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Author(s): M. F. Atiyah and I. M. Singer

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The index of elliptic operators: IV

By M. F. ATIYAH and I. M. SINGER*

Introduction

In this paper we develop an *index-theory for families of elliptic operators*. For a single elliptic operator P on a compact manifold the index is an integer defined by

$$\text{index } P = \dim \text{Ker } P - \dim \text{Coker } P$$

and, in the earlier papers of this series [5], [6], we obtained an explicit formula for index P in purely topological terms. This formula involved the symbol of P and was expressed in terms of K -theory [5] or in terms of cohomology [6]. For a *family* P of elliptic operators P_y , parametrized by the points y of a compact space Y , we can define a more general index which is not an integer but is an element of $K(Y)$. Roughly speaking this is defined as the difference

$$\text{index } P = \text{Ker } P - \text{Coker } P.$$

Here $\text{Ker } P$ stands for the family of vector spaces $\text{Ker } P_y$ and similarly for $\text{Coker } P$. If $\dim \text{Ker } P_y$ is constant (independent of $y \in Y$) $\text{Ker } P$ is a vector bundle. The same holds for $\text{Coker } P$, and index P is then well-defined as an element of $K(Y)$, the Grothendieck group generated by vector bundles over Y . In general, when $\dim \text{Ker } P_y$ varies, this definition has to be modified slightly (see [1; Appendix] or [2] and also § 2).

The problem which we pose and solve in this paper is that of giving a topological description of this index of a family of elliptic operators. Both the formulation and the proof follow closely the lines of [5]. In fact, as explained in [5], the proof presented there was of such a type that it lent itself naturally to the sort of generalization we are considering now. To explain the situation in more detail we shall begin therefore by recalling briefly the main results of [5].

For an elliptic pseudo-differential operator P on the compact manifold X the highest order terms or “symbol” $\sigma(P)$ of P define a symbol class $[\sigma(P)] \in K(TX)$, where TX is the tangent bundle of X and K denotes K -theory with compact supports. The index of P is first shown to depend only

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on this symbol class and hence to define a homomorphism

$$K(TX) \longrightarrow \mathbf{Z}$$

called the analytical index. On the other hand we gave a purely K -theoretical construction of another homomorphism $K(TX) \rightarrow \mathbf{Z}$ which was called the topological index. The main theorem of [5] asserted that these two indices were in fact equal. The construction of the topological index goes as follows. We choose a smooth embedding $i: X \rightarrow V$ where V is some euclidean space and we let N be an open tubular neighborhood of X in V . Then N is a real vector bundle over X and TN can be identified with $\pi^*(N \otimes_{\mathbf{R}} \mathbf{C})$ —the complexification of N lifted up to TX by the projection $\pi: TX \rightarrow X$. We define a homomorphism

$$i_!: K(TX) \longrightarrow K(TV)$$

to be the composition of the Thom homomorphism

$$\varphi: K(TX) \longrightarrow K(TN)$$

together with the natural homomorphism

$$K(TN) \longrightarrow K(TV)$$

induced by the open inclusion $TN \subset TV$. Finally letting $j: A \rightarrow V$ be the inclusion of the origin A in V we have also

$$j_!: K(TA) \longrightarrow K(TV).$$

But $TA = A$ is a point, so $K(TA) = \mathbf{Z}$, and $j_!$ is the periodicity *isomorphism*. Thus we can define the *topological index*

$$K(TX) \longrightarrow \mathbf{Z}$$

as $(j_!)^{-1} \circ i_!$. It is independent of the choice of the embedding $X \rightarrow V$.

The generalization of all this to families over Y is fairly clear. What we have to do is to make all the preceding constructions “fibrewise” over Y . An elliptic family P of operators on X parametrized by points of Y will have a symbol class $[\sigma(P)] \in K(Y \times TX)$. The index of P in $K(Y)$ will depend only on the symbol class and hence we will obtain a homomorphism

$$K(Y \times TX) \longrightarrow K(Y)$$

which is the analytical index (for families over Y). On the topological side given an embedding $X \rightarrow N \rightarrow V$ we simply take the cartesian product with Y throughout and obtain a homomorphism

$$i_!: K(Y \times TX) \longrightarrow K(Y \times TV)$$

and the periodicity isomorphism

$$j_! : K(Y) \longrightarrow K(Y \times TV) .$$

Combining these we define the topological index as $(j_!)^{-1} i_!$ and our main theorem will again assert that the topological and analytical indices coincide.

So far we have had in mind a family of operators P_y on a fixed manifold X . In fact it is more natural to allow the manifold X to vary with y also, so that P_y is an operator on a manifold X_y . Of course we must assume suitable regularity of the family of manifolds X_y : they should form a fibre bundle Z over the parameter space Y and the structure group should be the group of diffeomorphisms of X . Moreover the vector bundles over X where P_y acts will now also vary with y . Thus P_y should be a linear operator, $C^\infty(X_y; E_y) \rightarrow C^\infty(X_y; F_y)$ where E_y, F_y are smooth vector bundles over X_y which vary continuously with y . The precise definitions will be given in § 1.

Having a fibre bundle Z instead of the product $Y \times X$ does not essentially alter the preceding discussion concerning the analytical and topological indices. Since Z is locally a product over Y the analysis is just the same except that we must make sure that various function spaces which we use are left invariant by diffeomorphisms of X . For the symbol $\sigma(P)$ we have to introduce the tangent bundle along the fibres of Z : we denote it by TZ (since Z is only differentiable in the fibre direction no confusion should arise). The symbol class $[\sigma(P)]$ will then be an element of $K(TZ)$ and the analytical index is now a homomorphism

$$K(TZ) \longrightarrow K(Y) .$$

The details will be given in § 2. On the topological side the only difference is that we have to embed Z in $Y \times V$ compatibly with the projections onto Y . It is easy to prove that such embeddings exist, and from this we obtain the topological index. Details are given in § 3.

To prove the main theorem (equality of topological and analytical index) we consider, as in [5], the diagram

$$\begin{array}{ccccc} & & K(TN) & & \\ & \nearrow \varphi & \downarrow \delta & \searrow \varepsilon & \\ K(TZ) & & K(TS) & & K(Y \times TV) \\ & \searrow \alpha & \downarrow \beta & \swarrow \gamma & \\ & & K(Y) & & \end{array}$$

where N is now a bundle (over Y) of tubular neighborhoods along the fibre, S is the double of N , and α, β, γ are given by the analytical index. To prove

the theorem we must prove that the two triangles commute and that γ is the inverse of the periodicity isomorphism. This last part follows at once from

(i) the case when Y is a point, proved in [5], and

(ii) the fact that the analytical index is a homomorphism of $K(Y)$ -modules (see the definition in § 2).

The commutativity $\beta\delta = \gamma\varepsilon$ is the *excision property* of the index and the commutativity $\alpha = \beta\delta\varphi$ is the *multiplicative property*. Both of these are established exactly as in [5]. We shall discuss them briefly in § 3 and § 4.

From this introduction it should now be apparent that there is very little to be done to extend the proof of [5] to the case of families. For this reason we shall be brief and restrict ourselves to commenting on the new points that arise. These are mainly technical and of a fairly routine kind. A couple of lengthy technical arguments are relegated to an appendix.

In the final section of the paper we give a cohomological form for the index of elliptic families and we also discuss the relation of the equivariant index theorem of [5] with the index theorem of this paper.

