Annals of Mathematics

The index of elliptic operators: I*

By M. F. ATIYAH and I. M. SINGER

Introduction

This is the first of a series of papers which will be devoted to a study of
the index of elliptic operators on compact manifolds. The main result was
announced in [6] and, for manifolds with boundary[1], in [5]. The long delay
between these announcements and the present paper is due to several factors.
On the one hand, a fairly detailed exposition has already appeared in [14].
On the other hand, our original proof, reproduced with minor modifications
in [14], had a number of drawbacks. In the first place the use of cobordism,
and the computational checking associated with this, were not very enlighten-
ing. More seriously, however, the method of proof did not lend itself to
certain natural generalizations of the problem where appropriate cobordism
groups were not known. The reader who is familiar with the Riemann-Roch
theorem will realize that our original proof of the index theorem was modelled
closely on Hirzebruch's proof of the Riemann-Roch theorem. Naturally
enough we were led to look for a proof modelled more on that of Grothendieck.
While we have not completely succeeded in this aim, we have at least found
a proof which is much more natural, does not use cobordism, and lends itself
therefore to generalization. In spirit, at least, it has much in common with
Grothendieck's approach.

A further essential feature of our new approach is the elimination of
cohomology from the picture, at least in the first instance. The algebraic
topology that is really relevant to the study of elliptic operators is K-theory.[2]
This is not surprising since K-theory may be roughly described as "the alge-
braic topology of linear algebra". Thus, in the present paper, no homology
or cohomology is used, either explicitly or implicitly. Our main theorem, giv-
ing a formula for the index of an elliptic operator, is expressed purely in K-
theoretical terms. This is especially significant for generalizations which will
be dealt with in subsequent papers. The cohomological formula given in [6]
can be obtained quite simply from the K-theory formula given here. This,

[1] jointly with R. Bott.

[2] See for example [1] for an elementary treatment. See also [2] for a non-technical
exposition of the relation between K-theory and elliptic operators in euclidean space.

of course, is simply an exercise in algebraic topology, translating from one set of topological invariants to another, and it will be dealt with in paper III of this series. Which formula actually provides the "best answer" is partly a matter of taste. It depends on which invariants one is more familiar with, or can most easily compute.

One generalization of the index problem plays a minor role in our proof and is therefore included in this first paper. This concerns an elliptic operator which is invariant under a compact Lie group G. The index in this case is a character of G. The appropriate algebraic topology here is K_G-theory[3] (or equivariant K-theory), and our formula for the index is expressed in those terms. In a subsequent paper, II of this series,[4] we shall show how this formula can be expressed in terms of the fixed-point sets of elements of G. The character formula obtained in this way is closely related to the generalized Lefschetz fixed-point formula of [4], and is a special case of it when the fixed-points are all isolated. As a particularly interesting special case of this character formula (for general fixed-point sets) one can derive the formula of Langlands [12] for the dimension of spaces of automorphic forms. This will be dealt with in a separate paper.[5]

Besides the topics referred to above, subsequent papers of this series will deal with

(1) Families of elliptic operators, parametrized by a space X. In this case one can define an index in $K(X)$.

(2) Real elliptic operators. For example, if D is a real skew-adjoint elliptic operator, dim Ker D (mod 2) is a deformation invariant. This can be interpreted as an index, and a general formula will be obtained which includes this as a special case.

(3) Manifolds with boundary.

(4) Operators which are "elliptic relative to a group action"; i.e., their symbols are elliptic in the directions transversal to the orbits.

In order to keep this paper to a reasonable and readable size, we shall not attempt to develop *ab initio* all the relevant topology and analysis. Instead we shall summarize the relevant basic material. Thus § 2 contains a review of K- and K_G-theory, with special emphasis on those parts of the theory which will be needed. In § 5 we review pseudo-differential operators, and in § 6 we review the basic analytical properties of the index of elliptic operators. All of this is essentially standard material.

---

[3] See [16].

[4] by Atiyah and Segal.

[5] See also [7].

The main ideas of the proof are explained in a non-technical manner in §1. In §3 we define the *topological index* using the basic facts about K-theory explained in §2. Then in §4 we set up some axioms for "index functions" and prove a uniqueness theorem (4.6), so that any index function satisfying these axioms must coincide with the topological index of §3. The elementary properties of the analytical index given in §6 show that this is an index function. At this stage we can formulate our main theorem (6.7) which simply asserts that the analytical index is equal to the topological index. In view of the uniqueness theorem for index functions established in §4 it is enough to show that the analytical index satisfies the axioms. This is done in §8 and §9. In §7 there is a digression dealing with elliptic complexes, and we show how (6.7) implies a formula (7.1) for the Euler characteristic of an elliptic complex.

Many of the analytical ideas and devices which we use here are due to R. Seeley. He first introduced the "excision" property, and also carried through for us the technical treatment of operators on product manifolds. The final presentation of the analysis in §§5-9 owes a great deal to the help of L. Hörmander with whom we have had many illuminating discussions. Finally we are glad to acknowledge our indebtedness to R. Palais whose *Annals of Mathematics Study* [14] gave such a thorough presentation of our first proof, and has been of considerable assistance to us in preparing the present paper.

1. Idea of proof

In this section, we shall try to give some intuitive ideas of the nature of the proof. In our more formal treatment in the remaining sections, we shall present things in a slightly different manner for technical reasons.

Let X be a submanifold of a manifold Y, both being compact. Denote by $i: X \to Y$ the inclusion. Let A be an elliptic operator on X. Then the main step in our proof is the construction (on the symbolic level) of an elliptic operator $i_!(A)$ on Y such that

(1.1) $$\text{index } A = \text{index } i_!(A) .$$

Once this has been done, we can take Y to be a sphere, and the general index problem is reduced to the case of operators on the sphere. For these the problem is easily solved. In fact we can go one step further. Thus, if $i: X \to S$ is an embedding, and if $j: P \to S$ denotes the inclusion of a point, we can show that there is an elliptic operator B on the point P so that $j_!(B) = i_!(A)$ (up to a suitable equivalence preserving the index). Thus

$$\text{index } A = \text{index } i_!(A) = \text{index } j_!(B) = \text{index } B \ ,$$

and the problem is reduced to a point, where of course it is trivial!

The construction of $i_!(A)$ and the verification of (1.1) is thus the heart of the problem. Now in a tubular neighborhood U of X in Y, we may consider the "transversal de Rham complex" C; i.e., in each normal plane of X we consider the de Rham complex of that plane. The "tensor product" $A \otimes C$ can then be defined as an elliptic complex over U, at least on the symbolic level. It turns out that there is a natural trivialization of the symbol of C on the boundary of U, and this enables us to extend the symbol of $A \otimes C$ to the whole of Y. This is the definition of $i_!(A)$.

To compute the index of $i_!(A)$, we use an excision property,[6] observed by Seeley [15], which shows that

$$\text{index } i_!(A) = \text{index } k_!(A) \ ,$$

where $k: X \rightarrow Z$ is the inclusion of X in "the double" of U. Thus Z is fibered over X by spheres. Moreover the symbol of $k_!(A)$ is the tensor product of the symbol of A on X and a symbol "elliptic along the fibres" which may be roughly described as "one-half" of the de Rham complex along the fibres. Now if Z were a product bundle $X \times S$, we could use the multiplicative property of the index [14; Ch. XIV] and obtain

(1.2) $$\text{index } k_!(A) = \text{index } A \cdot \text{index } j_!(1) \ ,$$

where $j: P \rightarrow S$ is the inclusion of a point, and 1 is an operator of index 1 on P. This would reduce us to proving (1.1) with $X = P$, $Y = S$, which is fairly simple by direct computation (and can moreover be reduced, by the multiplicative property, to the cases dim $S = 1, 2$). In the general case when Z is not a product bundle, we have to generalize the multiplicative property of the index to deal with fibre bundles.[7] One does not always obtain a formula as simple as (1.2), but one does obtain this formula when the "operator along the fibres" is rigid in a certain sense. Fortunately the de Rham complex is rigid in this sense, and so (1.2) holds and the proof is then complete. This rigidity of the de Rham complex comes (in one point of view) from the fact that its cohomology groups are the cohomology groups of the underlying space and so have an *integral basis*. The space of solutions of a general elliptic operator does not normally have such a discrete structure, and it can therefore "vary" in a continuous fashion.

It is perhaps worth making a remark about the construction of the

[6] A very similar idea was suggested to us independently by P. J. Cohen.
[7] It is here that it becomes helpful to introduce groups into the index problem.

operator $i_!(A)$. The trivialization of the de Rham complex used in extending the symbol of $A \otimes C$ from U to the whole of Y corresponds to the existence of a natural boundary-value problem.[8] The use of such a boundary-value problem in U may be more conceptual but, on the other hand, boundary questions involve more delicate considerations both analytical and topological. For this reason, we have kept away from boundary-value problems for the present.

Finally, it may be worth pointing out that the proof given in this paper could be technically simplified in a number of ways if one only wants the index theorem of [6], particularly if one is content to compute the index of differential operators only. However, since our aim is to present a proof which will generalize in various directions, we have not attempted to make these simplifications. The ideas in any case are the same.

2. Review of K-theory

A general reference for the results summarized here is [1]. The basic periodicity theorem is also in [3], while more detailed facts about K_G can be found in [16]. It should be emphasized that the only[9] non-trivial result which is involved in all this is the periodicity theorem.[10]

Let X be a compact space. The isomorphism classes of complex vector bundles over X form an abelian semi-group under $\oplus$, and the associated abelian group is denoted by $K(X)$. If E is a vector bundle over X, the corresponding element of $K(X)$ is denoted by $[E]$. $K(X)$ is a commutative ring under $\otimes$. A continuous map $f: X \to Y$ induces a natural ring homomorphism $f^*: K(Y) \to K(X)$ which depends only on the homotopy class of f. $K(\text{Point})$ is naturally isomorphic to the ring $\mathbf{Z}$ of integers. If X is a space with a given base point P, then $\tilde{K}(X)$ is defined as the kernel of the homomorphism

$$K(X) \longrightarrow K(P)$$

induced by the inclusion $P \to X$. We have, moreover, a natural decomposition

$$K(X) \cong \tilde{K}(X) \oplus K(P) \cong \tilde{K}(X) \oplus \mathbf{Z} .$$

If Y is a closed subspace of X, we denote by X/Y the space with base point obtained by collapsing Y to a point (if $Y = \varnothing$ we take $X/\varnothing = X + P$ disjoint sum). We define

---

[8] See [5] for the relation of boundary conditions to the trivialization of an elliptic symbol.

[9] In K_G-theory we also need the basic (Peter-Weyl) theorem about representations of G.

[10] Further comments on the periodicity theorem and the proofs in [1] and [3] will be made later in this section.

$$K(X, Y) = \tilde{K}(X/Y) .$$

If X is a locally compact space we define $K(X) = \tilde{K}(X^+)$ where X^+ is the one-point compactification of X: it will be a contravariant functor of X for *proper* maps, because only these extend to X^+. This is "K-theory with compact supports" and there is an alternative way to describe it which is very convenient. We take, as basic objects, complexes of vector bundles over X, that is sequences

$$0 \longrightarrow E^0 \xrightarrow{\alpha} E^1 \xrightarrow{\alpha} \cdots \xrightarrow{\alpha} E^n \longrightarrow 0$$

of vector bundles E^i and homomorphisms α with $\alpha^2 = 0$. The *support* of such a complex is the set of points $x \in X$ for which

$$0 \longrightarrow E^0_x \longrightarrow E^1_x \longrightarrow \cdots \longrightarrow E^n_x \longrightarrow 0$$

is *not* exact. We consider only *complexes with compact support.* Two such complexes E, F are said to be homotopic if there is a complex G over $X \times I$ (with I the unit interval) such that $G \,|\, X \times \{0\} \cong E$ and $G \,|\, X \times \{1\} \cong F$. The homotopy classes of complexes over X, we denote by $C(X)$. This is an abelian semi-group under $\oplus$, and it contains a subsemigroup $C_\varphi(X)$ represented by complexes with empty support. As a second definition of $K(X)$ we can then take the quotient $C(X)/C_\varphi(X)$; although apparently only a semi-group, it turns out to be a group. The proof that our two definitions of $K(X)$ agree can be found in[11] [16]; in fact, we shall only use non-compact spaces in a very mild way (e.g. the total space of a vector bundle over a compact base) and the reader should have little difficulty in eliminating them entirely from the presentation. The formalism is, however, much simpler and more conceptual in the locally compact theory.

If E is a complex of vector bundles with compact support as above, we denote by $[E]$ its class in $K(X)$. If X is actually compact, then we have

(2.1) $$[E] = \sum (-1)^i [E^i] .$$

Two complexes E and F which involve the same bundles, and whose homomorphisms α_E and α_F coincide outside a compact set, are homotopic, and so $[E] = [F]$. In fact we can define the class $[E]$ of E even if the homomorphisms α are not defined on some compact set L; we simply multiply by $1 - \varphi$ where φ is a continuous function with compact support and vanishing in a neighborhood of L.

Note that $K(X)$ is a ring without identity unless X is compact. If U is

---

[11] For X, Y compact the analogous definition of $K(X, Y)$ by complexes on X with support in $X - Y$ is treated in [1]. The version described here follows from this and the continuity property (2.2): we consider the groups $K(\bar{U}, \bar{U} - U)$.

an open set of the locally compact space X, we have a natural map

$$X^+ \longrightarrow X^+/(X^+ - U) = U^+$$

and so a natural homomorphism

$$i_*: K(U) \longrightarrow K(X) .$$

Thus if $\{U_\alpha\}$ denotes the directed family of all open relatively compact subsets of X, the groups $K(U_\alpha)$ form a direct system of groups (with the homomorphisms $i_*^{\alpha\beta}: K(U_\alpha) \to K(U_\beta)$ if $U_\alpha \subset U_\beta$). Then K has the following continuity property

(2.2) $K(X) = \mathrm{dir\ lim}\ K(U_\alpha) .$

The proof of this is quite elementary (cf. [16]).

If we work only with complexes of fixed length $n \geq 1$, then it is still true that $K(X)$ can be defined as $C^n(X)/C_\varphi^n(X)$. The advantage of using complexes of arbitrary finite length lies in the multiplicative structure. Thus if E is a complex on X, F a complex on Y, then $E \boxtimes F$ (the external tensor product) is a complex on $X \times Y$, and it has compact support if E and F have compact support. This induces a product

$$K(X) \otimes K(Y) \longrightarrow K(X \times Y) .$$

As a simple example of multiplication of complexes, let $0 \longrightarrow E^0 \overset{\alpha}{\longrightarrow} E^1 \longrightarrow 0$ and $0 \longrightarrow F^0 \overset{\beta}{\longrightarrow} F^1 \longrightarrow 0$ be complexes with compact support on spaces X, Y respectively. Their external tensor product is the complex

$$0 \longrightarrow E^0 \boxtimes F^0 \overset{\varphi}{\longrightarrow} E^1 \boxtimes F^0 \oplus E^0 \boxtimes F^1 \overset{\psi}{\longrightarrow} E^1 \boxtimes F^1 \longrightarrow 0 ,$$

where $\varphi = \alpha \boxtimes 1 + 1 \boxtimes \beta$, $\psi = -1 \boxtimes \beta + \alpha \boxtimes 1$. Now, using metrics in the bundles, we can exhibit a complex of length one representing the same class in $K(X \times Y)$ as the complex $E \boxtimes F$, namely the complex

$$0 \longrightarrow E^0 \boxtimes F^0 \oplus E^1 \boxtimes F^1 \overset{\theta}{\longrightarrow} E^1 \boxtimes F^0 \oplus E^0 \boxtimes F^1 \longrightarrow 0 ,$$

where

$$\theta = \begin{pmatrix} \alpha \boxtimes 1, & -1 \boxtimes \beta^* \\ 1 \boxtimes \beta, & \alpha^* \boxtimes 1 \end{pmatrix} ,$$

and α^*, β^* denote the adjoints of α, β with respect to the metrics (for the proof that this is equivalent to $E \boxtimes F$ see [1; 2.6.10]). This particular representative for the product of two complexes will be needed later in § 9.

For the application to elliptic operators, we shall be particularly concerned with the groups $K(V)$ where V is a *real* vector bundle over a space[12] X.

[12] In the applications X will be a smooth manifold and $V = TX$ will be its cotangent bundle.

