The index of elliptic operators: III

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Introduction

In [7], paper I of this series, the index of an elliptic operator was computed in terms of K-theory. In this paper, we carry out what is essentially a routine exercise by passing from K-theory to cohomology. In this way, we end up with (2.12) the explicit cohomological formula for the index announced in [6].

In [7] we also considered elliptic operators (or complexes) compatible with a compact group G of transformations. The index in this case is a character of G, and the main theorem of [7] gave a construction for this in K_G-theory. In [5], paper II of this series, the value of this index-character at an element $g \in G$ was expressed as the index of a new "virtual operator" on the fixed-point set of g. This was referred to as a Lefschetz fixed-point formula. By combining this formula with the cohomological formula for the index, we obtain finally an explicit cohomological formula (3.9) for the index-character. We shall describe this formula in detail for a number of important operators. In particular we draw attention to the "integrality theorems" obtained in this way for actions of finite groups on manifolds. Most of these do not depend on the analysis in [7], but are a consequence of combining the purely topological results of [5] and the present paper.

We begin in § 1 with a brief review of the theory of characteristic classes and of the relation between cohomology and K-theory. The index formula (2.12) is then derived in § 2, and the more general index-character or Lefschetz formula (3.9) is given in § 3. The proofs are simple formal consequences of the results of papers I and II. We also give in (2.17) a rather more explicit form of the index theorem for the kind of operator which arises naturally from differential-geometric structures.

The bulk of the paper is actually devoted to examples and applications of the main theorem. The explicit formula (3.9) is of a rather formidable kind, and it seemed worthwhile to show what it reduces to in various important special cases. Thus in § 4 we discuss the Riemann-Roch theorem (4.3) for compact complex manifolds, and also the corresponding Lefschetz formula (4.6) for finite groups of automorphisms. In § 5 we examine briefly the Dirac operator of a Spin manifold. In some ways the most interesting elliptic operator is the one, discussed in § 6, the index of which is the Hirzebruch signature of a manifold. Because of its significance for differential topology, we examine this case in considerable detail. Besides obtaining the original theorem (6.6) of Hirzebruch, we also derive the corresponding Lefschetz formula (6.12). One noteworthy feature here is that this formula is of interest for all even-dimensional (oriented) manifolds, not just those with dimension divisible by 4, as is the case with (6.6).

In § 7 we show how the Lefschetz formula of the preceding sections can be used to define invariants for free G-actions (on odd-dimensional manifolds). Some of these are cobordism-type invariants, but the operator of § 6 gives rise to a more refined invariant, and we devote most of § 7 to this case.

In § 8 we consider the infinitesimal case of our general Lefschetz formula, and we derive results similar to those of Bott [9] [10] connecting characteristic numbers with zeros of vector fields.

Finally in § 9 we make some simple deductions from the general index and Lefschetz formulas. Under various special assumptions, these formulas simplify considerably, and, in many situations, one gets a zero index. Most of these special cases are already mentioned in [6].

As far as the ordinary index theorem goes, the particular cases of §§ 4, 5, 6, are quite adequately described in [19]. Our reason for reproducing this material again here is partly for the sake of completeness, and partly because we want to go on to discuss Lefschetz numbers. Our treatment however is somewhat briefer than that in [19], and the reader can refer to it for more detail.

1. Cohomology and characteristic classes

In this section we shall, for the benefit of the reader, give a brief summary of the theory of characteristic classes. For further details we refer to [8].

For the category of compact spaces, the most natural cohomology theory is Čech cohomology. This is also the one that fits in with vector bundles, K-theory, and general fibre bundle theory. For locally compact spaces, it is then convenient to take cohomology with compact supports, which amounts to taking the reduced cohomology of the one-point compactification. For differentiable manifolds, where our interest lies, the Čech cohomology groups (with real coefficients) can, by sheaf theory, be identified with the de Rham groups; i.e., the cohomology of the complex of exterior differential forms.

Let G be a compact Lie group. A *characteristic class* of G can be defined as a functor which assigns to every (compact) principal G-bundle P a cohomology class[1] of $X = P/G$ (with coefficients $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$ or $\mathbf{C}$ as the case may be). The set of all characteristic classes forms a ring which we denote by[2] H_G^*, or $H_G^*(A)$ when we want to specify the coefficient ring A. The following facts about H_G^* are known.

(1.1) If $G = T$ a torus, $H_T^2(Z)$ is naturally isomorphic to the character group (or Pontrjagin dual) $\hat{T}$ of T and $H_T^*(A)$ is the (completed) symmetric algebra of $\hat{T}$ over A; thus, if $x_1, \cdots, x_n$ are a basis for $\hat{T}$,

$$H_T^*(A) = A[[x_1, \cdots, x_n]]$$

is the ring of formal power series in $x_1, \cdots, x_n$.

(1.2) If G has maximal torus T and Weyl group W, the natural homomorphism

$$H_G^*(A) \longrightarrow H_T^*(A) \qquad \text{for } A = \mathbf{Q}, \mathbf{R}, \text{ or } \mathbf{C}$$

identifies $H_G^*(A)$ with the invariants of W acting on $H_T^*(A)$.

(1.3) If $G = U(n)$ is the unitary group, then the conclusion of (1.2) also holds for $A = \mathbf{Z}$. Thus, for any A, if $x_1, \cdots, x_n$ denote the standard characters of the maximal torus T of $U(n)$ (so that W is the permutation group S_n) we have

$$\begin{aligned} H_{U(n)}^*(A) &\cong A[[x_1, \cdots, x_n]]^{S_n} \\ &\cong A[[c_1, \cdots, c_n]] , \end{aligned}$$

where c_i is the i^{th} elementary symmetric function of $x_1, \cdots, x_n$.

The characteristic classes c_i of $U(n)$ are called *Chern classes*. Since a principal $U(n)$-bundle defines and (up to isomorphism) is defined by a complex vector bundle of dimension n, we can regard the Chern classes as functors from vector bundles to cohomology. Thus if E is a complex vector bundle over X, we have

$$c_i(E) \in H^{2i}(X; \mathbf{Z}) \qquad i = 1, \cdots, n = \dim E .$$

Introducing an indeterminate t, we define the Chern polynomial

$$c(E) = \sum c_i(E)t^i \qquad (c_0 = 1) .$$

Then one has

$$c(E \oplus F) = c(E) \cdot c(F) . \tag{1.4}$$

For a trivial bundle E, one has $c(E) = 1$. Together with (1.4), this shows that the Chern classes can be defined on $K(X)$.

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[1] We allow a cohomology class to be inhomogeneous, i.e., to be an element of the infinite product $\prod_q H^q(X)$.

[2] The usual notation is $H^{**}(B_G)$ where B_G is the classifying space. For brevity we shall avoid introducing B_G.

The power series

$$\sum_{i=1}^{n} e^{x_i} = n + \sum x_i + \frac{\sum x_i^2}{2!} + \cdots$$

can be expressed uniquely in terms of the elementary symmetric functions c_i. This defines a characteristic class (over $\mathbf{Q}$) called the *Chern character* and denoted by ch. It has the formal properties

$$\operatorname{ch}(E \oplus F) = \operatorname{ch} E + \operatorname{ch} F$$
$$\operatorname{ch}(E \otimes F) = (\operatorname{ch} E)(\operatorname{ch} F) .$$

It extends to a ring homomorphism

$$\operatorname{ch}: K(X) \longrightarrow H^*(X; \mathbf{Q}) .$$

Suppose now that G has maximal torus S, and that $\rho: G \to U(n)$ is a representation of G with $\rho(S) \subset T$ the standard maximal torus of $U(n)$. Then ρ induces a homomorphism

$$\hat{\rho}: \hat{T} \longrightarrow \hat{S}$$

and the elements $y_i = \hat{\rho}(x_i) \in \hat{S}$ are called the weights of ρ. The naturality of the isomorphism (1.2) implies

(1.5) If $\rho^*: H^*_{U(n)} \to H^*_G$ is the homomorphism, induced by ρ, then

$$\rho^* c = \prod (1 + y_i t)$$
$$\rho^* \operatorname{ch} = \sum e^{y_i} ,$$

where $c = \sum c_i t^i$, and t is an indeterminate. If M is the G-module defined by ρ (i.e., $M = \mathbf{C}^n$ with action $g(x) = \rho(g)x$), we shall also write $\operatorname{ch} M$ for the characteristic class

$$\rho^* \operatorname{ch} = \sum e^{y_i} .$$

Then $M \mapsto \operatorname{ch} M$ defines a ring homomorphism

$$\operatorname{ch}: R(G) \longrightarrow H^*_G(\mathbf{Q}) ,$$

the "universal" Chern character. If T is the maximal torus of G, we have a commutative diagram

$$\begin{tikzcd} R(G) \arrow[r, "\operatorname{ch}"] \arrow[d] & H^*_G(\mathbf{Q}) \arrow[d] \\ R(T) \arrow[r, "\operatorname{ch}"] & H^*_T(\mathbf{Q}) . \end{tikzcd}$$

For the torus T, we have $R(T) = \mathbf{Z}[\hat{T}]$ (the integral group ring of $\hat{T}$), and the Chern character is the ring homomorphism defined by

$$x_i \longmapsto e^{x_i} ,$$

where $x_1, \cdots, x_n$ are a basis for $\hat{T}$.

If E is a real vector bundle over X, the *Pontrjagin classes*

$$p_i(E) \in H^{4i}(X; \mathbf{Z})$$

are defined by

$$p_i(E) = (-1)^i c_{2i}(E \otimes_{\mathbf{R}} \mathbf{C}) .$$

Using (1.3) and (1.5), one finds

$$H^*_{O(n)}(A) = A[[p_1, \cdots, p_m]] \qquad m = \left[\frac{n}{2}\right], A = \mathbf{Q}, \mathbf{R}, \text{ or } \mathbf{C} .$$

If $x_1, \cdots, x_m$ are the basic characters of the standard maximal torus of $O(n)$, the p_i are given by

$$\begin{aligned} \sum (-1)^i p_i t^{2i} &= \prod_1^m (1 + x_i t)(1 - x_i t) \\ &= \prod (1 - x_i^2 t^2) , \end{aligned}$$

so that the p_i are the elementary symmetric functions of $x_1^2, \cdots, x_m^2$. For the special orthogonal group $SO(2m)$ there is, in addition, a further invariant of the Weyl group, namely

(1.6) $$e = \prod_1^m x_i .$$

Like the p_i, this is in fact the image of an integral characteristic class called the *Euler class*. This may be defined as follows.

Let E be an oriented real vector bundle of dimension n over X. Then we have a Thom isomorphism

(1.7) $$\psi \colon H^*(X; \mathbf{Z}) \longrightarrow H^*(E; \mathbf{Z}) ,$$

given by $\psi(u) = u \cdot \psi(1)$. Thus $H^*(E; \mathbf{Z})$ is a free $H^*(X; \mathbf{Z})$-module on the one generator $\psi(1)$. If $n = 2m$, then the Euler class $e(E)$ is defined by

(1.8) $$e(E) = i^*\psi(1) \qquad \in H^n(X; \mathbf{Z}) ,$$

where $i^*\colon H^*(E; \mathbf{Z}) \to H^*(X; \mathbf{Z})$ is induced by the zero-section $i\colon X \to E$.

If V is a complex vector bundle of dimension m over X, and E is the underlying real oriented vector bundle, then one has

(1.9) $$c_m(V) = e(E) .$$

In rational cohomology, this follows at once from (1.3), (1.6), and the fact that $U(m)$ and $SO(2m)$ have the same maximal torus. The result for integer cohomology then follows from the fact (1.3) that

$$H^*_{U(n)}(\mathbf{Z}) \longrightarrow H^*_{U(n)}(\mathbf{Q})$$

is injective.

We conclude this survey of characteristic classes with a quick look at the role of curvature. We suppose now that we are in the differentiable framework, i.e., all spaces are differentiable manifolds and all bundles are differentiable. Let G be a compact Lie group, $\mathfrak{g}$ its Lie algebra, T a maximal torus of G, $\mathfrak{t}$ its Lie algebra. Then the inclusion $\mathfrak{t} \to \mathfrak{g}$ induces an isomorphism

$$S(\mathfrak{g}^*)^G \longrightarrow S(\mathfrak{t}^*)^W \tag{1.10}$$

from the polynomials on $\mathfrak{g}$ invariant under the adjoint action of G, onto the polynomials on $\mathfrak{t}$ invariant under the Weyl group W. $S(\mathfrak{t}^*)$ is just the polynomial algebra over $\mathbf{R}$ generated by $\hat{T}$, and so by (1.2)

$$S(\mathfrak{t}^*)^W \cong H_G^*(\mathbf{R}) \,. \tag{1.11}$$

Thus (1.10) and (1.11) imply that the real characteristic ring of G may be identified with the invariant polynomials on its Lie algebra.

Suppose now that P is a principal G-bundle over a compact manifold X. Let α be a connection on P. Then one can define the curvature $\theta(\alpha)$ of α. This is an exterior differential 2-form on X with coefficients in the Lie algebra bundle. More precisely, let $\mathfrak{P} = P \times_G \mathfrak{g}$ be the vector bundle associated with P by the adjoint action of G on $\mathfrak{g}$. Then, for any x,

$$\theta(\alpha)_x \in \Omega_x^2 \otimes \mathfrak{P}_x \,,$$

where Ω^2 is the bundle of 2-forms on X.

If $f \in S(\mathfrak{g}^*)^G$ is an invariant polynomial of degree k, then $f(\theta(\alpha))$ is a well-defined exterior differential form on X of degree $2k$. It is closed, and its (de Rham) cohomology class is independent of the choice of the connection α. Thus we have a differentiable characteristic class

$$[f(\theta(\alpha))] \in H^{2k}(X; \mathbf{R})$$

for each $f \in S(\mathfrak{g}^*)^G$. Except for a numerical factor (involving π), this is precisely the characteristic class corresponding to f by the isomorphism

$$S(\mathfrak{g}^*)^G \cong H_G^*(\mathbf{R}) \,.$$

Thus, in the differentiable situation, characteristic classes can be represented by explicit differential forms constructed out of curvature.

For manifolds, the Thom isomorphism can be re-interpreted in a somewhat simpler manner. Assume that X is an oriented manifold of dimension N, and let E be an oriented real vector bundle over X of dimension n. Then we have the Poincaré duality isomorphisms

$$H^q(X; \mathbf{Z}) \cong H_{N-q}(X; \mathbf{Z})$$
$$H^p(E; \mathbf{Z}) \cong H_{N+n-p}(E; \mathbf{Z}) \,,$$

where the homology is singular homology. The Thom isomorphism then transforms into the homology isomorphism

$$H_{N-q}(X; \mathbf{Z}) \longrightarrow H_{N-q}(E; \mathbf{Z})$$

induced by the zero-section inclusion $X \to E$ (which is, of course, a homotopy equivalence).

Note that the singular homology of X is homology with compact support, but there is also homology with arbitrary support, and the fundamental class $[X]$ belongs to this latter group. If $u \in H^N(X; \mathbf{Z})$, we can evaluate u on $[X]$. For real coefficients if u is represented by a closed N-form with compact support, this value is simply the integral

$$u[X] = \int_X u .$$

If $Du \in H_0(X; \mathbf{Z})$ is the dual homology class, then one has

$$u[X] = \varepsilon(Du) ,$$

where the augmentation $\varepsilon: H_0(X; \mathbf{Z}) \to \mathbf{Z}$ is induced by the projection $X \to$ point. The interpretation of the Thom isomorphism *via* Poincaré duality then implies

$$\psi(u)[E] = u[X] \qquad u \in H^n(X; \mathbf{Z}) . \tag{1.12}$$

2. Cohomological form of the index theorem[3]

We begin by comparing the Thom isomorphisms

$$\varphi_V: K(X) \longrightarrow K(V)$$
$$\psi_V: H^*(X; \mathbf{Q}) \longrightarrow H^*(V; \mathbf{Q})$$

of K-theory and cohomology, where V denotes a complex vector bundle over X. We suppose first that X is compact so that $K(X)$ has an identity element 1. We can then consider the cohomology class

$$\mu(V) = \psi_V^{-1} \operatorname{ch} \varphi_V(1) \qquad \in H^*(X; \mathbf{Q}) .$$

Because φ and ψ are both natural,

$$V \longmapsto \mu(V)$$

is a functor from complex vector bundles to rational cohomology. It is therefore given by an element μ of the characteristic ring (cf. (1.3))

$$H^*_{U(n)}(\mathbf{Q}) = \mathbf{Q}[[c_1, \cdots, c_n]] = \mathbf{Q}[[x_1, \cdots, x_n]]^{S_n} .$$

Now if $i^*: H^*(E; \mathbf{Q}) \to H^*(X; \mathbf{Q})$ is the restriction homomorphism, then for any $u \in H^*(X; \mathbf{Q})$, we have by (1.8)

$$i^*\psi_V(u) = i^*(u \cdot \psi_V(1)) = ui^*(\psi_V(1)) = u \cdot e(V) .$$

[3] The calculations in this section are essentially the same as those in [4] and are, by now, fairly routine.

Taking $u = \mu(V)$, and recalling that the fundamental class

$$\lambda_V = \varphi(1) \in K(E)$$

restricts to

$$\lambda_{-1}(V) = \sum (-1)^i \lambda^i(V) \qquad \in K(X) ,$$

we get

$$\begin{aligned} \mu(V)\cdot e(V) &= i^*\psi_V \cdot \psi_V^{-1} \operatorname{ch} \varphi_V(1) \\ &= i^* \operatorname{ch} \lambda_V \\ &= \operatorname{ch} \lambda_{-1}(V) . \end{aligned}$$

Since this holds for all V, we have an identity

$$\mu\cdot e = \operatorname{ch} \lambda_{-1} \in H^{*}_{U(n)}(\mathbf{Q}) . \tag{2.1}$$

By (1.9) we have $e = c_n$, and by (1.5) we have

$$\operatorname{ch} \lambda_{-1} = \prod (1 - e^{x_i}) ,$$

Since $c_n = \prod x_i$ is not a zero divisor in $H^{*}_{U(n)}(\mathbf{Q})$, we deduce

$$\mu = \prod \left(\frac{1 - e^{x_i}}{x_i}\right) . \tag{2.2}$$

If we expand (2.2) in terms of the elementary symmetric functions c_i of $x_1, \cdots, x_n$, we find $\mu = \sum \mu_k$ where the μ_k are explicit polynomials in the c_i. For any V, we then have $\mu(V) = \sum \mu_k(c_1(V), \cdots, c_k(V))$. Note that $\mu_0(V) = (-1)^n$; and so if V is trivial, $\mu(V) = (-1)^n$.