In paper V of this series we shall extend the index theorem to real operators and the index of real families will then have an interesting application to the mod 2 index of skew adjoint real elliptic operators (see [7]).

In conclusion we should point out that a somewhat weaker version of the index theorem for families was announced by W. Shih in [11]. The proof sketched there was a generalization of the original proof of the index theorem as given in [10]. The new proof in [5] generalizes in a much more straightforward manner.

1. Continuous families of elliptic operators

It is clear what a continuous family of differential operators on R^n parametrized by $y \in Y$ should be: namely $\sum_{|\alpha| \leq k} a_\alpha(x, y) \partial/\partial x^\alpha$ where $\partial a_\alpha/\partial x^\beta$ are continuous functions on $R^n \times Y$. In this section we extend this notion. First, we replace R^n by a manifold X . Then we replace $X \times Y$ by a fibre bundle over Y . We replace differential operators by pseudo-differential operators (and their closure in a suitable topology which we need for tensor products).

Let X be a C^∞ compact manifold and let $\text{Diff}(X)$ denote the group of diffeomorphisms of X endowed with the topology of uniform convergence for each derivative. It is well known [9] that $\text{Diff}(X)$ is a topological group, and $(\text{Diff } X) \times X \rightarrow X$ is continuous.

Definition (1.1). Let Y be a Hausdorff space. Then Z is a manifold over Y if Z is a fibre bundle over Y with fibre X and structure group $\text{Diff}(X)$. In particular, if $\pi: Z \rightarrow Y$ is the projection, there exists a covering \mathfrak{U} of Y and $\varphi_U: \pi^{-1}(U) \xrightarrow{\sim} U \times X$, $U \in \mathfrak{U}$ so that $\varphi_U \circ \varphi_V^{-1}: U \cap V \times X \rightarrow U \cap V \times X$ is given by $(y, x) \mapsto (y, f_{U,V}(y)(x))$ with $y \mapsto f_{U,V}(y)$ a continuous map from $U \cap V \rightarrow \text{Diff}(X)$.

Next, we want to explain what we mean by a vector bundle over Z which is C^∞ along the fibres. Let E be a C^∞ vector bundle over X and let $\text{Diff}(X, E)$ denote the group of diffeomorphisms of E which map fibres to fibres linearly. Note the homomorphism $h: \text{Diff}(X, E) \rightarrow \text{Diff}(X)$ with kernel $\text{Aut}(E)$. If $\Phi \in \text{Diff}(X, E)$, then $\Phi = \alpha_\Phi h(\Phi)$, uniquely, where $\alpha_\Phi: h(\Phi)^*E \xrightarrow{\sim} E$. We say Φ is a lift of $h(\Phi)$ to E by α_Φ . Using this decomposition, it is easy to make $\text{Diff}(X, E)$ a topological group. Specifically, choose a connection c on E so that E_{x_1} can be identified with E_{x_2} if x_2 is close to x_1 by parallel translation of E_{x_1} to E_{x_2} along the unique geodesic from x_1 to x_2 . Let C denote this isomorphism. Then a neighborhood of $I \in \text{Diff}(X, E)$ is given by those Φ where $h(\Phi)$ is in a small neighborhood of $I \in \text{Diff}(X)$ and $\alpha_\Phi \circ C$ is in a neighborhood of $I \in \text{Aut}(E)$ (in the C^∞ topology).

Definition (1.2). Let Z be a manifold over Y , and let \tilde{E} be a vector bundle over Z with projection p . Then \tilde{E} is a smooth vector bundle over Z if $\tilde{E} \xrightarrow{\pi \circ p} Y$ is a fibre bundle over Y with fibre E (a C^∞ vector bundle over X) and with group $\text{Diff}(X, E)$.

Remarks. 1. The bundle Z over Y is associated to the bundle \tilde{E} over Y by the homomorphism $h: \text{Diff}(X, E) \rightarrow \text{Diff}(X)$.

2. There are other (equivalent) ways of defining smooth vector bundles over Z . In particular a map $f: Z \rightarrow G$ (a grassmannian) defines a smooth vector bundle over Z provided all fibre derivatives of f are continuous.

Associated to the above fibre bundles are bundles of pseudo-differential operators which we now discuss. Let $\mathcal{P}^m(X; E, F)$ denote the space of m^{th} -order pseudo-differential operators on X from $C^\infty(X; E)$ to $C^\infty(X; F)$ introduced in [5, § 5]. This space is a Fréchet space given by the following of pseudo-norms. (We do the case for $E = F = 1$ and leave the slight generalization to the reader.) For each coordinate neighborhood U with coordinates x_1, \dots, x_n and each C^∞ function f with support in U , let

$$\|P\|_{U, f, \alpha, \beta} = \sup_{\substack{x \in U \\ \xi \in \mathbb{R}^n}} \left| \frac{\partial}{\partial x^\alpha} \frac{\partial}{\partial \xi^\beta} \frac{p_f(x, \xi)}{(1 + |\xi|)^{m - |\beta|}} \right|$$

where $p_f(x, \xi) = e^{-i\langle x, \xi \rangle} P(e^{i\langle x, \xi \rangle} f)$.

As in [5, § 5] let $\overline{\mathcal{P}^m}(X; E, F)$ denote the completion of $\mathcal{P}^m(X; E, F)$ relative to the family of pseudo-norms $\|P\|_s$ (the norms of the bounded operators $P_s: H_s(X, E) \rightarrow H_{s-m}(X, F)$ induced by P on the Sobolev spaces). The usual estimate showing that P_s is bounded implies that the injection $\mathcal{P}^m \rightarrow \overline{\mathcal{P}^m}$ is continuous.

Suppose now \tilde{E} and \tilde{F} are two smooth vector bundles over Z . We wish to construct new fibre bundles $\overline{\mathcal{P}^m}(Z; \tilde{E}, \tilde{F})$ over Y with fibre $\overline{\mathcal{P}^m}(X; E, F)$ (and $\mathcal{P}^m(Z; \tilde{E}, \tilde{F})$ over Y with fibre $\mathcal{P}^m(X; E, F)$). First, let H be the closed subgroup of $\text{Diff}(X, F) \times \text{Diff}(X, E)$ consisting of $[(\Psi, \Phi), h(\Psi) = h(\Phi)]$. Note that H operates on $\mathcal{P}^m(X; E, F)$ by $P \rightarrow \Psi^{-1}P\Phi$ (the operator $\Psi^{-1}P\Phi$ still lies in $\mathcal{P}^m(X; E, F)$ because the class of pseudo-differential operators is invariant under diffeomorphisms). Since Φ (and Ψ) induce bounded operators on all $H_s(E)$ (and $H_s(F)$), the action of H on \mathcal{P}^m extends to $\overline{\mathcal{P}^m}$. To construct the desired bundles we now need the following result whose proof we give in the appendix.

