SIGNATURES OF HERMITIAN FORMS

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Abstract. Signatures of quadratic forms have been generalized to hermitian forms over algebras with involution. In the literature this is done via Morita theory, which causes sign ambiguities in certain cases. The main result of this paper consists of a method for resolving this problem, using properties of the underlying algebra with involution.

1. Introduction

Signatures of quadratic forms over formally real fields have been generalized in [BP2] to hermitian forms over central simple algebras with involution over such fields. This was achieved by means of an application of Morita theory and a reduction to the quadratic form case. A priori, signatures of hermitian forms can only be defined up to sign, i.e., a canonical definition of signature is not possible in this way. In [BP2] a choice of sign is made in such a way as to make the signature of the form which mediates the Morita equivalence positive. A problem arises when that form actually has signature zero or, equivalently, when the rank one hermitian form represented by the unit element over the algebra with involution has signature zero, for it is not then possible to make a sign choice.

In this paper, after introducing the necessary preliminaries (Section 2), we review the definition of signature of hermitian forms and study some of its properties, before proposing a method to address the problem mentioned above (Sections 3 and 4). Our main result (Theorem 4.6) shows that there exists a finite number of rank one hermitian forms over the algebra with involution, having the property that at any ordering of the base field at least one of them has nonzero signature. These rank one forms are used in an algorithm for making a sign choice, resolving the problem formulated above.

In Section 5 we show that the resulting total signature map associated to any hermitian form is continuous. Finally, in Section 6 we show, using signatures, that in general there is no obvious connection between torsion in the Witt group of an algebra with involution and sums of hermitian squares in this algebra.

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2. Preliminaries

2.1. **Central Simple Algebras with Involution.** The general reference for this section is [KMRT, 2.A, 2.C]. Let F be a field of characteristic not two and let A be a *central simple F-algebra*, i.e., $Z(A) = F$ and A has no nontrivial two-sided ideals. We always assume that $\dim_F(A)$ is finite. It can be shown that $\dim_F(A)$ is a square. We call $\deg(A) := \sqrt{\dim_F(A)}$ the *degree* of A. Let $m = \deg(A)$.

An *involution* σ on A is an anti-automorphism of A of period two. Throughout the paper we assume that σ is F-linear, i.e., the restriction of σ to F is the identity on F. Such involutions are also said to be *of the first kind*. Let

$$\mathrm{Sym}(A, \sigma) = \{a \in A \mid \sigma(a) = a\} \text{ and } \mathrm{Skew}(A, \sigma) = \{a \in A \mid \sigma(a) = -a\}.$$

Then σ is either *orthogonal* (or, *of type* $+1$) if $\dim_F \mathrm{Sym}(A, \sigma) = m(m + 1)/2$, or *symplectic* (or, *of type* -1) if $\dim_F \mathrm{Sym}(A, \sigma) = m(m - 1)/2$. By the Skolem-Noether theorem, two F-linear involutions σ and τ on A differ by an inner automorphism:

$$\tau = \mathrm{Int}(u) \circ \sigma$$

for some $u \in A^\times$ such that $\sigma(u) = \pm u$. Here $\mathrm{Int}(u)(x) := uxu^{-1}$ for $x \in A$. The involutions τ and σ are of the same type if and only if $\sigma(u) = u$.

We denote by $\mathrm{Sym}(A, \sigma)^\times$ and $\mathrm{Skew}(A, \sigma)^\times$ the sets of invertible elements in $\mathrm{Sym}(A, \sigma)$ and $\mathrm{Skew}(A, \sigma)$, respectively.

Examples 2.1.

(1) (F, id_F): the field F is trivially a central simple F-algebra. The identity map id_F is an orthogonal involution.
(2) $(M_n(F), t)$: the algebra of $n \times n$-matrices with entries from F is a central simple F-algebra. The transposition map t is an orthogonal involution.
(3) $((a, b)_F, -)$: the quaternion algebra determined by $a, b \in F^\times$ with F-basis $\{1, i, j, k\}$ satisfying $i^2 = a$, $j^2 = b$ and $ij = -ji = k$ is a central simple F-algebra. It is a division algebra if and only if the (quadratic) norm form $\langle 1, -a, -b, ab\rangle$ is anisotropic over F. Quaternion conjugation $-$, determined by $\bar{i} = -i$, $\bar{j} = -j$, and thus $\bar{k} = -k$ is the unique symplectic involution on $(a, b)_F$. Quaternion conjugation is often denoted by γ instead of $-$.
(4) $((a, b)_F, \vartheta)$: the involution ϑ defined on the quaternion algebra $(a, b)_F$ by $\vartheta(i) = -i$, $\vartheta(j) = j$, $\vartheta(k) = k$ is orthogonal.

2.2. **ε-Hermitian Spaces and Forms.** The general reference for this section is [K, Chap. I]. Treatments of the general and division cases can also be found in [G-B] and [L2], respectively.

Let A be a central simple F-algebra, equipped with an F-linear involution σ. Let $\varepsilon \in \{-1, 1\}$. An *$\varepsilon$-hermitian space* over (A, σ) is a pair (M, h), where M is a finitely generated right A-module and $h : M \times M \longrightarrow A$ is a sesquilinear form such that $h(y, x) = \varepsilon\sigma(h(x, y))$ for all $x, y \in M$. We call (M, h) a *hermitian space* when $\varepsilon = 1$ and a *skew-hermitian space* when $\varepsilon = -1$. Consider the left A-module $M^* = \mathrm{Hom}_A(M, A)$

as a right A-module via the involution σ. The form h induces an A-linear map $h^* : M \longrightarrow M^*, x \longmapsto h(x,\cdot)$. We call (M,h) *nonsingular* if h^* is an isomorphism. All spaces occurring in this paper are assumed to be nonsingular. If it is clear what M is, we simply write h instead of (M,h) and speak of a *form* instead of a space.

Witt cancellation and Witt decomposition hold for ε-hermitian spaces (M,h) over (A,σ). Furthermore, if $A = D$ is a division algebra (so that $M \cong D^n$ for some integer n) such that $(D,\sigma,\varepsilon) \neq (F, \mathrm{id}_F, -1)$, then h can be diagonalized: there exist elements $a_1,\dots,a_n \in \mathrm{Sym}(D,\sigma)^\times$ such that

$$h(x,y) = \sum_{i=1}^{n} \sigma(x_i)a_i y_i, \ \forall x,y \in D^n.$$

In this case we use the shorthand notation

$$h = \langle a_1,\dots,a_n\rangle_\sigma,$$

which resembles the notation used for diagonalized quadratic forms. If A is central simple we can certainly consider diagonal hermitian forms defined on free A-modules of finite rank, but some hermitian forms over (A,σ) may not be diagonalizable.

Let $S_\varepsilon(A,\sigma)$ denote the commutative monoid of isometry classes of ε-hermitian spaces over (A,σ) under orthogonal sum. In this paper we consider ε-hermitian spaces (M,h) up to isometry, and so identify them with their class in $S_\varepsilon(A,\sigma)$. Let $W_\varepsilon(A,\sigma)$ denote the Witt group (or, more precisely, the $W(F)$-module) of Witt classes of ε-hermitian spaces over (A,σ). When $\varepsilon = 1$ we drop the subscript and simply write $S(A,\sigma)$ and $W(A,\sigma)$.

2.3. **Adjoint Involutions.** The general reference for this section is [KMRT, 4.A]. Let A be a central simple F-algebra, equipped with an F-linear involution σ. Let (M,h) be an ε-hermitian space over (A,σ). The algebra $\mathrm{End}_A(M)$ is again central simple over F since M is finitely generated [KMRT, 1.10]. The involution ad_h on $\mathrm{End}_A(M)$, defined by

$$h(x, f(y)) = h(\mathrm{ad}_h(f)(x), y), \ \forall x,y \in M, \forall f \in \mathrm{End}_A(M)$$

is called the *adjoint involution* of h. The involution ad_h is F-linear and

$$\mathrm{type}(\mathrm{ad}_h) = \varepsilon\ \mathrm{type}(\sigma).$$

Furthermore, every F-linear involution on $\mathrm{End}_A(M)$ is of the form ad_h for some ε-hermitian form h over (A,σ) and the correspondence between ad_h and h is unique up to a multiplicative factor in $F^\times$ in the sense that $\mathrm{ad}_h = \mathrm{ad}_{\lambda h}$ for every $\lambda \in F^\times$.

By a theorem of Wedderburn there exists an F-division algebra D (unique up to isomorphism) and a finite-dimensional right D-vector space V such that $A \cong \mathrm{End}_D(V)$. Thus $A \cong M_m(D)$ for some positive integer m. Furthermore, if there is an F-linear involution σ on A, then there is an F-linear involution $-$ on D and an ε_0-hermitian form φ_0 over $(D,-)$ with $\varepsilon_0 \in \{-1,1\}$ such that (A,σ) and $(\mathrm{End}_D(V), \mathrm{ad}_{\varphi_0})$ are isomorphic as algebras with involution. In matrix form ad_{φ_0} is described as follows:

$$\mathrm{ad}_{\varphi_0}(X) = \Phi_0 \overline{X}^t \Phi_0^{-1}, \ \forall X \in M_m(D),$$

where $\Phi_0 \in \mathrm{GL}_m(D)$ is the Gram matrix of φ_0. Thus $\overline{\Phi_0}^t = \varepsilon_0 \Phi_0$.

2.4. **Hermitian Morita Theory.** We refer to [BP1, §1], [FM], [G-B, Chap. 2–3], [K, Chap. I, §9], or [L1] for more details. Let (M, h) be an ε-hermitian space over (A, σ). One can show that the algebras with involution $(\mathrm{End}_A(M), \mathrm{ad}_h)$ and (A, σ) are Morita equivalent: for every $\mu \in \{-1, 1\}$ there is an equivalence between the categories $\mathscr{H}_\mu(\mathrm{End}_A(M), \mathrm{ad}_h)$ and $\mathscr{H}_{\varepsilon\mu}(A, \sigma)$ of non-singular μ-hermitian forms over $(\mathrm{End}_A(M), \mathrm{ad}_h)$ and non-singular $\varepsilon\mu$-hermitian forms over (A, σ), respectively (where the morphisms are given by isometry), cf. [K, Thm. 9.3.5]. This equivalence respects isometries, orthogonal sums and hyperbolic forms. It induces isomorphisms

$$S_\mu(\mathrm{End}_A(M), \mathrm{ad}_h) \cong S_{\varepsilon\mu}(A, \sigma) \text{ and } W_\mu(\mathrm{End}_A(M), \mathrm{ad}_h) \cong W_{\varepsilon\mu}(A, \sigma)$$

of commutative monoids and $W(F)$-modules, respectively. The Morita equivalence and the isomorphisms are not canonical. One of the reasons is that $\mathrm{ad}_h = \mathrm{ad}_{\lambda h}$ for any $\lambda \in F^\times$, as observed above.