Since φ, ψ, and ch are all module homomorphisms, it follows that, for any $u \in K(X)$,

$$\psi_V^{-1} \operatorname{ch} \varphi_V u = \operatorname{ch} u \cdot \mu(V) . \tag{2.3}$$

More generally the same proof shows that (2.3) also holds for $u \in K(X, Y)$ where Y is a closed subspace of X. Suppose now that X, Y are differentiable manifolds with X compact, and let $i\colon X \to Y$ be an embedding with normal bundle N. Then, as in [7; § 3] we can identify TN with the complex vector bundle $\pi^*(N \otimes_{\mathbf{R}} \mathbf{C})$ over TX, where $\pi\colon TX \to X$ is the projection. The homomorphism $i_!\colon K(TX) \to K(TY)$ of [7; § 3] is defined as the composition

$$K(TX) \xrightarrow{\varphi} K(TN) \xrightarrow{k_*} K(TY) ,$$

where φ is the Thom homomorphism, and k_* is the natural map induced by the open inclusion $k\colon TN \to TY$ (N being identified with a tubular neibourhood of X in Y). Applying the relative form of (2.3) to an element

$$u \in K\big(B(X), S(X)\big) = K(TX)$$

($B(X)$ and $S(X)$ denoting the unit ball and unit sphere bundles of TX), and with $V = \pi^*(N \otimes_{\mathbf{R}} \mathbf{C})$, we get

$$\mathrm{ch}\, \varphi_V(u) = \psi_V(\mathrm{ch}\, u \cdot \mu(N \otimes_{\mathbf{R}} \mathbf{C})) \,. \tag{2.4}$$

Here we regard $H^*(TX)$ as a module over $H^*(X)$ in the usual way. For any manifold X we can regard TX as an almost complex manifold, on the same lines as in [7; § 3]. Locally we have

$$X \cong \mathbf{R}^n, \qquad TX \cong T\mathbf{R}^n \cong \mathbf{C}^n$$

the last isomorphism being given by $(x, \xi) \mapsto x + i\xi$ where $x \in \mathbf{R}^n$ and $\xi \in T_x$. In particular this gives TX a definite orientation and so defines a fundamental class. Note that, if $X \subset Y$, the orientation of the normal bundle of TX in TY induced by our orientation of TX and TY coincides with the orientation of the complex vector bundle $\pi^*(N \otimes_{\mathbf{R}} \mathbf{C})$. If we now evaluate (the top-dimensional component of) (2.4) on the fundamental class of TN and use (1.12), we obtain

$$\mathrm{ch}\, \varphi_V(u)[TN] = \mathrm{ch}\, u \cdot \mu(N \otimes_{\mathbf{R}} \mathbf{C})[TX] \,. \tag{2.5}$$

By naturality, on the other hand, we have

$$\begin{aligned} \mathrm{ch}\, \varphi_V(u)[TN] &= \mathrm{ch}\, k_*\, \varphi_V(u)[TY] \\ &= \mathrm{ch}\, i_!(u)[TY] \,. \end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6) therefore we obtain

$$\mathrm{ch}\, i_!(u)[TY] = \mathrm{ch}\, u \cdot \mu(N \otimes_{\mathbf{R}} \mathbf{C})[TX] \,. \tag{2.7}$$

We now consider the two inclusions $i: X \to E$, $j: P \to E$ where E is a euclidean space. Let $\dim X = n$, $\dim E = n + q$, then the trivial case of (2.7) with i replaced by j gives

$$\begin{aligned} \mathrm{ch}\, j_!(v)[TE] &= (-1)^{n+q}\, \mathrm{ch}\, v[TP] \\ &= (-1)^{n+q} v \qquad\qquad \text{for } v \in K(P) \cong \mathbf{Z} \,. \end{aligned}$$

In other words, the inverse of the isomorphism $j_!$ is given by

$$j_!^{-1}(w) = (-1)^{n+q}\, \mathrm{ch}\, w[TE] \qquad\qquad w \in K(TE) \,. \tag{2.8}$$

Recalling that the *topological index*

$$\text{t-ind}\colon K(TX) \longrightarrow \mathbf{Z}$$

was defined [7; § 3] by $\text{t-ind} = j_!^{-1} \circ i_!$, (2.7) and (2.8) yield the formula

$$\text{t-ind}\, u = (-1)^{n+q}\, \mathrm{ch}\, u \cdot \mu(N \otimes_{\mathbf{R}} \mathbf{C})[TX] \,. \tag{2.9}$$

It is convenient to express (2.9) in terms of the tangent bundle T rather than the normal bundle N of X. For this, we observe that, as a consequence of (2.2), we have

$$\mu(E \oplus F) = \mu(E)\mu(F)$$

for any two complex vector bundles. Moreover

$$\mu(E) = \pm 1 + \text{higher terms} \qquad \in H^*(X; \mathbf{Q}) ,$$

and so is invertible. Moreover for a trivial bundle W, we have $\mu(W) = (-1)^{\dim W}$. Thus

$$\mu(N \otimes_{\mathbf{R}} \mathbf{C}) = \mu(T \otimes_{\mathbf{R}} \mathbf{C})^{-1} = \mu^{-1}(T \otimes_{\mathbf{R}} \mathbf{C})$$

where μ^{-1} is the characteristic class of $U(n)$ given by

$$\mu^{-1} = \prod_1^n \frac{x_i}{1 - e^{x_i}} .$$

Let us now write

$$\mathcal{T}^* = (-1)^n \mu^{-1} = \prod_1^n \frac{-x_i}{1 - e^{x_i}}$$

$$\mathcal{T} = \prod \frac{x_i}{1 - e^{-x_i}} .$$

The class $\mathcal{T}$ is called the *Todd class*, and $\mathcal{T}^*$ may be called the dual Todd class. In fact, if V^* is the dual of V, then we have

$$\mathcal{T}^*(V) = \mathcal{T}(V^*) .$$

In particular, when $V = E \otimes_{\mathbf{R}} \mathbf{C}$ is the complexification of a real bundle, then $V \cong V^*$ and so $\mathcal{T}(V) = \mathcal{T}^*(V)$.

The functor $E \mapsto \mathcal{T}(E \otimes_{\mathbf{R}} \mathbf{C})$ defines a characteristic class of $O(n)$, the image of $\mathcal{T}$ in the homomorphism

$$H^*_{U(n)}(\mathbf{Q}) \longrightarrow H^*_{O(n)}(\mathbf{Q}) .$$

We shall call this class the *Index class* and write it as $\mathcal{I}$. Thus

$$\mathcal{I}(E) = \mathcal{T}(E \otimes_{\mathbf{R}} \mathbf{C}) .$$

If $y_1, \cdots, y_m$ are the usual basic characters for the maximal torus of $O(n)$ (where $m = [n/2]$) then (1.5) shows that

$$\mathcal{I} = \prod \frac{-y_i}{1 - e^{y_i}} \prod \frac{y_i}{1 - e^{-y_i}} . \tag{2.10}$$

Expressing this in terms of the elementary symmetric functions p_i of $y_1^2, \cdots, y_m^2$, we obtain an expansion

$$\mathcal{I} = \sum \mathcal{I}_{k,m}(p_1, \cdots, p_k)$$

where the $\mathcal{I}_{k,m}$ are explicit polynomials of weight k. Moreover $\mathcal{I}_{k,m}$ is independent of m provided $k \leqq m$. If we write $\mathcal{I}_k$ for this common value, then we always have

$$\mathcal{I} = \sum \mathcal{I}_k(p_1, \cdots, p_k) = 1 - \frac{p_1}{12} \cdots$$

provided we agree to put $p_k = 0$ for $k > m$. This is standard in this sort of formalism.

Finally, we follow the usual convention, and write

$$p_i(X) = p_i(TX) \ ,$$

calling these the Pontrjagin classes of X. Correspondingly we define the Index class of X by

$$\mathcal{I}(X) = \mathcal{I}(TX) = \sum_k \mathcal{I}_k(p_1(X), \cdots, p_k(X)) \ .$$

With this notation, (2.9) gives the following cohomological formula for the topological index.

PROPOSITION (2.11). *Let X be a compact differentiable manifold of dimension n and let the Index class*

$$\mathcal{I}(X) = \sum_k \mathcal{I}_k(p_1(X), \cdots, p_k(X)) \qquad \in H^*(X; \mathbf{Q})$$

be defined as the Todd class of the complexification of the tangent bundle. Let

$$\text{t-ind}\colon K(TX) \longrightarrow \mathbf{Z}$$

be the topological index as defined in [7; § 3]. *Then for any $u \in K(TX)$, we have*

$$\text{t-ind}\, u = (-1)^n\{\operatorname{ch} u \cdot \mathcal{I}(X)\}[TX]$$

where, on the right, we evaluate the top-dimentional component of $\operatorname{ch} u \cdot \mathcal{I}(X)$ *on the fundamental homology class of TX, and TX is oriented as an almost complex manifold with "horizontal part real and vertical part imaginary".*

Now the main theorem (6.7) of [7], for the special case when the group G is reduced to one element, asserts that

$$\text{a-ind} = \text{t-ind} \ .$$

Combined with (2.11), this then gives the cohomological form of the index theorem.

INDEX THEOREM (2.12). *Let P be an elliptic operator over a compact manifold X, and let $u \in K(TX)$ be the symbol class of P. Then the index of P is given by*

$$\text{index}\, P = (-1)^n\{\operatorname{ch} u \cdot \mathcal{I}(X)\}[TX]$$

where $\mathcal{I}(X)$ is the Index class of X, TX is oriented as in (2.11) *and* $n = \dim X$.

Remarks. 1. The factor $(-1)^n$ could have been eliminated by giving TX

the dual almost complex structure, as in [19]. There are reasons however to prefer the convention we have adopted here.

2. As explained in [7; § 7], we can in (2.12) replace the single operator P by a complex E and the index of P by the Euler characteristic of E. We shall do this in future.

If X is an oriented manifold, then the evaluation on TX in (2.12) can be replaced by an evaluation on X by using (1.12). We have to watch sign conventions however because the orientation of TX induced from that of X differs from our orientation. In fact, to pass from one fundamental class to the other, we need a sign $(-1)^{n(n-1)/2}$. Thus if $\psi\colon H^*(X) \to H^*(TX)$ is the Thom isomorphism[4] (2.12) can be replaced by

$$\text{(2.13)} \qquad \operatorname{index} P = (-1)^{n(n+1)/2}\{(\psi^{-1} \operatorname{ch} u)\cdot \mathcal{J}(X)\}[X] \,.$$

This is now the form of the index theorem announced in [6], except for notation and for conventions which were not made explicit in [6].

If X is not orientable, then a formula like (2.13) can also be written down, but we must use twisted coefficients, both for the fundamental class of X and for $\psi^{-1} \operatorname{ch} u$.

In the remaining sections we shall apply (2.13) to a number of particularly interesting elliptic complexes associated with some geometrical structure on X. All these are in fact examples of operators associated to an H-structure and, for these, (2.13) can be made even more explicit. We proceed to describe this class of operators.

Let H be a compact Lie group, V a fixed real oriented H-module. Then an *H-structure* on X will mean a principal H-bundle P over X together with an isomorphism (of oriented bundles)

$$P \times_H V \cong T(X)$$

where $P \times_H V$ denotes, as usual, the vector bundle over X associated to P by the H-module V. We then have a natural homomorphism

$$\alpha_P\colon K_H(V) \to K_H(P \times V) = K(TX) \,.$$

If $v \in K_H(V)$, the image

$$u = \alpha_P(v) \in K(TX)$$

may be called an elliptic symbol class associated to the H-structure. The element v, which may be called a "universal" elliptic symbol class for H-structures, usually arises in the following way. Let $M^0, \cdots, M^n$ be a sequence of complex H-modules, and let

[4] For n odd, it is important to specify whether the Thom isomorphism is $u \mapsto u\psi(1)$ or $u \mapsto \psi(1)u$; we have chosen the first alternative, hence the sign $(-1)^{n(n-1)/2}$.

$$\varphi_i\colon V \longrightarrow \operatorname{Hom}(M^i, M^{i+1})$$

be an H-map[5] so that for $\xi \in V$, $\xi \neq 0$,

$$0 \longrightarrow M^0 \xrightarrow{\varphi_0(\xi)} M^1 \longrightarrow \cdots \xrightarrow{\varphi_{n-1}(\xi)} M^n \longrightarrow 0$$

is exact. This sequence may be viewed as a complex of H-vector bundles on V with compact support [7; § 2], and so represents an element $v \in K_H(V)$.

We shall now show how to compute $\operatorname{ch} u$ when u is associated to an H-structure. To do this we consider any principal H-bundle P and put $T = P \times_H V$. Then we observe that

$$P \longmapsto \psi_T^{-1} \operatorname{ch} \alpha_P(v) \tag{2.14}$$

is a functor from principal H-bundles to rational cohomology, and so is given by an element of $H_H^*(\mathbf{Q})$. Now let $\dim V = 2l$, and let $\rho^*(e) \in H_H^*(\mathbf{Q})$ be the image of the Euler class e under the homomorphism

$$\rho^*\colon H^*_{\mathrm{SO}(2l)}(\mathbf{Q}) \longrightarrow H_H^*(\mathbf{Q})$$

induced by $\rho\colon H \to \mathrm{SO}(2l)$. Using (1.8), we then deduce

$$[\psi_T^{-1} \operatorname{ch} \alpha_P(v)]\cdot \rho^*(e)(P) = \operatorname{ch} \alpha_P(i^* v) \tag{2.15}$$

where $i^*\colon K_H(V) \to K_H(\text{point}) = R(H)$ is restriction to the origin, and α_P is also used for the homomorphism

$$\alpha_P\colon R(H) \longrightarrow K(X)$$

induced by $M \mapsto P \times_H M$ (for H-modules M). If $\rho^*(e) \neq 0$ in $H_H^*(\mathbf{Q})$, then (2.15) can be used to express the characteristic class (2.14) in terms of $i^*(v)$. If v is defined by a sequence of H-modules M^i and homomorphisms as described above, then

$$i^*(v) = \sum (-1)^i M^i \in R(H)\ .$$

By (1.5) the functor $P \mapsto \operatorname{ch} \alpha_P M$ gives the characteristic class

$$\operatorname{ch} M = \sum e^{y_i} \in H_H^*(\mathbf{Q})$$

where the y_i are the weights of M. Hence

$$\operatorname{ch} \alpha_P i^*(v) = \sum (-1)^i \operatorname{ch} M^i\ .$$

Using (2.15) and assuming $\rho^*(e) \neq 0$, we then get

$$\psi_T^{-1} \operatorname{ch} \alpha_P(v) = \frac{\sum (-1)^i \operatorname{ch} M^i}{\rho^*(e)}(P)\ . \tag{2.16}$$

To examine the condition $\rho^*(e) \neq 0$, we choose a maximal torus S of H map-

[5] φ_i is not necessarily linear although, as we shall see, the most interesting examples occur with φ_i linear.

ping into the standard maximal torus T of $SO(2l)$. If $x_1, \cdots, x_l \in \hat{T}$ are the basic characters, we have $e = \prod_1^l x_i$, and so $\rho^*(e) = \prod_1^l y_i$ where $y_i \in \hat{S}$ is the character of S induced by x_i. Thus $\rho^*(e) = 0$ is equivalent to the vanishing of one of the y_i or, in other words, to the existence of a fixed non-zero vector for S in R^{2l}.

Inserting (2.16) into the index formula (2.13), we then deduce

PROPOSITION (2.17). *Let $\rho: H \longrightarrow SO(2l)$ be a homomorphism such that the maximal torus of H has no fixed non-zero vector in R^{2l}. Then $\rho^*(e) \neq 0$, where $e \in H^*_{SO(2l)}(\mathbf{Q})$ is the Euler class, and*

$$\rho^*: H^*_{SO(2l)}(\mathbf{Q}) \longrightarrow H^*_H(\mathbf{Q})$$

is induced by ρ. Now let X be a compact oriented manifold of dimension $2l$ with an H-structure, i.e., *we have a principal H-bundle P over X, and TX is associated to P via ρ. Let $M^0, \cdots, M^N$ be complex H-modules, $E^0, \cdots, E^N$ the associated vector bundles over X, and suppose*

$$0 \longrightarrow \mathfrak{D}(E^0) \longrightarrow \cdots \longrightarrow \mathfrak{D}(E^N) \longrightarrow 0$$

is an elliptic complex whose symbol class in $K(TX)$ is associated to the H-structure. Then the index of this complex is given by

$$(-1)^l \left\{ \frac{\sum (-1)^i \operatorname{ch} M^i}{\rho^*(e)} (P) \mathcal{G}(X) \right\} [X] .$$

Remarks. 1. This formula shows that, for H-structures (under the assumptions of (2.17)), the index depends only on the bundles E^i, or rather on the H-modules M^i. In other words, two complexes where the M^i are the same, but the operators are different, have the same index, provided always that the symbol class is associated to the H-structure.

2. The group H in (2.17) should not be confused with the group G in [7; (6.7)]. In (2.17) we are concerned with the ordinary integer-index not the more general character-index. Of course one could envisage an H-structure as in (2.17) invariant under a further group G. In particular, suppose G acts trivially on X, then it follows that the action of G on the principal H-bundle P is given by some homomorphism $\rho: G \longrightarrow$ Centre (H). Let χ^i be the character of G induced by ρ from the character of the H-module M^i. Then the Lefschetz number $L(g, E)$ is given by the formula in (2.17) with $\operatorname{ch} M^i$ replaced by $\chi^i(g) \cdot \operatorname{ch} M^i$. The proof is essentially the same as that of (2.17).

3. Lefschetz fixed-point formula

In Theorem (2.12) of [5] we gave a "Lefschetz fixed-point formula" which computed Lefschetz numbers of automorphisms of elliptic complexes in terms of the index of associated elliptic symbol classes on fixed-point sets. Simply by inserting the cohomological formula for the index given in the preceding section into Theorem (2.12) of [5], we shall then obtain a cohomological form of the Lefschetz fixed-point formula for elliptic complexes.

It will be convenient to adopt the following convention. Let X be a trivial G-space so that

$$K_G(X) \cong K(X) \otimes R(G) ,$$

and let $g \in G$.

The homomorphism

$$K_G(X) \longrightarrow H^*(X; \mathbf{C})$$

given by

$$u = \sum a_i \otimes \chi_i \longmapsto \sum \chi_i(g) \cdot \mathrm{ch}\, a_i$$

will be denoted by ch $u(g)$. With this notation, Theorem (2.12) of [5] and (2.11) of this paper, give the formula

$$L(g, E) = (-1)^n \left\{ \frac{\mathrm{ch}\, i^* u(g)}{\mathrm{ch}\, \lambda_{-1}(N^g \otimes_{\mathbf{R}} \mathbf{C})(g)} \mathscr{I}(X^g) \right\} [TX^g] . \tag{3.1}$$

Here E is an elliptic complex acted on by a topologically cyclic group G, with symbol class $u \in K_G(TX)$, X^g is the fixed point set of a generator g of G, i^* is the restriction to $K_G(TX^g)$, N^g is the normal bundle of X^g in X, and $n = \dim X^g$.

