INDEX THEORY
FOR SKEW-ADJOINT FREDHOLM OPERATORS

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§ 1. Introduction.

Let H be a separable infinite-dimensional complex Hilbert space and let $\mathscr{F}(H)$ denote the space of all Fredholm operators on H, i.e. bounded linear operators with finite-dimensional kernel and cokernel. Then any $A\in\mathscr{F}(H)$ has an *index* defined by

$$\text{index } A = \dim \operatorname{Ker} A - \dim \operatorname{Coker} A$$

If we give $\mathscr{F}(H)$ the uniform (or norm) topology then the mapping $\mathscr{F}(H)\to\mathbf{Z}$ given by $A\mapsto\text{index } A$ is constant on components and maps the components bijectively onto $\mathbf{Z}$. More generally for any compact space X and any continuous map

$$A : X \to \mathscr{F}(H)$$

we can define a homotopy invariant

$$\text{index } A\in K(X)$$

where K(X) is the Grothendieck group of vector bundles over X ([1, Appendix] or [2]). We first deform A so that $\dim \operatorname{Ker} A_x$ is a locally constant function of x, then we put

$$\text{index } A = [\operatorname{Ker} A] - [\operatorname{Coker} A] \in K(X).$$

Here Ker A is the vector bundle over X whose fibre at $x\in X$ is $\operatorname{Ker} A_x$ and similarly for $\operatorname{Coker} A_x$. It is then a theorem ([1], [2]) that this index invariant defines a bijection

$$[X, \mathscr{F}(H)] \to K(X)$$

where [,] denotes the homotopy classes of mappings. This theorem completely identifies the homotopy type of the space $\mathscr{F}(H)$: it is a classifying space for the functor K.

Quite similarly if H_R is a real Hilbert space we have a bijection

$$\text{index} : [X, \mathscr{F}(H_R)] \to KR(X)$$

where KR(X) is the Grothendieck group of real vector bundles over X.

The main purpose of this paper is to develop an analogous theory for *skew-adjoint*

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Fredholm operators. In particular we want to show that, for a real skew-adjoint Fredholm operator A, the dimension modulo 2 of Ker A is the appropriate " index ".

Let $\hat{\mathcal{F}}(H_R)$ denote the space of real skew-adjoint Fredholm operators. Then our first theorem determines the homotopy type of this space:

Theorem A. — *Define a map*

$$\alpha : \hat{\mathcal{F}}(H_R) \to \Omega\mathcal{F}(H_R)$$

by assigning to each $A \in \hat{\mathcal{F}}$ *the path from* $+1$ *to* -1 *on* $\mathcal{F}$ *given by*

$$\cos \pi t + A \sin \pi t, \qquad 0 \leqslant t \leqslant 1.$$

Then α *is a homotopy equivalence, and so* $\hat{\mathcal{F}}(H_R)$ *is a classifying space for the functor* KR^{-1}.

In the complex case there is a similar but slightly different result.

Theorem B. — *The space* $\hat{\mathcal{F}}(H)$ *has three components* $\hat{\mathcal{F}}_+(H)$, $\hat{\mathcal{F}}_-(H)$ *and* $\hat{\mathcal{F}}_*(H)$ *characterized by*

$$A \in \hat{\mathcal{F}}_+(H) \Leftrightarrow i^{-1}A \text{ is essentially positive}$$

$$A \in \hat{\mathcal{F}}_-(H) \Leftrightarrow i^{-1}A \text{ is essentially negative}$$

$$A \in \hat{\mathcal{F}}_*(H) \Leftrightarrow A \notin \hat{\mathcal{F}}_\pm(H)$$

The components $\hat{\mathcal{F}}_+(H)$ *and* $\hat{\mathcal{F}}_-(H)$ *are contractible. The map*

$$\alpha : \hat{\mathcal{F}}_*(H) \to \Omega\mathcal{F}(H)$$

defined as in Theorem A, *is a homotopy equivalence, so that* $\hat{\mathcal{F}}_*(H)$ *is a classifying space for the functor* K^{-1}.

Note. — To say that iA is essentially positive means that iA is positive on some invariant subspace of H of finite codimension.

A slight generalization of these theorems produces inductively classifying spaces for the functors KR^{-k} and K^{-k}, from which the Bott periodicity theorems follow. For this, let C_k be the (real) Clifford algebra, generated by $e_1, \ldots, e_k$ subject to $e_i^2 = -1$, $e_ie_j = -e_je_i$ for $i \neq j$, and $e_i^* = -e_i$. Assume that H_R is a $*$-module for C_{k-1}, i.e. we have a $*$ representation $\rho : C_{k-1} \to$ bounded operators on H_R with $J_i = \rho(e_i)$ and

$$J_i^2 = -1, \quad i = 1, \ldots, k-1, \quad J_iJ_j = -J_jJ_i \quad i \neq j, \quad \text{and} \quad J_k^* = -J_k.$$

When C_{k-1} is simple, this representation is unique up to equivalence. When C_{k-1} is not simple, it is the direct sum of two simple algebras. We assume that ρ restricted to each simple subalgebra has infinite multiplicity, which again determines ρ up to equivalence. In any case, our ρ can be extended to a $*$ representation of $C_{k+1} \supset C_{k-1}$, thus assuring the existence of J_k and J_{k+1} satisfying the relations above. We use them to show that certain spaces we now define are not empty. Henceforth for simplicity we shall usually omit the symbol ρ and assume our Hilbert spaces are C_{k-1}-modules of the above type.

For $k \geqslant 1$ we now define $\mathscr{F}^k(H_R)$ to be the subset of $\hat{\mathscr{F}}(H_R)$ consisting of all A such that

$$AJ_i = -J_i A \qquad i = 1, \ldots, k-1.$$

Since we can take $A = J_k$ the space $\mathscr{F}^k(H_R)$ is non-empty. We now distinguish two cases, according to whether the algebra C_k is simple or not. If $k \not\equiv -1 \bmod 4$ (so that C_k is simple) put $\mathscr{F}^k_*(H_R) = \mathscr{F}^k(H_R)$. If $k \equiv -1 \bmod 4$ consider the operator

$$w(A) = J_1 J_2 \ldots J_{k-1} A.$$

This commutes with $J_1, \ldots, J_{k-1}$ and A. Also

$$w(A)^* = w(A).$$

As in Theorem B we now decompose $\mathscr{F}^k(H_R)$ into three parts $\mathscr{F}^k_+$, $\mathscr{F}^k_-$, $\mathscr{F}^k_*$ according as $w(A)$ is essentially positive, essentially negative or neither. If we take $A = J_k$ we see that

$$w(J_k)J_{k+1} = -J_{k+1}w(J_k)$$

so that $J_k \in \mathscr{F}^k_*(H_R)$. On the other hand if we take $A = J_k w(J_k)$ we see that $w(A) = 1$ so $J_k w(J_k) \in \mathscr{F}^k_+(H_R)$ and then $-J_k w(J_k) \in \mathscr{F}^k_-(H_R)$. Thus all three of these subspaces of $\mathscr{F}^k(H_R)$ are non-empty. It is also easily shown that each of them is open and therefore a union of components.

For $k = 1$ the space $\mathscr{F}^1(H_R)$ coincides with the whole space $\hat{\mathscr{F}}(H_R)$ of skew-adjoint operators. Moreover $\mathscr{F}^1_*(H_R) = \mathscr{F}^1(H_R)$. For $k = 0$ we adopt the convention that $\mathscr{F}^0(H_R) = \mathscr{F}(H_R)$ and that J_0 is the identity operator. Generalizing Theorem A we shall then prove

Theorem A(k). — *The spaces* $\mathscr{F}^k_\pm(H_R)$ — *defined for* $k \equiv -1 \bmod 4$ — *are contractible. For all* $k \geqslant 1$, *define a map*

$$\alpha : \mathscr{F}^k_*(H_R) \to \Omega(\mathscr{F}^{k-1}(H_R))$$

by assigning to each $A \in \mathscr{F}^k_*(H_R)$ *the path from* J_{k-1} to $-J_{k-1}$ *given by*

$$J_{k-1} \cos \pi t + A \sin \pi t, \qquad 0 \leqslant t \leqslant 1.$$

Then α *is a homotopy equivalence so that* $\mathscr{F}^k_*(H_R)$ *is a classifying space for the functor* KR^{-k}.

For $k = 1$, with the conventions already explained, Theorem A(k) coincides with Theorem A. For $k = 7$, C_{k-1} is the algebra of all 8×8 real matrices and $w(\mathscr{F}^7)$ consists of all self-adjoint Fredholm operators commuting with $\rho(C_{k-1})$. Then $H_R = \mathbf{R}^8 \otimes_{\mathbf{R}} H'_R$ and $\rho = \rho_0 \otimes I$ where ρ_0 is the standard representation of C_6 on $\mathbf{R}^8$. We can therefore identify $w(\mathscr{F}^7)$ with the space of all *self-adjoint* Fredholm operators. Thus as a corollary of Theorem A(k) we obtain

Corollary. — *The space of real self-adjoint Fredholm operators on Hilbert space has two contractible components consisting of essentially positive and essentially negative operators respectively. Their complement is a classifying space for the functor* KR^{-7}.