The algebras with involution (A, σ) and $(D, -)$ are also Morita equivalent. For future use, it will be convenient to decompose this Morita equivalence into three non-canonical equivalences of categories, the last two of which we will call *scaling* and *collapsing*. For computational purposes later on, we describe them in matrix form. We follow the approach of [LU2]:

$$\mathscr{H}_\varepsilon(A, \sigma) \longrightarrow \mathscr{H}_\varepsilon(M_m(D), \mathrm{ad}_{\varphi_0}) \xrightarrow{\text{scaling}} \mathscr{H}_{\varepsilon_0\varepsilon}(M_m(D), -^t) \xrightarrow{\text{collapsing}} \mathscr{H}_{\varepsilon_0\varepsilon}(D, -).$$

Scaling: Let (M, h) be an ε-hermitian space over $(M_m(D), \mathrm{ad}_{\varphi_0})$. Scaling is given by

$$(M, h) \longmapsto (M, \Phi_0^{-1} h). \tag{1}$$

Note that Φ_0^{-1} is only determined up to a scalar factor in $F^\times$ since $\mathrm{ad}_{\varphi_0} = \mathrm{ad}_{\lambda\varphi_0}$ for any $\lambda \in F^\times$.

Collapsing: Recall that $M_m(D) \cong \mathrm{End}_D(D^m)$ and that we always have $M \cong (D^m)^k \cong M_{k,m}(D)$ for some integer k. Let $h : M \times M \longrightarrow M_m(D)$ be an $\varepsilon_0\varepsilon$-hermitian form with respect to $-^t$. Then

$$h(x, y) = \overline{x}^t B y, \ \forall x, y \in M_{k,m}(D),$$

where $B \in M_k(D)$ satisfies $\overline{B}^t = \varepsilon_0\varepsilon B$, so that B determines an $\varepsilon_0\varepsilon$-hermitian form b over $(D, -)$. Collapsing is then given by

$$(M, h) \longmapsto (D^k, b).$$

3. Signatures of Hermitian Forms

In order to introduce signatures of hermitian forms, we deal with a special case first. Let F be a real closed field and let $\mathbb{H} = (-1, -1)_F$ denote Hamilton's quaternion division algebra over F. Let $-$ be quaternion conjugation on $\mathbb{H}$ and let $h \simeq \langle a_1, \dots, a_n \rangle_-$ be a hermitian form over $(\mathbb{H}, -)$. Now, $a_1, \dots, a_n \in \mathrm{Sym}(\mathbb{H}, -) = F$ and so we can consider the quadratic form $q = \langle a_1, \dots, a_n \rangle$. We define the *signature of* h, denoted $\operatorname{sign} h$, to be the signature of the quadratic form q. Note that this definition is independent of the choice of elements $a_1, \dots, a_n$ by a theorem of Jacobson [J].

For the general case we follow the approach of [BP2, §3.3,§3.4]: let F be a formally real field and let (A, σ) be a central simple F-algebra with F-linear involution. Let h be a hermitian form over (A, σ). Consider an ordering $P \in X_F$, the space of orderings of F. In order to define the signature of h at P we do the following: Extend scalars to the real closure F_P of F at P. The extended algebra with involution $(A \otimes_F F_P, \sigma \otimes \mathrm{id}_{F_P})$ is then Morita equivalent to an F_P-division algebra with F_P-linear involution (D_P, ϑ_P), where $\sigma \otimes \mathrm{id}_{F_P}$ is adjoint to an ε_P-hermitian form φ_P over (D_P, ϑ_P) and $\varepsilon_P \in \{-1, 1\}$. By a famous theorem of Frobenius the only division algebras with center the real closed field F_P are F_P itself and $\mathbb{H}_P$, Hamilton's quaternions over F_P. Furthermore, we may choose $\vartheta_P = \mathrm{id}_{F_P}$ (often simply denoted id) in the first case and $\vartheta_P = -$ in the second case by Morita theory (scaling). Thus we may take

$$(D_P, \vartheta_P) = (F_P, \mathrm{id}_{F_P}) \text{ or } (D_P, \vartheta_P) = (\mathbb{H}_P, -). \tag{2}$$

The Morita equivalence induces an isomorphism

$$\mathscr{M}_P : S(A \otimes_F F_P, \sigma \otimes \mathrm{id}_{F_P}) \xrightarrow{\sim} S_{\varepsilon_P}(D_P, \vartheta_P) \tag{3}$$

which is not canonical (for instance, the form φ_P is only determined up to a nonzero scalar factor). Since Morita equivalence preserves isometries of hermitian forms, we may define the signature of h at P to be equal to the signature of $\mathscr{M}_P(h \otimes F_P)$.

When $\varepsilon_P = 1$, the form $\mathscr{M}_P(h \otimes F_P)$ is either quadratic over F_P, in which case its signature is obtained in the usual way, or hermitian over $(\mathbb{H}_P, -)$, in which case the signature is computed as in the special case above.

When $\varepsilon_P = -1$, the forms φ_P and $\mathscr{M}_P(h \otimes F_P)$ are both skew-hermitian over $(\mathbb{H}_P, -)$ or alternating over F_P. Since skew-hermitian forms over $(\mathbb{H}_P, -)$ are always torsion [S2, Thm. 10.3.7] and alternating forms over F_P are always hyperbolic, it makes sense to define $\mathrm{sign}_P\, \mathscr{M}_P(h \otimes F_P) = \mathrm{sign}_P\, \varphi_P = 0$ in those cases. We call the orderings $P \in X_F$ for which $\varepsilon_P = -1$ the (A, σ)-*nil orderings* of F, or simply the *nil orderings* of F if the context is clear. We denote the set of (A, σ)-nil orderings of F by $\mathrm{Nil}(A, \sigma)$.

A different choice of Morita equivalence between $(A \otimes_F F_P, \sigma \otimes \mathrm{id}_{F_P})$ and (D'_P, ϑ'_P), say, may at most result in a sign change for the signature. This follows from the computations in [G-B, pp. 54–55]. (Note that such a sign change may occur, cf. Remark 3.3 below.) We fix a Morita equivalence for each ordering $P \in X_F$.

In light of these remarks we now make the following

Definition 3.1. We define the *signature of h at P*, denoted $\mathrm{sign}^\star_P\, h$, as follows:

$$\mathrm{sign}^\star_P\, h := \begin{cases} \mathrm{sign}_P\, \mathscr{M}_P(h \otimes F_P) & \text{if } \varepsilon_P = 1 \\ 0 & \text{if } \varepsilon_P = -1 \end{cases},$$

where the superscript $\star$ indicates the dependence on the choice of Morita equivalence discussed earlier.

Remark 3.2. An attempt to make this definition canonical is more problematic than suggested in [BP2, §3.3, §3.4], see Section 4.

The following table summarizes all the possibilities:

(D_P, ϑ_P) \ σ	orthogonal	symplectic
$(\mathbb{H}_P, -)$	$\varepsilon_P = -1$ φ_P, $\mathscr{M}_P(h \otimes F_P)$: skew-hermitian $\boxed{\operatorname{sign}^\star_P h := 0}$	$\varepsilon_P = 1$ φ_P, $\mathscr{M}_P(h \otimes F_P)$: hermitian $\boxed{\operatorname{sign}^\star_P h := \operatorname{sign}_P \mathscr{M}_P(h \otimes F_P)}$
(F_P, id_{F_P})	$\varepsilon_P = 1$ φ_P, $\mathscr{M}_P(h \otimes F_P)$: quadratic $\boxed{\operatorname{sign}^\star_P h := \operatorname{sign}_P \mathscr{M}_P(h \otimes F_P)}$	$\varepsilon_P = -1$ φ_P, $\mathscr{M}_P(h \otimes F_P)$: alternating $\boxed{\operatorname{sign}^\star_P h := 0}$

By the properties of Morita equivalence, the signature of a hyperbolic form will be zero and

$$\operatorname{sign}^\star_P(h_1 \perp h_2) = \operatorname{sign}^\star_P h_1 + \operatorname{sign}^\star_P h_2$$

for all hermitian forms h_1, h_2 over (A, σ). Thus $\operatorname{sign}^\star_P$ induces a homomorphism of additive groups $W(A, \sigma) \longrightarrow \mathbb{Z}$ for each $P \in X_F$.

Remark 3.3. Let h be a hermitian form over (A, σ), let $P \in X_F$ and let $\lambda_P \in F_P^\times$. If we replace φ_P by $\lambda_P \varphi_P$ in the computation of $\operatorname{sign}^\star_P h$ above, the final result is multiplied by the sign of λ_P. This follows from considering the scaling part of Morita equivalence, cf. (1).

Let h be a hermitian form over (A, σ) and let $P \in X_F \setminus \operatorname{Nil}(A, \sigma)$ (so that $\varepsilon_P = 1$). Let $\mathscr{B}$ be an F-basis of A. The isomorphism $\mathscr{M}_P$ in (3) can be decomposed into three isomorphisms as follows:

$$\begin{array}{ccccccc} S(A \otimes_F F_P, \sigma \otimes \mathrm{id}_{F_P}) & \xrightarrow{\xi_P^*} & S(M_m(D_P), \mathrm{ad}_{\varphi_P}) & \xrightarrow{\text{scaling}} & S(M_m(D_P), {\vartheta_P}^t) & \xrightarrow{\text{collapsing}} & S(D_P, \vartheta_P) \\ h \otimes F_P & \longmapsto & \xi_P^*(h \otimes F_P) & \longmapsto & \Phi_P^{-1} \xi_P^*(h \otimes F_P) & \longmapsto & \mathscr{M}_P(h \otimes F_P) \end{array} \tag{4}$$

Here ξ_P^* is the commutative monoid isomorphism induced by the isomorphism

$$\xi_P : (A \otimes_F F_P, \sigma \otimes \mathrm{id}_{F_P}) \xrightarrow{\sim} (M_m(D_P), \mathrm{ad}_{\varphi_P})$$

discussed in the context of Wedderburn's theorem in §2.3. The scaling matrix Φ_P is the matrix of the form φ_P with respect to the basis $\xi_P(\mathscr{B})$ of $M_m(D_P)$.

Example 3.4. We describe how to compute the signature of a diagonal hermitian form h over (A,σ) at an ordering $P \in X_F$. Assume that $h = \langle a_1, \dots, a_n\rangle_\sigma$ with respect to some F-basis $\mathscr{B}$ of A. Note that $a_1, \dots, a_n \in \mathrm{Sym}(A,\sigma)^\times$. By the properties of the signature we have

$$\mathrm{sign}_P^\star \langle a_1, \dots, a_n\rangle_\sigma = \sum_{i=1}^n \mathrm{sign}_P^\star \langle a_i\rangle_\sigma,$$

so it suffices to do the computation for a form $\langle a\rangle_\sigma$ of rank 1 (with $a \in \mathrm{Sym}(A,\sigma)^\times$).

If $\varepsilon_P = -1$, the ordering P is (A,σ)-nil and $\mathrm{sign}_P^\star \langle a\rangle_\sigma = 0$ for that ordering. Thus we assume that $\varepsilon_P = 1$.