The denominator in (3.1) can be made more explicit as we shall now see. We first examine the action of G on the normal bundle N^g. For each $x \in X^g$, the fibre N^g_x of N^g at x is a real G-module. Since G is cyclic, its irreducible real representations are of two types

(i) one-dimensional with $g \mapsto \pm 1$,

(ii) two-dimensional with

$$g \longmapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} .$$

In (ii) the representations given by θ and $-\theta$ are equivalent, and we may therefore restrict to the case $0 < \theta < \pi$. Such a two-dimensional real G-module has then a canonical complex structure in which g acts as the complex scalar $e^{i\theta}$. The real G-module N^g_x can therefore be written canonically as a direct sum

$$N^g_x = N^g_x(-1) \oplus \sum\nolimits_{0<\theta<\pi} N^g_x(\theta) . \tag{3.2}$$

The eigenvalue $+1$ does not occur, because N^g is normal to the fixed-point set X^g, and of course only a finite number of values of θ can occur (with non-zero space). Each space $N^g_x(\theta)$ has a natural complex structure in which g acts as $e^{i\theta}$. Since the decomposition can be defined by projection operators, (3.2) defines a direct sum decomposition of the vector bundle N^g (cf. [1; (1.6.2)] for the analogous proof in the complex case)

$$N^g = N^g(-1) \oplus \textstyle\sum_{0<\theta<\pi} N^g(\theta) . \tag{3.3}$$

The bundle $N^g(-1)$ is real, and each $N^g(\theta)$ is complex. *Thus $N^g(-1)$ has Pontrjagin classes, and each $N^g(\theta)$ has Chern classes.*

We propose now to express the denominator of formula (3.1) in terms of these characteristic classes. From (3.3), we get

$$N^g \otimes_{\mathbf{R}} \mathbf{C} = \big(N^g(-1) \otimes_{\mathbf{R}} \mathbf{C}\big) \oplus \textstyle\sum_\theta \big(N^g(\theta) \oplus N^g(\theta)^*\big)$$
$$\lambda_{-1}(N^g \otimes_{\mathbf{R}} \mathbf{C}) = \lambda_{-1}(N^g(-1) \otimes_{\mathbf{R}} \mathbf{C})\cdot\textstyle\prod_\theta \lambda_{-1}\big(N^g(\theta)\big)\cdot\prod_\theta \lambda_{-1}\big(N^g(\theta)\big)^* .$$

For the real bundle $N^g(-1)$, we have

$$\operatorname{ch} \lambda_{-1}(N^g(-1) \otimes_{\mathbf{R}} \mathbf{C})(g) = \varepsilon\{\textstyle\prod_{j=1}^r (1 + e^{x_j})(1 + e^{-x_j})\}N^g(-1) , \tag{3.4}$$

where $r = [s(-1)/2]$, $s(-1) = \dim N^g(-1)$, $\varepsilon = 1$ or 2 according as $s(-1)$ is even or odd, and

$$\textstyle\prod_j (1 + e^{x_j})(1 + e^{-x_j}) \in H^*_{O(s(-1))}(\mathbf{C}) .$$

For the complex vector bundle $N^g(\theta)$, we have

$$\operatorname{ch} \lambda_{-1}\big(N^g(\theta)\big)(g) = \{\textstyle\prod_{j=1}^{s(\theta)} (1 - e^{y_j+i\theta})\}N^g(\theta) \tag{3.5}$$

where $s(\theta) = \dim_{\mathbf{C}} N^g(\theta)$ and

$$\textstyle\prod_{j=1}^{s(\theta)} (1 - e^{y_j+i\theta}) \in H^*_{U(s(\theta))}(\mathbf{C}) .$$

Finally, for the dual $N^g(\theta)^*$, we have

$$\operatorname{ch} \lambda_{-1}\big(N^g(\theta)\big)^*(g) = \{\textstyle\prod_{j=1}^{s(\theta)} (1 - e^{-y_j-i\theta})\}N^g(\theta) . \tag{3.6}$$

In order to express these formulas in terms of Pontrjagin and Chern classes, we shall introduce the sequences of polynomials

$$\mathcal{R}_r(p_1, \cdots, p_r) \qquad \mathcal{S}_r^\theta(c_1, \cdots, c_r)$$

defined by the formal identities

$$\begin{aligned} \Big\{\textstyle\prod_j \Big(\frac{1 + e^{x_j}}{2}\Big)\Big(\frac{1 + e^{-x_j}}{2}\Big)\Big\}^{-1} &= \textstyle\sum \mathcal{R}_r(p_1, \cdots, p_r) \\ \Big\{\textstyle\prod_j \Big(\frac{1 - e^{y_j+i\theta}}{1 - e^{i\theta}}\Big)\Big(\frac{1 - e^{-y_j-i\theta}}{1 - e^{-i\theta}}\Big)\Big\}^{-1} &= \textstyle\sum \mathcal{S}_r^\theta(c_1, \cdots, c_r) , \end{aligned} \tag{3.7}$$

where the p_i are the elementary symmetric functions of the x_j^2, the c_i are the elementary symmetric functions of the y_j, and to define the r^{th} term of the sum, we take $\prod_j$ to be a product of r terms. More precisely, we put

$$\textstyle\prod_{j=1}^N = \sum \mathcal{R}_{r,N} ,$$

and observe that $\mathcal{R}_{r,N}$ is independent of N for $r \leqq N$ and similarly for $\sum \mathcal{S}_r^\theta$.

We put

$$\mathcal{R} = \sum \mathcal{R}_r \, , \qquad \mathcal{S}^\theta = \sum \mathcal{S}_r^\theta \, .$$

By (1.3) we can regard these as characteristic classes (over $\mathbf{C}$) of $O(n)$ and $U(n)$ respectively, for any n. For a real vector bundle E over a space Y, we have

$$\mathcal{R}(E) = \mathcal{R}(p_1(E), \cdots, p_n(E)) \qquad \in H^*(Y; \mathbf{C}) \, ,$$

and for a complex vector bundle F over Y,

$$\mathcal{S}^\theta(F) = \mathcal{S}^\theta(c_1(F), \cdots, c_n(F)) \qquad \in H^*(Y; \mathbf{C}) \, .$$

Note that $\mathcal{R}$ and $\mathcal{S}^\theta$ depend only on the stable classes of E, F, and not on the augmentation. Thus

$$\mathcal{R}(E \oplus 1_{\mathbf{R}}) = \mathcal{R}(E) \qquad \mathcal{S}^\theta(F \oplus 1_{\mathbf{C}}) = \mathcal{S}^\theta(F)$$

where $1_{\mathbf{R}}$ and $1_{\mathbf{C}}$ denote trivial line-bundles (real and complex respectively).

If we compare the formulas (3.7), we see that they differ from those in (3.4), (3.5), and (3.6) by a total factor

$$2^{s(-1)} \prod_\theta (1 - e^{i\theta})(1 - e^{-i\theta}) \, .$$

For any $x \in X^g$, this is just the value of

$$\det (1 - g \mid N_x^g) \, .$$

It is constant on each component of X^g, and can be regarded as an element of $H^0(X^g; \mathbf{C})$, we shall write this element as $\det (1 - g \mid N^g)$.

Combining (3.4), (3.5), and (3.6), we therefore get

$$\{\mathrm{ch}\, \lambda_{-1}(N^g \otimes_{\mathbf{R}} \mathbf{C})(g)\}^{-1} = \frac{\mathcal{R}(N^g(-1)) \prod_\theta \mathcal{S}^\theta(N^g(\theta))}{\det (1 - g \mid N^g)} \, . \tag{3.8}$$

Substituting this in (3.1), we finally obtain

LEFSCHETZ THEOREM (3.9). *Let g be a generator of the (topologically cyclic) compact Lie group G, X a compact G-manifold, E an elliptic complex on X on which G acts. Let X^g denote the fixed-point set of g, N^g the normal bundle of X^g in X, and*

$$N^g = N^g(-1) \oplus \sum_{0<\theta<\pi} N^g(\theta)$$

the decomposition of N^g determined by the action of G. Let $u \in K_G(TX)$ be the symbol class of E, $i^*u \in K_G(TX^g)$ its restriction to X^g. Let $\mathcal{I}(X) \in H^*(X; \mathbf{Q})$ denote the Index class of X where $\mathcal{I}$ is defined by (2.10), *and let $\mathcal{R}$, $\mathcal{S}^\theta$ be the characteristic classes of the orthogonal and unitary groups defined by* (3.7). *Then the Lefschetz number $L(g, E)$ is given by*

$$L(g, E) = (-1)^n\left\{\frac{\operatorname{ch} i^*u(g)\mathcal{R}(N^g(-1))\prod_{0<\theta<\pi}\mathcal{S}^\theta(N^g(\theta))\cdot\mathcal{I}(X^g)}{\det(1 - g \mid N^g)}\right\}[TX^g]$$

where $\det(1 - g \mid N^g) \in H^0(X^g; \mathrm{C})$ *assigns to the component of* X^g *containing* x *the value* $\det(1 - g \mid N^g_x)$, *and* n *assigns to each component its dimension.*

Remark. Theorem (3.9) gives a complete cohomological formula for the index character

$$\text{a-ind}: K_G(TX) \longrightarrow R(G)$$

for any compact Lie group G. To evaluate $\chi = \operatorname{ind} u$ on an element $g \in G$, we simply apply (3.9) to the closed cyclic group H generated by g.

The formula for $L(g, E)$ given in (3.9) involves the symbol u of E and the following cohomological invariants of (X, G):

(i) Pontrjagin classes of X^g,

(ii) Pontrjagin classes of $N^g(-1)$,

(iii) Chern classes of all the $N^g(\theta)$.

If X^g is oriented then, as in (2.13), we can replace the evalution on $[TX^g]$, by an evalution on $[X^g]$. We simply replace $(-1)^n \operatorname{ch} i^*(u)(g)$ in (3.9) by

$$(3.10) \qquad (-1)^{\frac{n(n+1)}{2}}\psi^{-1}(\operatorname{ch} i^*(u)(g)) ,$$

where $n = \dim X^g$, and

$$\psi: H^*(X^g; \mathrm{C}) \longrightarrow H^*(TX^g; \mathrm{C})$$

is the Thom isomorphism. If moreover $i^*(u)$ is associated to an H-structure on X^g, as in (2.17), we can compute (3.10) in terms of characteristic classes of this H-structure. This situation will be illustrated by some of the examples we consider in subsequent sections.

We should perhaps point out that in practice, for the "classical" operators, it is usually simpler to go back to the K-theory formulas of [5] rather than apply the formidable cohomological expression in (3.9).

4. The Riemann-Roch theorem

One of the most important examples of an elliptic complex arises in connection with complex manifolds. The index theorem (2.12) becomes in this case the Hirzebruch form of the celebrated Riemann-Roch theorem [13]. In fact this will be a case of an H-structure, and so we will be able directly to apply Proposition (2.17). We proceed to describe the details.

Let X be a compact complex manifold of dimension n, V a holomorphic vector bundle over X, $\mathcal{O}(V)$ the sheaf of germs of holomorphic sections of V. Moreover let $A(V)$ denote the Dolbeault complex of V

$$0 \longrightarrow A^{0,0}(V) \xrightarrow{\bar\partial} A^{0,1}(V) \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} A^{0,n}(V) \longrightarrow 0$$

where $A^{0,p}(V)$ denotes the differential forms of type $(0, p)$ with coefficients in V. The Dolbeault isomorphism [13; § 15] asserts that the cohomology groups of $A(V)$ are isomorphic to the sheaf cohomology groups $H^q(X, \mathcal{O}(V))$, and so $\sum(-1)^q \dim H^q(A(V)) = \sum(-1)^q \dim H^q(X, \mathcal{O}(V))$ is the Euler characteristic $\chi(X, V)$ which one wants to compute in the Riemann-Roch theorem.

The complex $A(V)$ is easily seen to be elliptic. In fact, let TX denote now the *complex* tangent bundle of X and choose a hermitian metric on X, so that we can identify $\bar{T}$ with the bundle of forms of type $(1, 0)$ and so T with the bundle of forms of type $(0, 1)$. Then we see that the symbol sequence of $A(V)$ is just the complex over TX defined by $V \otimes \Lambda^*(T)$, and hence is exact outside the zero section of TX. Moreover this shows that the symbol class $a(V)$ of $A(V)$ is given by

$$a(V) = [V]\lambda_T \in K(TX) , \tag{4.1}$$

where $\lambda_T \in K(TX)$ is the fundamental element, given by $\Lambda^*(T)$, and $K(TX)$ is regarded as a $K(X)$-module. Now λ_T is a symbol associated to the $U(n)$-structure of X (given by the hermitian metric). For the case $V = 1$ therefore, we are precisely in a position to apply (2.17), the M^i being the $U(n)$-modules $\lambda^i(\mathbf{C}^n)$. For the general case we either observe from (4.1) that $\operatorname{ch}(V)$ simply enters (2.12) as a multiplier or we formally give X a $U(n) \times U(m)$ structure where $m = \dim V$, and $U(m)$ acts trivially on X. This makes $V \cdot \lambda_T$ an associated symbol-class, and we can then apply (2.17) directly. Either way we obtain the formula

$$\chi(X, V) = (-1)^n \Big\{\operatorname{ch} V \frac{\prod(1 - e^{x_i})}{\prod x_i}(T) \cdot \mathcal{J}(X)\Big\}[X] \tag{4.2}$$

where

$$\frac{\prod(1 - e^{x_i})}{\prod x_i} \in H^*_{U(n)}(\mathbf{Q})$$

is applied to the complex tangent bundle T, and

$$\begin{aligned}\mathcal{J}(X) &= \mathcal{T}(T \otimes_{\mathbf{R}} \mathbf{C}) = \mathcal{T}(T \oplus T^*) \\ &= \prod\Big(\frac{x_i}{1 - e^{-x_i}} \cdot \frac{-x_i}{1 - e^{x_i}}\Big)(T) .\end{aligned}$$

Thus cancelling the factor $\prod x_i(1 - e^{-x_i})^{-1}$, we end up finally with

RIEMANN-ROCH THEOREM (4.3). *Let X be a compact complex manifold, V a holomorphic vector bundle over X. Let*

$$\mathcal{T}(X) = \prod \frac{x_i}{1 - e^{-x_i}}(T)$$

be the Todd class of X. Then the Euler characteristic $\chi(X, V)$ of the sheaf of germs of holomorphic sections of V is given by

$$\chi(X, V) = \{\mathrm{ch}\, V \cdot \mathcal{T}(X)\}[X] .$$

Remarks. 1. This theorem was previously known only for projective algebraic manifolds [13]. The proof we have given, using general elliptic operators, is the only one known at present for the general case.

2. For a complex manifold X, we know, by the Thom isomorphism in K-theory, that $K(TX)$ is a free $K(X)$-module generated by λ_T. In other words, there is a single elliptic symbol which, in a sense, generates everything. It is thus natural to replace the index homomorphism

$$K(TX) \longrightarrow \mathbf{Z}$$

of the general theory by the homomorphism

$$K(X) \longrightarrow \mathbf{Z}$$

given by $v \mapsto \mathrm{ind}\,(v\lambda_T)$. This is in effect what is usually done in algebraic geometry.

If now G is a finite[6] group of automorphisms of the pair (X, V), we can apply our general Lefschetz theorem (3.9). Alternatively we can return to Theorem (3.3) of [5] and combine that with the Riemann-Roch theorem (4.3). This gives

$$\sum (-1)^p \,\mathrm{Trace}\,(g \mid H^p(X, \mathcal{O}(V))) = \left\{\frac{\mathrm{ch}\,(V \mid X^g)(g)\mathcal{T}(X^g)}{\mathrm{ch}\,\lambda_{-1}((N^g)^*)(g)}\right\}[X^g] . \tag{4.4}$$

Now the complex vector bundle N^g has a decomposion

$$N^g = \sum N^g(\theta) ,$$

where $N^g(\theta)$ is the sub-bundle on which g acts as $e^{i\theta}$. Note that this notation differs slightly from that adopted in § 3 where N^g was a real bundle. Each $N^g(\theta)$ is a complex vector bundle and has Chern classes. Moreover we have

$$\mathrm{ch}\,\lambda_{-1}(N^g(\theta))^* = \prod_j (1 - e^{-x_j - i\theta})(N^g(\theta)) ,$$

where

$$\prod_j (1 - e^{-x_j - i\theta}) \in H^*_{U(m)}(\mathrm{C}) , \qquad m = \dim N^g(\theta) .$$

Let us therefore introduce for $0 < \theta < 2\pi$ the stable characteristic class

$$\mathcal{U}^\theta = \sum \mathcal{U}^\theta_r = \left\{\prod_j \left(\frac{1 - e^{-x_j - i\theta}}{1 - e^{-i\theta}}\right)\right\}^{-1} . \tag{4.5}$$

[6] This is perhaps more natural in the holomorphic case.

Each $\mathcal{U}_r^\theta$ is thus a polynomial with complex coefficients in the Chern classes. With this notation we have

$$\{\mathrm{ch}\,\lambda_{-1}(N^g(\theta))^*\}^{-1} = \frac{\mathcal{U}^\theta(N^g(\theta))}{(1-e^{-i\theta})^m}\,.$$

Thus, taking the product over all θ,

$$\{\mathrm{ch}\,\lambda_{-1}(N^g)^*\}^{-1} = \frac{\prod \mathcal{U}^\theta(N^g(\theta))}{\det_{\mathbf{C}}(1-g(N^g)^*)}\,,$$

where $\det_{\mathbf{C}}(1-g\mid(N^g)^*)\in H^0(X^g;\mathbf{C})$ assigns to the component of $x\in X^g$ the value $\det_{\mathbf{C}}(1-g\mid(N_x^g)^*)$. Substituting this in (4.4), we finally obtain.

HOLOMORPHIC LEFSCHETZ THEOREM (4.6). *Let X be a compact complex manifold, V a holomorphic vector bundle over X, and let G be a finite group of automorphisms of the pair (X, V). For any $g\in G$, let X^g denote the fixed point set of g, and let*

$$N^g = \sum N^g(\theta)$$

denote the (complex) normal bundle of X^g decomposed according to the eigenvalues $e^{i\theta}$ of g. Let $\mathcal{U}^\theta$ denote the characteristic class defined by (4.5). *Then we have*

$$\sum(-1)^p\,\mathrm{Trace}\,(g\mid H^p(X,\mathcal{O}(V)))$$
$$= \left\{\frac{\mathrm{ch}\,(V\mid X^g)(g)\cdot\prod_\theta \mathcal{U}^\theta(N^g(\theta))\cdot\mathcal{T}(X^g)}{\det(1-g\mid(N^g)^*)}\right\}[X^g]\,.$$

Using this and Theorem (3.5) of [5], we are then led to

RIEMANN-ROCH THEOREM FOR ORBIT SPACES (4.7). *Let G be a finite group of automorphisms of a compact complex manifold X, and let W be a holomorphic vector bundle over the complex space $Y = X/G$. Then we have*

$$\chi(Y, W) = \frac{1}{|G|}\sum_{g\in G}\mu(g)$$

where

$$\mu(g) = \left\{\frac{\mathrm{ch}\,(f^*W\mid X^g)(g)\prod_\theta \mathcal{U}^\theta(N^g(\theta))\mathcal{T}(X^g)}{\det_{\mathbf{C}}(1-g\mid(N^g)^*)}\right\}[X^g]$$

and $N^g(\theta)$, $\mathcal{U}^\theta$ are as in (4.6).

In [14] Hirzebruch used the Riemann-Roch theorem to compute the dimensions of certain spaces of automorphic forms. The extension (4.7) of the Riemann-Roch theorem to certain singular spaces can be used in the same way and leads to formulas extending those of [14] (cf. Langlands [16]). This will be explained elsewhere (see also [15]).

Theorem (4.6) implies certain "integrality conditions" for the fixed-point sets of G. In fact, the Lefschetz number is an algebraic integer, while the expression on the right-hand side of the formula in (4.6) is *a priori* only an algebraic number. The special case of isolated fixed-points is discussed in [2], and already this shows that these integrality theorems are non-trivial. Of course not every function from G to algebraic integers is a character. Thus the integrality theorems arising from (4.6) for a general finite group G are not just consequences of those for cyclic groups.