In the complex case we proceed quite similarly, the only difference being that the complexified Clifford algebras $C_k \otimes_{\mathbf{R}} \mathbf{C}$ are now simple only for even values of k. For any odd value of k the operator $\widetilde{w}(A)$ defined by

$$\begin{aligned} \widetilde{w}(A) &= w(A) && \text{if } k \equiv -1 \bmod 4 \\ &= i^{-1} w(A) && \text{if } k \equiv 1 \bmod 4 \end{aligned}$$

is then self-adjoint. We define the subspaces $\mathscr{F}^k_{\pm}(H)$, $\mathscr{F}^k_{*}(H)$ of $\mathscr{F}^k(H)$ according as $\widetilde{w}(A)$ is essentially positive, negative or neither. Then the complex analogue of Theorem $A(k)$ is:

Theorem $B(k)$. — *The spaces* $\mathscr{F}^k_{\pm}(H)$ — *defined for odd values of* k — *are contractible. For all* $k \geqslant 1$ *the map*

$$\alpha : \mathscr{F}^k_{*}(H) \to \Omega(\mathscr{F}^{k-1}(H))$$

defined as in Theorem $A(k)$ *is a homotopy equivalence. Thus* $\mathscr{F}^k_{*}(H)$ *is a classifying space for the functor* K^{-k}.

For $k=1$, Theorem $B(k)$ reduces to Theorem B.

Because the real Clifford algebra C_k is periodic in k with period 8 [4] we have homeomorphisms $\mathscr{F}^k_{*} \approx \mathscr{F}^{k+8}_{*}$ and so Theorem $A(k)$ implies:

Real Periodicity Theorem $KR \cong KR^{-8}$.

Similarly, since $C_k \otimes_{\mathbf{R}} \mathbf{C}$ has period 2 [4], Theorem $B(k)$ implies :

Complex Periodicity Theorem $K \cong K^{-2}$.

The proofs of the periodicity theorems obtained in this way are quite different from any earlier proofs. Whereas [6] uses Morse Theory, [3] uses elliptic operators and [1], [15] use polynomial approximation to make the proof amenable to algebraic techniques. In this paper, the theorem is stated in the original form as in [6], but the proofs use only standard spectral theory for normal operators on Hilbert space together with Kuiper's result [10] on the contractibility of the unitary group of Hilbert space.

In this presentation the periodicity theorems appear as corollaries of Theorems $A(k)$ and $B(k)$. In fact it is possible to reverse the situation and to deduce Theorems $A(k)$ and $B(k)$ from the periodicity theorems or rather from their Banach algebra versions given by Wood in [15]. This programme has been carried out by G. Segal [13] and, independently (in the framework of Banach Categories) by Karoubi [9].

In the Morse theory treatment given in [11] Milnor introduces a certain subspace Ω_k of the orthogonal group of Hilbert space. This may be defined as follows. Let M be a simple $*$-module for the Clifford algebra C_{k+1}, let $H_{\mathbf{R}}$ be a countable direct sum of copies of M and let $H_{\mathbf{R}}(n)$ be the sum of the first n copies. Then $\Omega_k(n)$, for $k \geqslant 1$, is defined to be the space of all orthogonal transformations A on $H_{\mathbf{R}}$ such that:

(i) $A^2 = -1$, $AJ_i = -J_i A$ $(i = 1, \ldots, k-1)$.

(ii) A preserves the subspace $H_{\mathbf{R}}(n)$ and coincides with J_k on the orthogonal complement $H_{\mathbf{R}}(n)^{\perp}$.

Clearly we have inclusions $\Omega_k(n) \subset \Omega_k(n+1)$, and the union (or limit) is Ω_k:

$$\Omega_k = \lim_{n \to \infty} \Omega_k(n).$$

For $k=0$ we put $\Omega_0 = \mathrm{O}(\infty) = \lim_{n\to\infty} \mathrm{O}(n)$, where $\mathrm{O}(n)$ denotes the orthogonal group of $\mathbf{R}^n$. The periodicity theorem proved in [11] is a consequence of a homotopy equivalence

$$\Omega_k \sim \Omega(\Omega_{k-1}).$$

The spaces Ω_k, or rather their uniform closures $\overline{\Omega}_k$, enter naturally in our treatment. In fact Theorems A(k) and B(k) are consequences of two intermediate equivalences

(i) $\Omega\mathscr{F}^k_* \sim \overline{\Omega}_k \quad (k \geqslant 0)$.

(ii) $\mathscr{F}^k_* \sim \overline{\Omega}_{k-1} \ (k \geqslant 1)$.

The first of these is established by exhibiting a fibration with contractible total space having base (equivalent to) $\mathscr{F}^k_*$ and fibre $\overline{\Omega}_k$. For the second we produce a suitable map from (a deformation retract of) $\mathscr{F}^k_*$ onto $\overline{\Omega}_{k-1}$ with contractible fibres. In both cases the contractibility of the spaces involved follow from Kuiper's theorem.

In several places we will perform deformations on self-adjoint (or skew-adjoint) operators whose continuity depends upon the following

Lemma. — *Let* $f: \mathbf{R} \times \mathrm{I} \to \mathbf{R}$ *be continuous, let* $f_t = f(\cdot, t)$, *and let* S *denote the space of self-adjoint operators. Then the map* $\mathrm{S} \times \mathrm{I} \to \mathrm{S}$ *given by* $(\mathrm{A}, t) \mapsto f_t(\mathrm{A})$ *is continuous.*

Proof. — Since, on a compact set of $\mathbf{R}$, the map $t \mapsto f_t$ is continuous in the supnorm topology, it suffices to show $\mathrm{A} \mapsto f_t(\mathrm{A})$ is continuous for a fixed t. Let X be a closed neighbourhood of the spectrum of A. Given $\varepsilon > 0$, choose a polynomial p such that $|p - f_t| < \varepsilon/3$ on X. Choose a neighbourhood N of A so that $\mathrm{B} \in \mathrm{N}$ implies $\mathrm{sp}(\mathrm{B}) \subset \mathrm{X}$ and $\|p(\mathrm{B}) - p(\mathrm{A})\| < \varepsilon/3$. Then $\|f_t(\mathrm{B}) - f_t(\mathrm{A})\| < \varepsilon$ by the usual $3.\varepsilon/3$ argument.

§ 2. Some elementary deformations.

Most of the spaces we shall encounter will, in the first place, be open sets in a Banach space. Such spaces will be denoted by script letters e.g. $\mathscr{B}$. It is usually convenient to replace these spaces by suitable deformation retracts which are closed in the Banach space. Such retracts will be denoted by the corresponding roman letter e.g. B will be a deformation retract of $\mathscr{B}$. In fact we shall be mainly concerned with the group of units $\mathscr{B}$ in a Banach algebra [1] $\mathscr{A}$ and various related subspaces. In particular if $\mathscr{A}$ is a C*-algebra (i.e. a complex Banach $*$-algebra in which $|x^*x| = |x|^2$) the group A of unitary elements (i.e. satisfying $x^*x = 1$) is a deformation retract of $\mathscr{A}$. The standard retraction is given by

$$(2.1) \qquad x_t = x((1-t)(\sqrt{x^*x})^{-1} + t\mathrm{I}), \qquad 0 \leqslant t \leqslant 1$$

[1] With identity.

where $\sqrt{x^*x}$ is the unique positive square root of x^*x. If a subspace $\mathscr{S}$ of $\mathscr{B}$ is stable under this retraction the intersection $S = \mathscr{S} \cap B$ will be a deformation retract of $\mathscr{S}$. For instance if $\mathscr{A}$ is the complexification of a real Banach $*$-algebra $\mathscr{A}_R$ (so that $\mathscr{A}_R$ is the subalgebra fixed under conjugation $x \mapsto \bar{x}$) then

$$x = \bar{x} \Rightarrow x_t = \bar{x}_t \quad \text{for all } t$$

and so $B_R = B \cap \mathscr{A}_R$ is a deformation retract of $\mathscr{B}_R = \mathscr{B} \cap \mathscr{A}_R$. This is called the orthogonal group of the algebra $\mathscr{A}_R$.

Our starting point is the C*-algebra $\mathscr{A}$ of bounded linear operators on the complex Hilbert space H. Its groups of units will be denoted by $\mathscr{L}$ (for linear group), its unitary subgroup by L. Let $\mathscr{K}$ denote the closed 2-sided ideal of $\mathscr{A}$ consisting of compact operators. Then the quotient $\mathscr{A}/\mathscr{K}$ is again a C*-algebra [8]. Its group of units will be denoted by $\mathscr{G}$, its unitary retract by G. The Fredholm operators on H can be characterized as those which are invertible modulo $\mathscr{K}$. Thus the space $\mathscr{F}$ of all Fredholm operators is the inverse image of $\mathscr{G}$ under the mapping

$$p : \mathscr{A} \to \mathscr{A}/\mathscr{K}$$

The restriction of p defines group homomorphisms

$$\mathscr{L} \to \mathscr{G}, \quad L \to G$$

whose kernels are denoted by $\mathscr{C}$, C respectively. Thus $\mathscr{C}$ consists of invertible operators of the form $1+T$ with T compact, and C consists of unitary operators of this form. Using the spectral theorem for compact operators it is easy to see that for $x \in \mathscr{C}$ the unitary retraction x_t lies in $\mathscr{C}$ for $0 \leq t \leq 1$ and so C is indeed a deformation retract of $\mathscr{C}$, as our notation implies.