We push $\langle a\rangle_\sigma$ through the sequence (4). The first step gives us:

$$\langle a \otimes 1\rangle_{\sigma\otimes\mathrm{id}} \longmapsto \xi_P^*(\langle a \otimes 1\rangle_{\sigma\otimes\mathrm{id}}) = \langle \xi_P(a \otimes 1)\rangle_{\mathrm{ad}_{\varphi_P}},$$

where

$$\xi_P(a \otimes 1) \in \mathrm{Sym}(M_m(D_P), \mathrm{ad}_{\varphi_P}).$$

For the second step, let Φ_p be the matrix of the form φ_P with respect to $\xi_P(\mathscr{B})$. It is easy to see that

$$\Phi_P^{-1}\langle \xi_P(a \otimes 1)\rangle_{\mathrm{ad}_{\varphi_P}} = \langle \Phi_P^{-1}\xi_P(a \otimes 1)\rangle_{\vartheta_P{}^t}.$$

For the third step, note that $\Phi_P^{-1}\xi_P(a \otimes 1) \in \mathrm{Sym}(M_m(D_P), \vartheta_P{}^t)$. Since (D_P, ϑ_P) is either (F_P, id_{F_P}) or $(\mathbb{H}_P, -)$, the matrix $\Phi_P^{-1}\xi_P(a\otimes 1)$ is either symmetric or hermitian and thus corresponds to a quadratic or a hermitian form ψ_P of dimension m over (D_P, ϑ_P). We then have

$$\mathrm{sign}_P^\star \langle a\rangle_\sigma := \mathrm{sign}_P\, \psi_P.$$

We give a simple illustration of this method (more elaborate examples will be given later in Propositions 6.4 and 6.5):

Example 3.5. Let F be the Laurent series field $\mathbb{R}(\!(x)\!)$. Then $X_F = \{P_1, P_2\}$, where $x >_{P_1} 0$ and $x <_{P_2} 0$. Consider the quaternion algebra $D = (-1, -x)_F$. This is a division algebra over F since its norm form $\langle 1, 1, x, x\rangle$ is anisotropic over F. Let $A := M_2(D)$ be equipped with the conjugate transpose involution $\sigma = \bar{\ }^t$, where $\bar{\ }$ denotes quaternion conjugation. Then σ is a symplectic involution. Consider the hermitian form $h = \langle (\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix})\rangle_\sigma$ over (A,σ). Now $A \otimes_F F_{P_1} \cong M_2(\mathbb{H}_{P_1})$ since x is a square in F_{P_1} and $A \otimes_F F_{P_2} \cong M_4(F_{P_2})$ since $-x$ is a square in F_{P_2}. We see that the ordering P_2 is $(D, -)$-nil, so that $\mathrm{sign}_{P_2}^\star h = 0$. Following the steps in Example 3.4 we get $\mathrm{sign}_{P_1}^\star h = \pm 2$ since σ becomes adjoint to the hermitian form $\varphi_{P_1} = \langle 1, 1\rangle_-$ over $(\mathbb{H}_{P_1}, -)$ after scalar extension to the real closure of F at P_1. (As observed in Remark 3.3, only knowing φ_{P_1} up to sign only gives us the signature up to sign. In Section 4 we will explain how a choice of sign can be made.)

Lemma 3.6. *Let $P \in X_F$ and let φ_P be as above. Then* $\mathrm{sign}_P^\star \langle 1\rangle_\sigma = \mathrm{sign}_P\, \varphi_P$.

Proof. This is trivially true for the (A,σ)-nil orderings of F. Thus assume that $P \in X_F$ is not nil. We extend scalars to the real closure of F at P, $\langle 1\rangle_\sigma \longmapsto \langle 1\rangle_\sigma \otimes F_P = \langle 1\otimes 1\rangle_{\sigma\otimes \mathrm{id}}$ and push $\langle 1\otimes 1\rangle_{\sigma\otimes\mathrm{id}}$ through the sequence (4), as illustrated in Example 3.4:

$$\langle 1\otimes 1\rangle_{\sigma\otimes\mathrm{id}} \longmapsto \xi_P^*(\langle 1\otimes 1\rangle_{\sigma\otimes\mathrm{id}}) = \langle \xi_P(1\otimes 1)\rangle_{\mathrm{ad}_{\varphi_P}} = \langle I_m\rangle_{\mathrm{ad}_{\varphi_P}} \longmapsto \Phi_P^{-1}\langle I_m\rangle_{\mathrm{ad}_{\varphi_P}} = \langle \Phi_P^{-1}\rangle_{\vartheta_P{}^t}.$$

(Note that $\xi_P(1\otimes 1) = I_m$, the $m\times m$-identity matrix in $M_m(D_P)$ since ξ_P is an algebra homomorphism.) The matrix Φ_P^{-1} now corresponds to a quadratic form over F_P or a hermitian form over $(\mathbb{H}_P, -)$. In either case Φ_P^{-1} is congruent to Φ_P. Thus $\mathrm{sign}_P^\star\langle 1\rangle_\sigma = \mathrm{sign}_P\,\varphi_P$. ∎

In [LT], Lewis and Tignol defined the *signature of the involution σ at $P \in X_F$* as follows:

$$\mathrm{sign}_P\,\sigma := \sqrt{\mathrm{sign}_P\,T_\sigma},$$

where T_σ is the *involution trace form* of (A,σ) which is a quadratic form over F defined by $T_\sigma(x) := \mathrm{Trd}_A(\sigma(x)x)$, for all $x \in A$. Here Trd_A denotes the reduced trace of A.

Examples 3.7.

(1) Let $(A,\sigma) = (M_n(F), t)$. Then $T_\sigma \simeq n^2 \times \langle 1\rangle$. Hence $\mathrm{sign}_P\,\sigma = n$ for all $P \in X_F$.
(2) Let $(A,\sigma) = ((a,b)_F, -)$. Then $T_\sigma \simeq \langle 2\rangle \otimes \langle 1,-a,-b,ab\rangle$. Hence $\mathrm{sign}_P\,\sigma = 2$ for all $P \in X_F$ such that $a <_P 0, b <_P 0$ and $\mathrm{sign}_P = 0$ for all other $P \in X_F$. Note that $N = \langle 1,-a,-b,ab\rangle$ is the norm form of A.

Remark 3.8. Let (A,σ) and (B,τ) be two central simple F-algebras with F-linear involution.

(1) Consider the tensor product $(A\otimes_F B, \sigma\otimes\tau)$. Then $T_{\sigma\otimes\tau} = T_\sigma \otimes T_\tau$ and so $\mathrm{sign}_P(\sigma\otimes\tau) = (\mathrm{sign}_P\,\sigma)(\mathrm{sign}_P\,\tau)$ for all $P \in X_F$.
(2) If $(A,\sigma) \cong (B,\tau)$, then $T_\sigma \simeq T_\tau$ so that $\mathrm{sign}_P\,\sigma = \mathrm{sign}_P\,\tau$ for all $P \in X_F$.

Remark 3.9. Pfister's local-global principle holds for algebras with involution (A,σ) and also for hermitian forms h over such algebras, [LU1]:

$$\mathrm{sign}_P\,\sigma = 0,\ \forall P \in X_F \Leftrightarrow (A,\sigma) \text{ is weakly hyperbolic}$$

(i.e., σ is the adjoint involution of a torsion form) and

$$\mathrm{sign}_P^\star\,h = 0,\ \forall P \in X_F \Leftrightarrow \text{the class of } h \text{ in } W(A,\sigma) \text{ is torsion.}$$

Remark 3.10. The map $\mathrm{sign}\,\sigma$ is continuous from X_F (equipped with the Harrison topology, see [Lam, Chapter VIII 6] for a definition) to $\mathbb{Z}$ (equipped with the discrete topology). Indeed: define the map $\sqrt{\ }$ on $\mathbb{Z}$ by setting $\sqrt{k} = -1$ if k is not a square in $\mathbb{Z}$. Since $\mathbb{Z}$ is equipped with the discrete topology, this map is continuous. Since T_σ is a quadratic form, the map $\mathrm{sign}\,T_\sigma$ is continuous from X_F to $\mathbb{Z}$ (by [Lam, Proposition 6.6]). Thus, by composition, $\mathrm{sign}\,\sigma = \sqrt{\mathrm{sign}\,T_\sigma}$ is continuous from X_F to $\mathbb{Z}$.

Lemma 3.11. *Let $P \in X_F$. Then*

$$\mathrm{sign}_P\,\sigma = \lambda_P\,|\mathrm{sign}_P\,\varphi_P|,$$

where $\lambda_P = 1$ *if* $(D_P, \vartheta_P) = (F_P, \mathrm{id}_{F_P})$ *and* $\lambda_P = 2$ *if* $(D_P, \vartheta_P) = (\mathbb{H}_P, -)$. *(If we want to indicate the dependence of* λ_P *on* (A, σ) *we will write* $\lambda_{P,A,\sigma}$*.) In particular, if* $P \in \mathrm{Nil}(A, \sigma)$*, then* $\mathrm{sign}_P\, \sigma = \mathrm{sign}_P\, \varphi_P = 0$.

Proof. This is a reformulation of [KMRT, 11.11] or of [LT, Theorem 1] and part of its proof. ∎

Lemma 3.12. *Let* (M, h) *be a hermitian space over* (A, σ)*. Let* $P \in X_F$*. Then*

$$\mathrm{sign}_P\, \mathrm{ad}_h = \lambda_P\, |\mathrm{sign}^\star_P\, h|,$$

with λ_P *as defined in Lemma 3.11. In particular,*

$$\mathrm{sign}^\star_P\, h = 0 \Leftrightarrow \mathrm{sign}_P\, \mathrm{ad}_h = 0.$$

Proof. Assume first that $P \in \mathrm{Nil}(A, \sigma)$. Then $\mathrm{sign}^\star_P\, h = 0$. Consider the adjoint involution ad_h on $\mathrm{End}_A(M)$. After extension of scalars to F_P we have Morita equivalences

$$\mathscr{H}(\mathrm{End}_A(M) \otimes_F F_P, \mathrm{ad}_h \otimes \mathrm{id}_{F_P}) \longrightarrow \mathscr{H}(A \otimes_F F_P, \sigma \otimes \mathrm{id}_{F_P}) \longrightarrow \mathscr{H}_{-1}(D_P, \vartheta_P)$$

and $\mathrm{ad}_h \otimes \mathrm{id}_{F_P}$ is adjoint to a skew-hermitian form over (D_P, ϑ_P). Note that ad_h and σ are of the same type since h is hermitian. Thus ad_h and ϑ_P are of opposite type since $P \in \mathrm{Nil}(A, \sigma)$. Therefore $P \in \mathrm{Nil}(\mathrm{End}_A(M), \mathrm{ad}_h)$. By Lemma 3.11 we conclude that $\mathrm{sign}_P\, \mathrm{ad}_h = 0$.