As a simple example of (4.6), let us work out the case when $\dim X = 2$, i.e., when X is a complex surface. Assuming that g acts non-trivially, and that X is connected, the fixed point set X^g consists of a finite set of points P_j and curves D_k. We shall, for simplicity, only work out the case when $V = 1$. Thus the Lefschetz number $L(g)$ will be given by

$$L(g) = \sum_j a(P_j) + \sum_k b(D_k) ,$$

where the numbers $a(P)$ and $b(D)$ are given as follows:

$$a(P) = \frac{1}{\det(1 - g \mid T_P)}$$

$$b(D) = \left\{\frac{1 + c/2}{1 - e^{-d-i\theta}}\right\}[D]$$

where c is the first Chern class of D, d is the first Chern class of the normal bundle of D, and $e^{i\theta}$ is the eigenvalue of g on the normal bundle. Evaluating $b(D)$ we get

$$\begin{aligned} b(D) &= \left\{\frac{1 + c/2}{1 - e^{-i\theta}(1 - d)}\right\}[D] \\ &= \left\{(1 - e^{-i\theta})^{-1}\Big(1 + \frac{c}{2}\Big)\Big(1 + \frac{de^{-i\theta}}{1 - e^{-i\theta}}\Big)^{-1}\right\}[D] \\ &= (1 - e^{-i\theta})^{-1}\left\{\frac{c[D]}{2} - \frac{e^{-i\theta}}{1 - e^{-i\theta}}d[D]\right\} . \end{aligned}$$

But $c[D] = 2(1 - \pi_D)$, where π_D is the genus of D, and $d[D] = D^2$ is the self-intersection number of D. Thus

$$b(D) = \frac{1 - \pi_D}{1 - e^{-i\theta}} - \frac{e^{-i\theta}}{(1 - e^{-i\theta})^2}D^2 .$$

In particular, if $g^2 = 1$, then $e^{i\theta} = -1$, $a(P) = 1/4$, $b(D) = (1 - \pi_D)/2 + (1/4)D^2$. Thus, as a very simple special case of (4.6), we get

PROPOSITION (4.8). *Let g be a non-trivial involution of a connected complex surface. Let the fixed-point set of g consist of N isolated points and M irreducible curves $\{D_k\}$. Then the holomorphic Lefschetz number $L(g)$ is* *given by*

$$L(g) = \frac{N}{4} + \sum_{k=1}^{M}\left(\frac{1-\pi_k}{2} + \frac{D_k^2}{4}\right),$$

where π_k *is the genus of* D_k *and* D_k^2 *is its self-intersection number.*

Remark. If X is algebraic, then the holomorphic differentials are birational invariants. Thus if we blow up an isolated fixed-point, g will induce $\tilde{g}$ on $\tilde{X}$, and $L(g) = L(\tilde{g})$. This checks with the formula in (4.8) because we lose one point and gain a rational curve with self-intersection -1. Another simple check is to take the involution $(x_0, x_1, x_2) \mapsto (-x_0, x_1, x_2)$ of the projective plane. There is one isolated fixed point, and one fixed line (so $\pi = 0$) with self-intersection $+1$. Thus $L(g) = 1/4 + 1/2 + 1/4 = 1$ which is correct, since there are no holomorphic differentials in positive dimension.

Since the Lefschetz number of an involution is always an integer, (4.8) implies that the number of curves D_k with odd self-intersection number has the same parity as N. This is a simple example of an "integrality theorem" involving higher-dimensional fixed-point sets.

5. The Dirac operator

An interesting elliptic operator called the Dirac[7] operator exists on spin-manifolds. We proceed to discuss this in detail and to examine the general theorems of §§ 2, 3 in this special case.

We recall that $SO(n)$ has a double covering Spin (n) which is its universal covering for $n > 2$. A spin-structure on an oriented compact manifold X of dimension n will mean a Spin (n)-structure in the sense of § 2. Thus we assume given a principal Spin (n)-bundle P over X and an isomorphism of oriented bundles

$$P \times_{\mathrm{Spin}(n)} \mathbf{R}^n \cong TX .$$

Equivalently P is a double covering of the princical $SO(n)$-bundle Q of X (for some riemannian metric) such that P_x is the Spin double covering of Q_x.

The group Spin (n) has a complex representation space Δ of dimension 2^n called the spin-representation[8]. Moreover Δ is a module for the Clifford algebra of $\mathbf{R}^n$ for the negative definite form $-\sum_1^n x_i^2$. In fact Spin (n) is defined as a subgroup of the group of units of the Clifford algebra, and the action of Spin (n) is induced by the Clifford multiplication. If $g \in$ Spin (n),

[7] Of course the original Dirac operator is defined for the indefinite relativistic metric and is not elliptic.

[8] For results about Spin groups and Clifford algebras, the reader may refer to [3] and [8].

$x \in \mathbf{R}^n$, $u \in \Delta$, we have therefore

$$g(xu) = gxg^{-1} \cdot g(u) = \rho(g)x \cdot g(u) ,$$

where $\rho: \mathrm{Spin}(n) \to SO(n)$ is the covering map. Thus the Clifford multiplication

$$\mathbf{R}^n \otimes \Delta \longrightarrow \Delta$$

is a homomorphism of Spin (n)-modules. If $n = 2l$ the representation Δ is the direct sum of two irreducible representations $\Delta^{\pm}$ of dimension 2^{n-1}. The decomposition $\Delta = \Delta^+ \oplus \Delta^-$ is interchanged by Clifford multiplication. Thus we have homomorphisms

$$\mathbf{R}^n \otimes \Delta^+ \longrightarrow \Delta^-$$
$$\mathbf{R}^n \otimes \Delta^- \longrightarrow \Delta^+ .$$

If $x_1, \cdots, x_l$ are the basic characters of the maximal torus T_0 of $SO(2l)$, they can also be considered as characters of the maximal torus T of Spin $(2l)$. Since T double covers T_0, the character group $\hat{T}_0$ is of index 2 in $\hat{T}$; an element in the other coset is $\frac{1}{2}(x_1 + \cdots + x_l)$. The weights of Δ^+ are the characters

$$\frac{1}{2}(\pm x_1 \pm x_2 \pm \cdots \pm x_l)$$

with an even number of minus signs. The weights of Δ^- are those with an odd number of minus signs. Thus we have

$$\text{(5.1)} \qquad \begin{aligned} \mathrm{ch}\,\Delta &= \textstyle\prod_{i=1}^{l} (e^{x/2} + e^{-x/2}) \\ \mathrm{ch}\,\Delta^+ - \mathrm{ch}\,\Delta^- &= \textstyle\prod_{i=1}^{l} (e^{x/2} - e^{-x/2}) . \end{aligned}$$

These characteristic classes may be viewed as characteristic classes of either Spin $(2l)$ or of $SO(2l)$, since over $\mathbf{Q}$ the characteristic rings of these two groups coincide.

If X is a Spin $(2l)$-manifold with principal bundle P we can form the associated complex vector bundles

$$E^{\pm} = P \times_{\mathrm{Spin}(2l)} \Delta^{\pm} , \qquad E = E^+ \oplus E^- .$$

The *Dirac operator* is a first order differential operator

$$D: \mathfrak{D}(E) \longrightarrow \mathfrak{D}(E)$$

defined as follows. Recall first that the riemannian metric defines a natural $SO(2l)$-connection, and this lifts to give a connection for P. We may therefore consider the covariant derivative

$$\partial: \mathfrak{D}(E) \longrightarrow \mathfrak{D}(E \otimes T^*) .$$

On the other hand $T \cong T^*$, and we have the bundle homomorphism

$$\mathfrak{D}(E \otimes T) \longrightarrow \mathfrak{D}(E)$$

induced by Clifford multiplication. We define D to be the composition of these two maps. Thus in terms of an orthonormal base e_i of T, we have

$$Ds = \sum e_i(\partial_i s)$$

where $\partial_i s$ is the covariant derivative of s in the direction e_i and $e_i(\ )$ denotes Clifford multiplication. The symbol of D is (up to a factor i) just Clifford multiplication, i.e., its value at $\xi \in T_x$ is the homomorphism

$$\Delta_x \longrightarrow \Delta_x$$

given by Clifford multiplication by ξ. Since $\xi(\xi(u)) = -\|\xi\|^2 \cdot u$ it follows that D is elliptic. Moreover because of the properties of Clifford multiplication D induces two operators

$$D^+ : \mathfrak{D}(E^+) \longrightarrow \mathfrak{D}(E^-) , \qquad D^- : \mathfrak{D}(E^-) \longrightarrow \mathfrak{D}(E^+)$$

each of which is elliptic. If we introduce the natural inner product on $\mathfrak{D}(E)$ defined by taking the inner product in each fibre, and then integrating over X with respect to the riemannian measure, we find that D is formally self-adjoint, and so D^- is the formal adjoint of D^+.

A solution of $Ds = 0$ will be called a *harmonic spinor*, and the space of harmonic spinors will be denoted by H. It decomposes:

$$H = H^+ \oplus H^- ,$$

where $H^+ = \operatorname{Ker} D^+$, $H^- = \operatorname{Ker} D^- \cong \operatorname{Coker} D^+$. Let us put $h^\pm = \dim H^\pm$, so that

$$\operatorname{index} D^+ = h^+ - h^- .$$

We shall refer to this as the *spinor index* of X, and write it as Spin (X).

Finally following Hirzebruch [14], let us define the stable characteristic class $\hat{\mathfrak{A}}$ by

$$\hat{\mathfrak{A}} = \prod_{i=1}^{l} \left(\frac{x_i}{e^{x/2} - e^{-x/2}}\right) \qquad \in H^{*}_{O(2l)}(\mathbf{Q}) . \tag{5.2}$$

For any real vector bundle E,

$$\hat{\mathfrak{A}}(E) = \sum \hat{\mathfrak{A}}_r(p_1(E), \cdots, p_r(E))$$

is an explicit polynomial in the Pontrjagin classes of E.

For a manifold X, we put

$$\hat{\mathfrak{A}}(X) = \hat{\mathfrak{A}}(T(X)) .$$

Since the operator D^+ has a symbol-class associated to the Spin-structure, we are in a position to apply the index theorem in the form given by Proposition (2.17). We get

$$\begin{aligned}\text{index } D^+ &= (-1)^l\Big\{\Big\{\prod\Big(\frac{e^{x/2}-e^{-x/2}}{x_i}\Big)\cdot\prod\Big(\frac{x_i}{1-e^{x_i}}\Big)\prod\Big(\frac{-x_i}{1-e^{-x_i}}\Big)(X)\Big\}\Big\}[X] \\ &= (-1)^l\hat{\mathfrak{A}}(X)[X]) .\end{aligned}$$

Since $\hat{\mathfrak{A}}$ involves only Pontrjagin classes we have

$$\begin{aligned}\text{index } D^+ &= 0 &&\text{if } \dim X \not\equiv 0\,(4) \\ &= \hat{\mathfrak{A}}(X)[X] &&\text{if } \dim X \equiv 0\,(4) .\end{aligned}$$

Thus we have established

THEOREM (5.3). *Let X be a Spin manifold of dimension $4k$, then the spinor index is equal to $\hat{\mathfrak{A}}(X)[X]$ where $\hat{\mathfrak{A}}$ is the characteristic class given by* (5.2).

Remarks. 1. The number

$$\hat{A}(X) = \hat{\mathfrak{A}}(X)[X]$$

is called the $\hat{A}$-genus of X.

2. It is easy to extend (5.3) to spinors with coefficients in a vector bundle V. One has a natural Dirac-type operator whenever V is associated to the tangent bundle. For other V, one has to choose first a specific connection.

3. Lichnerowicz [17] has shown that there are no harmonic spinors if the riemannian metric has strictly positive scalar curvature. In such circumstances (5.3) implies that the $\hat{A}$-genus is zero. The complex projective space $P_{2k}(\mathbf{C})$ is an example of a manifold with positive curvature and $\hat{A}$-genus non-zero; however $P_{2k}(\mathbf{C})$ has no Spin structure.

4. One also has a Dirac operator and a theorem analogous to (5.3) for Spin^c-manifolds where Spin^c is the complex spinor group of [3]. This case provides a natural unification between Spin and almost complex structures. It is in this context also that one has a fundamental element in $K(TX)$ which generates this freely as a $K(X)$-module. In particular for Spin manifolds the symbol class of D^+ is such a generator. This follows from the Thom isomorphism in K-theory for Spin (or Spin^c) bundles [3; 12.3].

Suppose now that X is a Spin $(2l)$-manifold and that the group G acts on the Spin-structure of X. This means not only that G acts on the manifold X and hence on the principal orthogonal bundle Q but also that we are given an action of G on P (the principal Spin-bundle) compatible with its action on Q. In these circumstances we can define the Spinor index of X as a character of G:

$$\text{Spin}\,(G, X) \in R(G) .$$

The value of this character at g (i.e., the Lefschetz number $L(g, D^+)$) will be denoted by Spin (g, X). Our general Lefschetz theorem (3.9) will of course give an explicit formula for this in terms of the fixed point set X^g. We shall compute it in the simple case when g is an *involution* on X. Note that g is not necessarily an involution on the Spin-structure: its action there can have order 4.

For simplicity, let us assume that X^g is orientable, and fix an orientation. Then over X^g we have a diagram of bundles

$$\begin{tikzcd} P_0 \arrow[r, phantom, "\subset"] \arrow[d] & P \arrow[d] \\ Q_0 \arrow[r, phantom, "\subset"] & Q \end{tikzcd}$$

where Q_0 is the principal $SO(2k) \times SO(2l - 2k)$ bundle given by the tangent and normal bundles of X^g (as usual $2k = \dim X^g$ depends on the component), and P_0 is a principal bundle with structure group

$$H_0 = \mathrm{Spin}\,(2k) \times_{Z_2} \mathrm{Spin}\,(2l - 2k)\ .$$

The element g acts on P_0 through the element $(1, \alpha) \in H_0$ where $\alpha \in \mathrm{Spin}\,(2l - 2k)$ is one of the two elements lying over $-1 \in SO(2l - 2k)$. If $u \in K_G(TX)$ is the symbol class of the Dirac operator D^+ (and G is the group generated by g), then $i^*u \in K_G(TX^g)$ is a symbol class associated to the H_0-structure P_0 of X^g and invariant under G. Using remark 2 after (2.17), and the character formula (5.1), we then find that the general Lefschetz formula (3.9) reduces to

$$\mathrm{Spin}\,(g, X) = \sum\nolimits_j \varepsilon_j i^{k(j)} \frac{\hat{\mathfrak{A}}(X^g_j)}{\mathrm{ch}\,\Delta(N^g_j)}[X^g_j]\ . \tag{5.4}$$

where the summation is over the connected components X^g_j of X^g and $\varepsilon_j = \pm 1$ depends in a rather subtle global manner on the particular component.

For a further more detailed discussion of this sign question, see [2; § 8] where the case of isolated fixed point ($k = 0$) is dealt with, each fixed point then contributes $\pm i^k/2^l$ in (5.4).

6. The signature

For any compact oriented manifold X of dimension $4k$, the cup-product defines a non-degenerate quadratic form on $H^{2k}(X; \mathbf{R})$. Let $p^+(p^-)$ denote the maximal dimension of a subspace on which this form is positive (negative) definite. The difference $p^+ - p^-$ is usually called the index of X and has been extensively studied by Thom and Hirzebruch. An alternative terminology is to call this invariant the *signature* and, to avoid a possible confusion with the index of elliptic operators, we shall adopt this alternative. For brevity we shall write it as Sign (X).

We propose to show that, in fact, Sign (X) *is* the index of a certain elliptic operator on X associated to the $SO(4k)$-structure (i.e., to the orientation and a riemannian metric). The index theorem in the form (2.17) will then give an explicit formula for Sign (X) in terms of Pontrjagin classes. This formula was originally obtained by Hirzebruch using Thom's results on cobordism. Our derivation of it is more direct and is independent of cobordism[9].

We shall also apply the general Lefschetz theorem (3.9) to the elliptic operator referred to above. This will then give a formula for the G-signature of a G-manifold X. This G-signature is a character of G defined by the action of G on $H^{2k}(X; \mathbf{R})$ and our formula will express this in terms of Pontrjagin classes and fixed-point sets. The particular interest of this case is that the result can be formulated purely in terms of differential topology. Our proof however involves analysis in an essential way. Although the original Hirzebruch signature formula was proved by purely topological means, it is not possible to reduce the problem of calculating the G-signature to the special case of the Hirzebruch signature. The reasons for this will become apparent later. Because of this, our formula for the G-signature seems to provide one of the best topological applications of the theory of elliptic operators. In fact a very special case, discussed in detail in [2] because it involves only isolated fixed-points, has already proved its worth in establishing a conjecture of Milnor about lens spaces. It is to be hoped that the more general case developed here will have further interesting applications.

For any compact manifold X of dimension n, we have the de Rham complex Ω of (complex-valued) exterior differential forms

$$0 \longrightarrow \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n \longrightarrow 0 .$$

The symbol of this is just the complex defined by the exterior algebra of the cotangent bundle T^*, and so it is certainly elliptic. Its cohomology groups, by the theorems of de Rham, are naturally isomorphic to the ordinary complex cohomology groups $H^q(X; \mathrm{C})$. Its Euler characteristic is therefore the usual Euler characteristic

$$E(X) = \sum (-1)^q \dim H^q(X; \mathrm{C}) .$$

If X is oriented and of even dimension the index theorem in this case merely asserts the well known fact that

$$E(X) = e(X)[X]$$

is obtained by evaluating the Euler class of X on $[X]$. This is not very excit-

[9] This was not true of the first method of proof of the index theorem as given in [19].

ing. Similarly, if a compact group G acts on X, it acts on the de Rham complex, and our Lefschetz theorem reduces to the classical Lefschetz fixed point formula, at least when the fixed-points are isolated. The general case is slightly more interesting, every component of the fixed-point set now contributes its Euler characteristic to the Lefschetz formula. We leave the details to the reader.

To get a really interesting index problem out of the de Rham complex we have to proceed differently. We assume from now on that $\dim X = 2l$, that X is oriented and that we have chosen a riemannian metric on X. This induces metrics on the bundles of i-forms and hence, by integration over X, inner products in the space Ω^i. The formal adjoint of d with respect to these inner products is denoted by d^*. The operator

$$\Delta = dd^* + d^*d$$

is the Laplace operator of the Hodge theory. It is an elliptic operator and preserves the degree of the differential forms, i.e., it maps each Ω^i into itself. The solutions of $\Delta u = 0$ are the *harmonic forms*, and the space H^i of harmonic i-forms is isomorphic to

$$H^i(\Omega) \cong H^i(X; \mathrm{C}) .$$

We consider now the first order operator $D = d + d^*$. It is formally self-adjoint, and we have (since $d^2 = (d^*)^2 = 0$)

$$\Delta = D^*D = D^2 .$$

From this, it follows that the solutions of $Du = 0$ coincide with the solutions of $\Delta u = 0$, i.e., with the harmonic forms. In fact, if $\Delta u = 0$, we have

$$0 = (\Delta u, u) = (Du, Du) ,$$

and so $Du = 0$.