From these groups we now turn to their Lie algebras. In particular we consider the space $\hat{\mathscr{A}}$ of bounded skew-adjoint operators (the Lie algebra of the unitary group). We adopt in general the notation $\hat{S}$ for the skew-adjoint elements in S, where S is any subspace of a C*-algebra. Thus $\hat{\mathscr{F}}$ denotes the space of skew-adjoint Fredholm operators, as in § 1, and $\hat{\mathscr{G}}$ is the space of skew-adjoint invertible elements in $\mathscr{A}/\mathscr{K}$. The map $\mathscr{F} \to \mathscr{G}$ induces a map $\hat{\mathscr{F}} \to \hat{\mathscr{G}}$ which is also surjective: if $p(f) = g = -g^*$ then $p\left(\frac{f-f^*}{2}\right) = g$. The essential spectrum of $f \in \hat{\mathscr{F}}$ coincides with the spectrum of $p(f) \in \hat{\mathscr{G}}$. From this it follows that the subsets $\hat{\mathscr{F}}_{\pm}$, $\hat{\mathscr{F}}_*$ defined in Theorem B are inverse images of corresponding subsets $\hat{\mathscr{G}}_{\pm}$, $\hat{\mathscr{G}}_*$. Moreover these subsets of $\hat{\mathscr{G}}$ are clearly both open and closed, and the same is therefore true of the subsets of $\hat{\mathscr{F}}$. Finally, for $f \in \hat{\mathscr{F}}_+$

$$f_t = ti + (1-t)f, \quad 0 \leq t \leq 1$$

provides a contraction of $\hat{\mathscr{F}}_+$ to the point i. Similarly $\hat{\mathscr{F}}_-$ contracts to the point $-i$.

We consider now the commutative diagram

(2.2)
$$\begin{array}{ccc} \hat{\mathscr{F}}_* & \xrightarrow{\alpha} & \Omega\mathscr{F} \\ \downarrow & & \downarrow \\ \hat{\mathscr{G}}_* & \xrightarrow{\beta} & \Omega\mathscr{G} \end{array}$$

where α is defined, as in Theorem B, by

$$\alpha(f) : t \mapsto \cos \pi t + f \sin \pi t \qquad 0 \leqslant t \leqslant 1$$

and β is given by a similar formula. Here Ω denotes the space of paths from $+1$ to -1. To see that β (and hence α) is well-defined we have only to observe that the spectrum of a skew-adjoint invertible element g has no real points, so that for any t

$$\cos \pi t + g \sin \pi t$$

is invertible, i.e. belongs to $\mathscr{G}$.

The map $\mathscr{F} \to \mathscr{G}$ has as fibres the cosets of $\mathscr{K}$. Moreover, according to a general theorem of Bartle and Graves [5], the map $p : \mathscr{A} \to \mathscr{A}/\mathscr{K}$ admits a continuous section ([1]) $s : \mathscr{A}/\mathscr{K} \to \mathscr{A}$. The restriction of s to $\mathscr{G}$ is then a homotopy inverse of $\mathscr{F} \to \mathscr{G}$: we retract $\mathscr{F}$ back onto $s(\mathscr{G})$ linearly by

$$f_t = tf + (1-t) s \circ p(f), \qquad 0 \leqslant t \leqslant 1.$$

Thus $\mathscr{F} \to \mathscr{G}$ is a homotopy equivalence and so therefore is $\Omega\mathscr{F} \to \Omega\mathscr{G}$. Similarly, $\hat{\mathscr{F}}_* \to \hat{\mathscr{G}}_*$ is a homotopy equivalence using the section $s_1 : g \to \dfrac{s(g)-s(g)^*}{2}$. Thus we have established

Lemma **(2.3)**. — *The maps*

$$\hat{\mathscr{F}}_* \to \hat{\mathscr{G}}_* \quad \textit{and} \quad \Omega\mathscr{F} \to \Omega\mathscr{G}$$

in diagram (2.2) *are homotopy equivalences. Hence α is a homotopy equivalence if and only if β is.*

Our next step is to retract $\mathscr{G}$ and $\hat{\mathscr{G}}_*$ onto their unitary parts G and $\hat{G}_*$. For this we must simply observe that, with x_t given by (2.1),

$$x^* = -x \Rightarrow x_t^* = -x_t$$

so that $\hat{\mathscr{G}}$ is stable under this retraction. The space $\hat{G}$ consists of elements $x \in G$ such that

$$x^* = -x, \; x^* x = 1 \quad \text{hence } x^2 = -1.$$

The subspaces $\hat{G}_+$, $\hat{G}_-$ consist of the single points $\{+i\}$, $\{-i\}$ respectively. The subspace $\hat{G}_*$ consists therefore of elements x such that

$$x^* = -x \quad \text{and} \quad \operatorname{Spec} x = \{\pm i\}.$$

([1]) Note that s is not required to be linear: in fact it is known that no continuous linear section exists.

Since $x^2=-1$ it follows that the restriction of β to $\hat{G}_*$ coincides with the exponential map

$$\beta_t(x)=\cos \pi t+x \sin \pi t=\exp(\pi tx)\in G.$$

Thus we have

Lemma **(2.4)**. — *The map* $\beta:\hat{\mathcal{G}}_*\to\Omega\mathcal{G}$ *is equivalent to the map* $\varepsilon:\hat{G}_*\to\Omega G$ *given by*

$$\varepsilon_t(x)=\exp(\pi tx) \qquad 0\leqslant t\leqslant 1.$$

Returning now to the diagram (2.2) we consider the map on the $\mathcal{F}$-spaces which lies over ε; this will be defined on $\hat{p}^{-1}(\hat{G}_*)$, the part of $\hat{\mathcal{F}}_*$ lying over $\hat{G}_*$, i.e. on the space of skew-adjoint operators whose essential spectrum consists of $\{\pm i\}$. For any such operator x, the path

$$t\mapsto \exp(\pi tx), \qquad 0\leqslant t\leqslant 1$$

lies in the orthogonal group L: it starts at the identity and ends at the point

$$\exp \pi x\in -C.$$

We recall that C is the kernel of $L\to G$ and consists of orthogonal operators of the from $1+T$ with T compact: $-C$ denotes those of the form $-1+T$ with T compact. Thus our path defines a point of the relative loop space $\Omega(L, -C)$ — of loops in L which begin at the identity and end in $-C$. Covering the map ε of (2.4) we have therefore a map

(2.5) $$\hat{p}^{-1}(\hat{G}_*)\to\Omega(L, -C)$$

given by the same formula. Finally we shall replace $\hat{p}^{-1}(\hat{G}_*)$ by its subspace $\hat{F}_*$ consisting of elements of norm 1. Thus $\hat{F}_*$ consists of operators x such that

(i) $x^*=-x$

(ii) ess. spec $x=\{\pm i\}$.

(iii) $||x||=1$.

As the notation suggests, $\hat{F}_*$ is a deformation retract of $\hat{\mathcal{F}}_*$ which we now show. Note that inf | ess. spec A |, $A\in\hat{\mathcal{F}}_*$, is $||p(A)^{-1}||$, a continuous function of A. First retract $\hat{\mathcal{F}}_*$ onto the subspace M with inf | ess. spec | $=1$ by $A\mapsto A(1-t+t||p(A)^{-1}||)^{-1}$. Then choose a symmetric deformation retraction λ_t of the imaginary axis onto the closed interval $[-i, +i]$. Then $x\mapsto\lambda_t(x)$, $0\leqslant t\leqslant 1$, deforms M onto $\hat{F}_*$. By the same argument as in (2.3) we deduce

Lemma **(2.6)**. — *The map* $\hat{F}_*\to\hat{G}_*$ *is a homotopy equivalence.*

Replacing $\hat{p}^{-1}(\hat{G}_*)$ by $\hat{F}_*$ in (2.5) we then obtain the crucial commutative diagram

(2.7) $$\begin{array}{ccc} \hat{F}_* & \xrightarrow{\delta} & \Omega(L, -C) \\ \downarrow & & \downarrow \\ \hat{G}_* & \xrightarrow{\varepsilon} & \Omega G \end{array}$$

where the vertical arrows are induced by $p : \mathscr{A} \to \mathscr{A}/\mathscr{K}$ and ε, δ are both given by the same formula

$$x \mapsto \exp(\pi t x), \qquad 0 \leqslant t \leqslant 1.$$

In view of (2.3) (2.4) and (2.6) Theorem B will be a consequence of the following two Propositions:

Proposition **(2.8)**. — *The map*

$$\delta : \hat{\mathrm{F}}_* \to \Omega(\mathrm{L}, -\mathrm{C})$$

defined by $\delta(x) : t \mapsto \exp(\pi t x)$, $0 \leqslant t \leqslant 1$, *is a homotopy equivalence.*

Proposition **(2.9)**. — *The map*

$$\Omega(\mathrm{L}, -\mathrm{C}) \to \Omega\mathrm{G}$$

induced by the projection $\mathscr{A} \to \mathscr{A}/\mathscr{K}$ *is a homotopy equivalence.*

These two propositions will be proved in the next section. Both depend on Kuiper's Theorem about the contractibility of the unitary group of Hilbert space.