Next, assume that $P \in X_F \setminus \mathrm{Nil}(A, \sigma)$. Without loss of generality we may replace F by its real closure at P. Consider the Morita equivalence

$$\mathscr{H}(A, \sigma) \longrightarrow \mathscr{H}(D, \vartheta)$$

with $(D, \vartheta) = (\mathbb{H}, -)$ or $(D, \vartheta) = (F, \mathrm{id})$. Let (N, b) be the hermitian space over (D, ϑ) corresponding to (M, h) under this Morita equivalence. Then $\mathrm{sign}^\star h = \mathrm{sign}\, b$. By [BP1, Remark 1.4.2] we have $(\mathrm{End}_A(M), \mathrm{ad}_h) \cong (\mathrm{End}_D(N), \mathrm{ad}_b)$ so that $\mathrm{sign}\, \mathrm{ad}_h = \mathrm{sign}\, \mathrm{ad}_b$. By [LT, Theorem 1] or [KMRT, 11.11] we have $\mathrm{sign}\, \mathrm{ad}_b = \lambda\, |\mathrm{sign}\, b|$ with $\lambda = 1$ if $(D, \vartheta) = (F, \mathrm{id})$ and $\lambda = 2$ if $(D, \vartheta) = (\mathbb{H}, -)$. We conclude that $\mathrm{sign}\, \mathrm{ad}_h = \lambda\, |\mathrm{sign}^\star h|$. ∎

Remark 3.13. Since by Remark 3.10 the total signature of an involution on A is continuous, it follows from Lemma 3.12 that if h is any hermitian form over (A, σ), then the set $\{P \in X_F \mid \mathrm{sign}_P\, h = 0\}$ is clopen.

Corollary 3.14. *Let* (M, h) *be a hermitian space over* (A, σ) *and let* $a \in \mathrm{Sym}(A, \sigma)^\times$*. Consider the hermitian space* (M, ah) *over* $(A, \mathrm{Int}(a) \circ \sigma)$*. Let* $P \in X_F$*. Then*

$$\mathrm{sign}^\star_P(ah) = \pm\, \mathrm{sign}^\star_P\, h.$$

Proof. An easy computation shows that the involutions ad_h and ad_{ah} coincide on $\mathrm{End}_A(M)$. Hence they have the same signature at $P \in X_F$. The conclusion now follows from Lemma 3.12. ∎

In other words, scaling by an invertible element at most changes the sign of the signature. Scaling by -1 gives an instance where a sign change of the signature occurs. This is contrary to what is claimed in [BP2, p. 662].

Lemma 3.15. *Let $a \in \mathrm{Sym}(A,\sigma)^\times$. For any $P \in X_F$ we have*

$$\mathrm{sign}^\star_P \langle a\rangle_\sigma = \pm \frac{1}{\lambda_P}\,\mathrm{sign}_P(\mathrm{Int}(a^{-1})\circ\sigma),$$

with λ_P as defined in Lemma 3.11.

Proof. The involution $(\mathrm{Int}(a^{-1})\circ\sigma)\otimes \mathrm{id}_{F_P}$ on $A\otimes_F F_P$ is adjoint to some form φ_P. We have

$$\mathrm{sign}^\star_P\langle 1\rangle_{\mathrm{Int}(a^{-1})\circ\sigma} = \mathrm{sign}_P\,\varphi_P = \pm\frac{1}{\lambda_P}\,\mathrm{sign}_P(\mathrm{Int}(a^{-1})\circ\sigma),$$

by Lemmas 3.6 and 3.11. By Corollary 3.14, we have

$$\mathrm{sign}^\star_P\langle a\rangle_\sigma = \pm\,\mathrm{sign}_P\langle 1\rangle_{\mathrm{Int}(a^{-1})\circ\sigma}.$$

The result now follows. ∎

Lemma 3.16. *Let (A,σ) and (B,τ) be central simple F-algebras, equipped with F-linear involutions. Let $a \in \mathrm{Sym}(A,\sigma)^\times$ and $b \in \mathrm{Sym}(B,\tau)^\times$. For any $P \in X_F$ we have*

$$\mathrm{sign}_P\langle a\otimes b\rangle_{\sigma\otimes\tau} = \pm\mu_P\,\mathrm{sign}_P\langle a\rangle_\sigma\,\mathrm{sign}_P\langle b\rangle_\tau,$$

where $\mu_P = 4$ if $A\otimes_F F_P$ and $B\otimes_F F_P$ are both non-split, and $\mu_P = 1$ otherwise.

Proof. By Lemma 3.15 and the fact that the signature of involutions is multiplicative we have

$$\begin{aligned}
\mathrm{sign}_P\langle a\otimes b\rangle_{\sigma\otimes\tau} &= \pm\frac{1}{\lambda_{P,A\otimes B,\sigma\otimes\tau}}\,\mathrm{sign}_P\Big(\mathrm{Int}((a\otimes b)^{-1})\circ(\sigma\otimes\tau)\Big)\\
&= \pm\frac{1}{\lambda_{P,A\otimes B,\sigma\otimes\tau}}\,\mathrm{sign}_P\Big((\mathrm{Int}(a^{-1})\circ\sigma)\otimes(\mathrm{Int}(b^{-1})\circ\tau)\Big)\\
&= \pm\frac{1}{\lambda_{P,A\otimes B,\sigma\otimes\tau}}\,\mathrm{sign}_P(\mathrm{Int}(a^{-1})\circ\sigma)\,\mathrm{sign}_P(\mathrm{Int}(b^{-1})\circ\tau)\\
&= \pm\frac{\lambda_{P,A,\sigma}\lambda_{P,B,\tau}}{\lambda_{P,A\otimes B,\sigma\otimes\tau}}\,\mathrm{sign}_P\langle a\rangle_\sigma\,\mathrm{sign}_P\langle b\rangle_\tau.
\end{aligned}$$

Letting $\mu_P = \lambda_{P,A,\sigma}\lambda_{P,B,\tau}/\lambda_{P,A\otimes B,\sigma\otimes\tau}$, its value can be determined by a case analysis. ∎

4. An Algorithm for choosing the Sign of the Signature

In quadratic form theory the signature of the form $\langle 1\rangle$ is always 1 at any ordering of the ground field. In contrast, the signature of the hermitian form $\langle 1\rangle_\sigma$ over (A,σ) may not even always be positive and could very well be zero, cf. Lemma 3.6.

In order to pursue the analogy with the quadratic forms case, it seems natural to require of the signature map at P from $W(A,\sigma)$ to $\mathbb{Z}$ that the signature of $\langle 1\rangle_\sigma$ be positive. This is precisely the approach taken in [BP2, §3.3, §3.4], where the form in $\{\varphi_P, -\varphi_P\}$ is chosen, whose signature at P is nonnegative, cf. Remark (3.3). The effect of this choice is to make the signature of $\langle 1\rangle_\sigma$ positive by Lemma 3.6.

However, it is possible that the signature of $\langle 1\rangle_\sigma$ at P (i.e. $\mathrm{sign}_P\,\varphi_P$) is zero, in which case the approach taken in [BP2] no longer works and the signature map at P from $W(A,\sigma)$ to $\mathbb{Z}$ remains defined only up to sign.

In order to fill this gap, our approach consists of replacing the form $\langle 1\rangle_\sigma$ by a finite number of rank one hermitian forms $\langle b_1\rangle_\sigma, \dots, \langle b_\ell\rangle_\sigma$ over (A,σ), having the property that at any ordering $P \in X_F$ at least one of them has nonzero signature. We start by proving the existence of the elements $b_1, \dots, b_\ell$. As our proof makes use of Merkurjev's theorem [Me] we will deal with the case of multi-quaternion algebras with decomposable involution first.

Remark 4.1.

(1) Let (A,σ), (B,τ) and (C,υ) be central simple F-algebras with F-linear involution such that $(A,\sigma) \cong (B,\tau) \otimes_F (C,\upsilon)$, then

$$\mathrm{type}(\sigma) = \mathrm{type}(\tau) \cdot \mathrm{type}(\upsilon), \tag{5}$$

cf. [KMRT, 2.23].

(2) Assume that A is a biquaternion algebra with decomposable involution σ. If σ is orthogonal, then it is not difficult to see that there exist quaternion algebras with orthogonal involution (Q_1,σ_1), (Q_2,σ_2) and quaternion algebras with symplectic involution (Q_1',γ_1), (Q_2',γ_2) such that

$$(A,\sigma) \cong (Q_1,\sigma_1) \otimes_F (Q_2,\sigma_2) \cong (Q_1',\gamma_1) \otimes_F (Q_2',\gamma_2),$$

cf. [ST, §2]. On the other hand, if σ is decomposable symplectic, then it follows from (5) that one of the quaternion components has to be endowed with the canonical (symplectic) involution, and the other with an orthogonal involution.

Lemma 4.2. *Let $(A,\sigma) = (Q_1,\sigma_1) \otimes_F \cdots \otimes_F (Q_n,\sigma_n)$ be a multi-quaternion algebra with decomposable F-linear involution. Let $P \in X_F \setminus \mathrm{Nil}(A,\sigma)$. Then the number of indices $i \in \{1,\dots,n\}$ such that $P \in \mathrm{Nil}(Q_i,\sigma_i)$ is even.*

Proof. Recall that $P \in \mathrm{Nil}(Q_i,\sigma_i)$ if and only if $Q_i \otimes_F F_P \cong \mathbb{H}_P$ (resp. $M_2(F_P)$) in case σ_i is orthogonal (resp. symplectic). The statement now follows from an easy, but tedious, case analysis depending on the type of σ and the parity of n. ∎

Proposition 4.3. *Let $(A,\sigma) = (Q_1,\sigma_1)\otimes_F \cdots \otimes_F (Q_n,\sigma_n)$ be a multi-quaternion algebra with decomposable F-linear involution. There exists a finite subset $S = \{a_1,\dots,a_\ell\}$ of $\mathrm{Sym}(A,\sigma)^\times$ such that for every $P \in X_F \setminus \mathrm{Nil}(A,\sigma)$ there is an index $r \in \{1,\dots,\ell\}$ such that*

$$\mathrm{sign}_P(\mathrm{Int}(a_r)\circ\sigma) \neq 0.$$

Proof. We will carry out the proof in three steps.

(1) Assume that $n = 1$, so that $A = (a,b)_F$ for certain elements $a, b \in F^\times$. For a positive integer t and $a_1,\dots,a_t \in F^\times$, recall the Harrison set notation

$$H(a_1,\dots,a_t) := \{P \in X_F \mid a_1 >_P 0, \dots, a_t >_P 0\}.$$

Observe that

$$X_F = H(a,b) \dot\cup H(a,-b) \dot\cup H(-a,-b) \dot\cup H(-a,b).$$

If σ is symplectic, then $X_F \setminus \mathrm{Nil}(A,\sigma) = H(-a,-b)$ and thus $\mathrm{sign}_P\,\sigma = 2$ for all $P \in H(-a,-b)$ since T_σ is the norm form of A (up to a factor $\langle 2 \rangle$). Thus we may take $S = \{1\}$.