Now the riemannian metric induces a bundle isomorphism

$$*\colon \lambda^i(T^*) \longrightarrow \lambda^{2l-i}(T^*) .$$

If α, β are real i-forms, their inner product is then given by

$$(\alpha, \beta) = \int \alpha \wedge *\beta .$$

If we extend $*$ linearly to the complexified forms, we then find that, for $\alpha, \beta \in \Omega^i$, their hermitian inner product is given by

$$(\alpha, \beta) = \int \alpha \wedge \overline{*\beta} .$$

Since $*$ arises from the pairing $\alpha \otimes \beta \mapsto \alpha \wedge \beta$, it follows that

$$*(*\alpha) = (-1)^i\alpha \qquad \text{when } \alpha \in \Omega^i . \tag{6.1}$$

From this and the integral formula for the inner product, one then finds

$$d^*\alpha = -*d*\alpha . \tag{6.2}$$

We now introduce a map τ on differential forms defined by

$$\tau(\alpha) = i^{p(p-1)+l}*\alpha , \qquad \alpha \in \Omega^p .$$

Note that τ is real if l is even, and imaginary if l is odd. Then $\tau^2(\alpha) = i^\varepsilon *^2 \alpha$ where

$$\begin{aligned}\varepsilon &= p(p-1) + l + (2l-p)(2l-p-1) + l \\ &= 4l^2 - 4lp + 2p^2 \equiv 2p \qquad \text{mod } 4 .\end{aligned}$$

Since $*^2\alpha = (-1)^p\alpha$, it follows that $\tau^2\alpha = \alpha$, and so τ is an involution. We can therefore decompose the space $\Omega = \sum \Omega^i$ into the ± 1-eigenspaces $\Omega^\pm$ of τ. Since τ is a bundle homomorphism, we have a bundle decomposition

$$\Lambda^*(T^* \otimes_{\mathbf{R}} \mathbf{C}) = \Lambda_+ \oplus \Lambda_- ,$$

and $\Omega^\pm$ are the spaces of smooth sections of the bundle $\Lambda_\pm$. Using (6.1) and (6.2), one then verifies that

$$D\tau = -\tau D ,$$

and so D maps Ω^+ into Ω^- and Ω^- into Ω^+. We denote by $D^\pm$ the restriction of D to the appropriate subspaces of Ω. Thus

$$D^+ : \Omega^+ \longrightarrow \Omega^- , \qquad D^- : \Omega^- \longrightarrow \Omega^+ .$$

Each is elliptic, and they are formal adjoints of each other. Moreover the solutions of $D^+u = 0$ are just the harmonic forms in Ω^+, and similarly for Ω^-. We shall denote by H^+ and H^- these subspaces of harmonic forms, and we put

$$h^+ = \dim H^+ , \qquad h^- = \dim H^- .$$

Thus we have

$$\operatorname{index} D^+ = h^+ - h^- .$$

Since $\Delta\tau = \tau\Delta$, it follows that τ induces an involution on the harmonic forms; the ± 1 eigenspaces are precisely H^+ and H^-. Now the space

$$V_k = H^k \oplus H^{2l-k} \qquad 0 \leqq k < l$$

is stable under τ; in fact τ switches the two factors in this decomposition. Thus the dimensions of the $+1$ and -1 eigenspaces of τ in V_k are equal, and so V_k contributes zero to index D^+. Hence

$$\operatorname{index} D^+ = h^l_+ - h^l_-$$

where

$$h^l_\pm = \dim H^l_\pm$$

and $H^l_\pm$ is the ± 1-eigenspace of τ in H^l. We now distinguish two cases according to the parity of l. First suppose l is odd, then $\tau = i\sigma$ with σ real, and $\sigma^2 = -1$. Then the $\pm i$-eigenspaces of σ are conjugate, and so have the same dimension. But these are the ∓ 1-eigenspaces of τ, and so index $D^+ = 0$ in this case. Supose now $l = 2k$ is even, then for $\alpha \in H^{2k}$ we have $\tau\alpha = *\alpha$. Thus for $\alpha \in H^{2k}_+$ real and non-zero we have

$$\int \alpha \wedge \alpha = \int \alpha \wedge *\alpha = (\alpha, \alpha) > 0 ,$$

while if $\alpha \in H^{2k}_-$ is real and non-zero

$$\int \alpha \wedge \alpha = -\int \alpha \wedge *\alpha = -(\alpha, \alpha) < 0 .$$

Since $\alpha \mapsto \int \alpha \wedge \alpha$ is precisely the quadratic form given by the cup-product on $H^{2k}(X; \mathbf{R})$, it follows that

$$h^{2k}_+ = p^+ \qquad h^{2k}_- = p^- ,$$

and so index $D^+ = \operatorname{Sign}(X)$ is the Hirzebruch signature of X.

Now it is clear that the symbol of D^+ is associated to the $SO(2l)$-structure of X so that its index can be computed by applying the index theorem in the form of (2.17). It remains only to calculate the characteristic class

$$T \longmapsto \operatorname{ch} \Lambda_+(T) - \operatorname{ch} \Lambda_-(T) .$$

Now $\Lambda_\pm(R^{2l})$ are $SO(2l)$-modules, defined as above as ∓ 1-eigenspaces of τ. Also τ is multiplicative, i.e.,

$$\tau(u \wedge v) = \tau(u) \wedge \tau(v)$$

for $u \in \Lambda^*(C^{2l})$, $v \in \Lambda^*(C^{2k})$, $u \wedge v \in \Lambda^*(C^{2l+2k})$. This implies that the image of

$$\Lambda_+(R^{2l+2k}) - \Lambda_-(R^{2l+2k}) \qquad \in R(SO(2l+2k))$$

in $R(SO(2l)) \times R(SO(2k))$ is

$$\{\Lambda_+(R^{2l}) - \Lambda_-(R^{2l})\}\{\Lambda_+(R^{2k}) - \Lambda_-(R^{2k})\} .$$

Hence passing from $SO(2l)$ to its maximal torus S, we see that $\Lambda_+(R^{2l}) - \Lambda_-(R^{2l})$ restricts to $\prod_{i=1}^{l} f(x_i)$ where $x_1, \cdots, x_l$ are the basic characters of S and

$$f(x) = \Lambda_+(R^2) - \Lambda_-(R^2) \qquad \in R(SO(2)) .$$

If e_1, e_2 is the usual basis of R^2, then

$$*e_1 = e_2 , \qquad *e_2 = -e_1 , \qquad *1 = e_1 \wedge e_2 , \qquad *(e_1 \wedge e_2) = 1 .$$

Thus a basis for $\Lambda_+(R^2)$ is $1 + i(e_1 \wedge e_2)$ and $e_1 + ie_2$, while a basis for $\Lambda_-(R^2)$ is

$1 - i(e_1 \wedge e_2)$ and $e_1 - ie_2$. Hence we have

$$\Lambda_+(R^2) = 1 + x^{-1}$$
$$\Lambda_-(R^2) = 1 + x ,$$

where x is the basic character of $SO(2)$. Hence we have

$$\Lambda_+(R^{2l}) - \Lambda_-(R^{2l}) = \prod_{i=1}^{l} (x_i^{-1} - x_i)$$

and so, applying the universal Chern character,

$$\operatorname{ch}\left(\Lambda_+(R^{2l}) - \Lambda_-(R^{2l})\right) = \prod_{i=1}^{l} (e^{-x_i} - e^{x_i}) . \tag{6.3}$$

Putting this formula into (2.17), we obtain

$$\begin{aligned}\operatorname{index} D^+ &= (-1)^l \left\{\left\{\prod_{i=1}^{l}\left(\frac{e^{-x_i} - e^{x_i}}{x_i}\right)\prod \frac{x_i}{1 - e^{x_i}} \cdot \frac{-x_i}{1 - e^{-x_i}}\right\}(X)\right\}[X] \\ &= \left\{\prod_{i=1}^{l} x_i \frac{(e^{x_i} + 1)}{(e^{x_i} - 1)}(X)\right\}[X] \qquad (6.4) \\ &= \left\{\prod_{i=1}^{l} \frac{x_i}{\tanh x_i/2}(X)\right\}[X] .\end{aligned}$$

Since $\{x/\tanh(x/2)\}$ is an even function of x, this expression involves only Pontrjagin classes of X, and is zero if l is odd. This checks with the analysis. The interesting case is when l is even.

Since $\{x/\tanh(x/2)\} = 2$ when $x = 0$, the characteristic class occurring in (6.4) is not stable. If we want to express our answer in terms of a stable class, we must introduce

$$\mathfrak{L} = \sum \mathfrak{L}_r(p) = \prod \frac{x_i/2}{\tanh x_i/2} . \tag{6.5}$$

Formula (6.4) is then equivalent to

$$\operatorname{index} D^+ = 2^l \mathfrak{L}(X)[X] .$$

The expression on the right will be denoted by $L(X)$ and called the L-genus of X. This agrees with the definition of Hirzebruch [13] because

$$2^l \prod \frac{x_i/2}{\tanh x_i/2} \qquad \text{and} \qquad \prod \frac{x_i}{\tanh x_i}$$

have the same term in degree l.

We have now established, as a special case of our general index theorem:

HIRZEBRUCH SIGNATURE THEOREM (6.6). *Let X be a compact oriented manifold of dimension $4k$. Let* Sign (X) *denote the signature of the quadratic form in $H^{2k}(X; \mathbf{R})$ and let $L(X) = 2^{2k}\mathfrak{L}(X)[X]$ where $\mathfrak{L}$ is the stable characteristic class of the orthogonal group given by* (6.5). *Then*

$$\operatorname{Sign}(X) = L(X) .$$

We proceed now to consider the Lefschetz theorem corresponding to (6.6). We suppose therefore that X is oriented of dimension $2l$, and that a compact Lie group G acts differentiably on X *preserving the orientation.* Choosing a G-invariant riemannian metric on X, the operator D^+ will be G-invariant, since it is functorially defined by the metric and orientation.

On the topological side, the bilinear form B on $H^l(X; \mathbf{R})$ given by $B(x, y) = (xy)[X]$ is G-invariant. Note that this form is *symmetric* for l even, but *skew-symmetric* for l odd. In both cases, by Poincaré duality, it is non-degenerate.

Suppose now we choose a positive definite inner product $\langle\ ,\ \rangle$ on H^l, invariant under G, and define the operator A by

$$B(x, y) = \langle x, Ay\rangle .$$

Then A commutes with the action of G and $A^* = (-1)^l A$. Consider first the case when l is *even*, so that A is self-adjoint. Then the positive and negative eigenspaces of A give a decomposition $H^l = H^l_+ \oplus H^l_-$ invariant under G. Thus we have two real representations ρ^+ and ρ^- of G, and, up to isomorphism, they are independent of the choice of inner product. This follows from the following three facts[10]:

(a) the characters of ρ^+ and ρ^- are continuous functions of the inner product,

(b) the space of all (G-invariant) inner products is connected,

(c) the characters of the compact group G are discrete.

As an obvious generalization of the signature, we now define the G-signature of the G-manifold X to be

$$\text{Sign}\,(G, X) = \rho^+ - \rho^- \in RO(G) \subset R(G); \tag{6.7}$$

it is an element of the real representation ring $RO(G)$.

By evaluating the character of $\text{Sign}\,(G, X)$ on an element $g \in G$, we obtain a real number which we denote by $\text{Sign}\,(g, X)$. Note that $\text{Sign}\,(g, X)$ *is determined entirely by the action of g on the real cohomology of X.* Thus $\text{Sign}\,(g, X)$ depends only on the connected component of G containing g. On the other hand it can be computed by the use of harmonic forms. In fact G acts on the space H^{2k}_R of real harmonic forms in dimension $2k$, preserving both the indefinite bilinear form $\int\alpha\wedge\alpha$ and the inner product $\int\alpha\wedge *\alpha$. Hence

$$\text{Sign}\,(g, X) = \text{Trace}_{\mathbf{R}}\,(g \mid (H^{2k}_{\mathbf{R}})^+) - \text{Trace}_{\mathbf{R}}\,(g \mid (H^{2k}_{\mathbf{R}})^-)\ ,$$

[10] An alternative more direct argument is to observe that the H_+ space of one inner product is always complementary to the H_- space of any other inner product. However we want a general argument which will apply equally to the case when l is odd.

where $(H^{2k}_{\mathbf{R}})^{\pm}$ are the eigenspaces of $*$ on $H^{2k}_{\mathbf{R}}$. As we saw before

$$(H^{2k})^{+} = (H^{2k}_{\mathbf{R}})^{+} \otimes_{\mathbf{R}} \mathbf{C} = (\operatorname{Ker} D^{+}) \cap H^{2k}$$
$$(H^{2k}_{\mathbf{C}})^{-} = (H^{2k}_{\mathbf{R}})^{-} \otimes_{\mathbf{R}} \mathbf{C} = (\operatorname{Ker} D^{-}) \cap H^{2k} ,$$

so that $\operatorname{Sign}(g, X)$ is the contribution of H^{2k} to the Lefschetz number $L(g, D^{+})$. On the ther hand, if we put

$$V^{q} = H^{q} \oplus H^{4k-q} \qquad 0 \leqq q < 2k ,$$

and recall that the involution τ switches the two factors in this decomposition, we see that we have G-module isomorphisms

$$(V^{q})^{+} \cong H^{q} \cong (V^{q})^{-} ,$$

and so V^{q} contributes nothing to $L(g, D^{+})$. Thus we have

$$\operatorname{Sign}(g, X) = L(g, D^{+}) \tag{6.8}$$

when $L(g, D^{+})$ is the Lefschetz number of g for the operator D^{+}.

We return now and consider the case when l is odd. In this case the operator A is skew-adjoint, and if $(AA^{*})^{1/2}$ denotes the positive square root of AA^{*}, the operator $J = A/(AA^{*})^{1/2}$ satisfies $J^{2} = -1$, and so defines a complex structure on H^{l}. Since J commutes with the action of G, we obtain in this way a complex representation ρ of G. For the same reasons as in the We case of even l, we see that ρ is independent of the choice of inner product. now define the G-signature of the G-manifold X by

$$\operatorname{Sign}(G, X) = \rho - \rho^{*} \in R(G) . \tag{6.9}$$

By evaluating the character of $\operatorname{Sign}(G, X)$ on an element $g \in G$, we obtain a purely imaginary number which we denote by $\operatorname{Sign}(g, X)$. Thus

$$\operatorname{Sign}(g, X) = \rho(g) - \rho^{*}(g) = 2i \operatorname{Im} \rho(g) .$$

Note that $\operatorname{Sign}(g, X)$ *depends only on the action of g on the real cohomology* of X. As in the even case, we can compute it by harmonic forms. We identify $H^{l}(X; \mathbf{R})$ with the space $H^{l}_{\mathbf{R}}$ of real harmonic l-forms. G preserves both the skew form $\int \alpha \wedge \beta$ and the inner product $(\alpha, \beta) = \int \alpha \wedge * \beta$. The skew-adjoint operator A is given by.

$$\begin{aligned}(\alpha, A\beta) = \int \alpha \wedge \beta &= (-1)^{l} \int \alpha \wedge ** \beta && \text{since } *^{2} = (-1)^{l} \\ &= -(\alpha, *\beta) && \text{since } l \text{ is odd} ,\end{aligned}$$

and so $A = -*$. Since

$$AA^{*} = -A^{2} = -*^{2} = 1 ,$$

it follows that the complex structure J is given by $J = -*$. Now in the complexification

$$H^l = H^l_{\mathbf{R}} \otimes \mathbf{C} ,$$

we defined the involution τ by

$$\tau = i^{l^2} * = i* .$$

Hence on the $(+1)$-eigenspace H^l_+ of τ, we have $-* = i$, and so we have an isomorphism of *complex* vector spaces

$$H^l_+ \cong H^l_{\mathbf{R}} ,$$

where $H^l_{\mathbf{R}}$ is endowed with the complex structure of J above. Similarly

$$H^l_- \cong \overline{H^l_{\mathbf{R}}}$$

and both isomorphisms are compatible with the action of G. Hence

$$\text{(6.10)} \qquad \operatorname{Sign}(g, X) = \operatorname{Trace}(g \,|H^l_+) - \operatorname{Trace}(g \mid H^l_-) .$$

Now exactly as in the case of even l, we see that the spaces H^q for $q \neq l$ contribute nothing to the Lefschetz number of D^+, and so we obtain

$$\operatorname{Sign}(g, X) = L(g, D^+) .$$

Remarks. 1. Like the Hirzebruch signature our G-signature is multiplicative, i.e.,

$$\operatorname{Sign}(G, X) \cdot \operatorname{Sign}(G, Y) = \operatorname{Sign}(G, X \times Y) .$$

This is valid for all dimensions provided we put $\operatorname{Sign}(G, X) = 0$ if $\dim X$ is odd.

2. The G-signature defined on H^l is a purely algebraic notion, and it has the following important property. If M is any real G-module and M^* is its dual, then $M \oplus M^*$ has two natural bilinear forms; a symmetric one B^+, and a skew-symmetric one B^-, and both of these have zero G-signature. To see this, observe that multiplication by -1 on M^* defines an involution α of $M \oplus M^*$ which commutes with G, but anti-commutes with $B^\pm$. In other words α takes $B^\pm$ into $-B^\pm$ and so

$$\begin{aligned} \operatorname{Sign}(G, B^\pm) &= \operatorname{Sign}(G, -B^\pm) \\ &= -\operatorname{Sign}(G, B^\pm) \qquad \text{(from the definitions)} , \end{aligned}$$

showing that $\operatorname{Sign}(G, B^\pm) = 0$.

Having identified $L(g, D^+)$ as a homology-invariant (for both l even and l odd) we proceed to apply the Lefschetz theorem (3.9) to calculate $L(g, D^+)$ in terms of the fixed point set X^g of X. Observe first that each component of X^g is even-dimensional. This follows from the fact that g preserves orientation, so that the number of $+1$ (or -1) eigenvalues in the action of g on the tangent space of X at any point $x \in X^g$ is necessarily even. If u denotes the symbol of D^+ then, using the multiplicative formula (6.3), we find that $\operatorname{ch} i^*(u)(g)$ is the product of the following terms

$$\begin{aligned} A &= \psi\left\{\prod\left(\frac{e^{-x_j} - e^{x_j}}{x_j}\right)(TX^g)\right\} \\ B &= \prod(-e^{-x_j} + e^{x_j})\big(N^g(-1)\big) \\ C^\theta &= \prod(e^{-x_j - i\theta} - e^{x_j + i\theta})\big(N^g(\theta)\big)\,. \end{aligned}$$

Here $\psi\colon H^*(X^g; \tilde{\mathbf{Q}}) \to H^*(TX^g; \mathbf{Q})$ is the Thom isomorphism, where $\tilde{\mathbf{Q}}$ denotes the local coefficient system isomorphic to $\mathbf{Q}$ defined by the local orientations of X^g. If X^g is orientable, then once we choose an orientation, we get an isomorphism $\tilde{\mathbf{Q}} \cong \mathbf{Q}$. If X^g is not orientable, then we have to work with these "twisted" coefficients. Note that, since X and the $N^g(\theta)$ are all oriented, the bundles TX^g and $N^g(-1)$ have isomorphic twisted coefficients. Thus if $\tilde{\mathbf{Q}}$ is as above, the Euler class $e\big(N^g(-1)\big) \in H^*(X^g; \tilde{\mathbf{Q}})$.

The evalution on the fundamental class $[TX^g]$ in (3.9) can, as usual, be replaced by an evalution on the "twisted" fundamental class $[X^g]$. We simply replace A by $\psi^{-1}A$, and multiply by $(-1)^k$ (where $\dim X^g = 2k$) to allow for the difference between the two orientations of TX^g. If we cancel the common factors occurring in $\psi^{-1}A, B, C^\theta$ on the one hand, and $\mathcal{I}, \mathcal{R}, \mathcal{S}^\theta$, and $\det(1 - g \mid N^g)$ on the other, we find

$$L(g, D^+) = \{A_1 B_1 \textstyle\prod_\theta C_1^\theta\}[X]\,,$$

where

$$A_1 = \prod \frac{x_j}{\tanh x_j/2}(TX^g) = 2^t \mathfrak{L}(X^g)\,, \qquad 2t = \dim X^g$$

$$B_1 = \prod \tanh \frac{x_j}{2}\big(N^g(-1)\big) = 2^{-r}\mathfrak{L}\big(N^g(-1)\big)^{-1}\cdot e\big(N^g(-1)\big)\,, \qquad 2r = \dim N^g(-1)$$

$$C_1^\theta = \prod \frac{1}{\tanh \dfrac{x_j + i\theta}{2}}\big(N^g(\theta)\big)\,.$$

In order to express C_1^θ by a stable characteristic class of the unitary group, we define

$$\mathfrak{M}^\theta = \sum \mathfrak{M}_r^\theta(c_1, \cdots, c_r) = \prod \frac{\tanh i\theta/2}{\tanh \dfrac{x_j + i\theta}{2}} \tag{6.11}$$

so that

$$C_1^\theta = (i \tan \theta/2)^{-s(\theta)} \mathfrak{M}^\theta\big(N^g(\theta)\big)\,, \qquad s(\theta) = \dim_{\mathbf{C}}\big(N^g(\theta)\big)\,.$$

Putting all this together we obtain:

G-SIGNATURE THOREM (6.12). *Let X be a compact oriented manifold of dimension $2l$, and let the compact Lie group G act on X preserving the orientation. Then G acts on $H^l(X; \mathbf{R})$ preserving the bilinear form. Let* Sign (G, X) *be the character of G defined from this action by* (6.7) *for l even, and by* (6.9) *for l odd. Let* Sign (g, X) *be the value of* Sign (G, X) *on an element $g \in G$. Let X^g be the fixed-point set of g, N^g the normal bundle of X^g in X, and*

$$N^g = N^g(-1) \oplus \sum_{0<\theta<\pi} N^g(\theta)$$

the decomposition of N^g determined by the eigenvalues of g. Then $N^g(-1)$ is a real vector bundle of even dimension and $N^g(\theta)$ is a complex vector bundle. Let[11] $2t = \dim X^g$, $2r = \dim N^g(-1)$, $s(\theta) = \dim_{\mathbf{C}} N^g(\theta)$. *Finally let $\mathfrak{L}$ be the stable characteristic class of the orthogonal group given by* (6.5), *and $\mathfrak{M}^\theta$ the stable characteristic class of the unitary group given by* (6.11). *Then we have*

$$\begin{aligned}\mathrm{Sign}\,(g, X) = \{2^{t-r} \textstyle\prod_{0<\theta<\pi} (i \tan \theta/2)^{-s(\theta)} \mathfrak{L}(X^g) \mathfrak{L}(N^g(-1))^{-1} e(N^g(-1)) \\ \textstyle\prod_{0<\theta<\pi} \mathfrak{M}^\theta(N^g(\theta))\}[X^g]\,.\end{aligned}$$

Here $e(N^g(-1))$ denotes the "twisted" Euler class of $N^g(-1)$, and $[X^g]$ is the "twisted" fundamental class of X^g, both twistings being defined by the local coefficient system of orientations of X^g.