§ 3. Proof of (2.8) and (2.9).

Before we embark on the details of the proof we shall make a few general observations on homotopy equivalence. Following Milnor [12] we say that a map $f : \mathrm{X} \to \mathrm{Y}$ is a singular equivalence if, for every point $x_0 \in \mathrm{X}$,

$$f_* : \pi_n(\mathrm{X}, x_0) \to \pi_n(\mathrm{Y}, f(x_0)) \qquad n \geqslant 0$$

is bijective (here π_0 denotes the set of path-components). For spaces in the class $\mathscr{W}$-having the homotopy type of a CW-complex — a singular equivalence is actually a homotopy equivalence [12]. In [12] Milnor shows that a suitable local convexity property ensures that a space belongs to $\mathscr{W}$. In particular this applies to open subsets of a Banach space.

The spaces we are concerned with are, in the first instance, either

(i) open sets of a Banach space or

(ii) deformation retracts of such open sets

and hence belong to $\mathscr{W}$. The same applies to pairs of spaces. It then follows from [12] that the loop spaces occurring in (2.8) and (2.9) also belong to $\mathscr{W}$.

We start now on the proof of (2.9). First we establish:

Lemma **(3.2)**. — *The map* L→G *has a continuous local section* (i.e. *a right inverse defined in a neighbourhood of* $1 \in \mathrm{G}$).

Proof. — Recall that there is a continuous section $s : \mathscr{A}/\mathscr{K} \to \mathscr{A}$ for the projection $p : \mathscr{A} \to \mathscr{A}/\mathscr{K}$. Clearly we may assume $s(1) = 1$ (for if $s(1) = 1 + k$, $t(u) = s(u) - k$ is a new section with $t(1) = 1$). Restricting to open sets this gives a local section $s : \mathscr{G} \to \mathscr{L}$

for the groups of units of these Banach algebras. Let $r : \mathscr{L} \to \mathrm{L}$ be the unitary retraction, and define locally

$$t : \mathrm{G} \to \mathrm{L}$$

by $t(g) = rs(g)$. Denoting also by r the unitary retraction $\mathscr{G} \to \mathrm{G}$ and observing that we then have $rp = pr$, it follows that

$$pt(g) = prs(g) = rps(g) = r(g) = g \quad \text{for } g \in \mathrm{G}.$$

Hence t is a local section as required.

Lemma (3.1) implies that the image of L is an open subgroup of G, and therefore also closed. Since L is contractible [10] and in particular connected (1) it follows that its image in G is just the identity component G_* of G. The existence of a local section for $\mathrm{L} \to \mathrm{G}_*$ then implies, by [14],

Proposition **(3.2)**. — $\mathrm{L} \to \mathrm{G}_*$ *is a principal fibre bundle with group* C.

Since $\Omega\mathrm{G} = \Omega\mathrm{G}_*$, Proposition (2.9) follows from (3.2) by standard homotopy theory [14]. Moreover, since L is contractible [14], we also have

Corollary **(3.3)**. — *The map* $\Omega(\mathrm{L}, -\mathrm{C}) \to -\mathrm{C}$ *which assigns to each path in* $\Omega(\mathrm{L}, -\mathrm{C})$ *its end-point, is a (singular) homotopy equivalence.*

Thus, instead of (2.8), it will be equivalent to prove

Proposition **(3.3)**. — *The map*

$$\exp \pi : \hat{\mathrm{F}}_* \to -\mathrm{C},$$

given by $\mathrm{A} \mapsto \exp \pi\mathrm{A}$, *is a homotopy equivalence.*

We recall that $-\mathrm{C}$ consists of all unitary operators A of the form $-1+\mathrm{T}$ with $\mathrm{T}(=1+\mathrm{A})$ compact. Now define $-\mathrm{C}(n)$ to be the subspace of $-\mathrm{C}$ consisting of those operators A for which $\operatorname{rank}(1+\mathrm{A}) \leqslant n$, and let $\hat{\mathrm{F}}_*(n) = (\exp \pi)^{-1}\{-\mathrm{C}(n)\}$ be the corresponding subspace of $\hat{\mathrm{F}}_*$. Since the union of the spaces $-\mathrm{C}(n)$, for $n \to \infty$, is dense in $-\mathrm{C}$ it is reasonable to expect this sequence to approximate $-\mathrm{C}$ for homotopy, and similarly for the sequence $\hat{\mathrm{F}}_*(n)$. More precisely we shall now prove

Proposition **(3.4)**. — *For any m and any choice of base points* $a \in -\mathrm{C}(m)$, $b \in \hat{\mathrm{F}}_*(m)$, *the inclusion maps induce bijections*:

$$\lim_{n \to \infty} \pi_k(-\mathrm{C}(n), a) \to \pi_k(-\mathrm{C}, a) \qquad k \geqslant 0$$

$$\lim_{n \to \infty} \pi_k(\hat{\mathrm{F}}_*(n), b) \to \pi_k(\mathrm{F}_*, b) \qquad k \geqslant 0.$$

Proof. — Consider first the case of $\hat{\mathrm{F}}_*$. It will be enough to exhibit a deformation

$$h_t : \hat{\mathrm{F}}_* \to \hat{\mathrm{F}}_*, \qquad 0 \leqslant t \leqslant 1$$

such that

(i) for any compact subset X of $\hat{\mathrm{F}}_*$, there exists an integer n, so that $h_1(\mathrm{X}) \subset \hat{\mathrm{F}}_*(n)$;

(ii) $h_t(b) = b$ for all t.

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(1) All our subspaces are locally path-connected so that path-components are the same as components.

Now we recall that the spectrum of an operator $A \in \hat{F}_*$ lies in the interval $[-i, +i]$ and that the essential spectrum consists just of the two points $\{\pm i\}$. We shall construct h_t by a spectral deformation. Thus let $0 < \alpha < 1$ and let

$$h_t : [-i, +i] \to [-i, +i]$$

be a deformation which shrinks $[-i, -i\alpha]$ to $-i$ and $[i\alpha, i]$ to $+i$, and is the identity on $[-i(\alpha-\varepsilon), i(\alpha-\varepsilon)]$. Then h_t induces a deformation $\hat{F}_* \to \hat{F}_*$ given by $A \mapsto h_t(A)$. For any given $A \in \hat{F}_*$ choose $i\beta \notin \operatorname{Spec} A$ with $\alpha < \beta < 1$, and define

$$n_\beta(A) = \operatorname{rank}\left\{ \frac{1}{2\pi i} \int_{|\lambda|=\beta} \frac{d\lambda}{\lambda - A} \right\}$$

This gives the total multiplicity of Spec A in the interval $[-i\beta, i\beta]$. For all $B \in \hat{F}_*$ sufficiently close to A we then have $i\beta \notin \operatorname{Spec} B$ and $n_\beta(B) = n_\beta(A)$, and so $h_1(B) \in \hat{F}_*(n_\beta(A))$. Hence for any compact set X in $\hat{F}_*$ we can find a fixed n so that $h_1(X) \subset \hat{F}_*(n)$. To ensure condition (ii) above we have only to choose α sufficiently close to 1 so that all the m eigenvalues $\neq \pm i$ of the base operator b lie in the interval $[-i(\alpha-\varepsilon), i(\alpha-\varepsilon)]$. This completes the case of $\hat{F}_*$. For $-C$ the proof is the same. In fact the spectral deformation $h_t : \hat{F}_* \to \hat{F}_*$ clearly induces a corresponding spectral deformation $g_t : -C \to -C$ with similar properties.

In view of this proposition it is now enough to prove

Proposition **(3.5)**. — *For any integer* $n \geqslant 0$, *the map* $\exp \pi : \hat{F}_*(n) \to -C(n)$ *is a homotopy equivalence.*

We shall prove (3.5) by induction on n. Let $D(n)$ be the complement of $-C(n-1)$ in $-C(n)$, so that operators $A \in D(n)$ have the property that $\operatorname{rank}(1+A) = n$, then the inductive step of the proof will depend on the following lemma:

Lemma **(3.6)**. — *Over the space* $D(n)$, *the map* $\exp \pi$ *is a fibre bundle with a contractible fibre.*

Proof. — For $A \in D(n)$ we have $\operatorname{rank}(1+A) = n$, a constant. From this it follows easily that $\{\operatorname{Ker}(1+A)\}_{A \in D(n)}$ is a Hilbert space sub-bundle $\mathscr{H}$ of $D(n) \times H$, and so its orthogonal complement $\mathscr{H}^\perp$ is an n-dimensional vector bundle. An operator $T \in \hat{F}_*(n)$ with $\exp \pi T \in D(n)$ defines a unitary automorphism of square -1 on $\mathscr{H}$, while its action on $\mathscr{H}^\perp$ is determined by $\exp \pi T$. Thus, over $D(n)$, $\exp \pi$ is a fibre bundle with fibre the space of all unitary operators on Hilbert space of square -1 — and having both $\pm i$ as eigenvalues of infinite multiplicity. Thus the fibre is homeomorphic to the homogeneous space

$$L(H)/(L(H_1) \times L(H_2)), \qquad H = H_1 \oplus H_2, \qquad \dim H_1 = \dim H_2 = \infty.$$

The map

$$L(H) \to L(H)/(L(H_1) \times L(H_2))$$

has a local section and so is a fibre bundle [14]. By [10] both the total space $L(H)$ and the fibre $L(H_1)\times L(H_2)$ are contractible. Hence the base has trivial homotopy groups and so is contractible also. This completes the proof of the lemma.