PROPOSITION (1.3). *The maps $H \times \overline{\mathcal{P}^m} \rightarrow \overline{\mathcal{P}^m}$ and $H \times \mathcal{P}^m \rightarrow \mathcal{P}^m$ are continuous.*

Let $B_{\tilde{E}}, B_{\tilde{F}}$, and B denote the principal bundles over Y associated to \tilde{E} , \tilde{F} , and Z . Since $h: \text{Diff}(X, E) \rightarrow \text{Diff}(X)$ induces a map $B_{\tilde{E}} \rightarrow B$ and similarly $B_{\tilde{F}} \rightarrow B$, we can, over the diagonal of $Y \times Y$, reduce the group $\text{Diff}(X, F) \times \text{Diff}(X, E)$ of $B_{\tilde{F}} \times B_{\tilde{E}}$ to H and get a new principal bundle $B(H)$ over Y with group H . Now $\mathcal{P}^m(Z; \tilde{E}, \tilde{F})$ and $\overline{\mathcal{P}^m}(Z; \tilde{E}, \tilde{F})$ are the fibre bundles associated to $B(H)$ via Proposition (1.3).

We also have the Hilbert space bundles $H_s(Z, \tilde{E})$ over Y with fibre $H_s(X, E)$ and group $\text{Diff}(X, E)$, for the map $\text{Diff}(X, E) \times H_s(X, E) \rightarrow H_s(X, E)$ is continuous. Similarly, we have the fibre bundle $C^\infty(Z, \tilde{E})$ with fibre $C^\infty(X, E)$ and group $\text{Diff}(X, E)$. The bundle $C^\infty(Z, \tilde{E})$ is naturally injected into $H_s(Z, \tilde{E})$.

Definition (1.4). A family of pseudo-differential operators parametrized by Y is a continuous section P of $\overline{\mathcal{P}^m}(Z; \tilde{E}, \tilde{F})$. When P_y is elliptic for each $y \in Y$, then the family P is an elliptic family.

Note that a continuous section of $\mathcal{P}^m(Z; \tilde{E}, \tilde{F})$ gives a family, because the map $\mathcal{P}^m \rightarrow \overline{\mathcal{P}^m}$ is continuous. Note also that when $Z = X \times Y$, $\tilde{E} = E \times Y$, and $\tilde{F} = F \times Y$, then a family is simply a continuous map of Y into $\overline{\mathcal{P}^m}(X; E, F)$. In this case, we shall call the family a product family. In a sufficiently small neighborhood of each point, a family restricts to a product family.

We now pass to the symbol of a family. In [5, § 5] we had the surjective map $\sigma: \mathcal{P}^m(X; E, F) \longrightarrow \text{Symb}^m(X; E, F)$ which extended to a continuous map $\bar{\sigma}: \overline{\mathcal{P}^m} \longrightarrow \overline{\text{Symb}^m}$ with dense range. Since both $\bar{\sigma}$ and σ are equivariant under the action of H and since $H \times \overline{\text{Symb}^m} \longrightarrow \overline{\text{Symb}^m}$ is continuous, we can construct a fibre bundle $\overline{\text{Symb}^m}(Z; \tilde{E}, \tilde{F})$ over Y with fibre $\overline{\text{Symb}^m}(X; E, F)$ and group H . Clearly the map $\bar{\sigma}_Y: \overline{\mathcal{P}^m}(Z; \tilde{E}, \tilde{F}) \longrightarrow \overline{\text{Symb}^m}(Z; \tilde{E}, \tilde{F})$ defined fibre-wise is continuous.

Definition (1.5). The symbol σ_P of a continuous family P is the continuous cross section of $\overline{\text{Symb}^m}(Z; \tilde{E}, \tilde{F})$ given by $\bar{\sigma}_Y \circ P$.

Since $\bar{\sigma}$ is not surjective, not every continuous section of $\overline{\text{Symb}^m}(Z; \tilde{E}, \tilde{F})$ is the symbol of a continuous family. However, using the Stone-Weierstrass theorem, one can prove (see the appendix)

PROPOSITION (1.6). *Suppose Y is compact. The symbols of continuous families are dense (for the compact-open topology) in the space of continuous sections of $\overline{\text{Symb}^m}(Z; \tilde{E}, \tilde{F})$.*

2. The index of elliptic families

In this section we shall define the index of an elliptic family as an element of $K(Y)$, where the parameter space Y is assumed compact. We shall also establish its elementary properties and, in particular, we will show that the index P depends only on the symbol class $[\sigma(P)] \in K(TZ)$.

In [5, (6.6)] we proved that, for an elliptic operator $P \in \overline{\mathcal{Q}^m}$, any distributional solution of $Pu = 0$ is necessarily C^∞ . Essentially the same argument leads to the following slight extension which we shall require.

LEMMA (2.1). *Let $P \in \overline{\mathcal{P}^m}$ be elliptic and let $Pu = v$ with $v \in C^\infty(X, F)$, $u \in H_i(X, E)$. Then $u \in C^\infty(X, E)$.*

To define the index of a family we shall now prove

PROPOSITION (2.2). *Let $P \in \overline{\mathcal{P}^m}(Z; \tilde{E}, \tilde{F})$ be elliptic. Then there exists a finite number of sections (s_1, \dots, s_q) of $C^\infty(Z, \tilde{F})$ such that the map $Q_y: C^\infty(Z, \tilde{E})_y \oplus C^q \rightarrow C^\infty(Z, \tilde{F})_y$ given by*

$$Q_y(u; \lambda_1, \dots, \lambda_q) = P_y(u) + \sum_{i=1}^q \lambda_i s_i(y)$$

is surjective. The vector spaces $\text{Ker } Q_y$ then form a vector bundle $\text{Ker } Q$ over Y and the element $[\text{Ker } Q] - [Y \times C^q] \in K(Y)$ depends only on P and not on the choice of sections s_i .

PROOF. Locally, in a neighborhood of $y_0 \in Y$, our family P is a product family and so is given by a continuous map $Y \longrightarrow \overline{\mathcal{P}}^m(X; E, F)$. Passing to the Sobolev space this induces (locally) a continuous map $P_s: Y \longrightarrow \mathcal{F}(H_s(X, E), H_{s-m}(X, F))$ where \mathcal{F} denotes the space of bounded Fredholm operators with the norm topology. Let $V = \text{Ker } P_s^*$, then

$$T_s(y): H_s(X, E) \oplus V \longrightarrow H_{s-m}(X, F)$$

defined by $T_s(y)(u \oplus v) = P_s(y)u + v$ is surjective for $y = y_0$ and hence (by a standard argument see [1, Appendix]) also surjective for y near to y_0 . Now $V \subset C^\infty(X, F)$ (since the adjoint of P is also elliptic and in $\overline{\mathcal{P}}^m$) and (2.1) then shows that, for y near y_0

$$T(y): C^\infty(X, E) \oplus V \longrightarrow C^\infty(X, F)$$

is surjective. This proves the first part of the proposition locally. The global version now follows easily by extending the local sections of $C^\infty(Z, \tilde{F})$ and then using a partition of unity argument. Since $\text{Ker } T(y) = \text{Ker } T_s(y)$ the local triviality of the kernels in C^∞ follows from the corresponding fact for the Hilbert space which is standard (see [1, Appendix]). Thus $\text{Ker } Q$ is a vector bundle over Y . For the last part it is enough to show that adding one further section s_{q+1} does not alter the element in $K(Y)$. This is clear if $s_{q+1} = 0$ and the general case can be reduced to this by using the homotopy invariance of $K(Y)$ and multiplying s_{q+1} by a parameter t going from 0 to 1.

Definition (2.3). The element of $K(Y)$ given in the preceding proposition is called the *index of the elliptic family* P . We denote it by $\text{ind } P$.