Next assume that σ is orthogonal. In this case

$$X_F \setminus \mathrm{Nil}(A,\sigma) = H(a,b) \dot\cup H(a,-b) \dot\cup H(-a,b).$$

Consider the orthogonal involution ϑ, defined by

$$\vartheta(1) = 1,\ \vartheta(i) = -i,\ \vartheta(j) = j,\ \vartheta(k) = k,$$

where $\{1, i, j, k\}$ denotes the usual F-basis of A. Since any two involutions differ by an inner automorphism, there exists a $q \in \mathrm{Sym}(A,\sigma)^\times$ such that $\vartheta = \mathrm{Int}(q) \circ \sigma$. Consider also the involutions $\tau = \mathrm{Int}(j) \circ \vartheta$ and $\omega = \mathrm{Int}(k) \circ \vartheta$. After computing the involution trace forms of ϑ, τ and ω we see that

$$\begin{aligned} \mathrm{sign}_P\,\vartheta &= 2 \text{ for all } P \in H(-a,b),\\ \mathrm{sign}_P\,\tau &= 2 \text{ for all } P \in H(a,b),\\ \mathrm{sign}_P\,\omega &= 2 \text{ for all } P \in H(a,-b). \end{aligned}$$

Thus we may take $S = \{q, jq, kq\}$. This settles the case $n = 1$.

(2) Next assume that $n = 2$, so that $A = (a,b)_F \otimes_F (c,d)_F$ for certain elements $a,b,c,d \in F^\times$ and $\sigma = \sigma_1 \otimes \sigma_2$.

***σ* orthogonal (*n* = 2).** We may assume that σ_1 is orthogonal on $Q_1 = (a,b)_F$ and that σ_2 is orthogonal on $Q_2 = (c,d)_F$, cf. Remark 4.1(2). We have $P \in X_F \setminus \mathrm{Nil}(A,\sigma)$ if and only if $A \otimes_F F_P \cong M_4(F_P)$. Hence, $P \in X_F \setminus \mathrm{Nil}(A,\sigma)$ if and only if

$$Q_1 \otimes_F F_P \cong M_2(F_P) \quad \text{and} \quad Q_2 \otimes_F F_P \cong M_2(F_P)$$

or

$$Q_1 \otimes_F F_P \cong \mathbb{H}_P \quad \text{and} \quad Q_2 \otimes_F F_P \cong \mathbb{H}_P.$$

Thus

$$\begin{aligned} X_F \setminus \mathrm{Nil}(A,\sigma) &= \Big[\Big(H(a,b) \dot\cup H(a,-b) \dot\cup H(-a,b)\Big) \cap \Big(H(c,d) \dot\cup H(c,-d) \dot\cup H(-c,d)\Big)\Big]\\ &\qquad \dot\cup \Big[H(-a,-b) \cap H(-c,-d)\Big]\\ &= \Big[\Big(X_F \setminus \mathrm{Nil}(Q_1,\sigma_1)\Big) \cap \Big(X_F \setminus \mathrm{Nil}(Q_2,\sigma_2)\Big)\Big]\\ &\qquad \dot\cup \Big(\mathrm{Nil}(Q_1,\sigma_1) \cap \mathrm{Nil}(Q_2,\sigma_2)\Big). \end{aligned}$$

We first consider $\Big(X_F \setminus \mathrm{Nil}(Q_1,\sigma_1)\Big) \cap \Big(X_F \setminus \mathrm{Nil}(Q_2,\sigma_2)\Big)$. By the $n = 1$ case there exist involutions $\pi_{i,1}, \pi_{i,2}, \pi_{i,3}$ on Q_i for $i = 1, 2$ such that

$$\pi_{i,k} = \mathrm{Int}(a_{i,k}) \circ \sigma_i \text{ for some } a_{i,k} \in \mathrm{Sym}(Q_i,\sigma_i)^\times$$

for $k = 1, 2, 3$ and for every $P \in X_F \setminus \mathrm{Nil}(Q_i, \sigma_i)$, one of $\mathrm{sign}_P\, \pi_{i,1}, \mathrm{sign}_P\, \pi_{i,2}, \mathrm{sign}_P\, \pi_{i,3}$ is equal to 2. We consider all possible products

$$\varrho_{k,\ell} = \pi_{1,k} \otimes \pi_{2,\ell} = \mathrm{Int}(a_{1,k} \otimes a_{2,\ell}) \circ (\sigma_1 \otimes \sigma_2)$$

for $k, \ell \in \{1, 2, 3\}$. Note that $a_{1,k} \otimes a_{2,\ell} \in \mathrm{Sym}(A, \sigma)^\times$. Then for each $P \in \big(X_F \setminus \mathrm{Nil}(Q_1, \sigma_1)\big) \cap \big(X_F \setminus \mathrm{Nil}(Q_2, \sigma_2)\big)$, one of

$$\mathrm{sign}_P\, \varrho_{k,\ell} = (\mathrm{sign}_P\, \pi_{1,k})(\mathrm{sign}_P\, \pi_{2,\ell})$$

is equal to 4.

Secondly, we consider $\mathrm{Nil}(Q_1, \sigma_1) \cap \mathrm{Nil}(Q_2, \sigma_2)$. For $i = 1, 2$, let γ_i denote the unique symplectic involution on Q_i. Then there exist $a_i \in \mathrm{Skew}(Q_i, \sigma_i)^\times$ such that $\gamma_i = \mathrm{Int}(a_i) \circ \sigma_i$. Let

$$\gamma = \gamma_1 \otimes \gamma_2 = \mathrm{Int}(a_1 \otimes a_2) \circ (\sigma_1 \otimes \sigma_2).$$

Then $a_1 \otimes a_2 \in \mathrm{Sym}(A, \sigma)^\times$. Furthermore,

$$\mathrm{sign}_P\, \gamma = (\mathrm{sign}_P\, \gamma_1)(\mathrm{sign}_P\, \gamma_2) = 4$$

for all $P \in \mathrm{Nil}(Q_1, \sigma_1) \cap \mathrm{Nil}(Q_2, \sigma_2)$. Thus we may take $S = \{a_{1,k} \otimes a_{2,\ell} \mid k, \ell = 1, 2, 3\} \cup \{a_1 \otimes a_2\}$.

$\boldsymbol{\sigma}$ symplectic ($\boldsymbol{n = 2}$). We may assume that σ_1 is orthogonal on $Q_1 = (a, b)_F$ and that $\sigma_2 = \gamma_2$ is the unique symplectic involution on $Q_2 = (c, d)_F$, cf. Remark 4.1(2). We have $P \in X_F \setminus \mathrm{Nil}(A, \sigma)$ if and only if $A \otimes_F F_P \cong M_2(\mathbb{H}_P)$. Hence, $P \in X_F \setminus \mathrm{Nil}(A, \sigma)$ if and only if

$$Q_1 \otimes_F F_P \cong M_2(F_P) \quad \text{and} \quad Q_2 \otimes_F F_P \cong \mathbb{H}_P$$

or

$$Q_1 \otimes_F F_P \cong \mathbb{H}_P \quad \text{and} \quad Q_2 \otimes_F F_P \cong M_2(F_P).$$

Thus

$$\begin{aligned} X_F \setminus \mathrm{Nil}(A, \sigma) &= \Big[\big(H(a, b) \,\dot\cup\, H(a, -b) \,\dot\cup\, H(-a, b)\big) \cap H(-c, -d)\Big] \\ &\qquad \dot\cup \Big[H(-a, -b) \cap \big(H(c, d) \,\dot\cup\, H(c, -d) \,\dot\cup\, H(-c, d)\big)\Big] \\ &= \Big[\big(X_F \setminus \mathrm{Nil}(Q_1, \sigma_1)\big) \cap \big(X_F \setminus \mathrm{Nil}(Q_2, \gamma_2)\big)\Big] \\ &\qquad \dot\cup \big(\mathrm{Nil}(Q_1, \sigma_1) \cap \mathrm{Nil}(Q_2, \gamma_2)\big). \end{aligned}$$

We first consider $\big(X_F \setminus \mathrm{Nil}(Q_1, \sigma_1)\big) \cap \big(X_F \setminus \mathrm{Nil}(Q_2, \gamma_2)\big)$. By the $n = 1$ case there exist involutions π_1, π_2, π_3 on Q_1 such that

$$\pi_k = \mathrm{Int}(a_k) \circ \sigma_i \text{ for some } a_k \in \mathrm{Sym}(Q_1, \sigma_1)^\times$$

for $k = 1, 2, 3$ and for every $P \in X_F \setminus \mathrm{Nil}(Q_1, \sigma_1)$, one of $\mathrm{sign}_P\, \pi_1, \mathrm{sign}_P\, \pi_2, \mathrm{sign}_P\, \pi_3$ is equal to 2. Also, $\mathrm{sign}_P\, \gamma_2 = 2$ for every $P \in X_F \setminus \mathrm{Nil}(Q_2, \gamma_2)$. For $k = 1, 2, 3$ we consider, as before, all possible products

$$\pi_k \otimes \gamma_2 = \mathrm{Int}(a_k \otimes 1) \circ (\sigma_1 \otimes \gamma_2).$$

Note that $a_k \otimes 1 \in \mathrm{Sym}(A,\sigma)^\times$. We have that for each $P \in \big(X_F \setminus \mathrm{Nil}(Q_1,\sigma_1)\big) \cap \big(X_F \setminus \mathrm{Nil}(Q_2,\gamma_2)\big)$, one of

$$\mathrm{sign}_P(\pi_k \otimes \gamma_2) = (\mathrm{sign}_P\, \pi_k)(\mathrm{sign}_P\, \gamma_2)$$

is equal to 4.

Secondly, we consider $\mathrm{Nil}(Q_1,\sigma_1)\cap\mathrm{Nil}(Q_2,\gamma_2)$. Let γ_1 denote the unique symplectic involution on Q_1. Then there exists $a_1 \in \mathrm{Skew}(Q_1,\sigma_1)^\times$ such that $\gamma_1 = \mathrm{Int}(a_1) \circ \sigma_1$. Hence $\mathrm{sign}_P\, \gamma_1 = 2$ for all $P \in \mathrm{Nil}(Q_1,\sigma_1) = X_F \setminus \mathrm{Nil}(Q_1,\gamma_1)$.