Lemma (3.6) implies that, over $D(n)$, the map $\exp \pi$ is a homotopy equivalence. By inductive assumption we may assume $\exp \pi$ is an equivalence over $-C(n-1)$. We want now to put these parts together and deduce that $\exp \pi$ is an equivalence over the whole of $-C(n)$. For this we need two things: first a general abstract lemma giving conditions under which such patching together of homotopy equivalences will work and secondly the verification that these conditions are satisfied in our case. We proceed to deal with these two questions in turn.

It is convenient to introduce the following definition. We say that an open set U in a space X is *respectable* if

(i) $X-U$ and $\partial U=\overline{U}\cap(X-U)$ both have the homotopy type of a CW-complex;

(ii) $(\overline{U}, \partial U)$ and $(X-U, \partial U)$ both have the HEP (Homotopy Extension Property).

We recall that, a sufficient condition for a closed subspace A of a *normal* space B to have the HEP is that it possess a neighbourhood of which it is a deformation retract. We are now ready for our abstract lemma

Lemma (**3.7**). — *Let* $f: X'\to X$ *be a map,* $A\subset X$, $A'=f^{-1}(A)$. *Assume that*

(i) $f: A'\to A$ *is a homotopy equivalence*;

(ii) $f: X'-A'\to X-A$ *is a fibre bundle with contractible fibre*;

(iii) A *has a respectable open neighbourhood* U *so that* $U'=f^{-1}(U)$ *is respectable and so that* $A\to\overline{U}$, $A'\to\overline{U}'$ *are homotopy equivalences.*

Then f is a homotopy equivalence.

Proof. — Since $X'-U'\to X-U$ and $\partial U'\to\partial U$ are both fibre bundles with contractible fibre it follows that both maps are singular equivalences and hence homotopy equivalences (since all spaces are assumed to have the homotopy type of a CW-complex). Also hypotheses (i) and (iii) imply that $\overline{U}'\to\overline{U}$ is a homotopy equivalence. Because of hypothesis (iii) the triads $(X-U, \overline{U}, \partial U)$, $(X'-U', U', \partial U)$ are Mayer-Vietoris triads in the terminology of [7, p. 240]. Hence by [7, (7.4.1)] f is a homotopy equivalence.

We want to apply this abstract situation to our particular case where

$$X'=\hat{F}_*(n), \qquad X=-C(n), \quad f=\exp\pi$$
$$A'=\hat{F}_*(n-1), \quad A=-C(n-1).$$

Property (i) of (3.7) is our inductive assumption, (ii) is just (3.6). It remains therefore to exhibit a respectable neighbourhood U satisfying (iii). To do this we introduce the function

$$\sigma: -C(n)\to[-1, 1]$$

defined by

$$\sigma(A) = -1 \quad \text{if} \quad A \in -C(n-1)$$
$$\sigma(A) = \text{Min Re}(\lambda) \text{ for } -1 \neq \lambda \in \text{Spec } A, \qquad \text{otherwise.}$$

Then σ is continuous and we define the open set U by $U = \sigma^{-1}[-1, 0)$, so that

$$\overline{U} = \sigma^{-1}[-1, 0], \qquad \partial U = \sigma^{-1}(0)$$

The sets

$$V_1 = \sigma^{-1}(-\tfrac{1}{2}, 0], \qquad V_2 = \sigma^{-1}[0, \tfrac{1}{2})$$

are neighbourhoods of ∂U in $\overline{U}$ and $X-U$ respectively and simple spectral deformation shows that ∂U is a deformation retract of each V_i. Thus $(\overline{U}, \partial U)$ and $(X-U, \partial U)$ have the HEP. Moreover, it is easy to show that $\partial U, X-U \in \mathscr{W}$ by constructing open sets in the space of operators which deform onto them. Thus U is a respectable open set. Moreover, if h_t is a deformation of the unit circle shrinking the semi-circle $\pi/2 \leqslant \theta \leqslant 3\pi/2$ to the point -1, it induces a deformation of U into A, showing that $A \to U$ is a homotopy equivalence. We now have to check the corresponding properties for U'. The arguments are essentially the same except that the interval $[-i, i]$ replaces the unit circle. This completes the proof of Proposition (3.5) which is all that was left in the proof of Theorem B.

The proof of Theorem A is quite similar. The only differences are the following :

1) The space $\hat{\mathscr{F}}(H_R)$ retracts as before onto the subspace of $\hat{\mathscr{F}}(H_R)$ consisting of operators with essential spectrum contained in $\{\pm i\}$, but now because we are dealing with real operators both $\pm i$ must occur. Thus the analogues of the components $\hat{\mathscr{F}}_{\pm}$ do not arise.

2) In the various spectral deformations we must now be careful to use only deformations (of $[-i, +i]$ or the unit circle) which are symmetrical under complex conjugation.

3) In the proof of (3.6) the space of orthogonal operators on H_R of square -1 is now homeomorphic with $L(H_R)/L(H_R; J)$ where $J^2 = -1$ and $L(H_R; J)$ is the subgroup of the orthogonal group $L(H_R)$ which commutes with J. Thus $L(H_R; J)$ is the unitary group of a complex Hilbert space and hence by Kuiper [10] both groups (and so also the homogeneous space) are contractible.

In the next section we shall give the appropriate modifications in the proofs of Theorems A, B to yield $A(k)$, $B(k)$.

§ 4. Proof of Theorems $A(k)$ and $B(k)$.

Except for the trivial parts of $A(k)$ and $B(k)$ dealing with the contractible components [1] the two theorems are formally similar and we shall prove them together. H will

(1) We leave these parts to the reader.

therefore stand for either a real or complex Hilbert space as the case may be, and k will be an integer $\geqslant 2$.

We recall first that H is assumed to be a Clifford module, i.e. we are given orthogonal transformations $J_1, J_2, \ldots, J_{k+1}$ of square -1 and anti-commuting. We shall denote by Γ_k the finite group of order 2^k generated by $J_1, \ldots, J_{k-1}$. Note that we have a homomorphism

$$\varepsilon : \Gamma_k \to \pm 1$$

given by $\varepsilon(J_i) = -1$ for all i. Thus, in the group algebra of Γ_k, we have the skew-averaging operator

$$\mu = \frac{1}{2^k} \sum_{\gamma \in \Gamma_k} \varepsilon(\gamma)\gamma.$$

In particular, making Γ_k act on the Banach algebra $\mathscr{A}$ of operators on H by conjugation, we have a skew-averaging operation on $\mathscr{A}$. Thus if $A \in \mathscr{A}$, $\gamma \in \Gamma_k$ then

$$\gamma\mu(A) = \varepsilon(\gamma)\mu(A).$$

Hence, if $\mathscr{A}^k$, $\mathscr{B}^k$, $\mathscr{K}^k$ denote the subsets of $\mathscr{A}$, $\mathscr{B} = \mathscr{A}/\mathscr{K}$, $\mathscr{K}$ which are skew-adjoint and anti-commute with $J_1, \ldots, J_{k-1}$, the projection $p : \mathscr{A} \to \mathscr{B}$ induces a projection

$$p^k : \mathscr{A}^k \to \mathscr{B}^k$$

with fibre $\mathscr{K}^k$. Restricting to

$$\mathscr{G}^k = \mathscr{G} \cap \mathscr{B}^k \qquad \mathscr{F}^k = \mathscr{F} \cap \mathscr{A}^k$$

we obtain a map

$$\mathscr{F}^k \to \mathscr{G}^k.$$

As in § 2 this admits a continuous section, has vector space fibres, and therefore is a homotopy equivalence. Moreover the map α of Theorems $A(k)$, $B(k)$ yields a commutative diagram

$$\begin{array}{ccc} \mathscr{F}^k_* & \xrightarrow{\alpha} & \Omega\mathscr{F}^{k-1} \\ \downarrow & & \downarrow \\ \mathscr{G}^k_* & \xrightarrow{\beta} & \Omega\mathscr{G}^{k-1} \end{array}$$

and β is therefore equivalent, for homotopy, to α (just as in (2.3)).

Next we consider the unitary (or orthogonal) retractions. We observe that, since Γ_k acts by orthogonal transformations

$$\gamma(x) = \pm x \Rightarrow \gamma(x^*) = \pm x^*$$

and so $\gamma(x^*x)=x^*x$. Thus the unitary retraction deforms $\mathscr{G}^k$ into $G^k=G\cap\mathscr{G}^k$. Let $G^k_*=G\cap\mathscr{G}^k_*$. Then, since elements $A\in G^k_*$ satisfy $A^2=-1$, $AJ_{k-1}=-J_{k-1}A$ and so $(AJ_{k-1})^2=-1$, the restriction of β to G^k_* can be written exponentially:

$$\begin{aligned}\beta_t(A)&=J_{k-1}\cos\pi t+A\sin\pi t\\&=J_{k-1}\exp(\pi tAJ_{k-1})\in G^{k-1}\end{aligned}$$

Thus we have the analogue of (2.4).