If W is another such bundle then composing the product

$$K(V) \otimes K(W) \longrightarrow K(V \times W)$$

with the map $K(V \times W) \to K(V \oplus W)$, induced by the diagonal inclusion $X \to X \times X$, we obtain a product

(2.3) $$K(V) \otimes K(W) \longrightarrow K(V \oplus W) .$$

In particular, taking $W = X$, we see that $K(V)$ is a $K(X)$-module.

Let E^i, E^{i+1} be complex vector bundles over X, and let $\pi^* E^i$ denote the induced bundle over V; the fibre $(\pi^* E^i)_v$ may then be identified with $E^i_{\pi(v)}$. A homomorphism $\alpha: \pi^* E^i \to \pi^* E^{i+1}$ is called (positively) homogeneous of degree m if

$$\alpha_{\lambda v} = \lambda^m \alpha_v \in \mathrm{Hom}\,(E^i_{\pi(v)}, E^{i+1}_{\pi(v)}) ,$$

for all $v \in V$ and all real positive scalars λ. Note that, if we fix a metric in V, and let $S(V)$ be the unit sphere bundle of V, then such a homogeneous α is determined by its restriction to $S(V)$. If all the homomorphisms α of the complex E over V are homogeneous of degree m, we say that E is homogeneous of degree m. If X is *compact* and if, outside the zero-section of V, E is exact, then it has compact support and so defines an element of $K(V)$. It is easy to show that $K(V)$ can in fact be defined using only homogeneous complexes (of fixed degree). More precisely let $^m C(V)$ denote the semigroup of (compactly supported) homogeneous complexes of degree m modulo homogenous (compactly supported) homotopies, and let $^m C_\varphi(V) \subset {}^m C(V)$ denote those elements represented by complexes whose restriction to the unit sphere $S(V)$ is induced by a complex on X, so that

$$\alpha_v = \|v\|^m \beta_{\pi(v)}$$

where

$$\cdots \longrightarrow E^i \xrightarrow{\ \beta\ } E^{i+1} \longrightarrow \cdots$$

is exact on X. Then

$$^m C(V)/{}^m C_\varphi(V) \cong C(V)/C_\varphi(V) \cong K(V) .$$

This is proved as follows. Given a complex E on V with compact support L, choose a ball bundle $B_\rho(V)$ of radius ρ containing L. The class $[E]$ depends only on the restriction of E to $B_\rho(V)$. Since X is a deformation retract of $B_\rho(V)$, it follows that, over $B_\rho(V)$, $E^i \overset{\gamma_i}{\cong} \pi^* F^i$, where F^i is the restriction of E^i to the zero section. Moreover this isomorphism is unique up to homotopy if we choose γ_i to be the identity on the zero-section. Putting

$$\alpha'_i = \gamma_{i+1} \alpha_i \gamma_i^{-1}$$

on the boundary $S_\rho(V)$ of $B_\rho(V)$, and extending it to V as a homogeneous function of degree m, we obtain a homogeneous complex E'. Associating E' to E then defines a homomorphism $C(V) \to {}^mC(V)$, and this induces the required isomorphism $C(V)/C_\varphi(V) \to C^m(V)/{}^mC_\varphi(V)$.

Again we can restrict ourselves to complexes of length one so that we can write

$$K(V) = {}^mC^1(V)/{}^mC_\varphi^1(V) \ .$$

The degree m above plays, of course, a quite harmless role. Multiplication by $\|v\|^{s-m}$ will map ${}^mC(V) \to {}^sC(V)$ isomorphically. Thus it would be natural to consider only the case $m = 0$. However this is inconvenient for the explicit description of products. If E, F are compactly supported homogeneous complexes of degree m over vector bundles V, W respectively, then $E \boxtimes F$ over $V \times W$ is again compactly supported (i.e., exact outside the zero-section) provided $m > 0$; if $m \leq 0$, there will be discontinuities in the homomorphisms of $V \boxtimes W$ at points $(v, 0)$ and points $(0, w)$. This fact will be of considerable analytical significance in § 9.

On the other hand, we cannot settle for $r > 0$ because in one situation we definitely must take $r = 0$. This occurs when X (the base of V) is non-compact. A homogeneous complex over V, as above, cannot now have compact support unless $r = 0$; when $r = 0$, it has compact support provided it is exact outside the zero-section of V and *in addition*, outside some compact set in X, the homomorphisms α are constant on the fibres of V. Actually, for non-compact X, we shall only explicitly need the following fact. Every element $a \in K(V)$ can be represented by a compactly supported homogeneous complex of degree zero

$$0 \longrightarrow \pi^*(E^0) \overset{\alpha}{\longrightarrow} \pi^*(E^1) \longrightarrow 0 \ ,$$

with E^0 and E^1 trivial outside a compact set in X. This is proved much as before. Since $K(V) = \tilde{K}(V^+)$, we can represent a by a complex

$$(2.4) \qquad\qquad 0 \longrightarrow F^0 \overset{\varphi}{\longrightarrow} F^1 \longrightarrow 0$$

in which, outside some compact $L \subset V$, we have isomorphisms

$$\beta_i \colon F^i \mid V - L \longrightarrow (V - L) \times \mathbf{C}^n \ ,$$

and, over $V - L$, $\varphi = \beta_1^{-1}\beta_0$. Let Y be open relatively compact in X containing $\pi(L)$, and let ρ be chosen so that the compact set L lies in the ball bundle $B_\rho(V) \mid \bar{Y}$. Then a is already determined by the restriction of the complex (2.4) to $B_\rho(V) \mid \bar{Y}$.
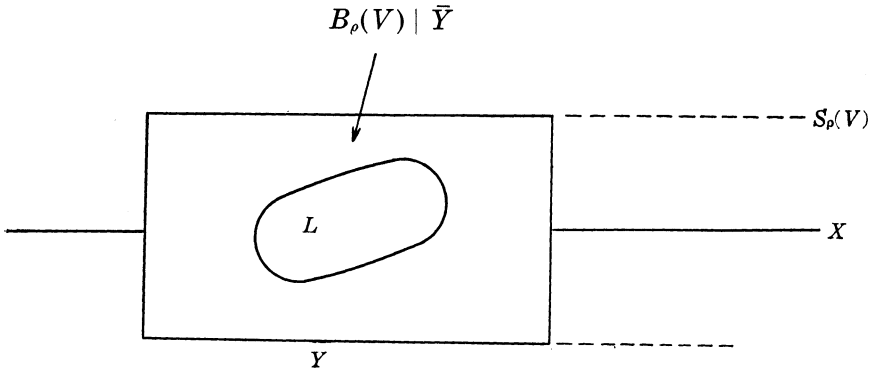
$$B_\rho(V) \mid \bar{Y}$$

FIG. 1

Now on this compact set, since $B_\rho(V) \mid \bar{Y}$ retracts onto $\bar{Y}$, we can find isomorphisms $\theta_i \colon F^i \to \pi^* E^i$ (where E^i is the restriction of F^i to the zero-section of V). Moreover we can assume the θ_i extend the isomorphisms over $\pi^{-1}(\bar{Y} - Y)$ induced by the β_i; more precisely this means that, if $x \in \bar{Y} - Y$ and $\pi(v) = x$, then the homomorphism $\theta_i(v) \colon F_v^i \to (\pi^* E^i)_v = F_x^i$ coincides with the composition $\beta_i(x)^{-1} \circ \beta_i(v)$. Now put $\alpha = \theta_1^{-1} \varphi \theta_0$ on

$$\partial\big(B_\rho(V) \mid \bar{Y}\big) = \big(S_\rho(V) \mid \bar{Y}\big) \cup \big(B_\rho(V) \mid (\bar{Y} - Y)\big),$$

extend it to all of $V \mid \bar{Y}$ so as to be homogeneous of degree zero and extend it outside $\bar{Y}$ in the obvious way (using the fact that the E^i are trivial there, and that α is isomorphic to the identity). We now have a representative for $a \in K(V)$ of the required type.

   Remark. If the base X is a smooth manifold, then the isomorphism classes of continuous and smooth vector bundles over X coincide (by standard approximation arguments). Similarly, if V is also smooth, we get the same group $K(V)$ if we take only smooth bundles over V in the above definitions by complexes.

   Suppose now that V is a complex vector bundle over X, and assume first that X is compact. Then the exterior algebra $\Lambda^*(V)$ defines in a natural way a complex of vector bundles over V, exact outside the zero-section (and homogeneous of degree 1). We shall call this the exterior complex of V, and denote it by $\Lambda(V)$. The class of $\Lambda(V)$ gives an element of $K(V)$ which we denote by λ_V. Since $K(V)$ is a $K(X)$-module, multiplication by λ_V induces a homomorphism

$$\varphi \colon K(X) \longrightarrow K(V).$$

We call this the *Thom homomorphism*. Note that if $i \colon X \to V$ denotes the zero-section, then by (2.1) we obtain

(2.5) $$i^* \varphi(x) = \Big\{ \sum (-1)^i \lambda^i(V) \Big\} x \qquad\qquad \text{for } x \in K(X).$$

For locally compact X, the exterior complex $\Lambda(V)$ does not have compact support. However for any complex E on X with compact support, the product $\Lambda(V) \otimes E$ has compact support, and

$$E \longmapsto \Lambda(V) \otimes E$$

defines again a Thom homomorphism $K(X) \to K(V)$.

Since for vector spaces V, W, we have a natural isomorphism of algebras

$$\Lambda^*(V \otimes W) \cong \Lambda^*(V) \otimes \Lambda^*(W) ,$$

it follows that, for vector bundles, V, W, over a compact space X, we have the multiplicative formula

(2.6) $$\lambda_V \cdot \lambda_W = \lambda_{V \oplus W} .$$

Here we use the product (2.3). The formula (2.6) implies that the Thom homomorphism

$$K(X) \longrightarrow K(V \oplus W)$$

coincides with the composition of the two Thom homomorphisms

$$K(X) \longrightarrow K(V)$$
$$K(V) \longrightarrow K(V \oplus W)$$

the latter being obtained by regarding $V \oplus W$ as a bundle over V.

The fundamental result of K-theory, the *Bott periodicity theorem*,[13] asserts that φ is an *isomorphism*. Note in particular the special case $X = \text{point}$; $V = \mathbf{C}^n$. Then we have an isomorphism

$$\varphi \colon K(\text{Point}) \longrightarrow K(\mathbf{C}^n) .$$

Thus $K(\mathbf{C}^n) \cong \mathbf{Z}$ and is generated by $\lambda_n = \lambda_{\mathbf{C}^n}$. The multiplicative property of λ shows that $\lambda_n = (\lambda_1)^n$.

There is a fairly straightforward generalization of K-theory to the category of G-spaces where G is a compact Lie group. Thus let X be a compact G-space, i.e., a compact space with a given action of G on X. By a G-vector bundle over X, we mean a G-space E which is also a vector bundle over X, the projection $E \to X$ being compatible with the group action, and such that for each $g \in G$ the map

$$E_x \longrightarrow E_{g(x)}$$

defined by g is linear. Note that if $X = \text{Point}$, then a G-vector bundle over X is just a complex representation space of G.

Starting from G-vector bundles over a G-space X we form a ring $K_G(X)$

---

[13] This is also called the Thom isomorphism theorem for K-theory; the Bott periodicity being restricted to the special case where V is a trivial bundle.

just as in the case where there is no group. All the elementary part of K-theory goes over without essential change to K_G (see [16]). The Bott periodicity theorem in the general case presents some new features, and not all proofs generalize automatically to K_G-theory. Thus the proof given in [1] applies only when the vector bundle V decomposes into a sum of line-bundles. This decomposable case would be adequate for our purposes because we shall explicitly need only the case when X is a point and G is an abelian group. However this is a rather artificial restriction and, in [3], a more fundamental proof of the periodicity theorem is given which applies directly to K_G. Since the proof in [3] uses indices of certain classical elliptic operators (on the sphere and projective space) it might appear that we were involved in a vicious circle. However this is not the case; the main theorem (6.7) of this paper is not used in [3]. In a sense, Theorem (6.7) expresses the index of a general elliptic operator in terms of the index of certain classical operators and, for these, one has explicit methods of computation which yield ultimately the concrete formulas in papers II and III of this series.

We shall in fact only use the special case of the Bott periodicity when X is a point, and so we state this explicitly:

(2.7) *Let G be a compact Lie group, V a complex G-module. Then the homomorphism*

$$\varphi\colon K_G(\text{point}) \longrightarrow K_G(V) \ ,$$

given by $\varphi(x) = x \cdot \lambda_V$, is an isomorphism.
Note that K_G (point) is the character or representation ring $R(G)$, and so is determined by its restriction to all abelian subgroups. It is for this reason that we could restrict ourselves to abelian groups and hence use only the periodicity theorem as proved in [1].

If G acts freely on X so that $X \to X/G$ is a principal fibre bundle, then we have a natural isomorphism

$$K_G(X) \cong K(X/G) \ .$$

More generally if $G \times H$ acts on X with H acting freely, then

$$K_{G \times H}(X) \cong K_G(X/H) \ .$$

In addition to the results reviewed so far, we shall need a small technical lemma which is not given explicitly in the literature, and which we shall therefore prove here.

Suppose W is a real G-module, $V = W \otimes_{\mathbf{R}} \mathbf{C}$ its complexification. Then the map

$$\psi\colon V \longrightarrow V$$

given by complex conjugation is a G-map. We want to investigate the induced homomorphism

$$\psi^*: K_G(V) \longrightarrow K_G(V) .$$

By (2.7) $K_G(V)$ is a free $R(G)$-module generated by the element λ_V, so it is sufficient to compute $\psi^*\lambda_V$. This can be done quite easily in general, but for our purposes the two cases given in following lemma will suffice.

LEMMA (2.8). *Let* $\psi: V \to V$ *be complex conjugation, where* $V = W \otimes_{\mathbf{R}} \mathbf{C}$ *is the complexification of the real G-module W. Then if ψ^* is the induced homomorphism of $K_G(V)$ we have*

(1) *If* $W = \mathbf{R}^1, G = O(1), \psi^*a = - a[V]$.
(2) *If* $W = \mathbf{R}^2, G = SO(2), \psi^*a = a$.

PROOF. Case (2) follows from the fact that ψ is G-homotopic to the identity. We define

$$\psi_t(u + iv) = u + ig_t(v) \qquad\qquad u, v \in W, 0 \leqq t \leqq 1$$

where $g_t \in G = SO(2)$ is rotation through πt. In case (1) we observe that the element $\psi^*\lambda_V + \lambda_V[V] \in K_{O(1)}(V)$ is represented by the complex

$$\mathbf{C} \oplus V \xrightarrow{\ \alpha_z\ } V \oplus \mathbf{C}$$

where

$$\alpha_z = \begin{pmatrix} \bar{z} & 0 \\ 0 & z \end{pmatrix} .$$

Here $\mathbf{C} = \Lambda^0(V)$ is a trivial $O(1)$-module, and we have used the natural isomorphisms $V \cong \bar{V}, V \otimes V \cong \mathbf{C}$. Let $g_t \in GL(2, \mathbf{C})$ be a path connecting the identity to the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$\begin{pmatrix} \bar{z} & 0 \\ 0 & 1 \end{pmatrix} g_t \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}$$

gives a homotopy from α_z to $\begin{pmatrix} 0 & z\bar{z} \\ 1 & 0 \end{pmatrix}$. Thus, on the unit circle $S(V)$, α_z is homotopic to a constant and so

$$\psi^*\lambda_V + \lambda_V[V] = 0 .$$

In view of (2.7), this completes the proof.

3. Symbols and the topological index

Let X be a differentiable manifold, TX its tangent bundle.[14] This is a

---

[14] Using a riemannian metric (G-invariant when appropriate) we will usually, in the topological sections, identify the tangent and cotangent bundles. In the analytical sections TX will denote the cotangent bundle.

real vector bundle. If X is a differentiable G-manifold (i.e. if the compact Lie group G acts differentiably on X) then so is TX and we may consider the group $K_G(TX)$.