This theorem, like the special case of it discussed in [2] when X^g is zero-dimensional, provides the most interesting application of our general Lefschetz theorem to differential topology. We shall therefore spend a little time discussing various special cases and corollaries. For brevity the expression on the right-hand side of the formula in (6.12) will be denoted by $L(g, X)$. Thus $L(1, X) = L(X)$ is the Hirzebruch L-genus.

First let us observe that the Euler class $e(N^g(-1))$ is in dimension $2r$ and $\dim X^g = 2t$. Thus if $r > t$, the formula in (6.12) shows that Sign $(g, X) = 0$. Thus we have

COROLLARY (6.13). *Assume, in the situation of* (6.12) *that (for all components of X^g) we have $r > t$. Then* Sign $(g, X) = 0$.

We shall now examine the particular case of an involution. If $\dim X = 4k + 2$, then Sign (G, X) is twice the imaginary part of a character, but for G of order 2 the characters are all real, and so Sign $(G, X) = 0$. Thus only the case $\dim X = 4k$ is of interest, and we apply (6.12) for this case. Since the only normal eigenvalue is now -1, we obtain

$$\mathrm{Sign}\,(g, X) = \{2^{t-r} \mathfrak{L}(X^g) \mathfrak{L}(N^g)^{-1} e(N^g)\}[X^g]\,. \tag{6.14}$$

To simplify this expression let us observe that, if $Y \subset X$ is any closed

[11] These numbers depend of course on the component of X^g. In order to keep the formula within bounds, we have not made this explicit.

submanifold, we can define its "self-intersection manifold". To do this, we take the inclusion map $i: Y \to X$ and replace it, in the manner of Thom, by a homotopic map $f: Y \to X$ transverse regular along $Y \subset X$. The inverse image $Z = f^{-1}(Y)$ is then the required self-intersection. Clearly the normal bundle of Z in Y is isomorphic to $N \mid Z$, where N is the normal bundle of Y in X. Hence the normal bundle of Z in X is isomorphic to $(N \oplus N) \mid Z$, and so has a natural orientation. Thus if X is oriented, so is Z. The oriented bordism class of Y represented by $j: Z \to Y$ is independent of the choice of the map f. In particular the oriented cobordism class of Z is independent of f. We call this the self-intersection of Y, and denote it by Y^2. Also the homology class of Y defined by Z is independent of f. In fact it is dual of the twisted Euler class $e(N)$. Hence for any $\xi \in H^*(Y)$, we have

$$\{\xi e(N)\}[Y] = j^*(\xi)[Z] .$$

We now return to our involution g, and apply this formula with $Y = X^g$, $\xi = 2^{t-r}\mathfrak{L}(X^g)\mathfrak{L}(N^g)^{-1}$. Since $\mathfrak{L}$ is multiplicative, and since $j^*(N^g)$ is the normal bundle of Z in X^g, we have

$$\begin{aligned}\{2^{t-r}\mathfrak{L}(X^g)\mathfrak{L}(N^g)^{-1}e(N^g)\}[Y] &= 2^{t-r}\mathfrak{L}(Z)[Z] \\ &= L(Z) \qquad\qquad (\text{since } \dim Z = 2(t-r)) .\end{aligned}$$

Combined with (6.14), this gives the following rather simple result.

PROPOSITION (6.15). *Let X be a compact oriented manifold of dimension $4k$, and let g be an orientation preserving involution with fixed point set X^g. Let $(X^g)^2$ denote the oriented cobordism class of the self-intersection of X^g in X. Then*

$$\operatorname{Sign}(g, X) = \operatorname{Sign}\big((X^g)^2\big) .$$

This rather attractive formulation of (6.12) for involutions was pointed out to us by F. Hirzebruch on the basis of the 8-dimensional case which we had explicitly computed.

Since $g^2 = 1$, we have

$$\begin{aligned}\operatorname{Sign}(g, X) &\equiv \dim H^{2k}(X, \mathbf{R}) \mod 2 \\ &\equiv E(x) \mod 2 ,\end{aligned}$$

where $E(X)$ is the Euler characteristic of X. Thus from (6.15) or (6.13), we deduce the following result proved differently by Conner and Floyd [12; (27.4)].

COROLLARY (6.16). *Let X be a compact oriented manifold of* dim $4k$ *with odd Euler characteristic. Let g be an involution of X preserving the orientation. Then* $\dim X^g \geqq \frac{1}{2} \dim X$ (i.e., *at least one component of X^g has such a dimension*).

Although Sign $(G, X) = 0$ when $\dim X \equiv 2 \bmod 4$, and G has order 1 or 2, this is certainly not so for other groups. The simplest example of a non-trivial G-signature in these dimensions is obtained as follows. Let w be a primitive cube root of unity, $X = \mathbf{C}/\{1, w\}$ the elliptic curve with periods $1, w$. Then $z \mapsto wz$ gives an automorphism g ("complex multiplication") of X of period 3. $H^1(X; \mathbf{C})$ is generated by the differentials dz, $d\bar{z}$, and we have

$$*dz = -idz \qquad *d\bar{z} = id\bar{z} .$$

Hence calculating Sign (g, X) by harmonic forms, we see that

$$\operatorname{Sign}(g, X) = w - \bar{w} = i\sqrt{3} \neq 0 .$$

For illustrative purposes, we shall now consider a few special cases of Theorem (6.12) in low dimensions.

As a simple check on (6.15) we can take $X = P_4(\mathbf{C})$ with either of the involutions

$$(x_0, x_1, x_2, x_3, x_4) \longmapsto (-x_0, x_1, x_2, x_3, x_4)$$
$$(x_0, x_1, x_2, x_3, x_4) \longmapsto (-x_0, -x_1, x_2, x_3, x_4) .$$

In both cases $\operatorname{Sign}(g, X) = 1$ because g acts trivially on the cohomology. In the first case, we have $X^g = P_3(\mathbf{C}) \cup P_0(\mathbf{C})$ so that $(X^g)^2 = P_2(\mathbf{C})$ and $\operatorname{Sign}\left((X^g)^2\right) = 1$; in the second case, $X^g = P_2(\mathbf{C}) \cup P_1(\mathbf{C})$ and $(X^g)^2 =$ point, has signature 1.

Returning now to Theorem (6.12), let us suppose that g is of *odd order*. Then $N^g(-1) = 0$, and so the formula for Sign (g, X) simplifies to

(6.17) $\quad \operatorname{Sign}(g, X) = \left(2^t \prod_{0<\theta<\pi} (i \tan \theta/2)^{-s(\theta)} \mathfrak{L}(X^g) \prod_{0<\theta<\pi} \mathfrak{M}^\theta\left(N^g(\theta)\right)\right\}[X^g] .$

Note that in this case X^g has a natural orientation induced from that of X and the complex orientation of the bundles $N^g(\theta)$. One must be careful however not to confuse this orientation of X^g with other "natural" orientations. For example, if $\dim X^g = 0$ so that X^g consists of points, the orientation to be used in (6.17) may not coincide with the usual orientation of a point. Suppose now that X is 4-dimensional, (and connected), and that g is non-trivial. The fixed-point set X^g then consists of points $\{P_j\}$ and 2-manifolds $\{Y_k\}$. The contribution of P_j to Sign (g, X) is then

$$-\varepsilon \cot \frac{\alpha_j}{2} \cot \frac{\beta_j}{2} ,$$

where $\exp(\pm i\alpha_j), \exp(\pm i\beta_j)$ are the eigenvalues of $g \mid T_{P_j}$, $0 < \alpha_j < \pi$, $0 < \beta_j < \pi$, and $\varepsilon = \pm 1$ is the difference between the two "natural" orientations of T_{P_j}. In other words if, in a basis of T_{P_j} (oriented relative to the orientation of X), $g \mid T_{P_j}$ is represented by the matrix

$$\begin{pmatrix} \cos \alpha_j & -\sin \alpha_j \\ \sin \alpha_j & \cos \alpha_j \end{pmatrix} \oplus \begin{pmatrix} \cos \beta_j & -\sin \beta_j \\ \sin \beta_j & \cos \beta_j \end{pmatrix} ,$$

then the contribution of P_j to Sign (g, X) is

$$-\cot\frac{\alpha_j}{2}\cot\frac{\beta_j}{2}\,.$$

Since

$$\mathfrak{M}_\theta = 1 + \frac{ic_1}{\sin\theta} + \cdots\,,$$

the contribution of the component Y_k to Sign (g, X) is

$$\left\{2\cdot -i\cot\theta_k/2\frac{ic_1(N_k^g)}{\sin\theta_k}\right\}[Y_k] = \frac{Y_k^2}{\sin^2\theta_k/2}\,,$$

where N_k^g is the complex normal bundle of Y_k and Y_k^2 is the self-intersection number of Y_k. Thus we have established

PROPOSITION (6.18). *Let X be a connected compact oriented 4-manifold, g an automorphism of X of odd order (necessarily preserving orientation). Let the fixed-point set X^g consist of isolated points $\{P_j\}$ and connected 2-manifolds $\{Y_k\}$. For each j, let the action of g on the tangent space at P_j be given by the matrix*

$$\begin{pmatrix}\cos\alpha_j & -\sin\alpha_j\\ \sin\alpha_j & \cos\alpha_j\end{pmatrix}\oplus\begin{pmatrix}\cos\beta_j & -\sin\beta_j\\ \sin\beta_j & \cos\beta_j\end{pmatrix}$$

relative to an oriented basis. For each k, let $\exp(\pm i\theta_k)$ denote the normal eigenvalues of g along Y_k, and let Y_k^2 denote the self-intersection number of Y_k. Then

$$\mathrm{Sign}\,(g, X) = \sum\nolimits_j -\cot\frac{\alpha_j}{2}\cdot\cot\frac{\beta_j}{2} + \sum\nolimits_k \operatorname{cosec}^2\frac{\theta_k}{2}\cdot Y_k^2\,.$$

As a check on (6.18), consider the transformation $(x_0, x_1, x_2)\mapsto(x_0, wx_1, wx_2)$ where $w = \exp(i\alpha)$ is a primitive q^{th} root of unity (q odd). Then we have one fixed point which contributes $-(\cot\alpha/2)^2$, and one fixed line which contributes $(\operatorname{cosec}\alpha/2)^2$. Thus

$$\mathrm{Sign}\,(g, X) = -\left(\cot\frac{\alpha}{2}\right)^2 + \left(\operatorname{cosec}\frac{\alpha}{2}\right)^2 = 1$$

which is correct, because g acts trivially on the cohomology of X.

We conclude this section by pointing out that one can also consider the operator D^+ extended (using a connection) to act on differential forms with coefficients in any vector bundle V. This no longer has any connection with ordinary cohomology, and so it is not as interesting as the case we have been considering. However one gets formulas generalizing (6.6) and (6.12). If we denote the Lefschetz number of the extension of D^+ to V by Sign (g, X, V), then we have

$$\text{Sign}(g, X, V) = \{\text{ch}(V \mid X^g)(g)A\}[X^g]\,, \tag{6.19}$$

where $\{A\}$ is the expression occurring in the formula of (6.12). The right hand side of (6.19) will for brevity be denoted by $L(g, X, V)$. For $g = 1$, we put $L(1, X, V) = L(X, V)$, and Sign $(1, X, V) =$ Sign (X, V).

Formula (6.19) is of interest because, as we vary V, it provides a whole set of "integrality theorems" for Pontrjagin numbers.

Finally we should perhaps point out that if X is a spin-manifold, then the operators of this section are closely related to the Dirac operator of § 5. This connection arises from the isomorphism of Spin $(2l)$-modules

$$(-1)^l(\Delta^+ - \Delta^-) = \Lambda_+(R^{2l}) - \Lambda_-(R^{2l})\,.$$

7. Invariants for free actions

It is well-known that the characteristic numbers of a manifold are invariants of the appropriate cobordism group. Thus the Pontrjagin numbers are invariants of oriented cobordism (SO-cobordism), and the Chern numbers are invariants of almost complex cobordism (U-cobordism). For a G-manifold, one can similarly define G-characteristic numbers using the various fixed-point sets of elements of G, and these will be G-cobordism invariants. For example the formulas for the index-character of the various operators of § 4, 5, 6 are all expressed in terms of such G-characteristic numbers. Moreover the fact that these particular combinations of characteristic numbers are characters of G provides "integrality theorems" analogous to the well-known integrality theorems of [4] and [8] for the usual characteristic numbers.

These remarks apply to even-dimensional manifolds, but by a change of view-point, they can be exploited for odd-dimensional manifolds on which G acts *freely*. Suppose for example that X is a compact oriented free G-manifold of dimension $2n - 1$, and suppose we can find an oriented G-manifold Y (not necessarily G-free) of dimension $2n$, with boundary X. Let $v \in K_{SO(2n)}(R^{2n})$ be any "universal symbol class", and let $u \in K_G(TY)$ be the element defined by v. If Y were *without boundary*, then we could for any $g \in G$ apply the index theorem in the form (2.17) and obtain an explicit expression, say $v(g, Y)$, involving the evaluation of certain characteristic classes on the fixed-point set Y^g of g. Since in our case Y has a boundary, this does not apply, but since X is G-free, we have

$$Y^g \cap \partial Y = \varnothing \qquad \text{for } g \neq 1$$

and so the expression $v(g, Y)$ still makes sense. We obtain in this way a function

$$g \longmapsto v(g, Y)$$

from the non-identical elements of G to $\mathbf{C}$. We denote this function by $v(Y)$. Note that it is just a complex-valued function, and not necessarily given by a character of G. Suppose now that Y' is another oriented G-manifold with boundary X, and that we put

$$Z = Y \cup_X (-Y')$$

(where $-Y'$ is Y' with the opposite orientation), so that Z is a closed oriented manifold without boundary. Because $v(g, Y)$ is computed from Y^g, and because

$$Z^g = Y^g + (-Y')^g \qquad \text{(disjoint)}$$

we have

$$v(g, Z) = v(g, Y) - v(g, Y') .$$

Now $v(g, Z)$ *is* given by a character because we can apply (2.17) to the *closed* G-manifold Z. Hence the residue class of $v(Y)$ *modulo characters* depends only on X, and we denote it by $v(X)$. It is clearly an invariant of free G-cobordism. Formally the reader will notice the close analogy with the invariant originally used by Milnor to distinguish exotic 7-spheres.

The invariants $v(X)$ we have just defined are definitely non-trivial as simple examples show. They can be used to derive many of the results of Conner-Floyd [12] on G-cobordism (cf. [2]). In fact we reverse the procedure of [12] in which G-cobordism is developed first, and then applied to fixed-point theory.

The detailed application of the invariants $v(X)$ to G-cobordism will not be developed here. Instead we shall concentrate on one particular case where much stronger results can be obtained. If we take v to be the universal symbol (say L) giving rise to the operator D^+ of § 6, then for a closed manifold Y of dimension $2n$ we have

$$L(g, Y) = \operatorname{Sign}(g, Y) ,$$

and this can be computed from the action of G on $H^n(Y; \mathbf{R})$. Because of its connection with cohomology, we can use L to define a more refined invariant than before.

First of all however we need to discuss an additivity property of the signature due to S. P. Novikov.[12] Suppose Y is an oriented G-manifold of of dimension $2n$ with boundary X, and let $\hat{H}^n(Y)$ denote the image of the

[12] We are indebted to Hirzebruch for drawing our attention to Novikov's result.

natural homomorphism

$$\varphi\colon H^n(Y, X) \longrightarrow H^n(Y) .$$

By Poincaré duality $H^n(Y)$ is dual to $H^n(Y, X)$, and hence the bilinear form B on $\hat{H}^n(Y)$ defined by

$$B(\varphi(a), \varphi(b)) = ab[Y]$$

is non-degenerate. B is symmetric for n even, skew-symmetric for n odd, and, as for closed manifolds, we can now define Sign (G, Y). Suppose now Y' is another oriented G-manifold with boundary $-X$ and that $Z = Y \cup_X Y'$. The additivity property is then

PROPOSITION (7.1). Sign $(G, Z) =$ Sign $(G, Y) +$ Sign (G, Y').

PROOF. Consider the cohomology sequences of (Z, Y) and (Z, Y')

$$H^n(Y', X) \xrightarrow{\alpha'} H^n(Z) \xrightarrow{\beta} H^n(Y)$$
$$H^n(Y') \xleftarrow{\beta'} H^n(Z) \xleftarrow{\alpha} H^n(Y, X)$$

where we replace $H^n(Z, Y)$ by $H^n(Y', X)$, and similarly with Y, Y' interchanged. By Poincaré duality, these two sequences are duals of each other. Thus $A = \operatorname{Im}\alpha$ and $A' = \operatorname{Im}\alpha'$ are mutual annihilators for the bilinear form $B(Z)$ on $H^n(Z)$. Hence $A \cap A'$ is the annihilator of $A + A'$, and so $H^n(Z)/A + A' \cong (A \cap A')^*$. On the other hand

$$\begin{aligned}(A + A')/A \cap A' &\cong A/A \cap A' \oplus A'/A \cap A' \cong \operatorname{Im}\beta\alpha \oplus \operatorname{Im}\beta'\alpha' \\ &\cong \hat{H}^n(Y) \oplus \hat{H}^n(Y') .\end{aligned}$$

Thus, splitting the filtration $A \cap A' \subset A + A' \subset H^n(Z)$ (G-invariantly), we get a decomposition of G-modules

$$H^n(Z) \cong (A \cap A') \oplus \hat{H}^n(Y) \oplus \hat{H}^n(Y') \oplus (A \cap A')^* ,$$

and with respect to this decomposition the bilinear form $B(Z)$ is represented by a matrix of the form

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & B(Y) & 0 & * \\ 0 & 0 & B(Y') & * \\ (-1)^n & * & * & * \end{pmatrix} .$$

By a transformation $T^t(\ )T$, i.e., by a change of splitting, all the terms $*$ can be eliminated. Then

$$B(Z) \cong B(Y) \oplus B(Y') \oplus C ,$$

where C is the natural bilinear form on $(A + A') \oplus (A + A')^*$. As remarked in § 5, the G-signature of such a C is always zero. Hence

$$\mathrm{Sign}\,(G, Z) = \mathrm{Sign}\,(G, Y) + \mathrm{Sign}\,(G, Y')$$

as required.