Next we define the subspace F^k_* of $\mathscr{F}^k_*$ to consist of those operators A such that

(i) $$p^k(A)\in G^k_*$$

(ii) $$||A||=1$$

and consider the path

$$\delta_t(A)=J_{k-1}\exp(\pi tAJ_{k-1}),\qquad 0\leqslant t\leqslant 1,\ A\in F^k_*.$$

This path lies in

$$L^{k-1}=L\cap\mathscr{A}^{k-1}$$

because
$$\begin{aligned}J_i\delta_t(A)&=-J_{k-1}J_i\exp(\pi tAJ_{k-1})\qquad i=1,\ldots,k-2\\&=-J_{k-1}\exp(\pi tAJ_{k-1})J_i\\&=\delta_t(A)J_i\end{aligned}$$

and
$$\begin{aligned}\delta_t(A)^*&=\exp(\pi tAJ_{k-1})^*J^*_{k-1}\\&=\exp(-\pi tAJ_{k-1})\,.\,(-J_{k-1})\\&=-J_{k-1}\exp(\pi tAJ_{k-1})\\&=-\delta_t(A).\end{aligned}$$

For $t=0$ we have $\delta_0(A)=J_{k-1}$. For $t=1$, $\delta_1(A)=J_{k-1}\exp(\pi AJ_{k-1})=T$ say satisfies

(i) T is orthogonal;

(ii) T anti-commutes with $J_1,\ldots,J_{k-2}$;

(iii) $T^2=-1$;

(iv) $T\equiv J_{k-1}$ mod compact operators.

The space of such T we will denote [1] by $\overline{\Omega}_{k-1}$. It is clearly contained in a fibre of the map $L^{k-1}\to G^{k-1}$. Let $L^{k-1}_\blacksquare$, $G^{k-1}_\blacksquare$ denote the components of L^{k-1}, G^{k-1} containing J_{k-1} and its image in G^{k-1} respectively. Then, proceeding as in § 2, we see that Theorems A(k), B(k) will follows from:

Proposition **(4.1)**. — $L^{k-1}_\blacksquare\to G^{k-1}_\blacksquare$ *is a fibre bundle with contractible total space and fibre* $\overline{\Omega}_{k-1}$.

Proposition **(4.2)**. — *The map* $F^k_*\to-\overline{\Omega}_{k-1}$ *given by* $A\mapsto J_{k-1}\exp(\pi AJ_{k-1})$ *is a homotopy equivalence.*

Proof of (4.1). — Let L_{k-1} be the subgroup of L which commutes with $J_1,\ldots,J_{k-2}$. Then L_{k-1} acts by conjugation on L^{k-1} and the isotropy group of J_{k-1} is precisely L_k.

(1) The reasons for this notation are alluded to in § 1.

Moreover it is easy to see that the map $L_{k-1} \to L^{k-1}$ given by $A \mapsto AJ_{k-1}A^{-1}$ has a continuous local section [15]. Thus the orbit of J_{k-1} is homeomorphic to the homogeneous space L_{k-1}/L_k. It is therefore open and similarly all other orbits are open. Each orbit is therefore also closed. Now by our assumptions on H the group L_{k-1} is a Kuiper group (a sum of orthogonal groups of infinite dimensional Hilbert spaces over **R**, **C** or **H**) and so contractible [10] and in particular *connected*. Thus

$$L_{k-1}/L_k = L_{\blacksquare}^{k-1}.$$

Moreover, the existence of a local section shows this is a fibre bundle. Consider now the map $L_{\blacksquare}^{k-1} \to G_{\blacksquare}^{k-1}$. By a similar argument to that in (3.1) this has a continuous local section, hence so does the composite map $L_{k-1} \to G_{\blacksquare}^{k-1}$. Since L_{k-1} also acts transitively on $G_{\blacksquare}^{k-1}$ this implies that $G_{\blacksquare}^{k-1}$ is (homeomorphic to) a coset space of L_{k-1} and [14] that $L_{\blacksquare}^{k-1} \to G_{\blacksquare}^{k-1}$ is a fibre bundle. By our hypothesis on the Clifford algebra structure of H each of the groups L_{k-1}, L_k is a Kuiper group and so contractible. The base $L_{\blacksquare}^{k-1}$ of the fibre bundle $L_{k-1} \to L_{\blacksquare}^{k-1}$ is therefore also contractible. Finally the fibre of $L_{\blacksquare}^{k-1} \to G_{\blacksquare}^{k-1}$ through the point J_{k-1} is contained in the space $\overline{\Omega}_{k-1}$. To show that it is equal to it, and hence complete the proof of (4.1), we need the following lemma:

Lemma (**4.3**). — *If* $J \in \overline{\Omega}_{k-1}$, i.e. *if* J *is orthogonal*, $J^2 = -1$, J *anticommutes with* $J_1, \ldots, J_{k-2}$, *and* $J \equiv J_{k-1}$ mod compact operators, *then* J *is conjugate to* J_{k-1} *by an element of* L^{k-1} *and hence (since* L^{k-1} *is connected)* J *can be joined to* J_{k-1} *by a path in* $J_{k-1}C(k-1)$.

Proof. — The result is obvious if the Clifford algebra C_{k-1} is simple, so suppose it is a sum of two simple algebras. By hypothesis the representation of C_{k-1} on H defined by $J_1, \ldots, J_{k-1}$ contains both simple C_{k-1}-modules with infinite multiplicity. If the same is true of the representation given by $J_1, \ldots, J_{k-2}$, J, then they are clearly conjugate by an orthogonal transformation T: thus $J = TJ_{k-1}T^{-1}$ and $TJ_i = J_iT$ for $i = 1, \ldots, k-2$ that is $T \in L^{k-1}$. We will assume therefore that one of the simple C_{k-1}-modules occurs with only finite multiplicity in the representation defined by $J_1, \ldots, J_{k-2}, J$. Thus if w is the central projection in C_{k-1} corresponding to this simple module its image $w(J_1, \ldots, J_{k-2}, J)$ in this representation has finite rank and so is compact. But $J \equiv J_{k-1}$ modulo compact operators, hence

$$w(J_1, \ldots, J_{k-1}) \equiv w(J_1, \ldots, J_{k-2}, J)$$

is also compact. But this is a contradiction because $w(J_1, \ldots, J_{k-1})$ is a projection operator on H of infinite rank. This completes the proof of the lemma and hence of (4.1).

It remains now to examine (4.2). For this it is convenient to translate both the spaces involved, i.e. F_*^k and $\overline{\Omega}_{k-1}$, by multiplying with the fixed operator J_{k-1}. Let $B = AJ_{k-1}$, then $A \in F_*^k$ if and only if B satisfies:

(i) B is Fredholm and skew-adjoint;

(ii) B commutes with $J_1, \ldots, J_{k-2}$;

(iii) B anti-commutes with J_{k-1} or equivalently (in view of (i)) $J_{k-1}BJ_{k-1}^{-1}=B^*$;
(iv) $||B||=1$ and $BB^*=I+C$ with C compact;
(v) If $k\equiv -1 \bmod 4$, $J_1J_2\ldots J_{k-2}B$ is neither essentially positive nor essentially negative. If $k\equiv 1 \bmod 4$ (and H is complex), $iJ_1\ldots J_{k-2}B$ is neither essentially positive nor essentially negative.

If $T=-J_{k-1}S$ then $T\in\overline{\Omega}_{k-1}$ if and only if S satisfies:

a) S is orthogonal;
b) S commutes with $J_1, \ldots, J_{k-2}$;
c) $(J_{k-1}S)^2=-1$ or equivalently (in view of *a)*) $J_{k-1}SJ_{k-1}^{-1}=S^*$;
d) $S\equiv -1$ modulo compact operators.

With these alterations Proposition (4.2) is equivalent to

Proposition **(4.3)**. — *The map* $B\mapsto \exp \pi B$ *is a homotopy equivalence from the space of operators* B *satisfying* (i)-(v) *above to the space of operators* S *satisfying* a)-d) *above.*

If we omit conditions (ii), (iii), *b)*, *c)* (and put $k=1$ in (v)) Proposition (4.3) reduces to Proposition (3.3). The proof of (4.3) is essentially the same as that of (3.3). All we have to do is to observe that the various steps in the proof are compatible with the symmetry conditions (ii), *b)* (commuting with $J_1, \ldots, J_{k-2}$) and the skew conditions (iii), *c)* (conjugation by J_{k-1} giving the adjoint). Condition (v) is just the appropriate condition to guarantee that the fibres occurring in the analogue of (3.6) are quotients of two Kuiper groups and so contractible. This completes the proof of Theorems A(k), B(k).

§ 5. Periodicity Theorems.

In this section we deduce the Bott periodicity theorems from Theorems A(k) and B(k). Then we shall define the index map $\mathrm{ind}_k : \mathscr{F}^k_* \to A_k$, where A_k is the Grothendieck group of graded C_k-modules modulo those extendable to C_{k+1}-modules. We discuss the meaning of the index for various k and the properties of ind_k under multiplication. We shall also interpret ind_k as the index of an appropriate family of Fredholm operators. In a subsequent paper this will allow us to reduce the index theorem for elliptic operators in $\mathscr{F}^k_*$ to the index theorem for families.