Suppose now that X, Y are G-manifolds with $X \subset Y$ and X compact. Let $i: X \to Y$ denote the inclusion. Then we shall first define an $R(G)$-homomorphism

$$i_! : K_G(TX) \longrightarrow K_G(TY) .$$

Choose a G-invariant riemannian metric on Y (in the neighborhood of X), and let N be an open tubular neighborhood of X in Y. Then N is a G-manifold, and it may be identified with the normal bundle of X in Y (which is, of course, a real G-vector bundle). The tangent bundle TX is then a closed G-submanifold of TY, and the tubular neighborhood TN of TX in TY may be identified with the vector bundle over TX obtained by lifting $N \oplus N$. The simple verification is left to the reader, but since there are two copies of N involved, we must be explicit about our identification. We shall agree that the first factor corresponds to a point of Y, and the second factor to tangents to Y (along the fibres of N). Moreover we shall identify $N \oplus N$ with $N \oplus iN = N \otimes_{\mathbf{R}} \mathbf{C}$, so that the neighborhood TN of TX in TY is identified with $\pi^*(N \otimes_{\mathbf{R}} \mathbf{C})$, $\pi: TX \to X$ denoting the projection. Since this is a complex G-vector bundle, we have the Thom homomorphism

$$\varphi: K_G(TX) \longrightarrow K_G(TN) .$$

Since TN is open in TY, we have a natural homomorphism

$$k_* : K_G(TN) \longrightarrow K_G(TY)$$

induced by the inclusion. Combining φ and k_* we obtain the homomorphism

$$i_! : K_G(TX) \longrightarrow K_G(TY)$$

which we wanted to construct. It is easy to check that it is independent of the metric used in the construction and of the choice of tubular neighborhood. The fact that it is functorial (i.e., that if $X \xrightarrow{i} Y \xrightarrow{j} Z$ are inclusions, then $(ji)_! = j_! i_!$) is a consequence of the transitivity of the Thom homomorphism (see § 2). Formula (2.5) implies the following

(3.1) $i^* i_!(x) = \{\sum (-1)^i \lambda^i (N \otimes_{\mathbf{R}} \mathbf{C})\} x$ for $x \in K_G(TX)$

where $i^*: K_G(TY) \to K_G(TX)$ is the restriction homomorphism, and $K_G(TX)$ is regarded as a $K_G(X)$-module in the usual way.

Using the homomorphisms $i_!$ just constructed we shall now proceed to define a homomorphism

$$K_G(TX) \longrightarrow R(G) .$$

This homomorphism will be called the *topological index* and will be written as t-ind.

Thus let X be a compact differentiable G-manifold, and let $i: X \to E$ be a differentiable G-embedding of X in a real representation space E of G. The existence of such embeddings is essentially a consequence of the Peter-Weyl theorem (for a proof see [13]). Let $j: P \to E$ denote the inclusion of the origin P in E. Then we have homomorphisms

$$K_G(TX) \xrightarrow{i_!} K_G(TE) \xleftarrow{j_!} K_G(TP) = R(G) .$$

But $j_!$ is just the Thom homomorphism for the vector space $E \otimes_{\mathbf{R}} \mathbf{C}$ (regarded as G-bundle over the point P) and hence, by (2.7), it is an isomorphism. Thus we may define

$$\text{t-ind}: K_G(TX) \longrightarrow R(G)$$

by

$$\text{t-ind} = (j_!)^{-1} \circ i_! .$$

To see that this is independent of the choice of the embedding $i: X \to E$, let $i': X \to E'$ be another embedding and consider the diagonal embedding

$$k: X \longrightarrow E \oplus E', \quad k(x) = i(x) \oplus i'(x) .$$

It will be sufficient to show that i and k give the same answer for t-ind (the same will then be true for i' and k). Now we have a G-homotopy of embeddings $X \to E \oplus E'$ given by

$$k_s(x) = i(x) \oplus s i'(x) \qquad\qquad 0 \leqq s \leqq 1$$

and t-ind depends only on the G-homotopy class. Thus it will be enough to compare the embedding $i: X \to E$ with $k_0: X \to E \oplus E'$. If N is the normal bundle of $i(X)$, then the normal bundle of $k_0(X)$ is $N \oplus E'$ A similar result holds, rather trivially with X replaced by the origin P. The transitivity of the Thom homomorphism then gives rise to the commutative diagram

$$
\begin{array}{ccc}
& K_G(TX) & \\
{\scriptstyle i_!} \swarrow & & \searrow {\scriptstyle (k_0)_!} \\
K_G(TE) & \xrightarrow{\;\psi\;} & K_G(T(E \oplus E')) \\
{\scriptstyle j_!} \nwarrow & & \nearrow {\scriptstyle l_!} \\
& K_G(TP) & \\
& \| & \\
& R(G) &
\end{array}
$$

where ψ is the Thom homomorphism for $E' \otimes_{\mathbf{R}} \mathbf{C}$ (as G-bundle over TE), and j, l denote the inclusions $P \rightarrow E, P \rightarrow E \oplus E'$. By (2.7), $j_!$ and $l_!$ are isomorphisms, and so therefore is ψ. It now follows from the diagram that the two ways from $K_G(TX)$ down to $R(G)$ coincide, i.e., formally

$$l_!^{-1}(k_0)_! = j_!^{-1} \psi^{-1} \psi i_! = j_!^{-1} i_! .$$

Thus i and k_0 yield the same value of t-ind. We have then established that t-ind is independent of the choice of the embedding $X \rightarrow E$ used in its definition.

An important element $\rho_X \in K(TX)$ on any compact manifold X is the class ρ_X of the "de Rham symbol." This is defined as follows.[15] We consider the exterior algebra $\Lambda^*(T)$ of the tangent bundle $T = TX$. Lifted to T this gives a complex of real vector bundles, exact outside the zero section. The complexification of this thus defines an element $\rho_X \in K(TX)$. If X is a G-manifold, $\Lambda^*(T)$ is acted on naturally by G and so $\rho_X \in K_G(TX)$.

Remark. The element ρ_X is closely related to (but not to be confused with) the elements λ_V of § 2. Their relation is as follows. If $T^c = T \otimes_{\mathbf{R}} \mathbf{C}$, the element

$$\lambda_{T^c} \in K_G(T^c X) .$$

If $i: TX \rightarrow T^c X$ is the inclusion, then

$$\rho_X = i^* \lambda_{T^c} .$$

If we take $X = S^n = \mathbf{R}^n \cup \infty$, $G = O(n)$, then, in particular, we get an element $\rho_{S^n} \in K_{O(n)}(TS^n)$. This element will play a fundamental role in our proof. Actually a more fundamental object is the element $j_!(1) \in K_{O(n)}(TR^n)$ where $j: P \rightarrow \mathbf{R}^n$ is the inclusion of the origin, but, as the following lemma shows, this element is in some sense just "half" of ρ_X.

LEMMA (3.2). *Let $j^0: P^0 \rightarrow S^n$, $j^\infty: P^\infty \rightarrow S^n$ be the inclusions of the origin and the point at infinity, and let $\theta: TS^n \rightarrow TS^n$ be multiplication (of tangent vectors) by -1. Then we have*

$$\rho_{S^n} = j_!^0(1) + \theta^* j_!^\infty(1) \in K_{O(n)}(TS^n) .$$

PROOF. S^n may be identified with the union $B_0^n \cup B_\infty^n$ of two copies of the unit ball $B^n \subset \mathbf{R}^n$, and this identification is compatible with the action of $O(n)$. Then we have an $O(n)$-isomorphism

$$TS^n \cong (B_0^n \times \mathbf{R}^n) \cup (B_\infty^n \times \mathbf{R}^n) ,$$

where we identify points over the equator $S^{n-1} = \partial B_0^n = \partial B_\infty^n$ by

---

[15] We will omit a factor i, which is more natural from the point of view of differential operators but for our present purposes it is irrelevant.

$$(x, v) \mapsto (x, h_x v) \qquad\qquad x \in S^{n-1}, \, v \in \mathbf{R}^n \,,$$

h_x denoting the reflection in the hyperplane of $\mathbf{R}^n$ orthogonal to x. Passing to exterior algebras, we then have an $O(n)$-isomorphism of complexes

$$\pi^* \Lambda^* T^c \cong \left(B_0^n \times \mathbf{R}^n \times \Lambda^*(\mathbf{C}^n)\right) \cup \left(B_\infty^n \times \mathbf{R}^n \times \Lambda^*(\mathbf{C}^n)\right) \,,$$

where $\pi: TS^n \to S^n$ is the projection and, on the right, we make the identification

(i) $(x, v, w) \mapsto (x, h_x v, h_x(w))$,

$h_x(\ )$ denoting the action on $\Lambda^*(\mathbf{C}^n)$ induced by the reflection h_x. Suppose now that $0 \leq s \leq 1$, and define a new complex A_s of vector bundles over TS^n by changing the bundle homomorphisms as follows. We define

$$B_0^n \times \mathbf{R}^n \times \Lambda^i(\mathbf{C}^n) \longrightarrow B_0^n \times \mathbf{R}^n \times \Lambda^{i+1}(\mathbf{C}^n)$$

by

(ii) $(x, v, w) \mapsto (x, v, (v - isx) \wedge w)$ and

$$B_\infty^n \times \mathbf{R}^n \times \Lambda^i(\mathbf{C}^n) \longrightarrow B_\infty^n \times \mathbf{R}^n \times \Lambda^{i+1}(\mathbf{C}^n)$$

by

(iii) $(x, v, w) \mapsto (x, v, (v + isx) \wedge w)$.

Since $h_x(x) = -x$, (i) implies that (ii) and (iii) agree over S^{n-1}, and thus define a complex of vector bundles over TS^n as asserted. Now observe the following facts about the complex A_s

(a) for all s, it is exact outside the zero section of TS^n;

(b) for $s = 0$, it is the original complex $\pi^* \Lambda^* T^c$;

(c) for $s = 1$, it is exact outside P_0 and P_∞.

By (a) and (b), the element $\rho_{S^n} \in K_{O(n)}(TS^n)$ is equally well defined by A_s, for any value of s. By (c), the complex A_1 defines an element $a = a^0 + a^\infty$ in

$$K_{O(n)}\left(T(S^n - S^{n-1})\right) = K_{O(n)}(T^0) \oplus K_{O(n)}(T^\infty) \,,$$

where $T^0 = T(B_0^n - S^{n-1})$, $T^\infty = T(B_\infty^n - S^{n-1})$. From their definitions[16] we see that

$$a^0 = k_!^0(1) \qquad a^\infty = \theta^* k_!^\infty(1) \,,$$

where $k^0: P^0 \to B_0 - S^{n-1}$, $k^\infty: P^\infty \to B_\infty - S^{n-1}$ are the inclusions. Applying the natural homomorphism

$$K_{O(n)}\left(T(S^n - S^{n-1})\right) \longrightarrow K_{O(n)}(TS^n)$$

a becomes ρ_{S^n}, $k_!^0(1)$, and $k_!^\infty(1)$ become $j_!^0(1)$, $j_!^\infty(1)$ respectively, and so we obtain

$$\rho_{S^n} = j_!^0(1) + \theta^* j_!^\infty(1)$$

as required.

---

[16] There is actually a scalar factor i between the complexes involved, but this does not affect the class in K.

4. Axioms for the index

In the preceding section we constructed an $R(G)$-homomorphism

$$\text{t-ind}: K_G(TX) \longrightarrow R(G) .$$

In this section we shall give axioms which will uniquely characterize this homomorphism. In the analytical part of the paper we shall then introduce the *analytical index* and show that it verifies the axioms.

We suppose now that, for every compact differentiable G-manifold X, we are given an $R(G)$-homomorphism

$$\text{ind}_G^X: K_G(TX) \longrightarrow R(G) .$$

(We shall write ind instead of ind_G^X when there is no possibility of confusion.) We assume this is functorial with respect to diffeomorphisms in X and homomorphisms in G. More precisely if $f: X \to Y$ is a G-diffeomorphism, then the diagram

$$\begin{array}{ccc}
K_G(TX) & \xrightarrow{\ f^*\ } & K_G(TY) \\
& \searrow\ \text{ind}_G^X \quad \text{ind}_G^X\ \swarrow & \\
& R(G) &
\end{array}$$

commutes, and if $\varphi: G' \to G$ is a homomorphism, the diagram

$$\begin{array}{ccc}
K_G(TX) & \xrightarrow{\ \varphi^*\ } & K_{G'}(TX) \\
\text{ind}_G^X \downarrow & & \downarrow \text{ind}_{G'}^X \\
R(G) & \xrightarrow{\ \varphi^*\ } & R(G')
\end{array}$$

commutes. Such a functorial homomorphism, we will refer to briefly as an *index function*.

We introduce the following two axioms for index functions.

(A1) *If X is a point*, ind *is the identity of $R(G)$.*

(A2) ind *commutes with the homomorphism $i_!$ of § 3.*

The meaning of (A1) is clear; when X is a point, $TX = X$, and $K_G(TX)$ is naturally isomorphic to $R(G)$. In (A2) we mean that, for any inclusion $i: X \to Y$ with X, Y compact G-manifolds, the diagram

$$\begin{array}{ccc}
K_G(TX) & \xrightarrow{\ i_!\ } & K_G(TY) \\
& \searrow\ \text{ind}_G^X \quad \text{ind}_G^X\ \swarrow & \\
& R(G) &
\end{array}$$

commutes.

The topological index t-ind satisfies (A1) and (A2). The first is trivial,

and the second follows from the transitivity of $i_!$.

The following is then a rather trivial consequence of § 3:

PROPOSITION (4.1). *Let* ind *be an index function satisfying* (A1) *and* (A2). *Then we have* ind $=$ t-ind.

PROOF. Given X we take an embedding $i: X \to E$ where E is a real G-module. Let E^+ be the one-point compactification of E. Since we can assume G acts orthogonally on E, it is clear that G acts differentiably on the sphere E^+; i.e., E^+ is a G-manifold. Now let $i^+: X \to E^+$ be the embedding given by i. Similarly if P is the origin of E, we have $j: P \to E, j^+: P \to E^+$. Consider the following diagram

$$
\begin{array}{ccccc}
& & K_G(TE) & & \\
& {}^{i_!}\nearrow & \downarrow & \nwarrow{}^{j_!} & \\
& \nearrow & \downarrow & \nwarrow & \\
K_G(TX) & \xrightarrow{\ i_!^+\ } & K_G(TE^+) & \xleftarrow{\ j_!^+\ } & K_G(TP) = R(G)\ . \\
& {}_{\mathrm{ind}_G^X}\searrow & \downarrow{\mathrm{ind}_G^{E^+}} & \swarrow{}_{\mathrm{ind}_G^X} & \\
& & R(G) & &
\end{array}
$$

The top two triangles commute because of the way $i_!$ and $j_!$ are defined. The bottom two triangles commute by (A2). By (A1), ind_G^P is the identity. Now $j_!$ is an isomorphism, and t-ind: $K_G(TX) \to R(G)$ is defined by t-ind $= j_!^{-1} i_!$. The diagram then shows that it coincides with ind_G^X.

Axiom (A2) is not easy to verify, and our next aim is to show that it follows from a number of other more elementary axioms. First we describe an *excision axiom*.

(B1) *Let* U *be a (non-compact) G-manifold,*

$$ j: U \longrightarrow X, \qquad j': U \longrightarrow X' $$

two open G-embeddings into compact G-manifolds X, X'. *Then the following diagram commutes*

$$
\begin{array}{ccccc}
& & K_G(TX) & & \\
& {}^{j*}\nearrow & & \searrow{}^{\mathrm{ind}_G^X} & \\
K_G(TU) & & & & R(G)\ . \\
& {}_{j'*}\searrow & & \nearrow{}_{\mathrm{ind}_G^{X'}} & \\
& & K_G(TX') & &
\end{array}
$$

When (B1) is satisfied, we can define

$$\text{ind: } K_G(TU) \longrightarrow R(G)$$

as the composition $\text{ind} \circ j^*$, and it is independent of the choice of $j: U \to X$ provided that one such X exists[17]. In particular if E is a real G-module then

$$\text{ind: } K_G(TE) \longrightarrow R(G)$$

is defined. Our next axiom is, like (A1), a normalization axiom[18].

   (B2) *Let* $j_!: R(O(n)) \to K_{O(n)}(T\mathbf{R}^n)$ *be induced by the inclusion* $j: P \to \mathbf{R}^n$, *where P is the origin. Then* $\text{ind } j_!(1) = 1$.
Actually we shall also want to consider a weaker axiom, involving only the abelian groups $O(1)$ and $SO(2)$.