Suppose now that X is a closed oriented G-manifold of dimension $2n - 1$, and assume that we can find an oriented G-manifold Y with boundary X. Then, for any element $g \in G$ *having no fixed-points* on X, we can define

$$\sigma(g, X) = L(g, Y) - \mathrm{Sign}\,(g, Y) .$$

We shall now show that this is independent of the choice of Y. Suppose then that Y' is another choice, and that $Z = Y \cup_X (-Y')$ is the closed oriented G-manifold obtained by attaching Y and $-Y'$ along their common boundary X. By (7.1) we have

$$\mathrm{Sign}\,(g, Z) = \mathrm{Sign}\,(g, Y) - \mathrm{Sign}\,(g, Y') \tag{7.2}$$

and, since Z^g is the disjoint sum of Y^g and $(-Y')^g$, we have

$$L(g, Z) = L(g, Y) - L(g, Y') . \tag{7.3}$$

Subtracting (7.3) from (7.2), we obtain

$$\begin{aligned}\{L(g, Y) - \mathrm{Sign}\,(g, Y)\} - \{L(g, Y') - \mathrm{Sign}\,(g, Y')\}& \\ = L(g, Z) - \mathrm{Sign}\,(g, Z)& \\ = 0& \qquad \text{by (6.12)} .\end{aligned}$$

Thus our invariant $\sigma(g, X)$ is well-defined. It is a G-diffeomorphism invariant. In fact, a minor modification in the proof we have just given shows it is an invariant in a rather stronger sense. Thus suppose X' is another G-manifold (on which g has no fixed point), and suppose that X and X' are equivalent in the following sense:

(A) there exists an oriented G-manifold W of dimension $2n$ with $\partial W = X - X'$, $W^g = \varnothing$, and $\hat{H}^n(W; \mathbf{R}) = 0$.

Then if $\partial Y = X, \partial Y' = X'$ we put

$$Z = Y \cup_X W \cup_{X'} (-Y') .$$

Computing as before we find that W contributes nothing either to L or to Sign, so that we have

$$\begin{aligned}\sigma(g, X) - \sigma(g, X') &= L(g, Z) - \mathrm{Sign}\,(g, Z) \\ &= 0 .\end{aligned}$$

We state this in a theorem.

THEOREM (7.4). *Let G be a compact Lie group, and let X be a closed oriented G-manifold of dimension* $(2n - 1)$ *which bounds an oriented G-*

manifold Y. For any $g \in G$ having no fixed point on X, we define the complex number $\sigma(g, X)$ by

$$\sigma(g, X) = L(g, Y) - \operatorname{Sign}(g, Y) ,$$

where $L(g, Y)$ is the number occurring on the right side of the formula in (6.12) *(with X replaced by Y), and* Sign (g, Y) *is defined by the action of G on $\hat{H}^n(Y; \mathbf{R})$ (the image of $H^n(Y, X; \mathbf{R}) \to H^n(Y; \mathbf{R})$)—endowed with its bilinear form—as in* (6.7) *and* (6.9). *Then $\sigma(g, X)$ depends only on X and not on Y. Moreover if X' is another G-manifold, and if there exists an oriented G-manifold W with $\partial W = X - X'$, $W^g = \emptyset$ and $\hat{H}^n(W; \mathbf{R}) = 0$, then*

$$\sigma(g, X) = \sigma(g, X') .$$

Theorem (7.4) is particularly interesting when G is finite, X is simply-connected and G acts freely on X, so that X is the universal covering space of $M = X/G$. If $M' = X'/G$ is h-cobordant to M (i.e., if there exists an oriented V with $\partial V = M - M'$ and with $M \to V$, $M' \to V$ homotopy equivalences), then the universal covering space W of V provides an equivalence as in (7.4) so that $\sigma(g, X) = \sigma(g, X')$ for $g \neq 1$. Hence we have

COROLLARY (7.5). *Let M be an oriented $(2n - 1)$-manifold with finite fundamental group G. Assume its universal covering space X bounds some oriented G-manifold. Then $\sigma(g, X)$, for $g \neq 1$, is an h-cobordism invariant of M.*

Remarks. 1. It has been pointed out to us by F. Hirzebruch and C. T. C. Wall that, for free actions of finite groups, the invariant σ can be defined without the hypothesis in (7.5). To do this, one has to appeal to the free cobordism theory of Conner-Floyd [12] according to which (when $\dim X = 2n - 1$) some multiple NX always bounds an oriented free G-manifold Y. We can then define

$$\sigma(g, X) = \frac{-1}{N} \operatorname{Sign}(g, Y) . \qquad (g \neq 1) .$$

Although this is a simple and more general definition, the fixed-point version is still very useful for computing σ, as is shown by the case of lens spaces (see below).

2. When X is a homotopy $(4k - 1)$-sphere, and g is a fixed-point-free involution, F. Hirzebruch has shown that our invariant $\sigma(g, X)$ coincides with the invariant of Browder-Livesay [11].

A specially simple application of (7.5) is to the case of lens spaces, when X is a sphere and G acts orthogonally. This is the case studied in detail in [2] where it is shown that our invariant σ distinguishes the different lens spaces. Of course for lens spaces we can take Y to be a ball, there is a unique fixed point and so we do not need the general fixed-point theory developed in this paper.

Since a connected group acts trivially on cohomology, the main interest in $\mathrm{Sign}\,(g, X)$ is with finite groups. For connected groups however one obtains some interesting identities which will be discussed in the next section. In the meantime we shall prove

PROPOSITION (7.6). *The function σ in* (7.5) *is an analytic function on the open set $U \subset G$ consisting of elements having no fixed point on X. If G is connected, so that $R(G)$ is an integral domain, then σ is defined by a unique element of the field of fractions of $R(G)$.*

PROOF. For elements $g \in G$ with the same fixed-point set in Y (where $\partial Y = X$ as in (7.5)), the explicit nature of the formula

$$\sigma(g, X) = L(g, Y) - \mathrm{Sign}\,(g, Y)$$

shows at once that σ is analytic in g. To deal with the general case where the fixed-point set changes, we must go back to the properties of the index given in [7]. The excision property implies that we have a well-defined homomorphism

$$\mathrm{ind}^{\mathring{Y}}\colon K_G(T\mathring{Y}) \longrightarrow R(G)$$

where $\mathring{Y}$ is the open manifold $Y - X$. Now let γ be a conjugacy class in G having no fixed points in X, and consider the inclusions

$$(TY)^{\gamma} \xrightarrow{\ i\ } T\mathring{Y} \xrightarrow{\ j\ } TY\ .$$

By the general localization theorem[13] of [20], j induces an isomorphism $(j_*)_\gamma$ in the following diagram

$$\begin{tikzcd}
K_G(T\mathring{Y})_\gamma \arrow[r, "(j_*)_\gamma"] \arrow[d, "\mathrm{ind}^{\mathring{Y}}_\gamma"'] & K_G(TY)_\gamma \arrow[dl, dashed, "\mathrm{ind}^{Y}_\gamma"] \\
R(G)_\gamma &
\end{tikzcd}$$

and hence ind^Y_γ can be defined to make the diagram commute. If, however, we restrict from G to the closed subgroup generated by $g \in \gamma$, the homomor-

[13] Here we need the localization theorem for general groups G, whereas in [5] we only gave the proof for abelian groups.

phism ind_γ^{Y0} can be calculated in terms of the fixed point manifold Y^g. This was how we obtained our general Lefschetz formula. In particular, if we let $a_\gamma \in K_G(TY)_\gamma$ be the element defined by the symbol class $a \in K_G(TY)$ of the operator D^+ of § 6, we have

$$\mathrm{ind}_\gamma^Y a_\gamma(g) = L(g, Y) , \tag{7.7}$$

where the left side is obtained by the map $R(G)_\gamma \to \mathbf{C}$ (evaluation at g or γ). This alternative expression for $L(g, Y)$ is better for investigating the dependence on g. In fact our proposition now follows rather formally as we shall see. For brevity, we put

$$R = R(G), \quad M = K_G(T\mathring{Y}), \quad N = K_G(TY)$$

so that M, N are R-modules. We now consider the continuous map $G \xrightarrow{\varphi} \mathrm{Spec}\, R(G)$, where $\mathrm{Spec}\, R(G)$ is the affine scheme of $R(G)$ and $\varphi(g)$ is the prime ideal defined by g. We let $\tilde{R}$ be the sheaf on G induced by φ from the structure sheaf on $R(G)$. Thus the stalk $\tilde{R}_g$ is just the local ring $R(G)_\gamma$ (where γ is the conjugacy class of g), and $\tilde{R}$ may be identified with a subsheaf of the sheaf of germs of analytic functions on G. The R-modules M, N define $\tilde{R}$-sheaves $\tilde{M}, \tilde{N}$ on G, and the R-homomorphisms

$$\begin{array}{ccc} M & \xrightarrow{j_*} & N \\ \downarrow{\scriptstyle \mathrm{ind}} & & \\ R & & \end{array}$$

induce $\tilde{R}$-homomorphisms

$$\begin{array}{ccc} \tilde{M} & \xrightarrow{\tilde{j}_*} & \tilde{N} \\ \downarrow{\scriptstyle \widetilde{\mathrm{ind}}} & & \\ \tilde{R} \, . & & \end{array}$$

The localization theorem implies that $\tilde{j}_*$ *is an isomorphism over the open set* $U \subset G$, so that restricting to U we get a diagram

$$\begin{array}{ccc} \tilde{M}_U & \xrightarrow{\cong} & \tilde{N}_U \\ \downarrow{\scriptstyle \widetilde{\mathrm{ind}}_U} & & \\ \tilde{R}_U \, . & & \end{array}$$

The element $a \in N$ defines a section of $\tilde{N}$, and hence a section a_U of $\tilde{M}_U$. Let L_U be the section of $\tilde{R}_U$ given by

$$L_U = \widetilde{\mathrm{ind}}_U\,(a_U) \, .$$

Then L_U is an analytic function on U. On the other hand, by (7.7) it is clear that the value of L_U at $g \in U$ is just $L(g, Y)$. Thus $L(g, Y)$ is an analytic function of g in U. Since the other term Sign (g, Y) in the definition of $\sigma(g, X)$ is a character of G, it follows that σ is analytic in U. To prove the last part of the proposition when G is connected, we take an element $g \in U$ which generates a maximal torus T (i.e., the powers of g are dense in T). Since $R(G) \to R(T)$ is injective, it follows that only the zero of $R(G)$ vanishes at g. Hence, if γ denotes the conjugacy class of g, the local ring $\widetilde{R}_g = R(G)_\gamma$ is just the full field $F(G)$ of fractions of $R(G)$. In the neighborhood of g, the analytic function $L(\ , Y)$ is therefore given by an element of $F(G)$. If U were connected, we could appeal to analytic continuation to get the uniqueness of this element. In fact U need not be connected.[14] But we can argue as follows. Let $V \subset U \cap T$ be the set of all elements having the same fixed points as g (i.e., as T). Then V is open (because it consists of elements having no fixed points in $Y - Y^g$) and dense (because it contains all generators of T). For any element $h \in V$, the explicit fixed-point expression for $L(h, Y)$ shows at once that this is given by a unique element f of the field $F(T)$. Since V is open and dense the analytic function L is entirely determined by $f \in F(T)$. Since, in a neighborhood of g, L is given by an element in $F(G)$, it follows that $f \in F(G)$, and the proof is complete.

As the simplest illustration of (7.6), let us consider circle actions. Thus we assume the circle G acts on Y and has no fixed points (for the whole group) on the boundary X. Then we have

$$R(G) = \mathbf{Z}[t, t^{-1}]$$
$$F(G) = \mathbf{Q}(t) ,$$

and we can interpret t as the coordinate $e^{i\theta}$ of a "general point" of G. Let $Z \subset Y$ be the points fixed under the whole group G. The normal bundle N of Z in Y can be written as a (finite) direct sum $N = \sum_{k>0} N_k$, where N_k is the complex vector bundle on which t acts as t^k. The manifold Z inherits then a natural orientation from those of N and Y. Since the connected group G acts trivially on the cohomology of Y, the Lefschetz formula (6.12), applied to the general element t of G, gives

$$\sigma(t, X) = \left\{2^m \prod_k \left\{\prod_j \frac{t^k e^{x_j} + 1}{t^k e^{x_j} - 1}(N_k)\right\}\mathfrak{L}(Z)\right\}[Z] - \operatorname{Sign}(Y) , \tag{7.7}$$

where, as usual, the elementary symmetric functions of the $x_j(N_k)$ are the Chern classes of N_k, and $2m = \dim Z$. Formula (7.7) shows clearly that $\sigma(t, X)$

[14] If we had introduced the complexification G^c of G, which is an affine algebraic group, all our sheaves could have been defined over a Zariski open set W containing U, and L_U would have been the restriction of a rational function G^c having no poles in W.

is a rational function of t (with coefficients in $\mathbf{Q}$), and that its denominator is a product of factors $1 - t^k$. Thus $\sigma(t, X)$ can only have a pole at a point $t = \alpha \in G$ which has a larger fixed point set than Z. Proposition (7.6) however gives a stronger result, namely that α cannot be a pole if $Y^\alpha \cap X = \emptyset$. Moreover the value $\sigma(\alpha, X)$ at such a point α is given by applying the formula (6.12) to the fixed point set Y^α.

The function $\sigma(t, X)$ appears to be quite an interesting invariant of the circle action. A simple deduction from (7.7), pointed out by F. Hirzebruch, is that $\sigma(t, X)$ is finite as $t \longrightarrow \infty$, and that its value there is

$$\begin{aligned} \sigma(\infty, X) &= 2^m \mathfrak{L}(Z)[Z] - \operatorname{Sign}(Y) \\ &= \operatorname{Sign}(Z) - \operatorname{Sign}(Y)\,. \end{aligned} \tag{7.8}$$

If the circle acts *freely* on X, we can take Y to be the associated disc bundle with Z as zero-section. If $x \in H^2(Z)$ denotes the first Chern class of the circle bundle $X \longrightarrow Z$, the formula (7.7) for $\sigma(t, X)$ reduces to

$$\sigma(t, X) = \left\{2^{n-1}\left(\frac{te^x + 1}{te^x - 1}\right)\mathfrak{L}(Z)\right\}[Z] - \operatorname{Sign}(Y)\,. \tag{7.9}$$

Moreover, since $H^n(Y, X) \longrightarrow H^n(Y)$ can be identified with the homomorphism $H^{n-2}(Z) \longrightarrow H^n(Z)$ given by multiplication with x, we see that $\operatorname{Sign}(Y)$ is just the signature of the degenerate form on $H^{n-2}(Z)$ given by $(u, v) \longmapsto xuv[Z]$. It is zero for odd n, and for even n, it can be interpreted as the signature of the quadratic form on $H^{n-2}(Z^2)$ restricted to the image of $H^{n-2}(Z)$. Here Z^2 denotes a self-intersection manifold of Z in Y.

The only possible pole of $\sigma(t, X)$ in (7.9) is at $t = 1$. The value for $t = -1$ is, as we have seen in (6.15), given by

$$\sigma(-1, X) = \operatorname{Sign}(Z^2) - \operatorname{Sign}(Y)\,.$$

For general circle actions on X without fixed points (but not necessarily free), it is not clear that we can find a G-manifold Y with $\partial Y = X$. However as we have remarked before, the restriction σ_f of σ to all finite subgroups of G can be defined without assuming the existence of Y. Since the points of finite order are dense on the circle, it follows that there is at most one analytic function σ (defined for $t \neq 1$) equal to σ_f at all points of finite order. However it is conceivable that σ_f may not extend to an analytic (or even continuous) function. This would prove that X cannot bound a G-manifold. It would be interesting to know if such a situation can actually occur. The same questions arise of course for any compact Lie group, the elements of finite order being always dense.

8. Vector fields

Our general Lefchetz theorem, when applied to one-parameter groups, leads to interesting identities on characteristic classes which we shall now describe. There are two cases of special interest, namely the riemannian and hermitian.

Let X be a compact oriented riemannian manifold of dimension $2l$. We recall [18] that the group of isometries of X is a compact Lie group G. Suppose now that A is a vector field on X which is an infinitesimal isometry. This means that the corresponding one-parameter group $\exp(tA)$ is a subgroup of G. The zero set X^A of A is fixed under the whole group $\exp(tA)$. Moreover for $0 < |t| < \varepsilon$ the fixed-point set of $\exp(tA)$ is precisely X^A.

The operator D^+ of § 6 is invariant under the group G of isometries of X and in particular therefore it is invariant under the one-parameter group $\exp(tA)$. Applying the G-signature theorem (6.12) to the element $g_t = \exp(tA)$, we get

(8.1) $$\operatorname{Sign}(g_t, X) = L(g_t, X) .$$

More generally if V is any vector bundle associated to the riemannian structure, we apply (6.19) and obtain

(8.2) $$\operatorname{Sign}(g_t, X, V) = L(g_t, X, V) .$$

We now consider both sides of this formula as functions of t for $0 < t < \varepsilon$. Since $\operatorname{Sign}(g_t, X, V)$ is given by a character of G, it is an analytic function of t, and as $t \to 0$

(8.3) $$\operatorname{Sign}(g_t, X, V) \longrightarrow \operatorname{Sign}(1, X, V) = \operatorname{Sign}(X, V) .$$

On the other hand, $L(g_t, X, V)$ is given by evaluating a cohomology class over X^A and, if we examine the explicit formula for $L(g_t, X, V)$, we see that it is analytic in t but has a pole as $t \to 0$. More precisely, each component of X^A of dimension $2m$ has a pole of order (at most) $n - m$. Thus we have a Laurent series

(8.4) $$L(g_t, X, V) = \textstyle\sum_{i=-n}^{\infty} a_i t^i ,$$

where the coefficients a_i are given by evaluating certain explicit expressions in characteristic classes over X^A. From (8.2), (8.3), and (8.4), we deduce

(8.5) $$\begin{cases} a_i = 0 & -n \leqq i \leqq -1 \\ a_0 = \operatorname{Sign}(X, V) . \end{cases}$$

On the other hand, putting $t = 0$ in (8.1), we have $\operatorname{Sign}(X, V) = L(X, V)$ and so

(8.6) $$a_0 = L(X, V) .$$

Since both sides of (8.6) are linear in V, it follows that (8.6) still holds if we replace V by an element of $K_G(X)$ (associated to the riemmannian structure). The most convenient elements to take are those given by the following lemma.

LEMMA (8.7). *Let $f(t_1, \cdots, t_l)$ be any symmetric polynomial in l variables with integer coefficients. Then there exists $u_f \in R(SO(2l))$ so that in $H^*_{SO(2l)}(\mathbf{Q})$*

$$\operatorname{ch} u_f = f(x_1^2, \cdots, x_l^2) + \text{higher terms} .$$

PROOF. Since the Chern character is a ring homomorphism $R(SO(2l)) \to H^*_{SO(2l)}(\mathbf{Q})$, it is sufficient to show that u exists when f is an elementary symmetric function σ_k of degree k. We now define u_k to be the coefficient of t^k in

$$\sum_{i=0}^{2l} \lambda^i t^i (1-t)^{2l-i} ,$$

where $\lambda^i = \lambda^i(\mathbf{C}^{2l}) = \lambda^i(\mathbf{R}^{2l}) \otimes_{\mathbf{R}} \mathbf{C}$ is regarded as an element of $R(SO(2l))$. Then we have

$$\sum_{k=0}^{2l} u_k t^k = (1-t)^{2l} \sum \lambda^i \left(\frac{t}{1-t}\right)^i$$

and so

$$\begin{aligned} \operatorname{ch}\left(\sum u_k t^k\right) &= (1-t)^{2l} \prod_{i=1}^{l} \left(1 + \frac{t}{1-t} e^{x_i}\right)\left(1 + \frac{t}{1-t} e^{-x_i}\right) \\ &= \prod \left(1 + t(e^{x_i} - 1)\right)\left(1 + t(e^{-x_i} - 1)\right) . \end{aligned}$$

Hence equating coefficients

$$\begin{aligned} u_k &= \sigma_k(e^{x_i} - 1, \cdots, e^{x_l} - 1, e^{-x_i} - 1, \cdots, e^{-x_l} - 1) \\ &= \sigma_k(x_1, \cdots, x_l, -x_1, \cdots, -x_l) + \text{higher terms.} \end{aligned}$$

Thus $(-1)^k u_{2k} = \sigma_k(x_1^2, \cdots, x_l^2) + \cdots$, and so this is the required element of $R(SO(2l))$.