For tensor product purposes, it is neater to describe $\mathscr{F}^k_*(H_k)$ in terms of $\mathbf{Z}_2$-graded real Hilbert spaces. We adopt the notation of [4] for graded tensor products of C_k-modules. [4] also contains the relevant background material. Let $H=H^0\oplus H^1$ be a $\mathbf{Z}_2$-graded C_k-module. Consider the set

$$\{B\in\hat{\mathscr{F}}(H);\ B \text{ is of degree } 1 \text{ (i.e., } B:H^0\to H^1 \text{ and } H^1\to H^0) \text{ and } BJ_i=-J_iB, i=1, \ldots, k\}.$$

Now H^0 is a $C_k^0\simeq C_{k-1}$-module generated by $J_1J_k, \ldots, J_{k-1}J_k$. It is easy to verify that the map $B\mapsto J_kB|_{H^0}$ gives an isomorphism of the above set with $\mathscr{F}^k(H^0)$. In the *graded* situation, we can replace $\mathscr{F}^k(H^0)$ by the above set which we will also denote by $\mathscr{F}^k(H)$.

Theorem (**5.1**). — $\mathcal{F}_*^{k+8} \simeq \mathcal{F}_*^k$ *so that* $\Omega^8(\mathcal{F}_*^k) \simeq \mathcal{F}_*^k$ *in the real case.* $\mathcal{F}_*^{k+2} \simeq \mathcal{F}_*^k$ *so that* $\Omega^2(\mathcal{F}_*^k) \simeq \mathcal{F}_*^k$ *in the complex case.*

Proof. — We leave the complex case to the reader. In the real case, let $M = M^0 \oplus M^1$ be the basic Z_2-graded finite dimensional C_8-module which represents the element $1 \in A_8 = \mathbf{Z}$. Since $C_{k+8} \simeq C_k \hat{\otimes} C_8$ and H is a C_k-module, then $H \hat{\otimes} M$ is a $\mathbf{Z}_2$-graded C_{k+8}-module. Since $C_8 = \mathbf{R}(16)$, it is easy to verify that the map $A \mapsto A \otimes I$ gives an isomorphism of $\mathcal{F}_*^k(H)$ with $\mathcal{F}_*^{k+8}(H \hat{\otimes} M)$. Also $\mathcal{F}_*^{k+8}(H \hat{\otimes} M)$ is isomorphic with $\mathcal{F}_*^{k+8}(H)$, for H and $H \hat{\otimes} M$ are isomorphic C^{k+8}-modules. Any two isomorphisms are homotopic by Kuiper's Theorem. Hence the induced isomorphisms on $\mathcal{F}_*^{k+8}$ are homotopic.

Definition. — *Let* $\text{ind}_k : \mathcal{F}_*^k \to A_k$ *be the map* $A \mapsto \{\text{Ker}(A)\}$ *with* Ker A *a* C_k-module *representing the element* $\{\text{Ker } A\}$ *of* A_k.

Proposition (**5.1**). — *The map* ind_k *is continuous. Hence* ind_k *is constant on components of* $\mathcal{F}_*^k$, *and induces a map,* $\widetilde{\text{ind}}_k : \pi_0(\mathcal{F}_*^k) \to A_k$. *The map* $\widetilde{\text{ind}}_k$ *is a bijection.*

Proof. — Since elements of $\mathcal{F}_*^k$ are skew-adjoint Fredholm, they have 0 as an isolated point in their spectrum. Using scalar multiplication, it suffices to prove continuity at $B \in \mathcal{F}_*^k$ with $B^2 < -I$ on $(\text{Ker } B)^\perp$. Choose a neighborhood $\mathcal{N}$ of B with the property that $C \in \mathcal{N}$ implies C^2 has no spectrum in $[-1+\varepsilon, -\varepsilon]$ and $\| C^2 - B^2 \| < \varepsilon < 1/2$. Let Q be the spectral projection of C^2 on $[-\varepsilon, 0]$, and let E be its range. We claim that the orthogonal projection P of E on $\text{Ker } B = \text{Ker}(B^2)$ is an isomorphism. For suppose $H = \text{Ker } B \oplus V$, and $v \in E \cap V$ with $\|v\| = 1$. Then $\langle (C^2 - B^2)v, v \rangle = \langle C^2 v, v \rangle - \langle B^2 v, v \rangle \geq -\varepsilon + 1$ contradicting $\| C^2 - B^2 \| < \varepsilon$. Hence P is injective. It is surjective for otherwise there exists a $u \in \text{Ker } B \cap E^\perp$ with $\|u\| = 1$. Again $\langle (B^2 - C^2)u, u \rangle = -\langle C^2 u, u \rangle \geq 1 - \varepsilon$ gives a contradiction. Since B^2 and C^2 commute with C_k, the orthogonal projection P gives a C_k-module isomorphism of E with Ker B. Write E as $\text{Ker } C \oplus (\text{Ker } C)^\perp$ so that $\text{ind}_k B - \text{ind}_k C = \{(\text{Ker } C)^\perp\}$. But C is nonsingular and skew-adjoint on $(\text{Ker } C)^\perp$ so that $(\text{Ker } C)^\perp$ is a C_{k+1}-module using $J_{k+1} = C(-C^2)^{1/2}$. Hence $\text{ind}_k B = \text{ind}_k C$.

To show $\widetilde{\text{ind}}_k$ is surjective, let $M = M^0 \oplus M^1$ represent an element of A_k. Replace H by $H \oplus M$ and let $B = J_{k+1} \oplus 0 \in \mathcal{F}_k^*(H \oplus M)$. Then $\widetilde{\text{ind}}_k B = \{M\}$.

To show $\widetilde{\text{ind}}_k$ is injective, we must show that $\text{ind}_k B = \text{ind}_k C$ implies B and C lie in the same component. Write $H = \text{Ker } B \oplus V$. Since $B|_V$ is nonsingular, the polar decomposition retraction connects $B|_V$ with a skew-adjoint unitary R, so that B and $B' = 0 \oplus R$ lie in the same component of $\mathcal{F}_*^k$. Similarly with $H = \text{Ker } C \oplus W$, C is connected to $C' = 0 \oplus S$ with S a skew-adjoint unitary. Since $\text{ind}_k B = \text{ind}_k C$, there exists finite dimensional C_k-modules $V' \subset V$ and $W' \subset W$ invariant under R and S respectively so that $\text{Ker } B \oplus V'$ and $\text{Ker } C \oplus W'$ are isomorphic C_k-modules. Write $H = \text{Ker } B \oplus V' \oplus V'' = \text{Ker } C \oplus W' \oplus W''$. Then B' is connected to B'' which is 0 on $\text{Ker } B \oplus V'$ and R on V'' while C' is connected to C'' which is 0 on $\text{Ker } C \oplus W'$ and S on W''. But V'' using R is a C_{k+1}-module as is W'' using S. Hence there is an isomorphism $T: V'' \rightarrow W''$ such that $TJ_iT^{-1} = J_i$, $i = 1, \ldots, k$ and $TRT^{-1} = S$. Since $\operatorname{Ker} B \oplus V'$ and $\operatorname{Ker} C \oplus W'$ are isomorphic C_k-modules, we can extend $T: H \rightarrow H$ so that T is a C_k-module isomorphism and $TB''T^{-1} = C''$. But the C_k-module isomorphisms of H form a connected group (in fact it is a Kuiper group and so contractible) so that B'' and C'' lie in the same component.

Let us now interpret the index map for values $k = 0, 1, 2, 4$, i.e., $A_k \neq 0$ and $k < 8$. For $k \neq 0$, we revert to the *ungraded* interpretation of $\mathscr{F}^k_*$.

$k = 0$. — $B \in \mathscr{F}^0(H)$ is completely determined by B_0 and $\operatorname{Ker} B = \operatorname{Ker} B_0 \oplus \operatorname{Ker} B_0^*$. Then $\operatorname{ind}_0 B = \{\operatorname{Ker} B\} \in A_0 \cong \mathbf{Z}$ is given by $\dim \operatorname{Ker} B_0 - \dim \operatorname{Ker} B_0^*$, the usual index of B_0, for $\operatorname{Ker} B_0 \oplus \operatorname{Ker} B_0^*$ is a graded C_1-module if and only if $\operatorname{Ker} B_0$ has the same dimension as $\operatorname{Ker} B_0^*$.

$k = 1$. — $A \in \mathscr{F}^1_*$ means A is a skew-adjoint operator on real H. $\{\operatorname{Ker} A\} \in A_1 \cong Z_2$ is 0, i.e., $\operatorname{Ker} A$ is a $C_1 = \mathbf{C}$-module if and only if $\dim \operatorname{Ker} A \equiv 0 \bmod 2$. Hence $\operatorname{ind}_1 A \equiv \dim \operatorname{Ker} A \bmod 2$.

$k = 2$. — $A \in \mathscr{F}^2_*$ means A is a skew-adjoint operator on real H which anti-commutes with J_1. Using J_1, H becomes a complex Hilbert space. Then Ker A is a complex space and $\{\operatorname{Ker} A\} \in A_2 \cong Z_2$ is 0 if and only if Ker A is a C_2-module, i.e., Ker A is a quaternionic space. Hence $\operatorname{ind}_2 A \equiv \dim_{\mathbf{C}} \operatorname{Ker} A \bmod 2$. Thus ind_2 can be interpreted as the complex dimension mod 2 of a skew-adjoint *antilinear* Fredholm operator on a complex Hilbert space.