   (B2′) $\text{ind } j_!(1) = 1$ *where* $j_!$ *is either*

$$R(O(1)) \longrightarrow K_{O(1)}(T\mathbf{R}^1)$$

or

$$R(SO(2)) \longrightarrow K_{SO(2)}(T\mathbf{R}^2) \ .$$

Finally we want to introduce a *multiplicative axiom*. This is the most significant one, and it requires some care in setting up. The simplest kind of multiplicative axiom would be concerned with product manifolds $X \times Y$, but this is not enough. We need to consider not just products but fibre bundles.

   Suppose then that $P \to X$ is a compact differentiable principal fibre bundle with group H (a compact Lie group). Thus H acts freely on P, on the right, and $X = P/H$. If F is a compact differentiable H-manifold (H acting on the left), we can form the associated fibre bundle Y over X, given by

$$Y = P \times_H F;$$

i.e., Y is the quotient of $P \times F$ by the action of H: $h(p, f) = (ph^{-1}, hf)$. Since H acts on F, it also acts on TF, and $P \times_H TF$ is then a vector bundle over Y. This is usually called the tangent bundle along the fibres, and denoted by $T(Y/X)$; it is a sub-bundle of TY and (using a metric), we have a decomposition

$$TY = T(Y/X) \oplus \pi^* TX$$

where $\pi: Y \to X$ is the projection. Thus we have a multiplication

$$K(TX) \otimes K(T(Y/X)) \longrightarrow K(TY) \ .$$

On the other hand, we have homomorphisms

$$K_H(TF) \longrightarrow K_H(P \times TF) \cong K(P \times_H TF)$$
$$= K(T(Y/X)) \ .$$

---

   [17] In fact this assumption can be avoided, but we do not need this.
   [18] In (B2) we either assume (B1) so that ind is unambiguously defined on $K_{O(n)}(T\mathbf{R}^n)$, or else we define it by the standard embedding $\mathbf{R} \subset (\mathbf{R}^n)^+ = S^n$.

Combining these we thus obtain a mutiplication

$$K(TX) \otimes K_H(TF) \longrightarrow K(TY) \,.$$

Suppose now that a second compact Lie group G acts on all the preceding situation. Thus we assume that G acts (on the left) on P and F commuting with the action of H and so inducing an action on $X = P/H$ and $Y = P \times_H F$. Then we get a multiplication

$$K_G(TX) \otimes K_G\big(T(Y/X)\big) \longrightarrow K_G(TY)$$

and a homomorphism

$$K_{G \times H}(TF) \longrightarrow K_{G \times H}(P \times TF) \cong K_G(P \times_H TF)$$
$$= K_G\big(T(Y/X)\big) \,,$$

which combine to give

(4.2) $$K_G(TX) \otimes K_{G \times H}(TF) \longrightarrow K_G(TY) \,.$$

On the other hand, if V is any complex $G \times H$-module, $P \times_H V$ is a G-vector bundle over X. This extends to an $R(G)$-homomorphism

(4.3) $$\mu_P \colon R(G \times H) \longrightarrow K_G(X) \,.$$

Finally let us recall that $K_G(TX)$ is a $K_G(X)$-module. We are now ready to formulate our multiplicative axiom.

(B3) *For all G, H, P, F, as above we have*

$$\operatorname{ind}_G^Y(ab) = \operatorname{ind}_G^X\big(a \cdot \mu_P(\operatorname{ind}_{G \times H}^F(b))\big)$$

where $a \in K_G(TX)$, $b \in K_{G \times H}(TF)$ the product ab is taken as in (4.2) and μ_P is the homomorphism of (4.3).

Suppose, in particular, that $\operatorname{ind}_{G \times H}^F(b)$ lies in the subring $R(G)$ of $R(G \times H)$. Then, since μ_P and ind_G^X are both $R(G)$-homomorphisms, (B3) simplifies to

$$\operatorname{ind} ab = \operatorname{ind} a \cdot \operatorname{ind} b \in R(G) \,.$$

It is only this special case of (B3) which we shall need, and so we state this as

(B3') *If $\operatorname{ind}_{G \times H}^F(b) \in R(G) \subset R(G \times H)$, then, in the notation of (B3),*

$$\operatorname{ind}_G^Y(ab) = \operatorname{ind}_G^X a \cdot \operatorname{ind}_{G \times H}^F b \,.$$

Remark. Taking $F = \text{point}$, $H = 1$, we see that (B3') implies (A1) unless $\operatorname{ind} \equiv 0$; but, of course, this is a trivial formal observation.

Axioms (B3) and (B3') are for fibre bundles with group H. In particular therefore they apply to products, taking $H = 1$. Thus (B3') implies

(B3'') *If X, F are G-manifolds and $a \in K_G(TX)$, $b \in K_G(TF)$, then*

$$\operatorname{ind}_G^{X \times F}(ab) = \operatorname{ind}_G^X a \cdot \operatorname{ind}_G^F b \,.$$

It is of course possible to reformulate (B3'') in terms of "external products"

for different groups G. Thus, if X_i is a G_i-manifold and $a_i \in K_{G_i}(TX_i)$, we can form the product $a_1 a_2 \in K_G(TX)$ where $X = X_1 \times X_2, G = G_1 \times G_2$. Then applying (B3″) with $X_1 = X, X_2 = F$ we see that

$$\text{ind}_G^X a_1 a_2 = \text{ind}_{G_1}^{X_1} a_1 \cdot \text{ind}_{G_2}^{X_2} a_2 \, ,$$

where the latter product is given by:

$$R(G_1) \otimes R(G_2) \longrightarrow R(G) \, .$$

Our aim now will be to show that Axioms (B1), (B2′) and (B3′) imply (A2). First we prove

PROPOSITION (4.4). *Axioms* (B1), (B2′) *and* (B3″) *imply* (B2).

PROOF. Axiom (B1) enables us to extend (B3″) to open sets in compact manifolds. Thus if $a_i \in K_{G_i}(TU_i)$ with U_i open in X_i $(i = 1, \cdots, k)$, then

$$\text{ind} \prod a_i = \prod \text{ind} \, a_i \, .$$

(The point of (B1) is that ind $\prod a_i$ may be computed by any manifold compactifying $\prod U_i$, not necessarily $\prod X_i$.) In particular, this multiplicative property holds for $U_i = R^{n_i}$ and $G_i \subset O(n_i)$

$$a_i = j_!^i(1) \qquad j^i \colon P \longrightarrow R^{n_i} \, .$$

If all $n_i = 1$ or 2 and $G_i = O(1)$ or $SO(2)$, then (B2′) asserts that ind $a_i = 1$. Hence by (B3′) ind $\prod a_i = \prod$ ind $a_i = 1 \in R(\prod G_i)$. Now, since $j_!$ is multiplicative, it follows that $\prod a_i$ is the restriction of $a = j_!(1) \in K_{O(n)}(TR^n)$. Thus ind $a \in R(O(n))$ gives 1 when restricted to any subgroup $\prod G_i$ of $O(n)$ (with $G_i = O(1)$ or $SO(2)$). But these subgroups contain all cyclic subgroups of $O(n)$, and so are sufficient to determine a character of $O(n)$. Hence ind $a = 1 \in R(O(n))$, establishing (B2).

Remark. The verification (for the analytical index) of (B2) is not in fact much more difficult than (B2′), but it seems relevant to observe (as we have done in (4.4)) that (B2′) is sufficient.

We can now prove

PROPOSITION (4.5). *Axioms* (B1), (B2) *and* (B3′) *imply* (A2).

If we have an index function satisfying (B1), then as observed above (B3′) or (B3″) implies a corresponding result for open sets in compact manifolds. Thus (B3′) continues to hold when F is an open set (stable under $G \times H$) of some $G \times H$- compact manifold $\tilde{F}$, the space $Y = P \times_H F$ is then an open set of the compact manifold $P \times_H \tilde{F}$. In particular, we can take $F = R^n$, $\tilde{F} = (R^n)^+ = S^n$, $H = O(n)$, $b = j_!(1)$ where $j \colon A \to R^n$ is the inclusion of the origin A. Then P is a principal $O(n)$-bundle over the compact manifold X, and G acts on P, commuting with $O(n)$. We make G act trivially on R^n. The space $Y = P \times_{O(n)} R^n$ is then the

associated real vector bundle over X; it is a G-bundle. The homomorphism

$$K_G(TX) \longrightarrow K_G(TY)$$

given by $a \mapsto ab$ is just the homomorphism $i_!$ induced by the zero-section inclusion $i: X \to Y$. If ind satisfies (B2), then

$$\text{ind}_{O(n)}(b) = 1 \in R\big(O(n)\big) ,$$

and if we regard G as acting trivially on $\mathbf{R}^n$, the same formula holds with $G \times O(n)$ instead of $O(n)$ (using the functoriality of ind for the projection homomorphism $G \times O(n) \to O(n)$). Now applying (B3'), we obtain

$$\text{ind } i_!(a) = \text{ind } ab = \text{ind } a \cdot \text{ind } b$$
$$= \text{ind } a \in R(G) .$$

This establishes (A2) in the special case when Y is a real vector bundle over X. But for a general embedding

$$k: X \longrightarrow Z ,$$

the homomorphism

$$k_!: K_G(TX) \longrightarrow K_G(TZ)$$

is defined as the composition of

$$j_!: K_G(TX) \longrightarrow K_G(TN)$$

and the natural homomorphism

$$K_G(TN) \longrightarrow K_G(TZ) ,$$

where N is the normal bundle of X in Z. In the following diagram

$$K_G(TX) \longrightarrow K_G(TN) \longrightarrow K_G(TZ)$$

$$\text{ind}^X \searrow \quad \downarrow \text{ind}^N \quad \swarrow \text{ind}^Z$$

$$R(G)$$

the first triangle commutes by what we have just proved, and the second commutes by (B1). Thus

$$\text{ind}^Z k_!(a) = \text{ind}^X a$$

and (A2) is established in general.

Putting together Propositions (4.1), (4.4), and (4.5), we then obtain the following uniqueness theorem.

THEOREM (4.6). *Let* ind *be an index function satisfying* (A1), (B1), (B2') *and* (B3'). *Then* ind = t-ind.

Remark. It will be noticed that (B3) has not been used—only (B3'). It will

actually turn out in § 9 that the stronger axiom (B3) holds for the analytical index.

Axiom (B2′) can be replaced by another which is, in practice, more convenient to check.[19] Let us recall (cf. § 3) that, for any compact G-manifold X, we have the "de Rham symbol class" $\rho_X \in K_G(TX)$. We then introduce the following variant of (B2′):

(B2″). (i) ind $\rho_{S^2} = 2 \in R(SO(2))$;

(ii) ind $\rho_{S^1} = 1 - \xi \in R(O(1))$, where $\xi \colon O(1) \to U(1)$ is the standard representation.

(iii) ind $j_!(1) = 1 \in \mathbf{Z}$, where $j \colon P \to S^1$ is the inclusion of the origin.

Remark. The elements occurring here all belong to $K_{O(n)}$ ($n = 1$ or 2), but in (i) and (iii) we choose to restrict to the smaller groups $SO(n)$.

We shall now show that (B2′), can be replaced by (B2″).

LEMMA (4.7). *Let* ind *be an index function satisfying* (B2″). *Then it also satisfies* (B2′).

PROOF. By Lemma (3.2), we have

$$\rho_{S^n} = j_!^0(1) + \theta^* j_!^\infty(1) \in K_{O(n)}(TS^n) \, ,$$

where $j^0 \colon P^0 \to S^n$, $j^\infty \to S^n$ are the inclusions of the origin and the point at infinity, and θ is multiplication by -1 on tangent vectors. Let $f \colon S^n \to S^n$ be reflection in the equator so that f interchanges P^0 and P^∞ and commutes with θ. Then

$$f^*\big(\theta^* j_!^\infty(1)\big) = \theta^* j_!^0(1) \, .$$

By the functorial properties of an index function, this implies that

$$\text{ind } \theta^* j_!^\infty(1) = \text{ind } \theta^* j_!^0(1) \in R\big(O(n)\big) \, .$$

Thus

$$\text{ind } \rho_{S^n} = \text{ind } (1 + \theta^*) j_!^0(1) \, .$$

Now, by definition, $j_!^0$ (which we now write simply as $j_!$) factors through the group $K_{O(n)}(T\mathbf{R}^n)$ and, on $T\mathbf{R}^n = \mathbf{C}^n$, θ coincides with complex conjugation. Applying Lemma (2.8) therefore we deduce

$$\text{ind } \rho_{S^2} = \text{ind } 2j_!(1) = 2 \text{ ind } j_!(1) \in R\big(SO(2)\big)$$

$$\text{ind } \rho_{S^1} = \text{ind } (1 - \xi)j_!(1) = (1 - \xi) \text{ ind } j_!(1) \in R\big(O(1)\big) \, .$$

Parts (i) and (ii) of (B2″) then give

---

[19] Verifying (B2′) for the analytical index involves computing the index of a certain operator in euclidean space (see a forthcoming paper by Atiyah and Hörmander). In some ways this is more direct and natural than verifying (B2″), on the sphere. However, it involves more explicit analysis which we avoid by the topological deformations implied in (4.7).

$$2 \operatorname{ind} j_!(1) = 2$$
$$(1 - \xi) \operatorname{ind} j_!(1) = 1 - \xi .$$

Now the annihilator of $1 - \xi$ in

$$R(O(1)) = \mathbf{Z}[\xi]/(1 - \xi^2)$$

consists of integral multiples of $(1 + \xi)$. Hence we deduce

$$\operatorname{ind} j_!(1) = 1 \in RSO(2)$$
$$\operatorname{ind} j_!(1) = 1 + a(1 + \xi) \in R(O(1))$$

for some integer a. Restricting the second equation to the identity of $O(1)$, and applying (iii) of (B2''), we deduce

$$1 = 1 + 2a \quad \text{i.e.,} \ a = 0 .$$

Thus $\operatorname{ind} j_!(1) = 1$ both for $SO(2)$ and for $O(1)$. This is precisely Axiom (B2'), bearing in mind that ind on $K(T\mathbf{R}^n)$ is defined *via* the compactification $\mathbf{R}^n \to S^n$ (cf. footnote[18]).

To conclude this section, we might point out that the topological index of §3 clearly satisfies axioms (A1), (A2), and also (B2). It is not quite so obvious that it satisfies (B1) and (B3) but, with a little work, these can be established directly. On the other hand, once we have shown that the analytical index (to be introduced in §6) satisfies (B1), (B2), and (B3), the uniqueness theorem (4.6) will imply that t-ind = a-ind, and so t-ind will also satisfy these axioms.

5. Pseudo-differential operators

In this section, we shall review the basic analytical facts concerning pseudo-differential operators. Their application to the index of elliptic operators will be treated in §6. For proofs of the results stated here, we refer to Kohn-Nirenberg [11], Seeley [15], Hörmander [9][10], and Palais [14]. We shall however attempt to present the material so that only standard results are given without proof.

The term "pseudo-differential" is applied, in different places, to slightly different classes of operators. For our purposes any one of these classes would be equally good. In fact, we shall eventually form a closure of this class and, by that stage, any differences would disappear. Perhaps the largest and most natural class is that given in Hörmander [10], and we begin therefore by recalling his definition.[20]

---

[20] Actually [10] is concerned primarily with classes $L^m_{\rho,\delta}$, and we only use the case $\rho = 1, \delta = 0$. The other values of ρ, δ are used in [10] to treat certain classes of hypoelliptic operators. For these the index problem can be solved by reducing it to the standard elliptic case discussed here. This is proved in a forthcoming paper of Atiyah and Hörmander.

Consider first an open set U of $\mathbf{R}^n$, and let $x = (x_1, \cdots, x_n)$ be the standard coordinates. For any integer m, we denote by $S^m(U)$ the set of all smooth functions $p(x, \xi)$ on $U \times \mathbf{R}^n$ such that, for every compact $K \subset U$, and all multi-indices α, β, we have

$$| D_x^\beta D_\xi^\alpha p(x, \xi) | \leq C_{\alpha,\beta,K}(1 + |\xi|)^{m-|\alpha|} , \qquad x \in K, \xi \in \mathbf{R}^n .$$

Here D_ξ^α stands for the partial derivative

$$(-i\partial/\partial\xi_1)^{\alpha_1}(-i\partial/\partial\xi_2)^{\alpha_2} \cdots (-i\partial/\partial\xi_n)^{\alpha_n} ,$$

$|\alpha| = \sum \alpha_i$ and $C_{\alpha,\beta,K}$ is a constant depending on α, β, K, and p. For any such p, we define a linear operator

$$P: \mathcal{D}(U) \longrightarrow \mathcal{E}(U)$$

by the formula

$$Pu = (2\pi)^{-n} \int p(x, \xi)\hat{u}(\xi)e^{i\langle x,\xi\rangle}d\xi .$$

Here $\mathcal{E}$ denotes the smooth functions on U, $\mathcal{D}$ those with compact support, and $\hat{u}$ is the Fourier transform of u. If p is a polynomial in ξ of degree m with smooth coefficients, then $p \in S^m(U)$, and P is the differential operator associated to it in the usual way. For this reason, when we want to show the dependence of P on p, in the general case, we write

$$P = p(x, D) ,$$

where D stands formally for the vector with components $-i(\partial/\partial x_j)$.