Remark. If the family P happens to have the property that $\text{Ker } P_y$ has constant dimension independent of y , then we have two vector bundles (over Y) $\text{Ker } P$ and $\text{Coker } P$. It is then not difficult to prove that (with the definition of index P given in (2.3))

$$\text{index } P = [\text{Ker } P] - [\text{Coker } P].$$

To do this we find a trivial bundle $Y \times W$ and a surjective bundle homomorphism $Y \times W \xrightarrow{\varphi} \text{Coker } P$. We use the sections of $\text{Coker } P = \text{Ker } P^* \subset C^\infty(Z, \tilde{F})$ given by a basis of W to compute index P as in Proposition 2.2. We have

$$\begin{aligned} \text{index } P &= [\text{Ker } P \oplus \text{Ker } \varphi] - [Y \times W] \\ &= [\text{Ker } P] + [\text{Ker } \varphi] - \{[\text{Ker } \varphi] + [\text{Coker } P]\} \\ &= [\text{Ker } P] - [\text{Coker } P]. \end{aligned}$$

The first basic property of the index is its homotopy invariance. This follows at once from the homotopy invariance of $K(Y)$. We simply observe

that a path in the space of families over Y is just a family over $Y \times I$ (I the unit interval) and use the isomorphism $K(Y \times I) \cong K(Y)$.

From the homotopy invariance of the index it follows that $\text{ind } P$ depends only on the symbol $\sigma(P)$. From the density result (1.6) we can even define $\text{ind } \sigma$ for any elliptic $\sigma \in \overline{\text{Symb}}^m(Z; \tilde{E}, \tilde{F})$: we put $\text{ind } \sigma = \text{ind } \sigma(P)$ for $\sigma(P)$ sufficiently close to σ . This is independent of the choice of P because of the homotopy invariance of the index. Moreover $\text{ind } P$ will depend only on the homotopy class of $\sigma(P)$.

Next we need to show that the index is essentially independent of the degree of homogeneity of the symbol as in [5, (6.3)], but first we must introduce metrics. By a metric on \tilde{E} we will mean a (positive definite hermitian) metric on \tilde{E} which is smooth along the fibres. More technically if we form the smooth vector bundle $\text{Herm}(\tilde{E})$, whose fibre at any point $z \in Z$ is the space of hermitian forms on \tilde{E}_z , then a metric on \tilde{E} is a continuous positive definite section of $C^\infty(Y, \text{Herm } \tilde{E})$. By the usual partition of unity argument such metrics exist and any two are homotopic. A metric¹ on TZ is simply called a metric on Z . As in [5] we shall now fix a metric on Z and using this we will identify TZ with its dual T^*Z (the cotangent bundle along the fibres). The unit sphere bundle of TZ will be denoted by $S(Z)$.

We shall now prove the analogue of [5, (6.3)].

PROPOSITION (2.4). *Let $\mu \in \overline{\text{Symb}}^m(Z; \tilde{E}, \tilde{F})$ and $\tau \in \overline{\text{Symb}}^k(Z; \tilde{E}, \tilde{F})$ be elliptic and assume they coincide on $S(Z)$. Then $\text{ind } \mu = \text{ind } \tau$.*

PROOF. The proof of [5, (6.3)] requires a little modification here because in [5] we used the fact that a self-adjoint operator has index zero, a result which does not hold for families². However, a family of positive-definite operators clearly has zero index (the kernels being zero) and so we use this fact instead. Now although a self-adjoint symbol always represents some self-adjoint operator (using $\frac{1}{2}(A + A^*)$) the same is not so clear for positivity and, for this reason, a little care is necessary in our proof. Suppose now that $k > m$, and let $\sigma \in \text{Symb}^1(Z; \tilde{E}, \tilde{E})$ be equal to the identity on $S(Z)$, so that $\tau = \mu \circ \sigma^{k-m}$. We propose to construct a positive-definite family $P \in \mathcal{P}^1(Z; \tilde{E}, \tilde{E})$: note that its symbol $\sigma(P)$ is then self-adjoint and so is linearly homotopic to σ . Once we have done this we pick elliptic families $Q \in \mathcal{P}^m(Z; \tilde{E}, \tilde{F})$, $R \in \mathcal{P}^k(Z; \tilde{E}, \tilde{F})$ with symbols close to μ, τ respectively

¹ For real vector bundles we use, of course, euclidean metrics.

² For a complex self-adjoint family A , the homotopy $tA + i(1-t)I$ shows that A has index zero, but this does not work for real families.

and we form the composition $Q \circ P^{k-m}$. Since P is positive-definite we see at once that $\text{ind } Q \circ P^{k-m} = \text{ind } Q$. On the other hand $\sigma(Q \circ P^{k-m}) = \sigma(Q) \circ \sigma(P)^{k-m}$ is homotopic to $\mu \circ \sigma^{k-m} = \tau$. Hence $\text{ind } \tau = \text{ind } Q = \text{ind } \mu$ as required. It remains therefore to construct P . In fact it is enough to require that P be positive semi-definite because adding the identity will then make it positive-definite and this does not alter the symbol (since P has order 1 and the identity is of order 0).

To find the desired family P , we use the fact that the sum of positive semi-definite operators is positive semi-definite. A partition of unity argument on Y localizes the problem to this: Find a product elliptic family $P: N \rightarrow \overline{\mathcal{P}^1}(X; E, E)$ such that $P(y)$, $y \in N$, is positive semi-definite relative to the metric $\rho(y)$ on X and $r(y)$ on E . Now use a partition of unity on X (P positive semi-definite implies $\varphi P \varphi$ is positive semi-definite for φ a real C^∞ function on X) to further reduce the problem to a coordinate neighborhood U on which $E|_U \simeq \mathbf{R}^q$, and the metric $r(y) = (r_{ij}(x, y))$, $i, j = 1, \dots, q$, $y \in N$, $x \in U$. We must find $P(y) = P_{ij}(y)$ elliptic $\in \mathcal{P}^1(U; \mathbf{R}^q, \mathbf{R}^q)$ such that if $\tilde{u} = (u^1, \dots, u^q)$ and $u_i \in C_c^\infty(U)$, then

$$\langle P\tilde{u}, \tilde{u} \rangle = \sum_{i,j,k} \int_U (P_{ij}(y)u_j)(x) r_{ik}(x, y) \bar{u}_k(x) dv_{\rho(y)} \geq 0.$$

Here, $dv_{\rho(y)}$ is the volume element relative to the $\rho(y)$ metric on X .

Choose $P_{ij}(y)$ as follows. Let

$$K_{ij}(x, \xi, y) = \varphi(\xi) |\xi| (r(x, y)^{-1})_{ji} \left(\frac{dv_{\rho(y)}}{dx} \right)^{-1},$$

with $\varphi(\xi) \geq 0$ and C^∞ , $\varphi(\xi) = 0$ for $|\xi| < 1/2$, $\varphi(\xi) = 1$ for $|\xi| \geq 1$ where $|\xi|^2 = \sum_{i,j} \rho_{ij}(x, y) \xi_i \xi_j$. Then $(P_{ij}(y)u)(x) = \int e^{i\langle x, \xi \rangle} K_{ij}(x, \xi, y) \hat{u}(\xi) d\xi$. With this choice

$$\begin{aligned} \langle P\tilde{u}, \tilde{u} \rangle &= \sum_j \int_U \int_{\mathbf{R}^n} e^{i\langle x, \xi \rangle} \varphi(\xi) |\xi| \hat{u}_j(\xi) \bar{u}_j(x) d\xi dx \\ &= \sum_j \int_{\mathbf{R}^n} \varphi(\xi) |\xi| |\hat{u}_j(\xi)|^2 d\xi \geq 0. \end{aligned}$$

Clearly $(P_{ij})(y)$ is elliptic for its symbol on the unit sphere at x is

$$r(x, y)_{ji}^{-1} \frac{dv_{\rho(y)}}{dx}.$$

Also $(P_{ij})(y)$ is a family because $K(x, \xi, y)$ and derivatives in x and ξ vary continuously with y . This completes the proof of the proposition.

