric spectral theory of the Schrödinger difference operator was begun in the articles [45], [33] and received its completion in [79]. These results were used in the papers [23], [50], [51], [81], in which the discrete Peierls model was integrated and its perturbations were investigated.

Translator's Remark. In the literature list which follows, whenever a Russian work has been translated into English a reference to the translation has been included, and the title I have given is then simply the title of the English translation, unless (as is not infrequent!) the title of the translation is incorrect or differs significantly from the Russian title. In these cases I have supplied my own translation of the Russian title and have indicated how the title of the English translation differs.

However, I have not corrected one "mistake" which is nearly universal in translations of the subject matter treated in this article. It is the lazy translation "finite-zone" (a literal translation of the Russian term) for what English writers generally call *finite gap* (operators, potentials, etc.). Because "finite-zone" is so frequent (although it is found almost exclusively in translations from the Russian) I have left it unchanged in the English titles but wish to draw the reader's attention to it here.

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