Let τ_2 be an orthogonal involution on Q_2. Note that $\mathrm{Nil}(Q_2,\gamma_2) = X_F \setminus \mathrm{Nil}(Q_2,\tau_2)$. There exists an element $a_2 \in \mathrm{Skew}(Q_2,\gamma_2)^\times$ such that $\tau_2 = \mathrm{Int}(a_2) \circ \gamma_2$. By the case $n = 1$, there are $b_1, b_2, b_3 \in \mathrm{Sym}(Q_2,\tau_2)^\times$ such that for every $P \in \mathrm{Nil}(Q_2,\gamma_2)$ one of $\mathrm{sign}_P\, \mathrm{Int}(b_i) \circ \tau_2$ is equal to 2. For $k = 1, 2, 3$, let

$$\pi_k := \mathrm{Int}(b_k) \circ \tau_2 = \mathrm{Int}(b_k) \circ \mathrm{Int}(a_2) \circ \gamma_2 = \mathrm{Int}(b_k a_2) \circ \gamma_2$$

and note that $b_k a_2 \in \mathrm{Skew}(Q_2,\gamma_2)^\times$. We consider all possible products

$$\gamma_1 \otimes \pi_k = \mathrm{Int}(a_1 \otimes b_k a_2) \circ (\sigma_1 \otimes \gamma_2).$$

Observe that $a_1 \otimes b_k a_2 \in \mathrm{Sym}(A,\sigma)^\times$. We have that for each $P \in \mathrm{Nil}(Q_1,\sigma_1) \cap \mathrm{Nil}(Q_2,\gamma_2)$, one of

$$\mathrm{sign}_P(\gamma_1 \otimes \pi_k) = (\mathrm{sign}_P\, \gamma_1)(\mathrm{sign}_P\, \pi_k)$$

is equal to 4. Thus we may take $S = \{a_k \otimes 1 \mid k = 1, 2, 3\} \cup \{a_1 \otimes b_k a_2 \mid k = 1, 2, 3\}$.

(3) Assume finally that $n \geq 3$.

σ orthogonal ($n \geq 3$). We may assume that σ_i is orthogonal on Q_i for $i = 1, \ldots, n$, cf. Remark 4.1(2). We have $X_F \setminus \mathrm{Nil}(A,\sigma) = \{P \in X_F \mid A \otimes_F F_P \cong M_{2^n}(F_P)\}$. For $P \in X_F$, let

$$\delta_P = \big|\{i \in \{1,\ldots,n\} \mid Q_i \otimes_F F_P \cong \mathbb{H}_P\}\big|.$$

Then $X_F \setminus \mathrm{Nil}(A,\sigma) = \{P \in X_F \mid \delta_P \text{ is even}\}$. Since $X_F \setminus \mathrm{Nil}(A,\sigma)$ is a finite union of sets of the form $\{P \in X_F \mid \delta_P = 2m\}$ for certain $m \in \mathbb{N}$, it suffices to prove the theorem for a fixed $m \in \mathbb{N}$ and for the set of orderings $\{P \in X_F \mid \delta_P = 2m\}$. The general statement will then follow by taking the union of the different sets S obtained in this way. Therefore we only consider orderings in $Y = \{P \in X_F \mid \delta_P = 2m\}$ for a fixed $m \in \mathbb{N}$. After relabeling indices we may assume that $Q_i \otimes_F F_P \cong \mathbb{H}_P$ if and only if $1 \leq i \leq 2m$. After regrouping we can thus write

$$\begin{aligned}(A,\sigma) = {} & (Q_1 \otimes_F Q_2, \sigma_1 \otimes \sigma_2) \otimes_F \cdots \otimes_F (Q_{2m-1} \otimes_F Q_{2m}, \sigma_{2m-1} \otimes \sigma_{2m}) \\ & \otimes_F (Q_{2m+1}, \sigma_{2m+1}) \otimes_F \cdots \otimes_F (Q_n, \sigma_n).\end{aligned}$$

Observe now that $P \in Y$ implies that $P \in X_F \setminus \mathrm{Nil}(Q_{2i+1} \otimes_F Q_{2i+2}, \sigma_{2i+1} \otimes \sigma_{2i+2})$ for $i = 0, \ldots, m-1$ and $P \in X_F \setminus \mathrm{Nil}(Q_\ell, \sigma_\ell)$ for $\ell = 2m+1, \ldots, n$. We now use the cases $n = 1$ and $n = 2$ and products of involutions to settle this case.

σ symplectic ($n \geq 3$). We may assume that σ_i is orthogonal on Q_i for $i = 1, \ldots, n-1$ and that $\sigma_n = \gamma_n$ is symplectic on Q_n, cf. Remark 4.1(2). We have $X_F \setminus \mathrm{Nil}(A,\sigma) = \{P \in X_F \mid A \otimes_F F_P \cong M_{2^{n-1}}(\mathbb{H}_P)\}$. For $P \in X_F$, let

$$\delta_P = \big|\{i \in \{1,\ldots,n\} \mid Q_i \otimes_F F_P \cong \mathbb{H}_P\}\big|.$$

Then $X_F \setminus \mathrm{Nil}(A,\sigma) = \{P \in X_F \mid \delta_P \text{ is odd}\}$. By an argument similar to the one in the previous case, it suffices to successively consider the following two sets of orderings:

Case a: $\{P \in X_F \setminus \mathrm{Nil}(A,\sigma) \mid Q_n \otimes_F F_P \cong \mathbb{H}_P\}$.

Case b: $\{P \in X_F \setminus \mathrm{Nil}(A,\sigma) \mid Q_n \otimes_F F_P \cong M_2(F_P)\}$.

In Case a, after relabeling, we may assume that $Q_i \otimes_F F_P \cong \mathbb{H}_P$ if and only if $i \in \{1,\dots,2m\} \cup \{n\}$. After regrouping we may write

$$(A,\sigma) = (A_1,\tau_1) \otimes_F (Q_{2m+1},\sigma_{2m+1}) \otimes_F \cdots \otimes_F (Q_n,\sigma_n),$$

where $(A_1,\tau_1) = (Q_1 \otimes_F Q_2, \sigma_1 \otimes \sigma_2) \otimes_F \cdots \otimes_F (Q_{2m-1} \otimes_F Q_{2m}, \sigma_{2m-1} \otimes \sigma_{2m})$. We conclude by using the orthogonal $n = 2$ case for (A_1,σ_1) and the case $n = 1$ for the other components together with products of involutions.

In Case b, after relabeling, we may assume that $Q_i \otimes_F F_P \cong \mathbb{H}_P$ if and only if $i \in \{1,\dots,2m+1\}$. After regrouping we may write

$$(A,\sigma) = (A_1,\tau_1) \otimes_F (Q_{2m+1},\sigma_{2m+1}) \otimes_F \cdots \otimes_F (Q_n,\sigma_n),$$

where $(A_1,\tau_1) = (Q_1 \otimes_F Q_2, \sigma_1 \otimes \sigma_2) \otimes_F \cdots \otimes_F (Q_{2m-1} \otimes_F Q_{2m}, \sigma_{2m-1} \otimes \sigma_{2m})$. We conclude by using the orthogonal $n = 2$ case for (A_1,σ_1), the symplectic $n = 2$ case for $(Q_{2m+1} \otimes_F Q_n, \sigma_{2m+1} \otimes \sigma_n)$ and the orthogonal $n = 1$ case for the other components together with products of involutions. ∎

Proposition 4.4. *Let (A,σ) be a central simple F-algebra equipped with an involution of the first kind. There exists an integer k and a finite subset $\{b_1,\dots,b_\ell\}$ of $\mathrm{Sym}(M_k(A),\sigma\otimes t)^\times$ such that for every $P \in X_F \setminus \mathrm{Nil}(M_k(A),\sigma\otimes t)$ there is an index $r \in \{1,\dots,\ell\}$ such that*

$$\mathrm{sign}_P(\mathrm{Int}(b_r) \circ (\sigma \otimes t)) \neq 0.$$

Proof. Since σ is of the first kind, the exponent of A in the Brauer group of F is at most 2. Thus, by Merkurjev's theorem [Me], there exist $k, m \in \mathbb{N}$ such that $M_k(A) \cong M_m(Q) \cong Q \otimes_F M_m(F)$, where $Q = Q_1 \otimes_F \cdots \otimes_F Q_n$ is a multi-quaternion algebra. Extend σ to the involution $\sigma \otimes t$ on $M_k(A)$, where t denotes transposition. Then $\sigma \otimes t \cong \mathrm{Int}(u) \circ (\tau \otimes t)$ for an involution τ of the same type as σ on Q and an invertible element $u \in \mathrm{Sym}(Q \otimes_F M_m(F), \tau \otimes t)$. Without loss of generality we may assume that $\tau = \sigma_1 \otimes \cdots \otimes \sigma_n$, where σ_i is an involution on Q_i for $i = 1,\dots,n$.

Consider the elements $a_1,\dots,a_\ell \in Q$ whose existence is asserted by Proposition 4.3. For $i = 1,\dots,\ell$ let b_i be the element in $M_k(A)$ which is mapped to $(a_i \otimes I_m)u^{-1}$ under the isomorphism $M_k(A) \cong Q \otimes_F M_m(F)$, where I_m denotes the identity matrix in $M_m(F)$. Then each $b_i \in \mathrm{Sym}(M_k(A),\sigma\otimes t)^\times$.

Let $P \in X_F \setminus \mathrm{Nil}(M_k(A),\sigma\otimes t)$. Observe that $\mathrm{Nil}(M_k(A),\sigma\otimes t) = \mathrm{Nil}(Q,\tau)$ since $M_k(A) \cong M_m(Q)$ and $\sigma \otimes t$ and τ are of the same type. By Proposition 4.3 there exists an index $r \in \{1,\dots,\ell\}$ such that

$$\mathrm{sign}_P(\mathrm{Int}(a_r) \circ \tau) \neq 0.$$

Thus by Remark 3.8(2) we have

$$\begin{aligned}
\operatorname{sign}_P(\operatorname{Int}(b_r)\circ(\sigma\otimes t)) &= \operatorname{sign}_P\Big(\operatorname{Int}((a_r\otimes I_m)u^{-1})\circ(\operatorname{Int}(u)\circ(\tau\otimes t))\Big)\\
&= \operatorname{sign}_P(\operatorname{Int}(a_r\otimes I_m)\circ\operatorname{Int}(u^{-1})\circ\operatorname{Int}(u)\circ(\tau\otimes t))\\
&= \operatorname{sign}_P(\operatorname{Int}(a_r\otimes I_m)\circ(\tau\otimes t))\\
&= \operatorname{sign}_P((\operatorname{Int}(a_r)\circ\tau)\otimes t)\\
&= \operatorname{sign}_P(\operatorname{Int}(a_r)\circ\tau)\cdot\operatorname{sign}_P(t)\\
&= m\operatorname{sign}_P(\operatorname{Int}(a_r)\circ\tau)\\
&\neq 0,
\end{aligned}$$

which concludes the proof. ∎

Corollary 4.5. *The set of (A,σ)-nil orderings of F is clopen.*

Proof. We have $\operatorname{Nil}(A,\sigma)=\operatorname{Nil}(M_k(A),\sigma\otimes t)$ for any $k\in\mathbb{N}$ since σ and $\sigma\otimes t$ are of the same type. By Proposition 4.4

$$\operatorname{Nil}(M_k(A),\sigma\otimes t)=\bigcap_{r=1}^{\ell}\{P\in X_F\mid \operatorname{sign}_P(\operatorname{Int}(b_r)\circ(\sigma\otimes t))=0\},$$

which is clopen by Remark 3.10. ∎

Theorem 4.6. *Let (A,σ) be a central simple F-algebra equipped with an involution of the first kind. There exists a finite subset $\{b_1,\dots,b_\ell\}$ of $\operatorname{Sym}(A,\sigma)^\times$ such that for every $P\in X_F\setminus\operatorname{Nil}(A,\sigma)$ there is an index $r\in\{1,\dots,\ell\}$ such that*

$$\operatorname{sign}_P(\operatorname{Int}(b_r)\circ\sigma)\neq 0.$$

Proof. Assume first that A is split, i.e. $A\cong M_n(F)$. If σ is symplectic then $\operatorname{Nil}(A,\sigma)=X_F$ and there is nothing to prove. If σ is orthogonal, then there exists $a\in\operatorname{Sym}(A,\sigma)^\times$ such that $\sigma=\operatorname{Int}(a)\circ t$, where t is the transpose involution. It follows that

$$\operatorname{sign}_P(\operatorname{Int}(a^{-1})\circ\sigma)=\operatorname{sign}_P t=n\neq 0$$

for all $P\in X_F$.