Remark. An alternative way of constructing P is to take the positive square root of a laplacian for \tilde{E} . However we then need to know that our

spaces of operators \mathcal{P} are closed under extraction of roots and our explicit construction for P was designed to by-pass this question.

If $P \in \overline{\mathcal{P}}^m(Z; \tilde{E}, \tilde{F})$ then the triple $\{E, F, \sigma(P)\}$ defines an element of $K(B(Z), S(Z)) = K(TZ)$ where $B(Z)$ is the unit ball bundle of Z . Just as in [5] every element of $K(TZ)$ arises from some symbol (for this we need to know that every vector bundle over Z has a smooth structure: proved by the usual approximation arguments). We have now established all the basic properties of the index which, just as in [5], show that

PROPOSITION (2.5). $\sigma \rightarrow \text{ind } \sigma$ induces a homomorphism

$$\text{a-ind}: K(TZ) \longrightarrow K(Y).$$

3. The topological index

We shall now proceed to define the topological index as a homomorphism $K(TZ) \rightarrow K(Y)$. For this we must first embed Z in $Y \times V$ for some euclidean space V .

Fix first a number of smooth functions f_1, \dots, f_k on X which define a smooth embedding $f: X \rightarrow \mathbf{R}^k$. Next let $\{U_j\}$ be an open covering of Y so that Z is a product over each U_j , and shrink this covering to get a slightly smaller open covering $\{U'_j\}$ with $\bar{U}'_j \subset U_j$. Using the product structure of Z over U_j we can extend f_1, \dots, f_k to give functions f_{1j}, \dots, f_{kj} over $Z|U_j$. Now multiply by a continuous function on U_j which is equal to 1 on U'_j and has compact support. The resulting functions then extend to give smooth functions g_{1j}, \dots, g_{kj} on all of Z . The set g_{ij} of all these functions (varying i and j) gives a map $g: Z \rightarrow V = \mathbf{R}^m$ (where $m = kl$, l being the number of open sets U_j). The restriction of g to any fibre X_y is a smooth embedding. Hence $Z \rightarrow Y \times V$ given by $z \mapsto (\pi(z), g(z))$ is our required embedding.

The normal bundles N_y of X_y in V clearly form a vector bundle over Z which is smooth in the sense of Remark 2 of §1 and hence in the sense of Definition (1.2). A euclidean metric on V induces a metric on N and it is not difficult to show that the structure group of the bundle $N \rightarrow Y$ can be reduced to the subgroup of $\text{Diff}(X, N_o)$ which preserves the metric on N_o (o denotes a point of Y). This shows that the unit sphere bundle $S(N)$ and its suspension $S(N \oplus 1)$ are both manifolds over Y in the sense of Definition (1.1).

The topological index is now constructed from the embedding $i: Z \rightarrow Y \times V$ as explained in the Introduction. It is given by $\text{t-ind} = j_1^{-1} \circ i_1$ where $i_1: K(TZ) \rightarrow K(Y \times TV)$ is the composition

$$K(TZ) \xrightarrow{\varphi} K(TN) \longrightarrow K(Y \times TV)$$

and $j_! : K(Y) \rightarrow K(Y \times TV)$ is the periodicity isomorphism. The fact that ind is independent of the choice of embedding i is proved just as in [5]: it depends on the multiplicative property of the Thom homomorphism φ . Our aim now is to establish

THEOREM (3.1). *The analytical and topological indices coincide as homomorphism $K(TZ) \rightarrow K(Y)$.*

As explained in the Introduction the proof of the theorem depends on establishing excision and multiplicativity for the analytical index. The excision property can be stated quite generally as follows. Let $Z \rightarrow Y$, $Z' \rightarrow Y$ be 2 manifolds over Y (with compact fibres X, X' respectively) and let $U \subset Z$, $U' \subset Z'$ be two open sets with a smooth equivalence $U \xrightarrow{\cong} U'$ (compatible with the maps to Y). Identifying U' with U we then have two homomorphisms $K(TU) \rightarrow K(Y)$, defined by

$$\begin{array}{ccccc} & & K(TZ) & & \\ & \nearrow & & \searrow & \\ K(TU) & & & & K(Y) \\ & \searrow & & \nearrow & \\ & & K(TZ') & & \end{array}$$

The excision property asserts that this diagram is commutative. The proof goes much as in [5, § 8]. We first represent an element of $K(TU)$ by a symbol σ of order zero equal to the identity outside a compact set L of U . We then find a family of operators having symbol σ and equal to the identity outside L . We extend this by the identity to the whole of Z or Z' . To compute the indices of these two families (P, P' say) we must first choose sections s_1, \dots, s_q as in (2.2). Since our operators are the identity outside $L \subset U$ we can assume that our sections s_i have their supports inside U (see the construction of the s_i in the proof of (2.2)). It is then clear that the bundles $\text{Ker } Q, \text{Ker } Q'$ (notation of (2.2) with Q' for the case Z') coincide and similarly for the cokernels. Thus $\text{ind } P = \text{ind } P'$.

To complete the proof of Theorem (3.2) we need to investigate the multiplicative properties of the index of families. This will be done in the next section.

4. Multiplicativity of the index

In [5, § 4] we gave a rather general multiplicative property for the index (Axioms (B3), (B3'), (B3'')). For the proof of Theorem (3.1) we need the analogue

of Axiom (B 3') for families, with $G = 1$, $F = S^n$, $H = O(n)$ and b the fundamental symbol as in Axiom (B 2). The proof is essentially the same as that given in [5, § 9]. We proceed to explain the details. We begin with a smooth vector bundle N over Z as in § 1. For (3.1) this is the normal bundle of an embedding as explained in § 3. We then form the manifold $S = S(N \oplus 1)$ over Y whose fibre at y is the sphere bundle of $N_y \oplus 1$ (obtained by compactifying N_y with a section at ∞). To complete the proof of Theorem (3.1) we have to show that the following diagram commutes

$$\begin{array}{ccc} K(TZ) & \xrightarrow{i_1} & K(TS) \\ \text{a-ind} \searrow & & \swarrow \text{a-ind} \\ & K(Y) & \end{array}$$

where i_1 is constructed from $i: Z \rightarrow S$ as in § 3 (i_1 is the composition of the Thom homomorphism $K(TZ) \rightarrow K(TN)$ and the natural extension $K(TN) \rightarrow K(TS)$). Now let

$$b \in K_{O(n)}(TS^n)$$

be the equivariant symbol class given by $b = j_1(1)$ where $j: P \rightarrow S^n$ is the embedding of the origin in S^n . As proved in [5] we have

$$\text{a-ind}(b) = 1 \in R(O(n)).$$

This is the normalization axiom (B2). Moreover, as explained in [5], for any S^n -bundle Σ over X (with group $O(n)$), the homomorphism $K(TX) \xrightarrow{i_1} K(T\Sigma)$ is given (locally over X) by multiplication with b . The same constructor works for families so that $K(TZ) \xrightarrow{i_1} K(TS)$ can be constructed (locally) by multiplication with b . In [5, § 9] we carefully chose representative operators for this symbolic construction and were then able to verify that the analytical index was compatible with i_1 . The only additional complication in carrying out the same argument for families over Y is that, in order to define the index of a family of operators we must add some sections as in (2.2). To minimize these complications it is convenient first to prove the following simple lemma.