A *pseudo-differential operator* is one which is locally of the above type. Precisely, we denote by $L^m(U)$ the set of all mappings $P: \mathcal{D}(U) \to \mathcal{E}(U)$ such that, for all $f \in \mathcal{D}(U)$, there exists some $p_f \in S^m(U)$ with $P(fu) = p_f(x, D)u$ for all $u \in \mathcal{D}(U)$. An equivalent definition [10; § 2] is that P is continuous, and that the commutator

$$p_f(x, \xi) = e^{-i\langle x,\xi\rangle}P(fe^{i\langle x,\xi\rangle})$$

belongs to $S^m(U)$ for all $f \in \mathcal{D}(U)$.

In this paper we shall introduce a sub-class $\mathscr{P}^m \subset L^m$ consisting of operators P for which all the functions $p_f(x, \xi)$ above lie in a certain subspace $S_0^m(U \times \mathbf{R}^n)$ of $S^m(U \times \mathbf{R}^n)$. A function $p \in S_0^m$ if, for $\xi \neq 0$, the limit

$$\sigma(p)(x, \xi) = \lim_{\lambda \to \infty} \frac{p(x, \lambda\xi)}{\lambda^m}$$

exists. Then $\sigma(p)$ is a C^∞ function on $U \times (\mathbf{R}^n - 0)$, and it is homogeneous of degree m in ξ. For an operator $P \in \mathscr{P}^m$, we define the symbol $\sigma(P)$ by

$$\sigma(P)(x, \xi) = \sigma(p_f)(x, \xi) ,$$

where f is any function equal to 1 near x. It does not depend on the choice
of f. The symbol is a C^∞ function of (x, ξ) outside $\xi = 0$, and it is homo-
geneous of degree m. It follows from results of [10] that L^m and $\mathscr{P}^m$ are
invariant under diffeomorphisms of U, and hence corresponding classes of
operators can be defined globally for smooth vector bundles over smooth
(paracompact) manifolds (for the details of this, see [10] or [4]). If E and F
are smooth vector bundles over a smooth manifold X, we shall denote by
$\mathscr{P}^m(X; E, F)$ the space of pseudo-differential operators[21]

$$P: \mathscr{D}(X; E) \longrightarrow \mathscr{E}(X; F)$$

of type $\mathscr{P}^m$. Locally, with respect to a coordinate patch on X and bases in E,
F, such an operator P is given by a matrix $p_{ij}(x, D)$ of operators in euclidean
space as above.

The symbol $\sigma(P)$ of $\mathscr{P}^m(X; E, F)$ is globally well-defined as a smooth
homomorphism

$$\sigma(P): \pi^*E \longrightarrow \pi^*F$$

of vector bundles on the cotangent space TX (with the zero-section removed).
Here π denotes the projection $\pi: TX \to X$. On each fibre of TX, $\sigma(P)$ is
(positively) homogeneous of degree m. We denote by $\mathrm{Symb}^m(X; E, F)$ the
space of all homomorphisms $\pi^*E \to \pi^*F$ which are defined and smooth outside
the zero-section and, on each fibre of TX, are homogeneous of degree m. If
we choose the unit sphere bundle $S(X)$ of TX defined by some smooth
riemannian metric, we can clearly identify $\mathrm{Symb}^m(X; E, F)$ with the space of
smooth homomorphisms $\pi_S^*E \to \pi_S^*F$ where $\pi_S: S(X) \to X$ is the projection.

We recall briefly a few important properties of pseudo-differential opera-
tors. First of all they compose well; that is,

$$P \in \mathscr{P}^m, Q \in \mathscr{P}^q, f \in \mathscr{D}(X) \Longrightarrow PfQ \in \mathscr{P}^{m+q},$$

and

$$\sigma(PfQ) = \sigma(P)f\sigma(Q).$$

If X is a compact manifold then the function f may be omitted. Secondly
they transpose well; that is,

$$P \in \mathscr{P}^m(X; E, F) \Longrightarrow P^t \in \mathscr{P}^m(X; F', E'),$$

where $E' = \mathrm{Hom}\,(E, \Omega)$ (Ω denoting the volume bundle of X as in [4; § 5]),
and P^t is the transpose of P (so that the distributional extension of P^t
coincides with the dual of P). Moreover $\sigma(P^t) = \sigma(P)'$ where $\sigma \mapsto \sigma'$ is the

---

[21] We write $\mathscr{E}(X; E)$ for the spaces of smooth sections of E over X and $\mathscr{D}(X; E)$ for
those with compact support. When X is clearly understood, we shall omit it.

map of symbols induced by the isomorphism $\mathrm{Hom}\,(E, F) \to \mathrm{Hom}\,(F', E')$ of bundles. If we choose a smooth hermitian metric for E, and a smooth positive measure on X, we get an anti-linear isomorphism $E \to E'$. If we also have a metric for F, then the transpose $P^t \in \mathscr{P}^m(X; F', E')$ can be replaced by the "formal adjoint" $P^* \in \mathscr{P}^m(X; F, E)$. For the symbols, we then have $\sigma(P^*) = \sigma(P)^*$.

We come now to the Sobolev spaces. If E is a smooth vector bundle over a smooth manifold X, and if s is any non-negative integer[22], we denote by $H_s^{\mathrm{loc}}(X; E)$ the space of those distributional sections u of E for which $Du \in L_2^{\mathrm{loc}}$ for all differential operators

$$D: \mathscr{D}(X; E) \longrightarrow \mathscr{D}(X; 1)$$

with smooth coefficients, and of order $\leq s$. If, for local coordinates (x_j), and a local base (e_i) of E, we write $u = \sum u_i(x)e_i$, then (in the coordinate patch)

$$u \in H_s^{\mathrm{loc}} \iff \left(\frac{\partial}{\partial x}\right)^\alpha u_i \in L_2^{\mathrm{loc}} \qquad \text{for all } \alpha \text{ with } |\alpha| \leq s .$$

The space $H_s^{\mathrm{loc}}(X; E)$ has a natural topology given by a countable[23] set of semi-norms. It is a Fréchet space. We also consider

$$H_s^{\mathrm{comp}}(X; E) \subset H_s^{\mathrm{loc}}(X; E)$$

consisting of sections with compact support. It has its own natural topology as a direct limit (over compact $K \subset X$) of Hilbert spaces (this is not of course the induced topology from H_s^{loc}). We define $H_{-s}^{\mathrm{loc}}(X; E)$ as the dual of $H_s^{\mathrm{comp}}(X; E')$, and $H_{-s}^{\mathrm{comp}}(X; E)$ as the dual of $H_s^{\mathrm{loc}}(X; E')$ where $E' = \mathrm{Hom}\,(E, \Omega)$ as above.

If X is compact, then $H_s^{\mathrm{comp}} = H_s^{\mathrm{loc}}$, and we write it simply as H_s. An explicit Hilbert space norm for H_s can be defined in terms of a hermitian metric and connection for E, and a smooth positive measure on X as follows. For $s = 0$ we have the usual L_2-norm

$$\| u \| = \left(\int_X \langle u, u \rangle\right)^{1/2} .$$

For $s > 0$, we first introduce the positive definite operator $\Delta = 1 + D^*D$, where $D: \mathscr{D}(E) \to \mathscr{D}(E \otimes T)$ is the covariant derivative given by the connection, and then put

$$\| u \|_s = \left(\int_X \langle \Delta^s u, u \rangle\right)^{1/2} .$$

---

[22] These spaces are also defined for real s [8] but the integer case is enough for our purposes. The invariance properties for integral s are then rather trivial.

[23] All our manifolds are assumed paracompact so that we can take a countable set of coordinate patches.

For $s < 0$, we take the dual norm. Note that, if a compact group G acts differentiably on X and E, then an invariant measure on X, and an invariant connection will lead to Hilbert space norms on $H_s(X; E)$ invariant by G.

Pseudo-differential operators behave well with respect to the H_s-spaces, namely we have

(5.1) *A pseudo-differential operator*

$$P: \mathcal{D}(X; E) \longrightarrow \mathcal{E}(X; F)$$

in $\mathcal{P}^m(X; E, F)$ extends, for integer s, to a continuous linear operator

$$P_s: H_s^{\mathrm{comp}}(X; E) \longrightarrow H_{s-m}^{\mathrm{loc}}(X; F) \ .$$

Let $Op_s^m = Op_s^m(X; E, F)$ denote the space of all continuous linear maps $H_s^{\mathrm{comp}}(X; E) \to H_{s-m}^{\mathrm{loc}}(X; F)$ with the topology of bounded convergence. Then $P \mapsto P_s$ defines a map $\mathcal{P}^m \to Op_s^m$ whose image we denote by $\mathcal{P}_s^m$. If X is compact, Op_s^m is a Banach space (with norm denoted by $\| \ \|_s^m$) and the closure $\bar{\mathcal{P}}_s^m$ has a rather simple structure as described in the following result.

(5.2) *Let X be compact, then the symbol*

$$\sigma: \mathcal{P}_s^m(X; E, F) \longrightarrow \mathrm{Symb}^m\, (X; E, F)$$

is continuous for the sup *norm topology on the unit sphere bundle of TX; it extends by continuity to a map*

$$\sigma_s: \bar{\mathcal{P}}_s^m(X; E, F) \longrightarrow \overline{\mathrm{Symb}^m}\, (X, E, F)$$

which is surjective, and has the compact operators $H_s \to H_{s-m}$ as kernel.

Remark. We recall that $\mathrm{Symb}^m\, (X; E, F)$ is isomorphic (by restriction to the unit sphere bundle $S(X)$ in TX) to the space of smooth homomorphisms $\pi_S^* E \to \pi_S^* F$ where $\pi_S: S(X) \to X$ is the projection. The closure $\overline{\mathrm{Symb}^m}(X; E, F)$ can thus be identified with the space of continuous homomorphisms $\pi_S^* E \to \pi_S^* F$.

To obtain C^∞ results from the H_s-spaces, it is convenient to consider all s simultaneously. For this purpose, we introduce (for any X, not necessarily compact) the space $Op^m(X; E, F)$ of all linear operators $\mathcal{D}(X; E) \to \mathcal{D}'(X; F)$ which extend by continuity to operators in $Op_s^m(X; E, F)$ for all s. Now the Sobolev lemma implies that

$$\mathcal{D}(X; E) = \bigcap_s H_s^{\mathrm{comp}}(X; E) \ , \qquad \mathcal{E}(X; F) = \bigcap_s H_s^{\mathrm{loc}}(X; F)$$

(with the inverse limit topology), and so an operator in Op^m actually maps $\mathcal{D}(X; E) \to \mathcal{E}(X; F)$ (continuously). For each s we have an embedding of Op^m in Op_s^m, and hence of Op^m in $\prod_s Op_s^m$: its image is closed, and we give Op^m the induced topology making it a Fréchet space.

The space Op^m is a *local* space of operators in the sense that the Schwartz kernels on $X \times X$ are a local space of distributions. This means that an operator P is in Op^m if and only if, for every pair of C^∞ functions φ, ψ on X with compact support, $\varphi P \psi \in Op^m$. Reasoning as in [3; Appendix] it follows that, if $\{U_i\}$ is a coordinate covering of X so that $\{U_i \times U_i\}$ covers $X \times X$,

$$P \in Op^m(X) \Longleftrightarrow P \,|\, U_i \in Op^m(U_i) \qquad\qquad \text{for all } i \,.$$

Moreover the semi-norms in the U_i determine the semi-norms in X.

By (5.1) $\mathscr{P}^m \subset Op^m$, and we now form the closure $\bar{\mathscr{P}}^m$. Since $\mathscr{P}^m$ is by definition a local space of operators, and since Op^m is local, it follows that $\bar{\mathscr{P}}^m$ is local. Since $\mathscr{P}^m \to \mathscr{P}^m_s$ is, by definition, continuous, (5.2) implies that the symbol extends by continuity to give a diagram

$$\begin{array}{ccc} \bar{\mathscr{P}}^m & \longrightarrow & \bar{\mathscr{P}}^m_s \\ & \sigma \searrow \quad \swarrow \sigma_s & \\ & \mathrm{Symb}^m \,. & \end{array}$$

Note that, in this diagram, while σ_s is surjective, σ is not. In fact, the image of σ is rather difficult to describe, and we shall not go into the question. An operator $P \in \bar{\mathscr{P}}^m_s$ will be called elliptic if $\sigma_s(P)$ is invertible. An operator $P \in \bar{\mathscr{P}}^m$ will be called elliptic if $\sigma(P)$ is invertible (in the space Symb^m of continuous symbols); this means that

$$P \text{ elliptic} \Longleftrightarrow P_s \text{ elliptic for some } s \Longleftrightarrow P_s \text{ elliptic for all } s.$$

The most important reason for introducing the closure $\bar{\mathscr{P}}^m$ lies in the behavior of pseudo-differential operators for products of manifolds as we shall now explain. Thus let E, F be smooth vector bundles over X, and G a smooth vector bundle over Y. If P is a continuous linear map $\mathscr{D}(X; E) \to \mathscr{E}(X; F)$, we denote by $\tilde{P}$ the "lifted operator" from $\mathscr{D}(X \times Y; E \boxtimes G) \to \mathscr{E}(X \times Y; F \boxtimes G)$, that is the unique continuous linear map such that

$$\tilde{P}(u \otimes v) = Pu \otimes v \qquad\qquad u \in \mathscr{D}(X; E), v \in \mathscr{D}(Y; G) \,.$$

If $m \geq 0$, this lifting operation behaves well with respect to the Op^m spaces, that is $P \mapsto \tilde{P}$ defines a continuous map

(5.3) $$Op^m(X; E, F) \longrightarrow Op^m(X \times Y; E \boxtimes G, F \boxtimes G) \,.$$

To verify this, it is enough to check the case when X and Y are domains in euclidean space, and all bundles are trivial of dimension one. Suppose then that $P \in Op^m(X)$. Then for $f \in \mathscr{D}(X \times Y)$, compact sets $K \subset X$ $L \subset Y$ and $|\alpha| + |\beta| = s - m \geq 0$, we have

$$\int_{K \times L} | D_x^\alpha D_y^\beta \tilde{P} f |^2 dx \, dy = \int_L dy \int_K | D_x^\alpha \tilde{P} D_y^\beta f |^2 dx$$

$$\leqq C \int_L dy \int_K \sum_{|\gamma| \leqq |\alpha|+m} | D_x^\gamma D_y^\beta f |^2 dx$$

$$\leqq C \int_{K \times L} \sum_{|\beta|+|\gamma| \leqq s} | D_x^\gamma D_y^\beta f |^2 dx \, dy \ .$$

This shows that (for $s \geqq m$) $\tilde{P}$ is continuous $H_s^{\text{comp}}(X \times Y) \to H_{s-m}^{\text{loc}}(X \times Y)$, and that $P \mapsto \tilde{P}$ is continuous from $Op^m(X)$ to $Op_s^m(X \times Y)$. Passing to adjoints gives the corresponding result for $s \leqq 0$. Hence for $m = 0$ or 1, we are through and obtain the continuity of (5.3). For $m > 1$, another small argument is needed, but since $m = 1$ suffices for applications, we shall omit the proof.

Unfortunately it is *not* true in general that $P \in \mathscr{P}^m \Rightarrow \tilde{P} \in \mathscr{P}^m$. However for the closure $\bar{\mathscr{P}}^m$, we do have

(5.4) *For $m > 0$, if $P \in \bar{\mathscr{P}}^m(X; E, F)$, then the lifted operator*

$$\tilde{P} \in \bar{\mathscr{P}}^m(X \times Y; E \boxtimes G, F \boxtimes G) \ .$$

Moreover $\sigma(\tilde{P}) = \tilde{\sigma}(P)$ where $\tilde{\sigma}$ is the lift of σ defined by

$$\tilde{\sigma}_{(\xi,\eta)}(e \otimes g) = \sigma_\xi(e) \otimes g, \qquad \xi \in TX, \eta \in TY, e \in E, g \in G.$$

This is essentially property (S6) of [14; Ch. XI], but we shall recall the proof which is quite elementary.