$k = 4$. — Since [1] $C_3 = \mathbf{H} \oplus \mathbf{H}$, we can write $H = H_+ \oplus H_-$, two quaternionic Hilbert spaces. $A \in \mathscr{F}^3_*$ implies $A: H_\pm \rightarrow H_\mp$ is a quaternionic operator. Let $D = A|_{H^+}$ so that $\operatorname{Ker} A = \operatorname{Ker} D \oplus \operatorname{Ker} D^*$ is a C_3-module, i.e. a quaternion space. This is a C_4-module if and only if $\dim_{\mathbf{H}} \operatorname{Ker} D = \dim_{\mathbf{H}} \operatorname{Ker} D^*$. Hence ind_4 can be interpreted as the quaternionic index of a Fredholm operator over the quaternions.

From Proposition (5.1) and Theorem A(k) we have isomorphisms

$$A_k = \pi_0(\mathscr{F}^k_*) = \pi_k(\mathscr{F}^0)$$

On the other hand we have the isomorphism

$$\pi_k(\mathscr{F}^0) \cong \widetilde{KR}(S^k) = KR^{-k}(\text{point})$$

given by the index of a Fredholm family. Combining these we end up with an isomorphism

$$\gamma_k : A_k \rightarrow KR^{-k}(\text{point}).$$

Now in [4] there is a basic simple construction on Clifford modules giving rise to a homomorphism

$$\beta_k : A_k \rightarrow KR^{-k}(\text{point}).$$

(1) $\mathbf{H}$ stands for the quaternions.

If $M_0 \oplus M_1$ is a Z_2-graded C_k-module we assign to it the element of $\widetilde{KR}(S^k)$ which is $M^0 - \{M^1, u, M^0\}$ where the second vector bundle is M^1 in the upper hemisphere, M^0 in the lower hemisphere and the two are glued along the equator S^{k-1} by Clifford multiplication. We shall prove

Proposition (5.2). — *The homomorphisms*

$$\gamma_k, \beta_k : A_k \to KR^{-k}(\text{point})$$

coincide.

Corollary. — β_k *is an isormorphism.*

Proof. — In the graded situation the map $\mathscr{F}^k \to \mathscr{F}^{k-1}$ assigns to A the path $t \mapsto J_k \cos \pi t + A \sin \pi t$. By iteration we get a map $\varphi_A : D^k \to \mathscr{F}$ given by

$$\varphi_A(t) = \sum_{l=1}^{k} J_l(\cos \pi t_l . \prod_{j=1}^{l-1} \sin \pi t_j) + A \prod_{j=1}^{k} \sin \pi t_j, \qquad t = (t_1, \ldots, t_k)$$

as a map from H^0 to H^1. Since A anti-commutes with the J_l we have $\varphi_A(t) M^0 \subset M^1$, when $\text{Ker}\, A = M^0 \oplus M^1$.

Observe now:

1) $\text{Ker}\, \varphi_A(t) = 0$ except when $t = (1/2, \ldots, 1/2)$ in which case $\text{Ker}\, \varphi_A(t) = M^0$: this follows, inductively, from the fact that $\alpha I + \beta S$ (with S skew-adjoint) is invertible except when $\alpha = 0$.

2) Since $\partial D^k = \{t : \prod_{j=1}^{k} t_j(1 - t_j) = 0\}$ we see that, for $t \in \partial D^k$, $\varphi_A(t) = \sum_{l=1}^{k} \alpha_l J_l$ with $\sum_l \alpha_l^2 = 1$ $(\alpha_l = \cos \pi t_l . \prod_{j=1}^{l-1} \sin \pi t_j)$.

If $A = J_{k+1}$ then $\varphi_A(t)^2 = -I$ for all $t \in D^k$, and so in particular $\varphi_{J_{k+1}}(t)$ is invertible. As we see from 2) the maps φ_A all coincide on ∂D^k and so we can define a map

$$\psi_A : S^k = D^k_+ \cup D^k_- \to \mathscr{F}$$

by putting $\psi_A = \varphi_A$ on D^k_+ (the upper hemisphere) and $\psi_A = \varphi_{J_{k+1}}$ on D^k_- (the lower hemisphere). The map ψ_A represents the element of $\pi_k(\mathscr{F})$ which corresponds to the component of $\mathscr{F}^k_*$ containing A in the bijection $\pi_0(\mathscr{F}^k_*) \to \pi_k(\mathscr{F})$. We must now calculate as in [1] the index of the family ψ_A as an element of $\widetilde{KR}(S^k)$. To do this we must first choose a closed-subspace $\widetilde{H} \subset H^0$ of finite codimension which is transversal to all $\text{Ker}\, \psi_A(t)$, $t \in S^k$. We then replace the family $\psi_A(t)$ by the family $\widetilde{\psi}_A(t) = \psi_A(t) \circ \widetilde{P}$, where $\widetilde{P}$ is orthogonal projection on $\widetilde{H}$, and take $\text{Ker}\, \widetilde{\psi}_A - \text{Ker}\, \widetilde{\psi}_A^*$. Now from (1) above we see that we can take $\widetilde{H} = (M^0)^\perp$, so $\text{Ker}\, \widetilde{\psi}_A$ is the trivial bundle M^0. Since $\psi_A(t) M^0 \subset M^1$ for $t \in D^k_+$ we see that, over D^k_+, we have a natural isomorphism $\text{Ker}\, \widetilde{\psi}_A^* \cong M^1$. On the other hand, for $t \in D^k_-$, ψ_A is an isomorphism, and so we get a natural isomorphism (over D^k_-) $\text{Ker}\, \widetilde{\psi}_A^* \cong M^0$. The glueing over the equator ∂D^k is just the restriction of φ_A. To conclude the proof we have now only to observe that the map $\partial D^k \to \text{ISO}(M^0, M^1)$ given by φ_A is homotopic to that given by Clifford multiplication. This follows from the explicit formula for φ_A in (2) above using a linear homotopy.

We now verify the multiplicative properties of the index. Suppose $S \in \mathcal{F}^k_*$ and $T \in \mathcal{F}^m_*$ with S, T acting on H_S, H_T respectively. Let $S \ast T$ be the operator defined on $H_S \hat{\otimes} H_T$, by

$$(S \ast T)(x \otimes y) = Sx \otimes y + (-1)^{\deg x} x \otimes Ty.$$

Then $S \ast T$ is skew-adjoint and $(S \ast T)^2 = S^2 \otimes I + I \otimes T^2$ so that

$$\operatorname{Ker} S \ast T = \operatorname{Ker} S \otimes \operatorname{Ker} T$$

and $S \ast T$ is Fredholm. We claim that $S \ast T \in \mathcal{F}^{k+m}_*$. To show that $S \ast T \in \mathcal{F}^{k+m}$, we must show that $S \ast T$ anti-commutes with $J_S \in C_k$ and $J_T \in C_l$. Now

$$(S \ast T) J_S (x \otimes y) = S J_S x \otimes y - (-1)^{\deg x} J_S x \otimes T_y$$

while $J_S(S \ast T)(x \otimes y) = J_S Sx \otimes y + (-1)^{\deg x} J_S x \otimes T_y = -(S \ast T) J_S (x \otimes y)$. Also,

$$(S \ast T) J_T (x \otimes y) = (-1)^{\deg x} (S \ast T)(x \otimes J_T y) = (-1)^{\deg x} \{Sx \otimes J_T y + (-1)^{\deg x} x \otimes T J_T y\},$$

while

$$\begin{aligned} J_T (S \ast T)(x \otimes y) &= J_T \{Sx \otimes y + (-1)^{\deg x} x \otimes Ty\} \\ &= -(-1)^{\deg x} Sx \otimes J_T y + x \otimes J_T T y = -(S \ast T) J_T (x \otimes y). \end{aligned}$$

It is easy to check that $S \ast T \in \mathcal{F}^{k+m}_*$. Since tensor product gives the multiplication $A_k \otimes A_m \to A_{k+m}$, we have proved

Proposition **(5.3)**. — $\operatorname{ind}_{k+l} S \ast T = (\operatorname{ind}_k S)(\operatorname{ind}_l T)$.

The map $\mathcal{F}^k_* \times \mathcal{F}^l_* \to \mathcal{F}^{k+l}_*$ used above induces a multiplication in $KR^*(X)$. That this multiplication coincides with that defined via suspensions comes from the homotopy commutativity of the diagram

$$\begin{array}{ccc} \mathcal{F}^k_* \times \mathcal{F}^l_* & \longrightarrow & \mathcal{F}^{k+l}_* \\ \downarrow & & \downarrow \\ \Omega^k(\mathcal{F}) \times \Omega^l(\mathcal{F}) & \longrightarrow & \Omega^{k+l}(\mathcal{F}). \end{array}$$

We omit the proof.

Finally, we remark that the map giving the periodicity theorem is the usual one obtained by multiplication by the generator of $A_8 \cong KR^{-8}(\text{point})$. For the map of $\mathcal{F}^k_*(H) \to \mathcal{F}^{k+8}_*(H \hat{\otimes} M)$ is given by $S \mapsto S \ast T_0 = S \otimes I$ where T_0 is the zero map on M.

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