LEMMA (4.1). *The element $b \in K_{O(n)}(TS^n)$ can be represented by an operator B commuting with $O(n)$ and such that*

$$\text{Ker } B^* = 0, \quad \text{Ker } B = 1$$

(where 1 denotes the trivial 1-dimensional representation of $O(n)$).

PROOF. Since $\text{ind } b = 1 \in R(O(n))$ there exists an operator C with

$$[\text{Ker } C] - [\text{Ker } C^*] = 1 \in R(O(n)).$$

This means there exists a surjection

$$P: \text{Ker } C \longrightarrow \text{Ker } C^*$$

compatible with $O(n)$ -action and having 1 as kernel. Extend P to be zero on $(\text{Ker } C)^\perp$ and then put $B = C + P$. Clearly P is of order $-\infty$.

Remark. If $B: C^\infty(S^n, G^0) \rightarrow C^\infty(S^n, G^1)$ then a generator u of $\text{Ker } B$ will assign to any S^n -bundle Σ over X (with group $O(n)$) a section \tilde{u} of \tilde{G}_0 (the vector bundle over Σ induced by G^0).

If we use this operator B in³ [5, § 9] then $\text{Ker } \tilde{B}^* = 0$ and so $\text{Ker } D = \text{Ker } P_0$, $\text{Ker } D^* = \text{Ker } P_1$. Suppose now we modify the operator $A: C^\infty(X, E^0) \rightarrow C^\infty(X, E^1)$ by adding a finite-dimensional map $T: V \rightarrow C^\infty(X, E^1)$ to kill the cokernel. Simultaneously we modify the operator

$$D = \begin{pmatrix} \tilde{A} & -\tilde{B}^* \\ \tilde{B} & \tilde{A}^* \end{pmatrix}$$

by adding to the top-left entry a finite-dimensional map \tilde{T} derived from T as follows. For any $v \in V$, Tv is a section of E^1 over X . Lifting this to the total space (the Y of [5, § 9]) we get a section $(Tv)^\sim$ of \tilde{E}^1 . We put

$$\tilde{T}v = (Tv)^\sim \otimes \tilde{u}$$

where \tilde{u} is the section of \tilde{G}^0 described in the remark after (4.1). It is then easy to check that the modified D (written simply as $D + \tilde{T}$) has zero cokernel. Moreover, Lemma (2.1) implies that $D + \tilde{T}$ has the necessary regularity properties. Since $\tilde{B}\tilde{T} = 0$ we find, as in [5, § 9], that $(D + \tilde{T})^*(D + \tilde{T})$ is diagonal and hence the method of calculating index D given there can also be used for $D + \tilde{T}$, and yields

$$(1) \quad \text{Ker } (D + \tilde{T}) = \text{Ker } (A + T).$$

Since the cokernels have been killed and, since adding T or \tilde{T} does not affect the index, (1) implies

$$(2) \quad \text{index } D = \text{index } A.$$

In this form the argument extends immediately to families, (1) is now an isomorphism of vector bundles over the parameter space and (2) is an equation in K of the parameter space. This completes the proof of Theorem (3.1).

³ The notation of [5, § 9] is too lengthy to be reproduced here. We recommend the reader to have [5] at hand in reading what follows.

5. Further comments

In [6] we expressed the topological index of an elliptic operator in cohomological terms. This was obtained by applying the Chern character to the K -theory construction of the index. In precisely the same way we can treat the index of a family. The only differences are

- (i) we use “integration along the fibre”

$$\pi_* : H^*(TZ) \longrightarrow H^*(Y)$$

to generalize evaluation on the fundamental cycle of TZ (H^* denotes cohomology with compact supports),

- (ii) we compute $\text{ch} \circ \text{ind} \in H^*(Y; \mathbb{Q})$ and not the index itself: thus we lose torsion. The result is the following

THEOREM (5.1). *Let P be a family of elliptic operators parametrized by Y and let $u \in K(TZ)$ be the symbol class of P . Then we have*

$$\text{ch}(\text{index } P) = (-1)^n \pi_* \{ \text{ch } u \cdot \mathcal{I}(Z) \}$$

where $\mathcal{I}(Z)$ is the Todd class of the complexification of TZ (the tangent bundle along the fibres of $Z \rightarrow Y$), $n = \dim X$ is the dimension of the fibre and $\pi_* : H^*(TZ) \rightarrow H^*(Y)$ is integration along the fibre.

Note. Integration along the fibre in a fibre bundle is an operation which decreases dimension of cohomology by the dimension of the fibre. It can be defined in many ways, one of which is just the cohomological counterpart of our construction of the topological index: this is the convenient definition to use in proving (5.1). The terminology refers to another definition applicable when all spaces are smooth manifolds: we represent a cohomology class by a differential form and then integrate over the fibres to obtain a differential form on the base. An orientation on the fibre is necessary for this interpretation and in our case we orient the fibres TX as in [6, (2.11)].

In [5] we gave an equivariant index theorem involving a compact group G . This is related to the index theorem for families in the following way. Let P be an elliptic operator on the compact G -manifold X which commutes with the action of G , so that $\text{index } P \in R(G)$ as in [5]. Then, for any fibration ξ over a space Y with fibre X and group G , P defines a family $\xi(P)$ of elliptic operators parametrized by Y . The family $\xi(P)$ has kernel and cokernel of constant dimension and these are in fact the vector bundles over Y associated to ξ by the representations of G on $\text{Ker } P$, $\text{Coker } P$. Hence

$$\text{index } \xi(P) = \xi^*(\text{index } P)$$

where $\xi^*: R(G) \rightarrow K(Y)$ is the homomorphism induced by ξ . The index theorem (3.1), for families coming in this way from compact groups, is then a consequence of the index theorem of [5]. Of course, there are families of operators not arising in this way and for these we need (3.1). Particularly interesting cases of (3.1) are obtained by taking the various operators studied in [6]. For example if $\dim X = 4k$ and is oriented then, choosing a metric along the fibres of $Z \rightarrow Y$ we obtain the family of "signature operators", generalizing the signature operator of [6, § 6]. The index $\text{Sign}(Z)$ of this family is the element $[H^+] - [H^-] \in K(Y)$ where H_y^\pm are the subspaces of $H^{2k}(X_y; \mathbf{R})$ defined as in [6, § 6] by the metric and the cup-product. This element can be expressed in terms of the fundamental group $\pi_1(Y)$ as follows. The representation of $\pi_1(Y)$ on $H^{2k}(X; \mathbf{R})$ gives a homomorphism

$$\alpha: \pi_1(Y) \longrightarrow G$$

where G is the orthogonal group of the quadratic form on $H^{2k}(X; \mathbf{R})$ (given by $(\alpha, \beta) \mapsto \alpha\beta[X]$). In general G is non-compact and its maximal compact subgroup L is a product $O(p) \times O(q)$. The homomorphism α induces a map $f: Y \rightarrow B_G \sim B_L$ (\sim denotes homotopy equivalence). The two vector bundles on B_L associated to the representations of $O(p), O(q)$ on C^p, C^q can therefore be pulled back to give bundles on Y . These are just H^\pm and so $[H^+] - [H^-]$ is the pull-back of a universal class in $K(B_L) = K(B_G)$ which we will simply denote by Sign . Thus $\text{Sign } Z = f^*(\text{Sign})$.