Secondly assume that A is not split, so that $A\cong M_n(D)$ for some $n\in\mathbb{N}$ and some division algebra D. Since σ is an F-linear involution on A, there exists an F-linear involution ϑ on D. We first show that for every $P\in X_F\setminus\operatorname{Nil}(A,\sigma)$ there exists a $b_P\in\operatorname{Sym}(A,\sigma)^\times$ such that $\operatorname{sign}_p(\operatorname{Int}(b_P)\circ\sigma)\neq 0$.

Let $P\in X_F\setminus\operatorname{Nil}(A,\sigma)$. Assume for the sake of contradiction that $\operatorname{sign}_P\omega=0$ for every F-linear involution ω on A. Since such involutions are adjoint to n-dimensional hermitian forms over (D,ϑ) and this correspondence is one-to-one (up to a nonzero scalar factor), all hermitian forms of dimension n over (D,ϑ) have signature zero at P by Lemma 3.12. Let $d\in\operatorname{Sym}(D,\vartheta)^\times$ be arbitrary, then the n-dimensional hermitian form $n\times\langle d\rangle_\vartheta$ has signature zero at P. This implies $\operatorname{sign}_P^\star\langle d\rangle_\vartheta=0$ for all $d\in\operatorname{Sym}(D,\vartheta)^\times$. Hence all hermitian forms over (D,ϑ) have signature zero at P.

However, by Proposition 4.4 (with D in the role of A) there exists $k \in \mathbb{N}$ and an involution τ on $M_k(D)$ such that $\operatorname{sign}_P \tau \neq 0$. But τ is adjoint to some hermitian form over (D, ϑ) which should have zero signature at P, a contradiction with Lemma 3.12. We conclude that there exists a $b_P \in \operatorname{Sym}(A, \sigma)^\times$ such that $\operatorname{sign}_P(\operatorname{Int}(b_P) \circ \sigma) \neq 0$.

For $a \in \operatorname{Sym}(A, \sigma)^\times$ define

$$U(a) := \{P \in X_F \mid \operatorname{sign}_P(\operatorname{Int}(a) \circ \sigma) \neq 0\}.$$

By Remark 3.10 the set $U(a)$ is clopen in X_F. By the previous part of the proof we have

$$X_F \setminus \operatorname{Nil}(A, \sigma) = \bigcup_{P \in X_F} U(b_P).$$

Since $X_F \setminus \operatorname{Nil}(A, \sigma)$ is compact by Corollary 4.5, there exists $\ell \in \mathbb{N}$ and $b_1, \dots, b_\ell \in \operatorname{Sym}(A, \sigma)^\times$ such that

$$X_F \setminus \operatorname{Nil}(A, \sigma) = \bigcup_{i=1}^{\ell} U(b_i).$$

∎

Corollary 4.7. *An ordering $P \in X_F$ is (A, σ)-nil if and only if $\operatorname{sign}^\star_P h = 0$ for every hermitian form h over (A, σ) if and only if $\operatorname{sign}^\star_P \langle a \rangle_\sigma = 0$ for every $a \in \operatorname{Sym}(A, \sigma)^\times$.*

Proof. By Theorem 4.6,

$$\operatorname{Nil}(A, \sigma) = \bigcap_{i=1}^{\ell} \{P \in X_F \mid \operatorname{sign}_P(\operatorname{Int}(b_i) \circ \sigma) = 0\}.$$

The result then follows from the definition of nil-ordering and Lemma 3.15 (since $\operatorname{sign}_P(\operatorname{Int}(b_i) \circ \sigma) = |\operatorname{sign}^\star_P \langle b_i^{-1} \rangle_\sigma|$). ∎

The Algorithm. Fix some tuple of elements $(b_1, \dots, b_\ell)$ with properties as described in Theorem 4.6. Observe that for each $P \in X_F$ we have

$$|\operatorname{sign}^\star_P \langle b_i \rangle_\sigma| = \frac{1}{\lambda_P} \operatorname{sign}_P(\operatorname{Int}(b_i) \circ \sigma) \tag{6}$$

by Lemma 3.15 since $\langle b_i^{-1} \rangle_\sigma \simeq \langle b_i \rangle_\sigma$. By Theorem 4.6 this implies that for each $P \in X_F \setminus \operatorname{Nil}(A, \sigma)$ at least one of $\operatorname{sign}^\star_P \langle b_1 \rangle_\sigma, \dots, \operatorname{sign}^\star_P \langle b_\ell \rangle_\sigma$ is nonzero.

Therefore, for each $P \in X_F \setminus \operatorname{Nil}(A, \sigma)$ we decide if the signature computation is performed with φ_P or $-\varphi_P$ as follows:

(*i*) Let i be the least element in $\{1, \dots, \ell\}$ such that $\operatorname{sign}^\star_P \langle b_i \rangle_\sigma \neq 0$.

(*ii*) If $\operatorname{sign}^\star_P \langle b_i \rangle_\sigma >_P 0$, we keep using φ_P for the signature computation at this ordering. If $\operatorname{sign}^\star_P \langle b_i \rangle_\sigma <_P 0$, we replace φ_P by $-\varphi_P$ in the computation of signatures at P (which then makes $\operatorname{sign}^\star_P \langle b_i \rangle_\sigma >_P 0$).

Note that we may assume $b_1 = 1$, in which case our algorithm extends the algorithm in [BP2, §3.3, §3.4].

This algorithm depends on the choice of the tuple $(b_1, \dots, b_\ell)$ in Theorem 4.6. Once such a choice is made for the algebra with involution (A, σ) we can consider properties such as the continuity of the total signature function $\mathrm{sign}(h) : X_F \longrightarrow \mathbb{Z}$ associated to a hermitian form h over (A, σ).

Notation. In the remainder of the paper when writing sign_P instead of $\mathrm{sign}_P^\star$ we mean that we use the above algorithm for some fixed choice of a tuple $(b_1, \ldots, b_\ell)$.

5. Continuity of the Total Signature Map of Hermitian Forms

Lemma 5.1. *There is a finite partition of X_F into clopens*

$$X_F = \mathrm{Nil}(A, \sigma) \mathbin{\dot\cup} \dot{\bigcup_{i=1}^{s}} Z_i,$$

and there are $\alpha_1, \ldots, \alpha_s \in \mathrm{Sym}(A, \sigma)^\times$ such that $\mathrm{sign}\langle \alpha_i \rangle_\sigma$ is constant non-zero on Z_i.

Proof. Let $b_1, \ldots, b_\ell$ be as in Theorem 4.6 and, for $r = 1, \ldots, \ell$, let

$$Y_r := \{P \in X_F \mid \mathrm{sign}_P \langle b_i \rangle_\sigma = 0,\ i = 1, \ldots, r\}.$$

Observe that each Y_r is clopen since

$$Y_r = \bigcap_{i=1}^{r} \{P \in X_F \mid \mathrm{sign}_P(\mathrm{Int}(b_i) \circ \sigma) = 0\}.$$

We have $Y_0 := X_F \supseteq Y_1 \supseteq \cdots \supseteq Y_{\ell-1} \supseteq Y_\ell = \mathrm{Nil}(A, \sigma)$ and therefore,

$$X_F = (Y_0 \setminus Y_1) \mathbin{\dot\cup} (Y_1 \setminus Y_2) \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} (Y_{\ell-1} \setminus Y_\ell) \mathbin{\dot\cup} \mathrm{Nil}(A, \sigma).$$

Let $r \in \{0, \ldots, \ell - 1\}$ and consider $Y_r \setminus Y_{r+1}$. By (6) the map $\mathrm{sign}\langle b_{r+1} \rangle_\sigma$ is never 0 on $Y_r \setminus Y_{r+1}$ and only takes a finite number of values $k_1, \ldots, k_m$.

Claim: There exists a $\lambda \in \{1, 2\}$ such that

$$\mathrm{sign}\langle b_{r+1} \rangle_\sigma = \frac{1}{\lambda} \mathrm{sign}(\mathrm{Int}(b_{r+1}) \circ \sigma)$$

on $Y_r \setminus Y_{r+1}$.

Proof of claim: If σ is orthogonal and $P \notin \mathrm{Nil}(A, \sigma)$, then $(D_P, \vartheta_P) \cong (F_P, \mathrm{id}_{F_P})$. By Lemma 3.11 together with the definition of signature of a hermitian form (since $P \in Y_r \setminus Y_{r+1}$) we have

$$\mathrm{sign}_P \langle b_{r+1} \rangle_\sigma = \mathrm{sign}_P(\mathrm{Int}(b_{r+1}) \circ \sigma).$$

If σ is symplectic and $P \notin \mathrm{Nil}(A, \sigma)$, then $(D_P, \vartheta_P) \cong (\mathbb{H}_P, -)$. By Lemma 3.11 together with the definition of signature of a hermitian form (since $P \in Y_r \setminus Y_{r+1}$) we have

$$\mathrm{sign}\langle b_{r+1} \rangle_\sigma = \frac{1}{2} \mathrm{sign}(\mathrm{Int}(b_{r+1}) \circ \sigma).$$

So we simply take $\lambda = 2$ if σ is symplectic and $\lambda = 1$ if σ is orthogonal.

The claim gives us:

$$(\mathrm{sign}\langle b_{r+1} \rangle_\sigma)^{-1}(k_i) \cap (Y_r \setminus Y_{r+1}) = \Big(\mathrm{sign}(\mathrm{Int}(b_{r+1}) \circ \sigma)\Big)^{-1}(\lambda k_i) \cap (Y_r \setminus Y_{r+1}),$$

which is clopen by Remark 3.10. It follows that $Y_r \setminus Y_{r+1}$ is covered by finitely many disjoint clopen sets on which the map $\operatorname{sign}\langle b_{r+1}\rangle_\sigma$ has constant non-zero value. The result follows since the sets $Y_r \setminus Y_{r+1}$ for $r = 0, \dots, \ell - 1$ form a partition of $X_F \setminus \operatorname{Nil}(A, \sigma)$. ∎

Proposition 5.2. *Let h be a hermitian form over (A, σ). The total signature of h,*

$$\operatorname{sign} h : X_F \longrightarrow \mathbb{Z},\ P \longmapsto \operatorname{sign}_P h$$

is continuous.