Since $\bar{\mathscr{P}}^m$ is a local class of operators, it will be sufficient to deal with the case when $X = U \subset \mathbf{R}^n$ and $Y = V \subset \mathbf{R}^q$ are domains in euclidean space, and all bundles are trivial of dimension one. Moreover by the continuity of (5.3) it will be sufficient to prove that

$$P \in \mathscr{P}^m(U) \Longrightarrow \tilde{P} \in \bar{\mathscr{P}}^m(U \times V) \ .$$

Again, since $\bar{\mathscr{P}}^m$ is a local class, it will be sufficient to show that, for all $\varphi, \psi \in \mathscr{D}(U)$ and $\varphi_1, \psi_1 \in \mathscr{D}(V)$, $Q = \varphi\varphi_1\tilde{P}\psi\psi_1 \in \bar{\mathscr{P}}^m(U \times V)$. To do this we shall construct a family $R^t \in \mathscr{P}^0(U \times V)$, defined for $t > 0$, and such that

   (i) $Q \circ R^t \in \mathscr{P}^m(U \times V)$

   (ii) $Q \circ R^t \to Q$ in $Op^m(U \times V)$ as $t \to 0$.

First of all we choose a family of functions $\sigma^t(\xi, \eta)$, defined for $t > 0$, taking values in the unit interval and such that

   (a) σ^t is homogeneous of degree zero and C^∞ outside the origin,

   (b) $\sigma^t = 1$ for $|\xi| < t|\eta|$

         $= 0$ for $|\xi| > 2t|\eta|$.

Also let $\varphi(\lambda)$ be a C^∞ function of $\lambda \in \mathbf{R}$ such that

$$\varphi(\lambda) = 0 \qquad\qquad\qquad \text{for } |\lambda| \leqq 1$$

$$= 1 \qquad\qquad\qquad \text{for } |\lambda| \geqq 2 \ .$$

Then put

$$\rho^t(\xi, \eta) = 1 - \varphi\big(t(|\xi|^2 + |\eta|^2)^{1/2}\big)\sigma^t(\xi, \eta) .$$

The appearance of this function is indicated below: $\rho^t = 1$ in the horizontally shaded region, and $\rho^t = 0$ in the vertically shaded region
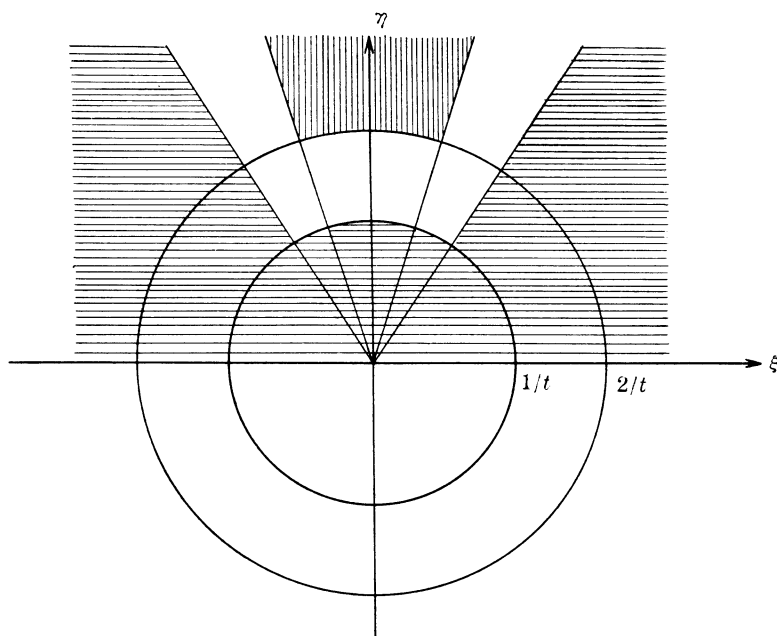

FIG. 2

Finally we define R^t to be convolution by the inverse Fourier transform of ρ^t so that

$$R^t u(x, y) = (2\pi)^{-n-q}\int \rho^t(\xi, \eta)\hat{u}(\xi, \eta)e^{i\langle x,\xi\rangle + i\langle y,\eta\rangle}d\xi d\eta .$$

Then the operator $Q^t = Q \circ R^t$ is given by the integral formula

$$(Q^t u)(x, y) = (2\pi)^{-n-q}\int \varphi_1(y)\varphi(x)p_\psi(x, \xi)\rho^t(\xi, \eta)\widehat{\psi_1 u}(\xi, \eta)d\xi d\eta .$$

The properties of the function ρ^t show that $p_\psi \rho^t \in S_0^m(U \times V \times \mathbf{R}^{n+q})$ so that $Q^t \in \mathscr{P}^m(U \times V)$. Now, as remarked in [11], the H_s estimates for pseudo-differential operators do not require regularity in (ξ, η). The fact that, for $m \geq 0$, $D_x^\beta p_\psi(x, \xi)/(1 + |\xi| + |\eta|)^m$ is bounded then implies that $Q \in Op^m$ (which we already know). Similarly for $m > 0$, the inequalities

$$\left| \frac{D_x^\beta\{p_\psi(x, \xi)(\rho^t(\xi, \eta) - 1)\}}{(1 + |\xi| + |\eta|)^m} \right| < C_\beta t^m ,$$

(which are easy consequences of the properties of ρ^t) show that, as $t \to 0$, $Q^t - Q \to 0$ in Op^m. As for the symbols we have

$$\sigma(Q^t) = \varphi_1 \varphi \sigma(p_\psi)(1 - \sigma^t)\psi_1 = \varphi_1 \varphi \tilde{\sigma}(P)(1 - \sigma^t)\psi \psi_1 \ ,$$

and, as $t \to 0$, this converges to $\varphi_1 \varphi \tilde{\sigma}(P)\psi \psi_1$. Since the symbol is also a local object, this implies that $\sigma(\tilde{P}) = \tilde{\sigma}(P)$ as required.

The final property of pseudo-differential operators we need concerns the situation of a group action. Thus let X be a compact manifold, and G a compact Lie group acting smoothly on X and on vector bundles E, F over X. Then G acts on the space $\bar{\mathscr{P}}^m(X; E, F)$ (because of the invariance of $\bar{\mathscr{P}}^m$ under diffeomorphism). If $g \in G$ and $P \in \bar{\mathscr{P}}^m$, we denote the action simply by $g(P)$. Note that, if u is a section of E, and $u \mapsto gu$ denotes the action of G on sections, then

$$g(P)u = gPg^{-1}u \ .$$

Then we shall need the following continuity property.[24]

(5.5) *For fixed $P \in \bar{\mathscr{P}}^m(X; E, F)$, the map $G \to \bar{\mathscr{P}}^m \subset Op^m$ given by $g \mapsto g(P)$ is continuous.*

Since G can be assumed to act unitarily on all H_s spaces, and since $\bar{\mathscr{P}}^m$ is the uniform closure of $\mathscr{P}^m$ it will be sufficient to prove (5.5) for $P \in \mathscr{P}^m$. Let A be an element of the Lie algebra of G, and let A_E, A_F denote the first order differential operators defined by the action of A on E, F, respectively. The symbol $\sigma(A_E)$ is given by

$$\sigma(A_E)\xi = A(\xi)I_E \ ,$$

where I_E is the identity of E, and $A(\xi)$ is inner product of the cotangent vector $\xi \in (TX)_x$ with the tangent vector A_x defined by the action of A on X. Hence

$$\sigma(P)\sigma(A_E) = \sigma(A_F)\sigma(P) \ ,$$

and so, by a basic property of pseudo-differential operators,

$$PA_E - A_F P \in \mathscr{P}^m \ .$$

For A in a bounded neighborhood of zero in the Lie algebra, it follows that we can find constants C_s so that

$$\| PA_E - A_F P \|_s^m < C_s \ .$$

Now, for $g_t = \exp tA$ and $u \in \mathscr{D}(E)$, we put

---

[24] The proof of (5.5) that follows we owe to L. Hörmander. In fact, the proof in [10] of the invariance of pseudo-differential operators under diffeomorphism also gives a certain uniformity which essentially includes (5.5) as a special case.

$$f_t = g_t(P)u = \exp tA_F \circ P \circ \exp (-tA_E)u \ .$$

Then

$$\left\| \frac{df_t}{dt} \right\|_{m-s} = \| \exp tA_F \circ (PA_E - A_F P) \circ \exp (-tA_E)u \|$$

$$< C_s \| u \|_s \ ,$$

since G can be assumed to act unitarily on all H_s spaces. Hence

$$\| f_t - f_0 \|_{m-s} < C_s t \| u \|_s$$

establishing the continuity of $P \mapsto g(P)$ at the identity element of G. Since each g acts continuously on $\bar{\mathscr{P}}^m$, the continuity everywhere follows.

In view of (5.5), if $P \in \bar{\mathscr{P}}^m$, and if dg denotes normalized Haar measure on G $\left(\text{so that} \int_G dg = 1\right)$, we can form the average

$$\text{Av}(P) = \int_G g(P) dg$$

and this will still be in[25] $\bar{\mathscr{P}}^m$.

Since $\sigma: \bar{\mathscr{P}}^m \to \overline{\text{Symb}}^m$ is continuous, it follows at once that we have

(5.6) $\sigma \, \text{Av}(P) = \text{Av}(\sigma P) \ .$

Similar results hold for the closure $\bar{\mathscr{P}}_s^m$ in each H_s-space.

6. The index of elliptic operators

Let X be a *compact* manifold, E and F smooth vector bundles over X. We recall that an operator $P \in \mathscr{P}^m(X; E, F)$ is said to be *elliptic* of order m if $\sigma(P)$ is invertible. One can then construct $Q \in \mathscr{P}^{-m}(X; F, E)$ such that $PQ - 1$ and $QP - 1$ both have smooth kernels (and in particular are compact). This then leads at once to the basic results on elliptic operators.

(6.1) *P has closed range,* Ker P *and* Coker P *are finite-dimensional and all distributional solutions of P and its adjoint are C^∞,* i.e. Ker P = Ker P_s *and* Coker $P \cong$ Coker $P_s \cong$ Ker P_s^* *for all s.*

The index of P is defined to be

$$\text{index } P = \dim \text{Ker } P - \dim \text{Coker } P \ .$$

In view of the last part of (6.1) we also have

index $P = $ index P_s for all s .

For operators P in $\bar{\mathscr{P}}_s^m$ with invertible symbols, we can, using (5.2), find $Q_{s-m} \in \bar{\mathscr{P}}_{s-m}^{-m}(X; F, E)$ such that $QP - 1$ and $PQ - 1$ are both compact. This

---

[25] Actually it is true that $P \in \mathscr{P}^m \Rightarrow \text{Av}(P) \in \mathscr{P}^m$, but this requires a little more work and is not necessary for our purposes.

implies that P is a Fredholm operator (from H_s to H_{s-m}), i.e., that it has closed range and Ker P, Coker P both have finite dimension. Index P is therefore defined for all elliptic $P \in \bar{\mathscr{P}}_s^m$. By standard properties of Fredholm operators in Hilbert space the index is a continuous (and so locally constant) function, and it is unchanged by addition of compact operators. Together with (5.2), this implies

(6.2) *The index depends only on the homotopy class of the symbol in the space of continuous invertible symbols of given order.*

In fact the order of an operator is not significant for purposes of the index as is shown by

(6.3) *If P, Q are elliptic operators in $\mathscr{P}^m, \mathscr{P}^n$ such that $\sigma(P)$ and $\sigma(Q)$ coincide on the unit sphere bundle of TX (for some metric), then* index $P =$ index Q.

To prove (6.3), we observe that $\sigma(P)/\sigma(Q)$ is self-adjoint, and so we can find a self-adjoint elliptic R so that

$$\sigma(P) = \sigma(Q)\sigma(R) .$$

This implies that index $P =$ index $Q +$ index $R =$ index Q, since index $R = 0$ for a self-adjoint operator.

Finally let us add two trivial formal properties of the index

(6.4) index $P \oplus Q =$ index $P +$ index Q

(6.5) index $P = 0$ if $P: \mathscr{D}(X; E) \to \mathscr{D}(X; F)$ is induced by a bundle isomorphism $E \to F$ over X.

If $P \in \bar{\mathscr{P}}_s^m(X, E, F)$ is elliptic, then $\sigma(P)$ defines, as in § 2, an element of $K(TX)$. Moreover our description of $K(TX)$ in § 2 by homogeneous complexes (of length one) and the properties (6.2) $-$ (6.5) of the index above show that index P *depends only on the class of $\sigma(P)$ in $K(TX)$*, and that $P \mapsto$ index P induces a homomorphism

$$K(TX) \longrightarrow \mathbf{Z} .$$

This will be called the *analytical index* and denoted by a-ind. Note that it does not depend on the integers m, s chosen above. The independence of m follows from (6.3) and the independence of s from the fact that $\mathscr{P}^m$ is dense in $\bar{\mathscr{P}}_s^m$, that we have regularity (6.1) for elliptic operators in $\mathscr{P}^m$, and that the index is continuous.

Let us now return and consider an elliptic operator $P \in \bar{\mathscr{P}}^m$, so that P_s is a Fredholm operator for all s. Since $\sigma(P_s)$ is independent of s, it follows from what has been said above, that index P_s is independent of s. But

$$\text{index } P_s = \dim \text{Ker } P_s - \dim \text{Ker } P_{m-s}^* \, .$$

Since $H_s \supset H_{s+1}$ and $H_{m-s} \subset H_{m-s-1}$, it follows that index P_s is a monotone decreasing function of s, and so can be constant only if Ker P_s and Ker P_{m-s}^* are independent of s. Hence we have established

(6.6) *Let $P \in \bar{\mathscr{P}}^m$ be elliptic, then all distributional solutions of $Pu = 0$ are C^∞ and the same holds for the adjoint P^*.*

This result, together with the fact that each P_s has closed range, easily implies that P has closed range; if f is in the closure of the range of P, it is in the closure of the range of P_s, and so $f = P_s g_s$ for $g_s \in H_s$. But (6.6) then implies $g_s - g_t \in C^\infty$ for all s, t, and so $g_s \in C^\infty$. Thus we can again compute index P_s on C^∞ sections,

$$\text{index } P_s = \text{index } P = \dim \text{Ker } P - \dim \text{Coker } P \, .$$

Finally we consider the situation of a group action. If the compact Lie group G acts smoothly on X and on vector bundles over X, and if $P \in \bar{\mathscr{P}}^m$, then we saw in § 5 that we can average over G to obtain $\text{Av}(P) \in \bar{\mathscr{P}}^m$, and that averaging commutes with taking symbols. In particular if $\sigma(P)$ is invariant, then

$$\sigma(\text{Av } P) = \sigma(P) \, .$$

Similar results hold for $P \in \bar{\mathscr{P}}_s^m$.