If Y is simply-connected then the above argument shows that $\text{Sign}(Z)$ is trivial (that is $\text{Sign}(Z) = (\text{Sign}(X) \cdot 1 \in K(Y))$). For examples with $\pi_1(Y)$ and $\text{Sign}(Z)$ non-trivial, together with a fuller discussion of the signature of fibre-bundles see [3].

As we have remarked, Theorem (3.1) can be deduced, in certain cases, from the index theorem of [5]. The converse is also true. Given a G -invariant operator P on the G -manifold X we constructed families $\xi(P)$ for each fibration $\xi: Z \rightarrow Y$ with fibre X and group G and observed that $\text{index } \xi(P) = \xi^* \text{index } P$. Letting Y run over all compact subsets of the classifying space B_G we see that a knowledge of all $\xi(P)$ determines $\alpha(\text{index } P)$ where

$$\alpha: R(G) \longrightarrow K(B_G)$$

assigns to each representation of G the associated vector bundle over B_G . Now it is proved in [4] that, for any compact Lie group G , $K(B_G) \cong \widehat{R(G)}$, the completion of $R(G)$ for the $I(G)$ -adic topology (where $I(G)$ is the augmentation ideal). Thus Theorem (3.1) determines the image of $\text{index } P$ under the completion $R(G) \rightarrow \widehat{R(G)}$. For many groups, including connected groups

and finite p -groups, completion is injective (that is, $\bigcap_{n=1}^{\infty} I(G)^n = 0$) and the equivariant index theorem then follows from (3.1). However, for a general finite group this is not true, so that (3.1) does not include the equivariant case.

It would of course be possible to combine the equivariant case and that of families and to introduce equivariant families. We leave this to the reader.

APPENDIX

We give here the proofs of Propositions (1.3) and (1.6). We begin by recalling the statement of (1.3).

PROPOSITION (1.3). *The maps $A : H \times \overline{\mathcal{P}^m}(X; E, F) \rightarrow \overline{\mathcal{P}^m}(X; E, F)$ and $B : H \times \mathcal{P}^m(X; E, F) \rightarrow \mathcal{P}^m(X; E, F)$ are continuous.*

PROOF. We shall treat the case of $\overline{\mathcal{P}}$ in detail. The case of \mathcal{P} is quite similar (see the comments at the end of the proof). It suffices to show that

- (a) the map $P \mapsto \Psi_0^{-1} P \Phi_0$ is continuous for each $(\Psi_0, \Phi_0) \in H$;
- (b) the map $(\Psi, \Phi) \mapsto \Psi^{-1} P_0 \Phi$ is continuous at the identity (I, I) in H for each fixed $P_0 \in \overline{\mathcal{P}^m}$;
- (c) the map A is continuous at $((I, I), 0)$.

For, granted (a), (b), (c) then first A is continuous if it is continuous at $(I, I) \times \mathcal{P}^m$: The map $((\Psi, \Phi), P) \mapsto \Psi^{-1} P \Phi$ equals

$$((\Psi, \Phi), P) \xrightarrow{A'} ((\Psi_0^{-1} \Psi, \Phi_0^{-1} \Phi), \Psi_0^{-1} P \Phi_0) \xrightarrow{A''} (\Psi_0^{-1} \Psi)^{-1} (\Psi_0^{-1} P \Phi_0) (\Phi_0^{-1} \Phi)$$

and A' is continuous because of (a) and the fact that H is a topological group (only continuity of left multiplication is used).

Next $\Psi^{-1} P \Phi - P_0 = (\Psi^{-1}(P - P_0)\Phi) + (\Psi^{-1} P_0 \Phi - P_0)$. The first term on the right is small because of (c) and the second term because of (b). Hence A is continuous.

Now both (a) and (c) are true for the following reason. Each $\Phi \in \text{Diff}(X, E)$ induces a bounded operator on $H_s(X, E)$ and its norm $\|\Phi\|_s$ can be estimated by a finite number of derivatives of Φ , i.e., there exists a neighborhood N of I in $\text{Diff}(X, E)$ such that $\sup_{\Phi \in N} \|\Phi\|_s = K < \infty$. Hence

$$(a): \quad \|\Psi_0^{-1}(P - P_0)\Phi_0\|_{s-m} \leq \|\Psi_0^{-1}\|_{s-m} \|\Phi_0\|_s \|P - P_0\|_s \quad \text{and}$$

$$(c): \quad \text{Choose } N_1 \times N_2 \cap H \text{ in } H \text{ such that } \sup_{\Phi \in N_1} \|\Phi\|_s = K < \infty \text{ and } \sup_{\Psi \in N_2} \|\Psi^{-1}\|_{s-m} = L < \infty.$$

$$\text{Then } \|\Psi^{-1} P \Phi\|_s \leq KL \|P\|_s.$$

We now prove (b). We must show that for each s , and $\varepsilon > 0$, there exists

a neighborhood $N_1 \times N_2 \cap H$ in H such that $\|\Psi^{-1} P_0 \Phi - P_0\|_s < \varepsilon$ for $\Phi \in N_1$, $\Psi \in N_2$. We reduce the problem to the case where E and F are trivial line bundles. Let $\{f_i\}$ be a finite partition of unity subordinate to a covering by coordinate neighborhoods U_i over which E and F are trivialized. We need only prove (b) for each $f_i P_0 f_j$. Choose a neighborhood of (I, I) in H so that (Ψ, Φ) in the neighborhood implies $h(\Psi^{-1})(\text{supp } f_i) \subset C_1$ (a fixed compact set of U_i) and $h(\Phi)(\text{supp } f_j) \subset C_2$ (a fixed compact set of U_j). Thus the computation of $\|\Psi^{-1}(f_i P_0 f_j) \Phi - f_i P_0 f_j\|_s$ involves only sections of E over U_j and sections of F over U_i . Hence we can assume E and F are trivial bundles.

With E trivial, then $\Phi \in \text{Diff}(X, E)$ implies $\Phi = h(\Phi)\alpha_\Phi$ where $\alpha_\Phi \in \text{Aut } E$. Choose a neighborhood of (I, I) such that α_Φ and α_Ψ are close to the identity. Then

$$\begin{aligned} \Psi^{-1} P_0 \Phi &= \alpha_\Psi^{-1} h(\Phi)^{-1} P_0 h(\Phi) \alpha_\Phi \\ &= (\alpha_\Psi^{-1} - I)(h(\Phi)^{-1} P_0 h(\Phi))\alpha_\Phi + h(\Phi)^{-1} P_0 h(\Phi)(\alpha_\Phi - I) + h(\Phi)^{-1} P_0 h(\Phi). \end{aligned}$$

Since α_Φ, α_Ψ close to the identity implies $\|\alpha_\Phi - I\|_s, \|\alpha_\Psi^{-1} - I\|_{s-m}$ are small, $\|h(\Phi)\|_s$ are uniformly bounded and since we know that $\|h(\Phi)\|_s$ are uniformly bounded for a neighborhood of $I \in \text{Diff}(X)$, we conclude: it suffices to prove

(b'): The map $\text{Diff}(X) \rightarrow \mathcal{P}^m(X, 1, 1)$ given by $\varphi \mapsto \varphi^{-1} P_0 \varphi$ is continuous at the identity.

