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BRIEF COMMUNICATIONS

The Complex Lagrangian Grassmannian

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The manifold of the Lagrange subspaces of the symplectic vector space $\mathbb{R}^{2n}$ is $U(n)/O(n)$ (see [1]). It is natural to try to construct complex and quaternionic versions of this manifold (see [2]). We prove below that the complex version is the manifold $U(n)$ of unitary matrices. This manifold is a complexification of the space of Hermitian matrices in the same sense in which the real Lagrangian Grassmannian compactifies the space of real symmetric matrices.

1. Complexification of the Darboux theorem. Instead of the complex symplectic structure in the complex vector space $\mathbb{C}^n$, we consider the following version of this structure.

Definition 1. By a $\mathbb{C}$*-symplectoidal structure* in $\mathbb{C}^n$ we mean a real symplectic structure Ω that is invariant under the multiplication of the vectors by complex numbers of modulus one, $\Omega(\lambda\xi, \lambda\eta) = \Omega(\xi, \eta)$ for $|\lambda| = 1$.

Theorem 1. *Any* $\mathbb{C}$*-symplectoidal structure can be written in the form* $\Omega = \sum \pm x_k \wedge y_k$ *in a suitable system of complex coordinates* $z_k = x_k + iy_k$ *in the vector space* $\mathbb{C}^n$.

Proof. The multiplications by e^{it} form a one-parameter group of linear symplectic operators in $\mathbb{R}^{2n}$, that is, a Hamiltonian flow with a quadratic Hamiltonian function H. By a real linear symplectic transformation, one can reduce the Hamiltonian function to the standard normal form

$$H = \sum(\pm(p_k^2 + q_k^2)/2), \qquad \Omega = \sum p_k \wedge q_k,$$

because the eigenfrequencies are equal to ± 1 since the oscillations are 2π-periodic.

The invariant coordinate 2-planes (p_k, q_k) are complex lines (because the oscillations act as the multiplications by e^{it}). However the symplectic $(p_k \wedge q_k)$ orientations can be sometimes opposite to the complex orientations (defined by frames of the form ξ, $i\xi$). If the complex coordinate on the plane (p_k, q_k) is $z_k = x_k + iy_k$, then $x_k \wedge y_k = c_k p_k \wedge q_k$. For $z_k(t) = e^{it}\xi_k$ we have $\dot{z}_k = iz_k$, $\dot{x}_k = -y_k$, and $\dot{y}_k = x_k$. For $H = \pm(p_k^2 + q_k^2)/2$ we obtain the Hamiltonian equations $\dot{p}_k = \mp q_k$ and $\dot{q}_k = \pm p_k$. Hence, $p_k \pm iq_k = c_k(x_k + iy_k) = c_k z_k$. A linear change of variables $Z_k = c_k z_k$ gives $p_k \pm iq_k = Z_k = X_k + iY_k$, so that $dp_k \wedge dq_k = \pm dX_k \wedge dY_k$, and thus we have reduced Ω to the desired canonical form.

2. Adapted symplectic structure in $\mathbb{C}^{2n}$. Consider the bilinear form $S(\xi, \eta) = \Omega(i\xi, \eta)$, where Ω is a $\mathbb{C}$-symplectoidal structure. The form S is symmetric because $S(\eta, \xi) = \Omega(i\eta, \xi) = \Omega(-\eta, i\xi) = -\Omega(\eta, i\xi) = \Omega(i\xi, \eta) = S(\xi, \eta)$. The signature of the quadratic form $S(\xi, \xi)$ is a (unique) invariant of the $\mathbb{C}$-symplectoidal structure (the signature is defined by the number of minus signs in the normal form in the above theorem).

Definition 2. A $\mathbb{C}$-symplectoidal structure Ω in $\mathbb{C}^{2n}$ is called an *adapted symplectic structure* if the signature of the quadratic form S vanishes (that is, if the number of minus signs in the normal form of Ω is equal to the number of plus signs).

By the above theorem on the normal form, the adapted structure is unique (up to a complex linear transformations of the space $\mathbb{C}^{2n}$).

Example. Denote the summands in the expansion of the vector $\xi \in \mathbb{C}^{2n} = \mathbb{C}_1^n \oplus \mathbb{C}_2^n$ by $x \in \mathbb{C}_1^n$ and $y \in \mathbb{C}_2^n$, and denote the similar summands for the vector η by $v \in \mathbb{C}_1^n$ and $w \in \mathbb{C}_2^n$.

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Denote by $\langle\ ,\ \rangle$ the Hermitian inner product of vectors in $\mathbb{C}_1^n$ by vectors of $\mathbb{C}_2^n$ (for instance, $\langle x, w\rangle = \sum x_k \overline{w_k}$).

Consider the Hermitian form $\omega(\xi,\eta) = \langle x, w\rangle + \langle y, v\rangle \in \mathbb{C}$ in $\mathbb{C}^{2n}$. It is clear that $\omega(\eta,\xi) = \overline{\omega}(\xi,\eta)$ and $\omega(\lambda\xi,\eta) = \lambda\omega(\xi,\eta)$. Hence, $\omega(\lambda\xi,\lambda\eta) = \omega(\xi,\eta)$ for $|\lambda| = 1$, and thus the form ω is S^1-invariant.

Denote the real and imaginary parts of the form ω by S and Ω, respectively:

$$\omega(\xi,\eta) = S(\xi,\eta) + i\Omega(\xi,\eta),$$

where $S(\xi,\eta) \in \mathbb{R}$ and $\Omega(\xi,\eta) \in \mathbb{R}$.