Now assume $l = 2k$ so that $\dim X = 4k$, and let f, u_f be as above with $\deg f = k$. We consider the element

$$u_f(TX) \in K_G(X)$$

associated to the tangent bundle of X. To calculate $L(g_t, X, u_f(TX))$ we shall need to calculate

$$\operatorname{ch}\left(i^* \cdot u_f(TX)\right)(g_t) ,$$

(where $g_t = \exp tA$). The normal eigenvalues $e^{\pm i\theta}$ of g_t are now of the form $e^{\pm it\alpha}$ where $\pm i\alpha$ are the eigenvalues of the skew adjoint transformation of the normal plane induced by the vector field A. Note in particular that (for small t) the eigenvalue -1 does not occur, so that $N^g(-1) = 0$. Now to calculate $\operatorname{ch}\left(i^* u_f(TX)\right)(g)$ on a component of X^A of dimension $2m$, we have only to adopt the following prescription. We take $\operatorname{ch} u_f$, replace the last $l - m$ of the variables x_j by $y_j + it\alpha_j$, and then make the symmetric functions of $(x_1^2, \cdots, x_m^2)$

act on TX^A, while the symmetric functions of the y_j corresponding to a given α act on $N_\alpha(X^A)$, the part of the normal bundle defined by α.

Thus we obtain the formula

$$
\begin{aligned}
L(g_t, X, u_f(TX)) \\
&= \left\{2^m \frac{\operatorname{ch} u_f(x_1, \cdots, x_m, \cdots, y_j + it\alpha_j, \cdots)}{\prod \tanh \frac{y_j + it\alpha_j}{2}} \mathfrak{L}(X^A)\right\}[X^A] \\
(8.8) \qquad &= \left\{2^l \frac{\operatorname{ch} u_f(x_1, \cdots, x_m, \cdots, y_j + it\alpha_j, \cdots)}{t^{l-m}} \prod \left(i\alpha_j + \frac{y_j}{t}\right)^{-1} Q(t)\right\}[X^A]
\end{aligned}
$$

where $Q(t)$ is a power series in t with constant term 1. Now the term in $\{\ \}$ of the right degree in (x, y) (namely of degee m) is easily seen to be holomorphic in t. Moreover the value for $t = 0$ involves only the constant term of $Q(t)$ and the first term, namely f, in $\operatorname{ch} u_f$. Thus putting $t = 0$, we deduce

$$
(8.9) \qquad L(X, u_f(TX)) = \{2^l f(x_1^2, \cdots, x_m^2, \cdots (y_j + i\alpha_j)^2, \cdots) \prod (i\alpha_j + y_j)^{-1}\}[X^A] .
$$

On the other hand computing $L(X, u_f(TX))$ directly, we find

$$
(8.10) \qquad L(X, u_f(TX)) = 2^l f(x_1^2, \cdots, x_m^2)[X] .
$$

Suppose now that we take f to be a polynomial of degree $q < k$. Then we can proceed as above to calculate

$$
L(g_t, X, \rho \cdot u_f(TX))
$$

where $\rho \in R(T)$ is any character of the torus T, generated by $\exp(tA)$, such that $\rho(g_t)$ is divisible by precisely $t^{2(k-q)}$. We then obtain a formula just like (8.9), except that the left-hand side is now zero. Formally this fits in with (8.10) because, when $\deg f < k$, we have

$$
f(x_1^2, \cdots, x_m^2)[X] = 0 .
$$

Thus we have established the following theorem.

THEOREM (8.11). *Let X be a compact oriented manifold of dimension $4k$, and let A be an infinitesimal isometry. Let X^A denote the zero-set of A, and let $\pm i\alpha_j$ be the eigenvalues of the skew-adjoint transformation N_A induced by A in the normal bundle of X^A (the α_j will be constant on each component). We orient the normal bundle so that, for an oriented basis, N_A is given by the matrix $\bigoplus_j \begin{pmatrix} 0 & -\alpha_j \\ \alpha_j & 0 \end{pmatrix}$ where $\alpha_j > 0$. We then take the induced orientation on X^A. Now let $f(t_1, \cdots, t_l)$ be a symmetric polynomial homogeneous of degree $q \leqq k$. Then we have*

$$\begin{aligned}\{f(\cdots, x_i^2, \cdots, (y_j + i\alpha_j)^2, \cdots) \textstyle\prod (i\alpha_j + y_j)^{-1}\}[X^A] & \\ = f(x_1^2, \cdots, x_{2k}^2)[X] \qquad & \textit{for } q = k \\ = 0 \qquad & \textit{for } q < k\,.\end{aligned}$$

Here the symmetric functions in the x_i^2 act on TX^A (and give its Pontrjagin classes), while the symmetric functions in the y_j (for a fixed α) give the Chern classes of $N_\alpha(X^A)$.

This theorem gives an expression for the various Pontrjagin numbers of X in terms of Pontrjagin numbers and normal eigenvalues of X^A. The simplest case occurs when X^A consists of a finite set of points $\{P\}$ so that (8.11) becomes

$$\begin{aligned}\sum \varepsilon(P)(-1)^k \frac{f(\alpha_1^2(P), \cdots, \alpha_{2k}^2(P))}{\prod \alpha_j(P)} & \\ = f(x_1^2, \cdots, x_{2k}^2)[X] \qquad & \text{for } \deg f = k \\ = 0 \qquad & \text{for } \deg f < k\end{aligned} \tag{8.12}$$

where $\varepsilon(P) = 1$ if our convention for orientation of T_P agrees with that of X and -1 otherwise.

Formula (8.12) has been proved by direct differential-geometric methods by Bott [9], and similar methods [10] also yield (8.11). Naturally Bott uses the curvature-description of characteristic classes (cf. § 1). Our proof is algebro-topological, because, despite appearances, we have in fact used no analysis. Essentially (8.11) is a simple consequence of the localisation theorem of [5], and computations with characteristic classes.

We pass now to the case of complex structures. If we have a vector field A preserving a complex structure on X, the one-parameter group $\exp(tA)$ lies in the group of all holomorphic automorphisms of X. Unfortunately the connected component of this group is a Lie group but not in general compact. We cannot therefore apply our methods without making a further restriction, whereas the differential-geometric methods of Bott [9] still work. On the other hand, our methods use only the *almost complex structure*: integrability is irrelevant for our topological considerations. Using the formula of (4.6) and calculations entirely analogous to those above, we obtain

PROPOSITION (8.13). *Let X be an almost complex hermitian manifold of dimension l, and let A be a vector field preserving the metric and the almost complex structure. Let X^A denote the zero-set of A. Then the tangent and normal bundles of X^A have natural complex structures, and the skew-hermitian transformation N_A induced by A on the normal bundle of X^A has eigenvalues $i\alpha_j$. Let $f(t_1, \cdots, t_l)$ be a symmetric polynomial homogeneous of degree $q \leqq l$. Then we have*

$$\begin{aligned}\{f(x_1, \cdots, y_j + i\alpha_j, \cdots) \textstyle\prod (i\alpha_j + y_j)^{-1}\}[X^A] & \\ = f(x_1, \cdots, x_l)[X] & \qquad \textit{for } q = l \\ = 0 & \qquad \textit{for } q < l\,.\end{aligned}$$

Here the symmetric functions in the x_i act on TX^A (and give its Chern classes), while the symmetric functions in the y_i (for fixed α) give the Chern classes of $N_\alpha(X^A)$.

In addition to results like (8.12) and (8.13), there are more precise results which use the analysis of [7]. Thus if we return to (8.1) and recall that Sign (g_t, X) depends only on the action of g_t on the cohomology of X, we see that

$$\operatorname{Sign}(g_t, X) = \operatorname{Sign}(1, X) = \operatorname{Sign}(X)$$

is actually *independent of t*. Hence, in addition to the identities of (8.5), we have in this case an infinite sequence of identities obtained by equating to zero all positive powers of t in the expansion (8.4) for $V = 1$.

Similar remarks apply to Kähler manifolds. Thus if G is a compact connected group of automorphisms of a Kähler manifold, then G acts trivially on the sheaf cohomology group $H^q(X, \mathcal{O})$—because these are canonically isomorphic to subspaces of the complex cohomology groups $H^q(X; \mathbf{C})$.

9. Miscellaneous special cases

The special cases of the index and Lefschetz theorems we have discussed in the preceding sections were all ones arising from the geometry of some structure. In this section we discuss a number of special cases not of this type.

Consider first the case of *differential operators on odd-dimensional manifolds*. The symbol σ of a differential operator of order r satisfies the symmetry condition

$$\sigma\big(\alpha(\xi)\big) = (-1)^r\sigma(\xi)\,, \tag{9.1}$$

where ξ is a cotangent vector, and $\alpha(\xi) = -\xi$ is the antipodal map on TX. For the class $[\sigma]$ in $K(TX)$, the factor $(-1)^r$ can be ignored; we have

$$\alpha^*[\sigma] = [\sigma] \qquad \in K(TX)$$

and so

$$\operatorname{ch} \alpha^*[\sigma] = \operatorname{ch}[\sigma] \qquad \in H^*(TX; \mathbf{Q})\,.$$

Hence by the Index Theorem (2.12), we have

$$\begin{aligned}\operatorname{index}\sigma &= (-1)^n \operatorname{ch}[\sigma]\cdot\mathcal{I}(X)[TX] \\ &= (-1)^n\alpha^* \operatorname{ch}[\sigma]\cdot\mathcal{I}(X)[\alpha_*(TX)] \\ &= (-1)^n \operatorname{ch}[\sigma]\cdot\mathcal{I}(X)\cdot(-1)^n[TX] \\ &= -\operatorname{index}\sigma\ ,\end{aligned}$$

and so index $\sigma = 0$. The crucial point in this is of course that the map α on TX changes the orientation if n is odd. Thus we have established

PROPOSITION (9.2). *The index of an elliptic differential operator on an odd dimensional compact manifold is zero.*

Remark. The full index theorem is not needed to prove (9.2). It is not difficult to prove in fact that, for a differential operator d on an odd-dimensional manifold, the symbol gives an element of finite order in $K(TX)$. As soon as one knows that the index is a homomorphism $K(X)\to\mathbf{Z}$, this implies that index $d = 0$.

For Lefschetz numbers, the analogue of (9.2) does not necessarily hold. In fact the map $\theta\mapsto-\theta$ of the circle has a non-trivial Lefschetz number (equal to 2) for the de Rham complex, i.e., for the ordinary derivative $f\mapsto df$. However if we consider only orientation preserving actions, the situation is different. In fact if X is oriented, and $g\colon X\to X$ preserves the orientation, then the bundle $N^g(-1)$ (i.e., the part of the normal bundle to X^g with eigenvalue -1) has even dimension. Since all the bundles $N^g(\theta)$ corresponding to other eigenvalues always have even dimension, it follows that

$$\dim X^g \equiv \dim X \mod 2\ .$$

If now u is the symbol of a G-invariant differential operator d, then the restriction i^*u to TX^g will still satisfy the symmetry condition (9.1). Hence, if $\dim X$ is odd, (9.2), together with the Lefschetz theorem (3.9), implies that the Lefschetz number $L(g, u) = 0$. Thus we have

PROPOSITION (9.3). *Let X be a compact oriented manifold of odd dimension, G a compact Lie group of orientation preserving actions of X, and d an elliptic G-invariant differential operator on X. Then the G-index of d is zero.*

In the remainder of this section we shall consider examples in which the vector bundles are trivial. In other words, we consider elliptic operators acting on *systems of functions*. The symbol of such an operator is then a continuous map

$$\sigma\colon S(X)\longrightarrow GL(N, \mathbf{C})\ ,$$

where $S(X)$ is the unit sphere bundle of X, and N is the rank of the system.

We can therefore consider the induced homomorphism in cohomology[15]

$$\sigma^*: H^*(GL(N, \mathrm{C})) \longrightarrow H^*(S(X)) .$$

We recall now a few facts about the cohomology of $GL(N, \mathrm{C})$ and the relation of σ^* with characteristic classes.

First we recall that $U(N)$ is deformation retract of of $GL(N, \mathrm{C})$ so that

$$H^*(GL(N, \mathrm{C})) \cong H^*(U(N)) .$$

Then $H^*(U(N))$ is an exterior algebra generated by elements

$$h_i^N \in H^{2i-1}(U(N)) \qquad i = 1, \cdots, N .$$

These elements have the following additional properties:

(i) the restriction homomorphism takes h_i^N to h_i^{N-1} (for $i < N$),

(ii) $h_N^N = \pi^*(u_N)$ where $\pi: U(N) \to U(N)/U(N-1) = S^{2N-1}$ is the natural map and $u_N \in H^{2N-1}(S^{2N-1})$ is the natural generator[14].

In view of (i) we write h_i instead of h_i^N.

Since the harmonic forms on $U(N)$ are just the bi-invariant forms, one can easily write explicit differential forms w_i representing h_i.

Suppose now that B is a closed subspace of a compact space A and that

$$f: B \longrightarrow GL(N, \mathrm{C})$$

is a continuous map. Then (extending f to a map of A into $\mathrm{End}(\mathrm{C}^n)$), we get a complex on A exact on B, and so an element

$$[f] \in K(A, B) .$$

The characteristic classes of $[f]$ are related to the classes f^*h_i by $c_i[f] = \delta f^*h_i$ where $\delta: H^*(B) \to H^*(A, B)$ is the coboundary homomorphism. This then implies that

$$\mathrm{ch}\,[f] = \delta\left\{\sum_{i=1}^N \frac{(-1)^{i-1} f^*h_i}{(i-1)!}\right\} .$$

Returning now to our symbol σ we see that

$$\mathrm{ch}\,[\sigma] = \delta\left\{\sum_{i=1}^N \frac{(-1)^{i-1}\sigma^*h_i}{(i-1)!}\right\} \qquad \in H^*(TX)$$

where

$$\delta: H^*(SX) \longrightarrow H^*(BX, SX) \cong H^*(TX)$$

is the coboundary, and BX is the unit ball bundle of TX. Substituting this formula for $\mathrm{ch}\,[\sigma]$ in the index theorem (2.12), and using the formula

[15] Although $GL(N, \mathrm{C})$ is not compact, we take cohomology here with arbitrary (not compact) supports.

[16] S^{2N-1} is oriented as the boundary of the ball in C^N.

$$\delta u[TX] = u[SX]$$

for any $u \in H^*(SX)$, we deduce

PROPOSITION (9.4). *Let $\sigma: SX \to GL(N, \mathbf{C})$ be the symbol of an elliptic $N \times N$-system d. Then*

$$\text{index } d = (-1)^n\Big\{\Big(\sum_{i=1}^{N} \frac{(-1)^{i-1}\sigma^* h_i}{(i-1)!}\Big)\mathcal{I}(X)\Big\}[SX]$$

where $h_i \in H^{2i-1}(GL(N, \mathbf{C}))$ are the generators normalized as in (ii) *above, $\mathcal{I}(X)$ denotes the Index class of X and* $\dim X = n$.

As a simple example, suppose X is a hypersurface in $\mathbf{R}^{n+1}$. Then all Pontrjagin classes of Y are zero, and so $\mathcal{I}(X) = 1$. Hence as a Corollary of (9.4), we have

COROLLARY (9.5). *Let X be a hypersurface in $\mathbf{R}^{n+1}$, σ the symbol of an elliptic $N \times N$-system d on X. Then*

$$\begin{aligned} \text{index } d &= (-1)^{N+n-1}\frac{\sigma^* h_n}{(n-1)!}[SX] && \textit{if } N \geqq n \\ &= 0 && \textit{if } N < n\,. \end{aligned}$$

Remarks. 1. If $N = n$ then, by property (ii) of h_n, we see that $\sigma^* h_n[SX]$ can be interpreted as the degree of the composite map

$$\pi \circ \sigma: SX \longrightarrow S^{2n-1}$$

so that (9.5) takes the simple form

$$\text{index } d = \frac{-\text{degree } (\pi \circ \sigma)}{(n-1)!}\,. \tag{9.6}$$

2. Since the first Pontrjagin class occurs in dimension 4, it follows that $\mathcal{I}(X) = 1$ if $\dim X \leqq 3$. Hence (9.5) and (9.6) also hold for any X of dimension $\leqq 3$.

In (9.4) the evaluation on $[SX]$ can be reduced as usual to an evaluation on the twisted fundamental class $[X]$. For this we introduce the homomorphism

$$\pi_*: H^*(SX; \mathbf{Q}) \longrightarrow H^*(X; \tilde{\mathbf{Q}})$$

usually called "integration over the fibre". Here $\tilde{\mathbf{Q}}$ denotes twisted coefficients, and π_* lowers dimension by $n-1$. One way of defining π_* is as the composition

$$H^*(SX; \mathbf{Q}) \xrightarrow{\delta} H^*(TX; \mathbf{Q}) \xrightarrow{\psi^{-1}} H^*(X; \tilde{\mathbf{Q}})$$

where δ is the coboundary of the pair (BX, SX), and ψ is the Thom isomorphism. The formula of (9.4) can now be rewritten

$$\text{index}\, d = (-1)^n \left\{ \left(\sum \frac{(-1)^{i-1} \pi_* \sigma^* h_i}{(i-1)!} \right) \mathcal{J}(X) \right\} [X] \,. \tag{9.7}$$

Since

$$\dim \pi_* \sigma^* h_i = 2i - n \ ,$$

the summation in (9.7) need only be taken over $n/2 \leqq i \leqq N$. In particular, if $N \leqq (n-1)/2$, the summation is empty and so index $d = 0$. If $n = 2N$, we have just one term namely $\pi_* \sigma^* h_N$ in dimension 0. Since $\mathcal{J}(X)$ involves only dimensions divisible by 4, it follows that index $d = 0$ unless n is divisible by 4. If n is divisible by 4, we consider the exact cohomology sequence

$$\begin{array}{ccccc} H^{n-1}(SX; \mathbf{Q}) & \xrightarrow{\delta} & H^n(TX; \mathbf{Q}) & \xrightarrow{\alpha} & H^n(BX; \mathbf{Q}) \\ & & \uparrow{\scriptstyle\psi} & & \| \\ & & H^0(X; \tilde{\mathbf{Q}}) & & H^n(X; \mathbf{Q}) \,. \end{array}$$

Assuming that X is connected, we have

$$\begin{aligned} H^0(X; \tilde{\mathbf{Q}}) &\cong \mathbf{Q} \qquad && \text{if } X \text{ is orientable} \\ &= 0 && \text{if } X \text{ is non-orientable} \,. \end{aligned}$$

Moreover if X is oriented, and 1 denotes the generator of $H^0(X; \mathbf{Q})$, we have

$$\alpha\psi(1) = e(X) \ ,$$

where $e(X)$ is as usual, the Euler class of X. Hence if $e(X) \neq 0$, we have

$$\operatorname{Im} \delta = \operatorname{Ker} \alpha = 0 \ ,$$

and so $\delta\sigma^* h_N = 0$. Thus

$$\pi_* \sigma^* h_N = \psi^{-1} \delta \sigma^* h_N = 0 \ ,$$

and so index $d = 0$. We therefore have the following Corollary of (9.4) giving conditions under which index $d = 0$.

COROLLARY (9.8). *Let d be an elliptic $N \times N$ system on a compact n-manifold where $N \leqq n/2$. Then* index $d = 0$ *unless $n = 2N = 4k$, X is orientable, and has Euler number zero.*

Corollary (9.8) applies in particular when $N = 1$ and $n > 1$. Thus *when* $\dim X > 1$, *the index of an elliptic operator acting on functions is always zero.*

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