It suffices to prove (b') for $P_0 \in \mathcal{P}^m$ because

$$\begin{aligned} \|\varphi^{-1} P_0 \varphi - P_0\|_s &\leq \|\varphi^{-1}(P_0 - P_j)\varphi\|_s + \|\varphi^{-1} P_j \varphi - P_j\|_s + \|P_j - P_0\|_s \\ &\leq C \|P_j - P_0\|_s + \|\varphi^{-1} P_j \varphi - P_j\|_s. \end{aligned}$$

Thus if $P_j \in \mathcal{P}^m$ and $P_j \rightarrow P_0$ and (b') holds for $P_j \in \mathcal{P}^m$, it holds for $P_0 \in \overline{\mathcal{P}^m}$.

Choose first a neighborhood \mathcal{U} of $I \in \text{Diff}(X)$ which is homeomorphic to a convex neighborhood \mathcal{U} of 0 in $C^\infty(X, T(X))$ via the local homeomorphism $\mathcal{E}: C^\infty(X, T(X)) \rightarrow \text{Diff}(X)$ given by a riemannian metric (see [9]). If $\varphi \in \mathcal{U}$ and $\varphi = \mathcal{E}_v$, let $\varphi_t = \mathcal{E}_{tv}$, $t \in [0, 1]$.

Let g_t denote the geodesic flow, a 1-parameter group, acting on $T(X)$, hence acting on $C^\infty(X, T(X))$. Let $f \in C^\infty(X)$ and let $h_t = (\varphi_t^{-1} P_0 \varphi_t) f$.

LEMMA 1. *dh_t/dt exists and equals $\varphi_t^{-1} [P_0, V_t] \varphi_t f$ where $V_t(x) = g_t(V(\varphi_t^{-1}(x)))$.*

PROOF.

$$\frac{h_t - h_{t_0}}{t - t_0} = \varphi_{t_0}^{-1} \frac{\varphi_{t_0} \varphi_t^{-1} - I}{t - t_0} P_0 \varphi_t f + \varphi_{t_0}^{-1} P_0 \frac{\varphi_t \varphi_{t_0}^{-1} - I}{t - t_0} \varphi_{t_0} f.$$

Let $\psi_t = \varphi_{t+t_0} \varphi_{t_0}^{-1}$, a 1-parameter family of diffeomorphisms. We need only show that the tangent vector to the curve $t \mapsto \psi_t x$ at x is $V_{t_0}(x)$. But

$t \mapsto \varphi_{t+t_0}(y) = \exp_y(t+t_0) V(y)$ has tangent vector $g_{t_0} V(y)$ at $\varphi_{t_0}(y)$. Put $y = \varphi_{t_0}^{-1}(x)$.

LEMMA 2. *The map $C^\infty(X, T(X)) \rightarrow \overline{\mathcal{P}^m}$ given by $V \mapsto [P_0, V]$ is continuous.*

PROOF. This is a linear map between Fréchet spaces, so it suffices to show the graph is closed, i.e., $V_n \rightarrow 0$ and $[P_0, V_n]$ converges in $\overline{\mathcal{P}^m}$ implies $[P_0, V_n] \rightarrow 0$. But if $f \in C^\infty(X)$, then $V_n f \rightarrow 0$ in $C^\infty(X)$ so that $P_0 V_n f \rightarrow 0$ and $V_n P_0 f \rightarrow 0$. Hence $[P_0, V_n] f \rightarrow 0$ in $C^\infty(X)$ and in all H_s . Hence $[P_0, V_n] \rightarrow 0$ in $\overline{\mathcal{P}^m}$.

From Lemma 1,

$$\|\varphi_t^{-1} P_0 \varphi_t - P_0\|_s \leq t \sup_{t_1 \leq t} \|\varphi_{t_1}^{-1} [P_0, V_{t_1}] \varphi_{t_1}\|_s \leq tC \sup_{t_1 \leq t} \|[P_0, V_{t_1}]\|_s.$$

Now (b') follows from Lemma 2 and the continuity of the maps $\text{Diff } X \times C^\infty(X, T(X)) \rightarrow C^\infty(X, T(X))$ and $(t, V) \mapsto g_t V$ at $(I, 0)$ and $(0, 0)$ respectively. This completes the proof of (1.3) for the spaces $\overline{\mathcal{P}^m}$. For \mathcal{P} the proof follows the same lines except for (c) which amounts to the uniformity in the asymptotic expansion for pseudo-differential operators (see [8]).

We recall next the statement of (1.6).

PROPOSITION (1.6). *Suppose Y is compact. The symbols of continuous families are dense (for the compact-open topology) in the space of continuous sections of $\overline{\text{Symb}^m}(Z; \tilde{E}, \tilde{F})$.*

PROOF. Using a partition of unity on Y we reduce to the case of a product family. A partition of unity argument on X then reduces us to studying families P_y given by the continuous map $P: Y \rightarrow \overline{\mathcal{P}^m}(U)$ where U is a domain in \mathbf{R}^n (and all bundles are trivial of dimension 1). Given a $\sigma: y \mapsto \sigma_y(x, \xi)$, where $\sigma_y(x, \xi)$ is continuous on $U \times (\mathbf{R}^n - \{0\})$, positively homogeneous of order m , and compactly supported in U , we must exhibit a family P_y such that $\sigma(P_y)$ approximates σ_y uniformly in y where $\|\sigma_y(x, \xi)\| = \sup_{x \in U, \xi=1} |\sigma_y(x, \xi)|$. By Stone-Weierstrass σ (restricted to $U \times S^{n-1}$) can be approximated by a continuous map $\mu: Y \rightarrow C^\infty(U \times S^{n-1})$. Let $\tilde{\mu}$ denote the extension of μ to $U \times (\mathbf{R}^n - \{0\})$ which is positively homogeneous of order m . It is sufficient to show that there exists a family P with $\sigma(P) = \tilde{\mu}$.

This amounts to the standard construction of a pseudo-differential operator with a given symbol, but carrying along the parameter space Y . The crucial point is this: the constructed family P_y is continuous because

$$\frac{\partial}{\partial x^\alpha} \frac{\tilde{\mu}_y(x, \xi)}{(1 + |\xi|)^m}$$

is continuous in y . In fact P_y is the transformation

$$u \longmapsto \left(\frac{1}{2\pi} \right)^n \int e^{i\langle x, \xi \rangle} K_y(x, \xi) \hat{u}(\xi) d\xi$$

where $K_y(x, \xi) = \varphi(\xi) \tilde{\mu}_y(x, \xi)$ and φ is C^∞ on \mathbf{R}^n , equal to 0 when $|\xi| \leq 1/2$, and equal to 1 when $|\xi| \geq 1$.

INSTITUTE FOR ADVANCED STUDY
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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