It remains now to show that a G-invariant Fredholm operator $P \colon H \to H'$, where H and H' are Hilbert spaces acted on by G, has an index in $R(G)$ with the usual properties. The definition of index P is clear, we put

$$\text{index } P = [\text{Ker } P] - [\text{Coker } P] \in R(G) \, ,$$

which makes sense because Ker P and Coker P are finite-dimensional G-modules. To prove that this is locally constant for the norm topology of P, we proceed as follows.[26] Choose $V \subset H$ to be any G-invariant closed subspace of finite co-dimension with $V \cap \text{Ker } P = 0$ (for example $V = (\text{Ker } P)^\perp$), and let $\tilde{P} \colon H/V \to H'/P(V)$ be induced by P. From the exact sequences

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & V & \longrightarrow & H & \longrightarrow & H/V & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle P} & & \downarrow{\scriptstyle \tilde{P}} & & \\
0 & \longrightarrow & P(V) & \longrightarrow & H' & \longrightarrow & H'/P(V) & \longrightarrow & 0 \, ,
\end{array}
$$

we deduce G-isomorphisms Ker $P \cong$ Ker $\tilde{P}$, Coker $P \cong$ Coker $\tilde{P}$. Hence

$$\text{index } P = [\text{Ker } P] - [\text{Coker } P] = [\text{Ker } \tilde{P}] - [\text{Coker } \tilde{P}]$$
$$= [H/V] - [H'/P(V)]$$

---

[26] We are simply giving here a proof of the invariance of the usual index which extends naturally to group actions.

by a simple property of the index for finite-dimensional spaces. Now the map $(PV)^{\perp} \oplus V \to H'$ given by $x \oplus y \mapsto x + Qy$ is an isomorphism for $Q = P$, and so also an isomorphism for $\| P - Q \|$ small. Hence, for such Q, we have $V \cap \operatorname{Ker} Q = 0$ and $(PV)^{\perp} \cong H'/Q(V)$. Hence introducing $\tilde{Q} \colon H/V \to H'/Q(V)$ we find as before

$$\begin{aligned}
\operatorname{index} Q = \operatorname{index} \tilde{Q} &= [H/V] - [H'/Q(V)] \\
&= [H/V] - [(PV)^{\perp}] \\
&= \operatorname{index} P .
\end{aligned}$$

Thus index $P \in R(G)$ is locally constant.[27] If K is compact and G-invariant, the homotopy $P + tK$ with $0 \leq t \leq 1$ then shows that index $P = \operatorname{index} (P + K)$.

As has already been indicated in the introduction, our main theorem will be

THEOREM (6.7). *The analytical index and the topological index coicide as homomorphisms $K_G(TX) \to R(G)$.*

This theorem gives in principle a complete topological answer to the problem of computing the index of G-invariant elliptic operators. Alternative and more explicit methods of computing the topological index will be derived in papers II and III.

In view of the axiomatic characterizations of the topological index given in § 4 we have only to show that the analytical index verifies the appropriate axioms. The fact that

$$\text{a-ind} \colon K_G(TX) \longrightarrow R(G)$$

is functorial for G-diffeomorphisms of X, and for homomorphisms of groups $G \to G'$, is immediate from the naturality of the construction.

Axiom (A1) is quite trivial for the analytical index. In fact an elliptic operator P on a point is just a G-linear map $P \colon V \to W$ of finite-dimensional G-modules, and we have

$$[\sigma(P)] = [V] - [W] = \operatorname{index} P \in R(G) .$$

The verification of the remaining axioms will be carried out in § 8 and § 9.

7. Elliptic complexes

In this section we digress briefly to discuss the notion of an *elliptic complex*. Let X be a compact manifold, E^i a sequence of smooth vector bundles over X, and

---

[27] Note that we did not really use the compactness of G. Everything holds for *unitary* representations of any group G provided we interpret $R(G)$ as the group generated by characters of finite-dimensional representations of G.

$$d_i \colon \mathfrak{D}(X;\, E^i) \longrightarrow \mathfrak{D}(X;\, E^{i+1})$$

a pseudo-differential operator of order m with symbol σ_i. The sequence

$$0 \longrightarrow \mathfrak{D}(E^0) \xrightarrow{\ d_0\ } \mathfrak{D}(E^1) \longrightarrow \cdots \xrightarrow{\ d_{n-1}\ } \mathfrak{D}(E^n) \longrightarrow 0$$

is called an *elliptic complex* (of order m) if

  (i) $d_{i+1}d_i = 0$,

  (ii) the symbol complex

$$0 \longrightarrow \pi^* E^0 \xrightarrow{\ \sigma_0\ } \pi^* E^1 \longrightarrow \cdots \xrightarrow{\ \sigma_{n-1}\ } \pi^* E^n \longrightarrow 0$$

over the cotangent bundle TX is exact outside the zero-section.

In [4] it is shown that[28] the homology groups $H^i(E) = \operatorname{Ker} d_i / \operatorname{Im} d_{i-1}$ are finite-dimensional, and so the Euler characteristic

$$\chi(E) = \sum (-1)^i \dim H^i(E)$$

is defined. If a compact group G acts on E (commuting with the d_i), then $H^i(E)$ is a G-module, and we can define $\chi(E)$ as an element of $R(G)$.

If the complex is of length one, then we have just one elliptic operator d_0 and $\chi(E)$ becomes just the index of d_0. Thus an elliptic complex is a natural generalization of a single elliptic operator. Moreover elliptic complexes occur naturally in differential geometry; the two significant examples being the de Rham complex and its complex analogue, the Dolbeault complex.

The symbol complex $\sigma(E)$ of E is, in the terminology of § 2, a compactly supported homogeneous complex over TX, and so it defines an element of $K(TX)$ or of $K_G(TX)$ in the group situation. The problem of computing $\chi(E)$ in terms of the class of $\sigma(E)$ in $K_G(TX)$ can be reduced to the problem of the index of a single operator by the following simple device.

Introduce G-invariant metrics in all the bundles and on X so that we obtain adjoints d_i^* for the d_i. Now consider the single operator

$$D \colon \mathfrak{D}\!\left(\bigoplus_i E^{2i}\right) \longrightarrow \mathfrak{D}\!\left(\bigoplus_i E^{2i+1}\right)$$

defined by $D = d + d^*$. More precisely

$$D(u_0, u_2, \cdots) = (d_0 u_0 + d_1^* u_2,\ d_2 u_2 + d_3^* u_4,\ \cdots).$$

Since $d^2 = 0$, we have $(d^*)^2 = 0$, and so

$$D^*D = \bigoplus_i \Delta_{2i}\,, \qquad D^*D = \bigoplus_i \Delta_{2i-1}\,,$$

where Δ_i denotes the "laplacian" on $\mathfrak{D}(E_i)$ given by

$$\Delta_i = d_{i-1}d_{i-1}^* + d_i^* d_i\,.$$

---

[28] Actually in [4] the degrees of the d_i are not assumed to be the same. Here for simplicity we have made this assumption.

Now the exactness of the symbol complex $\sigma(E)$ (off the zero-section) implies that

$$\sigma(\Delta_i) = \sigma_{i-1}\sigma^*_{i-1} + \sigma^*_i\sigma_i \in \text{Hom}\,(\pi^*E^i, \pi^*E^i)$$

is an isomorphism (off the zero-section). Hence Δ_i, and so also D, is elliptic. Using the regularity properties of elliptic operators, it then follows, as in the usual Hodge theory, that

$$\text{Ker}\,D = \bigoplus_i H^{2i} \cong \bigoplus_i H^{2i}(E)$$
$$\text{Coker}\,D = \bigoplus_i H^{2i+1} \cong \bigoplus H^{2i+1}(E)$$

where H^i stands for the space of "harmonic" sections of E^i, namely $\text{Ker}\,\Delta_i$. Hence

$$\text{index}\,D = \chi(E)\,.$$

Moreover $\sigma(D)$ and $\sigma(E)$ represent the same class in $K_G(TX)$ [1; 2.6.10]. Thus we are reduced to the case of a single operator, and our main theorem (6.7) yields immediately

THEOREM (7.1). *Let E be a G-invariant elliptic complex over the compact G-manifold X, and let $[\sigma(E)] \in K_G(TX)$ be the class of the symbol sequence of E. Then the Euler characteristic*

$$\chi(E) = \sum(-1)^i H^i(E) \in R(G)$$

is given by

$$\chi(E) = \text{t-ind}\,[\sigma(E)]\,.$$

Remark. [3] was concerned with a general endomorphism T of an elliptic complex (not necessarily arising from a compact group), and the introduction of metrics invariant under T is not in general possible. For this reason complexes are needed essentially in [3] but not here.

8. The excision and normalization axioms

In this section, we shall show that the analytical index of § 6 satisfies the excision axiom (B1) and the normalization axiom (B2″) of § 4.

Consider first the excision (B1). Thus we suppose that U is an open G-invariant subset of the compact G-manifold X, and we denote by $j: U \to X$ the inclusion. Then j induces

$$j_*: K_G(TU) \longrightarrow K_G(TX)\,.$$

Shortly we shall show that any element $a \in K_G(TU)$ is the symbol class of an elliptic G-invariant operator P which is "equal to the identity" outside a compact set. More precisely, this means that $P \in \bar{\mathscr{P}}^0(U; E, F)$ is G-invariant with $[\sigma(P)] = a$,

$$\alpha: E \mid U - L \longrightarrow (U - L) \times \mathbf{C}^n$$
$$\beta: F \mid U - L \longrightarrow (U - L) \times \mathbf{C}^n$$

are G-bundle isomorphisms outside some compact set L, and for $u \in \mathcal{D}(U-L; E)$, we have

(8.1) $$Pu = \beta^{-1}\alpha u \ .$$

For the moment assume we have such an operator P. Then it extends in an obvious fashion to a G-invariant operator $j_* P$ on X; we extend E, F trivially to bundles $j_* E, j_* F$ on X using α and β, and then extend P outside U by (8.1). Clearly we have

$$[\sigma j_*(P)] = j_*[\sigma(P)] = j_*(a) \in K_G(TX) \ .$$

On the other hand, if $u \in \mathcal{D}(X; j_* E)$, (8.1) shows that

$$(j_* P) u = 0 \Longrightarrow \text{supp } u \subset U \quad \text{and} \quad Pu = 0 \ .$$

Thus Ker $P \cong$ Ker $j_* P$, and similarly for the adjoint P^*. Hence

$$\text{index}^x j_* P = [\text{Ker } P] - [\text{Ker } P^*] \in R(G) \ .$$

This shows that $\text{index}^x j_* P$ can be computed from the operator P on U, and so does not depend on X. This is axiom (B1).

It remains now to show how to construct the operator P. As shown in § 2, we can represent $a \in K_G(TU)$ as the class of a complex over TU

$$0 \longrightarrow \pi^* E \overset{\sigma}{\longrightarrow} \pi^* F \longrightarrow 0 \ ,$$

where E, F are vector bundles on U, σ is homogeneous of degree zero and, outside a compact set L_1 of U, we have isomorphisms

$$\alpha: E \mid U - L_1 \longrightarrow (U - L_1) \times \mathbf{C}^n$$
$$\beta: F \mid U - L_1 \longrightarrow (U - L_1) \times \mathbf{C}^n \ ,$$

such that $\sigma = \pi^*(\beta^{-1}\alpha)$. Moreover, everything is G-invariant, and we can assume σ is smooth. Since the construction of a pseudo-differential operator with given symbol is done locally and then globalized by partitions of unity, it is clear that we can find $P_1 \in \mathcal{P}^0(U; E, F)$ with $\sigma(P_1) = \sigma$, and such that P_1 is equal to the identity (or precisely is induced by $\beta^{-1}\alpha$) outside some (G-invariant) compact $L \supset L_1$. The required operator P is then the average $\text{Av}(P_1)$. Strictly speaking we have only shown that averaging preserves the class $\bar{\mathcal{P}}^0$ on *compact* manifolds. Since U is non-compact, we should either extend the proof to non-compact manifolds (which is not hard), or we can argue as follows. We can certainly form $\text{Av}(P_1)$ in some weak sense (e.g., using the distribution topology of kernels). On the other hand, because P_1 is the identity outside a compact set, it is clear that averaging over G will

commute with extension from X to U, that is

$$\mathrm{Av}(j_* P_1) = j_* \mathrm{Av}(P_1) \ .$$

Since X is compact, $\mathrm{Av}\,(j_* P_1) \in \bar{\mathscr{P}}^0(X; j_* E, j_* F)$. Hence its restriction to U, namely $\mathrm{Av}\,(P_1)$, belongs to $\bar{\mathscr{P}}^0(U; E, F)$.

We now verify (B2''). For parts (i) and (ii) we have to consider the de Rham complex of exterior differential forms on the n-sphere for $n = 1, 2$; the symbol class of this is just the element ρ_{S^n} of Axiom (B2''). If we are prepared to use the de Rham theorems, which assert that the cohomology of this complex is naturally isomorphic to the ordinary cohomology groups $H^2(S^n, \mathbf{C})$, then (i) and (ii) of (B2'') become obvious. In fact we have

$$H^q(S^n, \mathbf{C}) = 0 \qquad\qquad\qquad \text{for } 0 < q < n$$
$$\dim H^0(S^n, \mathbf{C}) = \dim H^n(S^n, \mathbf{C}) = 1 \ .$$

The connected group $SO(2)$ acts trivially on $H^q(S^2, \mathbf{C})$, and so

$$\text{a-ind } \rho_{S^2} = 2 \in R\big(SO(2)\big) \ ,$$

proving (i). On the other hand, the generator of $O(1)$ changes the orientation of S^1, and so acts as -1 on $H^1(S^1, \mathbf{C})$ (but trivially on $H^0(S^1, \mathbf{C})$). Thus

$$\text{a-ind } \rho_{S^1} = 1 - \xi \in R\big(O(1)\big)$$

proving (ii).

A direct proof without appealing to the de Rham theorems (or to the cohomology of spheres!) is however quite easy. For the circle, the de Rham complex is just $f \mapsto df = (df/dx) \cdot dx$ where $x \bmod 1$ is a parameter for the circle. Hence $\mathrm{Ker}\, d$ consists of the constant functions, and $\mathrm{Coker}\, d$ is generated by dx. The generator of $O(1)$ is the map $x \mapsto -x$ which induces $dx \mapsto -dx$ on $\mathrm{Coker}\, d$ and, of course, the identity on $\mathrm{Ker}\, d$. Thus

$$\text{a-ind } \rho_{S^1} = 1 - \xi \ .$$

For S^2, the de Rham complex has three terms:

$$0 \longrightarrow \Omega^0 \xrightarrow{\ d^0\ } \Omega^1 \xrightarrow{\ d^1\ } \Omega^2 \longrightarrow 0 \ .$$

Again $\mathrm{Ker}\, d^0$ consists of constants. Taking adjoints, we see that $\mathrm{Coker}\, d^1$ is generated by the volume form in Ω^2. Since $SO(2)$ acts trivially on both these spaces, it remains to show that there is no contribution from Ω^1, i.e., that

$$dw = 0 \implies w = df \qquad\qquad w \in \Omega^1, f \in \Omega^0 \ .$$

Now[29] outside the north pole, the Poincaré lemma shows that $w = df_0$, and similarly $w = df_\infty$ outside the south pole. Then $d(f_0 - f_\infty) = 0$, and so

---

[29] The proof that follows is essentially a specially simple version of the proof of the de Rham theorems.

$f_0 - f_\infty =$ constant. Thus f_0 is actually defined everywhere, i.e., $f_0 \in \Omega^0$.

We come now to part (iii) of Axiom (B2''). Here the operator whose index we have to compute is not a differential operator, and the whole problem in this case is a matter of sign conventions.[30] In fact, if we make use of the multiplicative Axiom (B3'') we can compute $(\text{a-ind } j_1(1))^2$ on $S^1 \times S^1$ (or better on S^2). Using part (i) of (B2'') which we have just verified, one finds

$$\left(\text{a-ind } j_1(1)\right)^2 = 1 \,,$$

so that a-ind $j_1(1) = \pm 1$, and the whole problem is to show that, *with our choice of sign conventions*, we get the plus sign. This we now proceed to do.

Consider first the operator P on the circle $S^1 = \mathbf{R}/2\pi\mathbf{Z}$ defined by

$$\begin{aligned} Pe^{inx} &= e^{inx} && \text{for } n \geq 0 \\ &= 0 && \text{for } n < 0 \,. \end{aligned}$$

We shall show that $P \in \mathscr{P}^0$. Let $f \in \mathscr{D}(\mathbf{R})$ have support in an interval of length $< 2\pi$ (which we identify with its image in S^1). The Fourier coefficients of $f(x)e^{ix\xi}$ are $\hat{f}(n - \xi)$. Hence, in supp f,

$$\begin{aligned} p_f(x, \xi) &= e^{-ix\xi}P\big(f(x)e^{ix\xi}\big) = \sum_0^\infty \hat{f}(n - \xi)e^{ix(n-\xi)} \\ &= f(x) - \sum_{-1}^{-\infty} \hat{f}(n - \xi)e^{ix(n-\xi)} \,. \end{aligned}$$

As $\xi \to -\infty$, the sum $\sum_0^\infty$ converges to 0 faster than any power of $|\xi|$, and so do all its derivatives with respect to x and ξ. When $\xi \to +\infty$, the sum $\sum_{-1}^{-\infty}$ has the same properties. Thus P is pseudo-differential of order zero. Since

$$\begin{aligned} p_f(x, \xi) &\longrightarrow f(x) && \text{as } \xi \longrightarrow +\infty \\ &\longrightarrow 0 && \text{as } \xi \longrightarrow -\infty \end{aligned}$$

it follows that $P \in \mathscr{P}^0$, and that its symbol σ_P is given by

$$\begin{aligned} \sigma_P(x, \xi) &= 1 && \text{for } \xi > 0 \\ &= 0 && \text{for } \xi < 0 \,. \end{aligned}$$

Now define $A = e^{ix}P + (1 - P)$. This is then an operator in $\mathscr{P}^0$, and its symbol is given by

(8.2)
$$\begin{aligned} \sigma_A(x, \xi) &= e^{ix} && \text{for } \xi > 0 \\ &= 1 && \text{for } \xi < 0 \,. \end{aligned}$$

A is therefore elliptic. On the other hand, by definition

$$\begin{aligned} Ae^{inx} &= e^{i(n+1)x} && \text{for } n \geq 0 \\ &= e^{inx} && \text{for } n < 0 \,, \end{aligned}$$

---

[30] See [14, p. 281] for a discussion of the various places where a convention of sign is involved. Because we have no cohomology so far, our sign problem is a little less acute than in [14].

and so Ker $A = 0$ and Coker A is generated by the constant functions. Thus index $A = -1$.