Proposition 1. *The structure Ω is an adapted symplectic structure in $\mathbb{C}^{2n}$.*

By the above uniqueness theorem, our formula for ω provides a normal form for any adapted symplectic structure.

It follows from the definition of the form ω that

$$S(\xi,\eta) = \operatorname{Re} x \operatorname{Re} w + \operatorname{Im} x \operatorname{Im} w + \operatorname{Re} y \operatorname{Re} v + \operatorname{Im} y \operatorname{Im} v = S(\eta,\xi),$$
$$\Omega(\xi,\eta) = \operatorname{Im} x \operatorname{Re} w - \operatorname{Re} x \operatorname{Im} w + \operatorname{Im} y \operatorname{Re} v - \operatorname{Re} y \operatorname{Im} v = -\Omega(\eta,\xi),$$
$$\Omega(i\xi,\eta) = \operatorname{Re} x \operatorname{Re} w + \operatorname{Im} x \operatorname{Im} w + \operatorname{Re} y \operatorname{Re} v + \operatorname{Im} y \operatorname{Im} v = S(\xi,\eta),$$

and therefore

$$S(i\xi,\eta) = \Omega(-\xi,\eta) = -\Omega(\xi,\eta).$$

We see that Ω and S are nondegenerate real S^1-invariant bilinear forms, Ω is skew-symmetric, S is symmetric, and the signature of S is zero, which follows, for instance, from the formula

$$4S(\xi,\eta) = \|x+w\|^2 - \|x-w\|^2 + \|y+v\|^2 - \|y-v\|^2.$$

The proposition is thus proved.

Denote by X, Y, V, and W the n-component complex vectors given by the formulas

$$x - y = X, \quad x + y = Y, \quad v - w = V, \quad v + w = W.$$

We have constructed a new decomposition $\mathbb{C}^{2n} = C_3^n \oplus \mathbb{C}_4^n$ ($X \in \mathbb{C}_3^n$, $Y \in \mathbb{C}_4^n$, $V \in \mathbb{C}_3^n$, and $W \in \mathbb{C}_4^n$ are the new "components" of the vectors ξ and η). Note that $2x = X + Y$, $2y = -X + Y$, $2v = V + W$, $2w = -V + W$. Therefore, for the form ω, we obtain the relation

$$4\omega(\xi,\eta) = 4\langle x, w\rangle + 4\langle y, v\rangle = 2\langle Y, W\rangle - 2\langle X, V\rangle. \qquad (*)$$

3. Lagrangian planes. A complex subspace L of $\mathbb{C}$-dimension n in $\mathbb{C}^{2n}$ is said to be Ω-Lagrangian if $\Omega(\xi,\eta) = 0$ for any vectors ξ, η in L. It is said to be ω-Lagrangian if $\omega(\xi,\eta) = 0$ for any vectors ξ, η in L.

Proposition 2. *A subspace L is Ω-Lagrangian if and only if it is ω-Lagrangian.*

Proof. As we saw above, $\omega(\xi,\eta) = \Omega(i\xi,\eta) + i\Omega(\xi,\eta)$. Since the subspace L is complex, it follows that $i\xi$ belongs to L together with ξ. Therefore, if $\Omega = 0$ on L, then $\omega = 0$ on L. If $\omega = 0$, then its imaginary part Ω (as well as its real part S) also vanishes.

Proposition 3. *The intersection of any Lagrangian subspace L with the plane $x = y$ (on which $X = 0$) is the point* 0.

Proof. For any vector $\xi \in L \cap (x = y)$, we have $\omega(\xi,\xi) = 2\langle x, x\rangle = 0$, and hence $x = 0$.

Corollary 1. *Every Lagrangian subspace L in $\mathbb{C}^{2n}$ is the graph of a complex linear operator $U\colon \mathbb{C}_3^n \to \mathbb{C}_4^n$ (i.e., it is defined by the equation $Y = UX$).*

Indeed, by the previous proposition, L is a section of the fibration $\mathbb{C}^{2n} \to \mathbb{C}_3^n$ with fibers parallel to $\mathbb{C}_4^n$.

Theorem 2. *The graph of an operator U is a Lagrangian subspace if and only if the operator is unitary.*

Proof. By formula $(*)$ we have

$$4\omega(\xi,\eta) = 2\langle Y, W\rangle - 2\langle X, V\rangle.$$

For $Y = UX$ and $W = UV$, the condition $\omega = 0$ (which means that the graph is Lagrangian) becomes

$$\langle UX, UV\rangle = \langle X, V\rangle$$

for any X and V in $\mathbb{C}^n_3$. This means that the operator U is unitary and that the graph is Lagrangian for any unitary operator U.

Corollary 2. *The manifold of the Lagrangian subspaces in $\mathbb{C}^{2n}$ is diffeomorphic to the group $U(n)$ of unitary matrices.*

The diffeomorphism sends any matrix U to the graph of the corresponding operator.

Remark. If a complex subspace is the graph of an operator $y = Ax$, then the condition that the subspace is Lagrangian, $\omega(\xi, \eta) = \langle x, w\rangle + \langle y, v\rangle = 0$, becomes $\langle x, Av\rangle + \langle Ax, v\rangle = 0$, that is, $A^* = -A$ (the operator A is skew-Hermitian). The relation between the operators A and U is provided by the Cayley transform (which already occurred in this situation in [1]):

$$A = (U-1)/(U+1), \qquad U = (1+A)/(1-A).$$

Therefore, the unitary group compactifies the (real) vector space of the skew-Hermitian matrices (corresponding to the unitary matrices that have no eigenvalue equal to -1).

References

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