To establish axiom (B2″) of the analytical index, it remains therefore to show that the natural homomorphism

(8.3) $K(T\mathbf{R}^1) \longrightarrow K(TS^1)$

takes the element $-j_!(1)$ into the class of σ_A in $K(TS^1)$. To do this, we define a continuous symbol σ on R^1 by equation (8.2) for $0 \leqq x \leqq 2\pi$, and by $\sigma = 1$ for $x < 0$ or $x > 2\pi$. It is then clear that $[\sigma] \to [\sigma_A]$ in the homomorphism (8.3). Thus we have finally to check that

$$[\sigma] = -j_!(1) \in K(T\mathbf{R}^1) .$$

Now both these elements are defined by maps

$$T\mathbf{R}^1 - \text{(compact set)} \longrightarrow \mathbf{C} - (0) .$$

For σ, the compact set is the rectangle $0 \leqq x \leqq \pi$, $|\xi| \leqq 1$, while for $j_!(1)$, it is the disc $|x + i\xi| \leqq 1$.
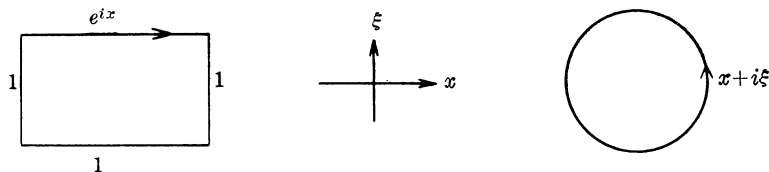

FIG. 3

On the rectangle, σ is e^{ix} on the top and 1 elsewhere, while $j_!(1)$ is $x + i\xi$ on the unit circle. An elementary homotopy then deforms σ into $-j_!(1)$ and the verification of Axiom (B2″) is then complete.

9. The multiplicative axiom

In this section, we show that the analytical index satisfies the multiplicative axiom (B3) (and hence also the weaker axiom (B3′)). Once this has been done we shall have established all the axioms necessary to apply Theorem (4.6), and deduce that the analytical and topological indices coincide. Thus our main theorem (6.7) will be proved.

We start by recalling the context of axiom (B3). Thus let $Y = P \times_H F$ be a fibre bundle over a compact manifold X where $P \to X$ is a principal bundle with compact Lie group H, and F is a compact manifold on which H acts on the left. In addition a second compact Lie group G acts on both P and F commuting with the action of H. Hence G acts on $X = P/H$ and $Y = P \times_H F$. In what follows, metrics are chosen on P, X, F, Y, and assorted

vector bundles over these spaces so as to be invariant under G and/or H.

Axiom (B3) is concerned with two elements

$$a \in K_G(TX) , \qquad b \in K_{G \times H}(TF)$$

and their product, in a suitable sense, which is an element

$$ab \in K_G(TY) .$$

What we have to do is to calculate the analytic index of ab in terms of the analytical indices of a and b. For this purpose we must first represent a, b, and ab, by elliptic operators. This we proceed to do rather carefully.

Consider first $a \in K_G(TX)$. We choose a representative smooth symbol α of order[31] 1 (and G-invariant). Let $A_1 \in \mathscr{P}^1$ have symbol α, but for the moment we do not insist that A_1 be G-invariant. Now take a covering $\{U_j\}$ of X with trivializations of P, and hence of Y, over each U_j. Let $\{\varphi_j^2\}$ be a smooth partition of unity on X for this covering, and consider the operator $A_1^j = \varphi_j A_1 \varphi_j$ on U_j. Let $Y_j = p^{-1}(U_j) \cong U_j \times F$, where $p \colon Y \to X$ is the projection, then the lifted operator $\tilde{A}_1^j$ on Y_j belongs by (5.4) to the class $\bar{\mathscr{P}}^1(Y_j)$. Because of the φ_j it can also be regarded as an operator on Y, and so it belongs to $\bar{\mathscr{P}}^1(Y)$. Now form the average over G

$$\tilde{A} = \mathrm{Av}(\textstyle\sum \tilde{A}_1^j) \in \bar{\mathscr{P}}^1(Y) .$$

Since $\sigma(\sum A_1^j) = \sum \varphi_j^2 \sigma(A_1) = \alpha$, and since taking symbols commutes both with lifting (5.4) and with averaging (5.6), it follows that

$$\sigma(\tilde{A}) = \tilde{\alpha} .$$

Note that because we have chosen invariant metrics everywhere, the lifting $\tilde{\alpha}$ of the symbol α is globally well-defined: if we split the cotangent space of Y into horizontal components ξ and vertical components η, then $\tilde{\alpha}(\xi, \eta) = \alpha(\xi)$ (where we identify ξ with the corresponding vector on the base X).

If we restrict $\tilde{A}_1^j$ to sections coming from the base, i.e., constant on the fibres of Y, we recover the original operator A_1^j. Hence the restriction of $\tilde{A}$ to such sections is just the G-invariant elliptic operator

$$A = \mathrm{Av}(\textstyle\sum A_1^j) \in \bar{\mathscr{P}}^1(X)$$

with symbol α.

We turn next to the element $b \in K_{G \times H}(TF)$. We choose a $G \times H$-invariant operator $B \in \bar{\mathscr{P}}^1(F)$ with $\beta = \sigma(B)$ in the class b. Let $\tilde{B}_1$ be the operator on $P \times F$ obtained by lifting B. Since it is $G \times H$-invariant, it induces a G-invariant operator $\tilde{B}$ on $Y = P \times_H F$. We simply restrict $\tilde{B}_1$ to sections constant along the fibres of $P \times F \to Y$. Since P is locally a product, it

_____

[31] As promised in § 5 we are only going to apply (5.4) for $m = 1$.

follows that, over the open sets $Y_j = p^{-1}(U_j) \cong U_j \times F$, the restriction $\tilde{B}_j$ of $\tilde{B}$ is just the lifting of B. It then follows from (5.4) that $\tilde{B}_j \in \bar{\mathscr{P}}^1(Y_j)$, and so $\tilde{B} \in \bar{\mathscr{P}}^1(Y)$. Let $\tilde{\beta} = \sigma(\tilde{B})$. It is invariantly defined by β, and in terms of horizontal (ξ) and vertical (η) components, we have $\tilde{\beta}(\xi, \eta) = \beta(\eta)$.

Finally we come to the element ab. For this we take the operator

$$D = \begin{pmatrix} \tilde{A} & -\tilde{B}^* \\ \tilde{B} & \tilde{A}^* \end{pmatrix} \in \bar{\mathscr{P}}^1(Y) \ .$$

It is G-invariant (since $\tilde{A}$ and $\tilde{B}$ are), and we have

$$\sigma(D) = \begin{pmatrix} \tilde{\alpha} & -\tilde{\beta}^* \\ \tilde{\beta} & \tilde{\alpha}^* \end{pmatrix} \ .$$

Remark. If $A \in \bar{\mathscr{P}}^1(X; E^0, E^1)$ and $B \in \bar{\mathscr{P}}^1(F; G^0, G^1)$, then

$$D \in \bar{\mathscr{P}}^1(Y; E^0 \boxtimes G^0 \oplus E^1 \boxtimes G^1, E^1 \boxtimes G^0 \oplus E^0 \boxtimes G^1) \ ,$$

and the two liftings of A, which occur in D, are liftings to different bundles. Thus the $\tilde{A}$ in the top-left is the lifting to G^0, while $\tilde{A}^*$ in the bottom right is (adjoint of) the lifting to G^1. However the position in the matrix makes clear which bundles are involved, and to simplify the notation we shall omit all reference to the actual bundles involved.

As explained in § 2, $\sigma(D)$ is a representative for the class ab. What we have to do now is to compute the index of the elliptic operator D.

We first show that $\tilde{A}$ and $\tilde{B}$ commute. From the local product representation $Y_j \cong U_j \times F$, it follows that $\tilde{B}_j$ commutes with $\tilde{A}_i^j$. Thus $\tilde{B}$ commutes with $\sum \tilde{A}_i^j$ and, because $\tilde{B}$ is G-invariant, it also commutes with $\tilde{A} = \mathrm{Av}\,(\sum \tilde{A}_i^j)$. Similar remarks apply with $\tilde{B}^*$ for $\tilde{B}$, and so the off-diagonal terms in D commute with the diagonal terms. Hence

$$D^*D = \begin{pmatrix} \tilde{A}^*\tilde{A} + \tilde{B}^*\tilde{B} & 0 \\ 0 & \tilde{A}\tilde{A}^* + \tilde{B}\tilde{B}^* \end{pmatrix} = \begin{pmatrix} P_0 & 0 \\ 0 & Q_0 \end{pmatrix} \qquad \text{say}$$

$$DD^* = \begin{pmatrix} \tilde{A}\tilde{A}^* + \tilde{B}^*\tilde{B} & 0 \\ 0 & \tilde{A}^*\tilde{A} + \tilde{B}\tilde{B}^* \end{pmatrix} = \begin{pmatrix} P_1 & 0 \\ 0 & Q_1 \end{pmatrix}$$

Computing the kernels (on smooth sections), it follows that

$$\mathrm{Ker}\,D = \mathrm{Ker}\,D^*D = \mathrm{Ker}\,P_0 \oplus \mathrm{Ker}\,Q_0$$

$$\mathrm{Ker}\,D^* = \mathrm{Ker}\,DD^* = \mathrm{Ker}\,P_1 \oplus \mathrm{Ker}\,Q_1 \ .$$

But by (6.6) (regularity for elliptic operators in $\bar{\mathscr{P}}^m$), it follows that we have $\mathrm{Ker}\,D^* \cong \mathrm{Coker}\,D$ and so

$$\mathrm{index}^Y D = (\mathrm{Ker}\,P_0 - \mathrm{Ker}\,P_1) + (\mathrm{Ker}\,Q_0 - \mathrm{Ker}\,Q_1) \in R(G) \ .$$

Consider now the operator $P_0 = \tilde{A}^*\tilde{A} + \tilde{B}^*\tilde{B}$ (on smooth sections). For any smooth section u we have

$$\langle P_0 u, u \rangle = \langle \tilde{A}u, \tilde{A}u \rangle + \langle \tilde{B}u, \tilde{B}u \rangle ,$$

and so $\operatorname{Ker} P_0 = \operatorname{Ker} \tilde{A} \cap \operatorname{Ker} \tilde{B}$. Since $\tilde{B}$ is the natural extension of the operator B on the fibres, it follows that $\operatorname{Ker} \tilde{B}$ consists of those smooth sections which, on each fibre Y_x, lie in $\operatorname{Ker} B_x$ (where B_x denotes the operator on Y_x corresponding to B on the standard fibre F). But this is just another way of saying that $\operatorname{Ker} \tilde{B}$ is the space of smooth sections of the vector bundle $K_B = P \times_H \operatorname{Ker} B$ over X. Since $\tilde{A}$ and $\tilde{B}$ commute, it follows that $\tilde{A}$ induces an operator C on the sections of K_B. Since $\tilde{A} = \operatorname{Av}(\sum \varphi_j \tilde{A}_1^j \varphi_j)$, we have $C = \operatorname{Av}(\sum \varphi_j C_j \varphi_j)$ where C_j is the operator induced on $K_B | U_j$ by $\tilde{A}_1^j$. But, by definition of $\tilde{A}_1^j$, this means that $C_j = A_1 \otimes \operatorname{Id}(K_B)$. Hence $C_j \in \mathscr{P}^1$, $C \in \bar{\mathscr{P}}^1$ and

$$\sigma(C) = \alpha \otimes \operatorname{Id}(K_B) .$$

Thus C is a G-invariant elliptic operator over X, and the class of $\sigma(C)$ in $K_G(TX)$ is the product $a[K_B]$ of the class $a \in K_B(TX)$ of the symbol α and the class $[K_B] \in K_G(X)$ of the vector bundle K_B. Since $\tilde{A}^* = (A^*)^\sim$, replacing $\tilde{A}$ by $\tilde{A}^*$ replaces P_0 by P_1 and C by C^*. Hence

$$\operatorname{Ker} P_0 - \operatorname{Ker} P_1 = \operatorname{Ker} C - \operatorname{Ker} C^* = \text{a-ind}^x a[K_B] \in R(G) .$$

Replacing B by B^*, we similarly get

$$\operatorname{Ker} Q_1 - \operatorname{Ker} Q_0 = \text{a-ind}^x a[L_B] \in R(G) ,$$

where $L_B = P \times_H \operatorname{Coker} B$. Hence finally we obtain

$$\text{index}^Y D = \text{a-ind}^x \big(a([K_B] - [L_B])\big) = \text{a-ind}^x \big(a \cdot \mu_P(\text{a-ind}_{G \times H}^F b)\big) \in R(G) .$$

Since the class of $\sigma(D)$ is the product ab, we have therefore established Axiom (B3) for the analytical index. The proof of Theorem (6.7) is thus complete.

MATHEMATICAL INSTITUTE, OXFORD UNIVERSITY, AND
INSTITUTE FOR ADVANCED STUDY
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

REFERENCES

1. M. F. ATIYAH, K-Theory, Benjamin, 1967.
2. ———, Algebraic topology and elliptic operators, Comm. Pure Appl. Math., 20 (1967), 237–249.
3. ———, Bott periodicity and the index of elliptic operators, Quart. J. Math. (Oxford) 1968 (to appear).
4. ——— and R. BOTT, A Lefschetz fixed point formula for elliptic complexes: I, Ann. of Math. 86 (1967), 374–407.

5. ————, "The index theorem for manifolds with boundary", in Differential Analysis (Proc. Bombay Symposium) Oxford, 1964.

6. M. F. ATIYAH and I. M. SINGER, *The index of elliptic operators on compact manifolds*, Bull. Amer. Math. Soc. 69 (1963), 422-433.

7. F. HIRZEBRUCH, *Elliptische Differentialoperatoren auf Mannigfaltigkeiten*, Arbeitsgemeinschaft für Forschung des Landes Nordrhein-Westfalen, 33 (1965), 563-608.

8. L. HÖRMANDER, Linear Partial Differential Operators, Springer, 1963.

9. ————, *Pseudo-differential operators*, Comm. Pure Appl. Math. 18, (1965), 501-517.

10. ————, "Pseudo-differential operators and hypo-elliptic equations", in Amer. Math. Symposia in Pure Mathematics, Vol. 10, 138-183.

11. J. J. KOHN and L. NIRENBERG, *An algebra of pseudo-differential operators*, Comm. Pure Appl. Math. 18 (1965), 269-305.

12. R. P. LANGLANDS, *The dimension of spaces of automorphic forms*, Amer. J. Math. 85 (1963), 99-125.

13. R. PALAIS, *Imbedding of compact, differentiable transformation groups in orthogonal representations*, J. Math. Mech., 6 (1957), 673-678.

14. R. PALAIS, Seminar on the Atiyah-Singer Index Theorem, Ann. of Math. Study 57, Princeton, 1965.

15. R. T. SEELEY, *Integro-differential operators on vector bundles*, Trans. Amer. Math. Soc. 117 (1965), 167-204.

16. G. B. SEGAL, Equivariant *K*-theory, Publ. Math. Inst. Hautes Etudes Sci., Paris, 1968 (to appear).