Proof. We use the notation and the conclusion of Lemma 5.1. Since $\operatorname{Nil}(A, \sigma)$ and the sets Z_i are clopen, it suffices to show that $(\operatorname{sign} h)|_{Z_i}$ is continuous for every $i = 1, \dots, s$.

Let $i \in \{1, \dots, s\}$ and let $k_i \in \mathbb{Z} \setminus \{0\}$ be such that $\operatorname{sign}\langle \alpha_i\rangle_\sigma = k_i$ on Z_i. Let $k \in \mathbb{Z}$. Then

$$\begin{aligned}
((\operatorname{sign} h)|_{Z_i})^{-1}(k) &= \{P \in Z_i \mid \operatorname{sign}_P h = k\} \\
&= \{P \in Z_i \mid k_i \operatorname{sign}_P h = k_i k\} \\
&= \{P \in Z_i \mid k_i \operatorname{sign}_P h = k \operatorname{sign}_P \langle \alpha_i\rangle_\sigma\} \\
&= \{P \in Z_i \mid \operatorname{sign}_P (k_i \times h \perp k \times \langle -\alpha_i\rangle_\sigma) = 0\}.
\end{aligned}$$

It follows from Lemma 3.12 that

$$((\operatorname{sign} h)|_{Z_i})^{-1}(k) = \{P \in Z_i \mid \operatorname{sign}_P \operatorname{ad}_{k_i \times h \perp k \times \langle -\alpha_i\rangle_\sigma} = 0\},$$

which is clopen by Remark 3.10. ∎

6. Torsion in Witt Groups and Sums of Hermitian Squares

Let F be a formally real field. It is well-known that the Witt ring of F is torsion-free if and only if F is pythagorean (i.e., every sum of squares in F is a square in F), see [Lam, VIII, Theorem 4.1].

Now let (A, σ) be a central simple F-algebra equipped with an F-linear involution. A *hermitian square* in (A, σ) is an element of A of the form $\sigma(x)x$ for some $x \in A$. We denote the set of hermitian squares in (A, σ) by $(A, \sigma)^2$ and the set of sums of hermitian squares in (A, σ) by $\Sigma(A, \sigma)^2$. It is clear that

$$(A, \sigma)^2 \subseteq \Sigma(A, \sigma)^2 \subseteq \operatorname{Sym}(A, \sigma).$$

We say that (A, σ) is *pythagorean* if $\Sigma(A, \sigma)^2 = (A, \sigma)^2$, i.e., if every sum of hermitian squares in (A, σ) is a hermitian square in (A, σ).

We denote the torsion subgroup of $W(A, \sigma)$ by $W_t(A, \sigma)$. A fundamental result of Pfister is that $W_t(F)$ is 2-primary. The torsion subgroup $W_t(A, \sigma)$ is 2-primary as well, see [S1, Cor. 6.1] or [Ma, Thm. 4.1].

In this section we will show that there is in general no obvious relation between the property ‘torsion-free Witt group’ ($W_t(A, \sigma) = 0$) and the property ‘pythagorean’.

An unsurprising exception is the following:

Proposition 6.1. *Let D be a quaternion division algebra over a formally real field F, equipped with quaternion conjugation $-$. Let N be the norm form of D.*

(1) *If F is pythagorean, then $(D,-)$ is pythagorean.*
(2) $(D,-)$ *is pythagorean and N is strongly anisotropic if and only if every weakly isotropic hermitian form over $(D,-)$ is isotropic and is of dimension at least two.*
(3) $(D,-)$ *is pythagorean and N is strongly anisotropic if and only if $W_t(D,-)=0$.*

(*Note that N is strongly anisotropic if and only if $\ell\times N$ is anisotropic for all $\ell\in\mathbb{N}$ if and only if every sum of nonzero hermitian squares is nonzero.*)

Proof. (1) Follows from a computation with the norm form N of D.

(2) Let $h=\langle a_1,\dots,a_n\rangle_-$ be a hermitian form over $(D,-)$. Note that $a_1,\dots,a_n\in \mathrm{Sym}(D,-)=F$. Assume that $\ell\times h$ is isotropic for some positive integer ℓ. Then there exist ℓ vectors $(x_{11},\dots,x_{1n}),\dots,(x_{\ell1},\dots,x_{\ell n})$ in D^n, not all zero, such that

$$a_1(\overline{x_{11}}x_{11}+\cdots+\overline{x_{\ell1}}x_{\ell1})+\cdots+a_n(\overline{x_{1n}}x_{1n}+\cdots+\overline{x_{\ell n}}x_{\ell n})=0.$$

Thus, by the hypotheses on $(D,-)$ there exist $y_1,\dots,y_n\in D$, not all zero, such that

$$a_1\overline{y_1}y_1+\cdots+a_n\overline{y_n}y_n=0,$$

i.e., h is isotropic. Note that $n\geq 2$ since D is a division algebra.

Conversely, let $\alpha=\overline{x}x+\overline{y}y$ with $x,y\in D^\times$ and note that $\alpha\in F$. Then the hermitian form $\langle 1,1,-\alpha,-\alpha\rangle_-=2\times\langle 1,-\alpha\rangle_-$ is isotropic. By the assumption the form $\langle 1,-\alpha\rangle_-$ is isotropic, so that there exists a $z\in D^\times$ such that $\alpha=\overline{z}z$. Note that $\alpha\neq 0$ since D is a division algebra. Furthermore, the strong anisotropy of N follows at once.

(3) Assume that $(D,-)$ is pythagorean and that N is strongly anisotropic. Let h be a torsion hermitian form over $(D,-)$. Since the torsion in $W(D,-)$ is 2-primary there exists a minimal positive integer ℓ such that $2^\ell\times h$ is hyperbolic. Let f be an anisotropic hermitian form which is in the Witt class of h in $W(D,-)$. Then $2^\ell\times f$ is hyperbolic, and thus in particular isotropic, which implies that f is isotropic by (2), a contradiction.

Conversely, assume that $W_t(D,-)=0$. Let $x_1,\dots,x_n\in D^\times$ and assume for the sake of contradiction that $N(x_1)+\cdots+N(x_n)=0$. Let k be an integer such that $2^k\geq n$. Then $2^k\times\langle 1\rangle_-$ is isotropic, and so the quadratic form $N\otimes(2^k\times\langle 1\rangle)$ is isotropic and thus hyperbolic since it is a Pfister form. By a theorem of Jacobson [J] the hermitian form $2^k\times\langle 1\rangle_-$ is hyperbolic. Since $W_t(D,-)=0$ we obtain that $\langle 1\rangle_-$ is hyperbolic, which is impossible.

Now let $\alpha=\overline{x}x+\overline{y}y$ with $x,y\in D^\times$ and note that $\alpha\in F^\times$. Then the hermitian form $\langle 1,1,-\alpha,-\alpha\rangle_-$ is isotropic. Hence the quadratic form $N\otimes\langle 1,1,-\alpha,-\alpha\rangle$ is isotropic, and thus hyperbolic since it is a Pfister form. But this implies that $2\times\langle 1,-\alpha\rangle_-=\langle 1,1,-\alpha,-\alpha\rangle_-$ is hyperbolic by Jacobson's theorem. Thus $\langle 1,-\alpha\rangle_-$ is hyperbolic by our assumption. Therefore α is a norm. ∎

Remark 6.2. The converse of Proposition 6.1(1) is not true. For example, $F=\mathbb{Q}$ is not pythagorean, but $((-1,-1)_{\mathbb{Q}},-)$ is pythagorean since every sum of four squares in $\mathbb{Q}$ is again a square in $\mathbb{Q}$.

Proposition 6.3. *Let F be a formally real field. Consider Hamilton's quaternion algebra $\mathbb{H}=(-1,-1)_F$ equipped with the orthogonal involution ϑ from Example 2.1*(4)*. Then:*

(1) $W(\mathbb{H},\vartheta) = W_t(\mathbb{H},\vartheta) \neq 0$.
(2) *If F is real closed, then* $\mathrm{Sym}(\mathbb{H},\vartheta) = (\mathbb{H},\vartheta)^2$ *and* $(\mathbb{H},\vartheta)$ *is pythagorean.*

Proof. (1) Let $h \in W(\mathbb{H},\vartheta)$. Since $\mathrm{Nil}(\mathbb{H},\vartheta) = X_F$ we have that $\mathrm{sign}_P\, h = 0$ for all $P \in X_F$. Thus h is a torsion form by Pfister's local-global principle (cf. Remark 3.9). Hence $W_t(\mathbb{H},\vartheta) = W(\mathbb{H},\vartheta) \neq 0$.

(2) Let $u_0 = \alpha_0 + \gamma_0 j + \delta_0 k \in \mathrm{Sym}(\mathbb{H},\vartheta)$. For $u = \alpha + \beta i + \gamma j + \delta k \in \mathbb{H}$ we have

$$\vartheta(u)u = (\alpha^2 + \beta^2 - \gamma^2 - \delta^2) + 2(\alpha\gamma + \beta\delta)j + 2(-\beta\gamma + \alpha\delta)k.$$

We will show that the equation $\vartheta(u)u = u_0$ has a solution $(\alpha,\beta,\gamma,\delta) \neq (0,0,0,0)$. Let $\gamma = 0$. An easy computation shows that $\vartheta(u)u = u_0$ if and only if $2\alpha\delta = \delta_0$, $2\beta\delta = \gamma_0$ and

$$\delta^4 + \alpha_0\delta^2 - \frac{1}{4}(\delta_0^2 + \gamma_0^2) = 0.$$

The last equation is quadratic in δ^2 with discriminant $\Delta = \alpha_0^2 + \delta_0^2 + \gamma_0^2$. Since F is real closed (and thus pythagorean), Δ is a square. Let ε be the positive square root of Δ. Then

$$\delta^2 = \frac{-\alpha_0 \pm \varepsilon}{2}.$$

Since $\Delta \geq \alpha_0^2$, we have $\varepsilon \geq \alpha_0$ and so $(-\alpha_0 + \varepsilon)/2 \geq 0$. Thus $\sqrt{(-\alpha_0 + \varepsilon)/2}$ exists, since F is real closed.

Finally, $(\mathbb{H},\vartheta)$ is pythagorean since $\Sigma(\mathbb{H},\vartheta)^2 \subseteq \mathrm{Sym}(\mathbb{H},\vartheta)$. ∎

This proposition shows that already for algebras with involution over a real closed base field, 'pythagorean' does not imply 'torsion-free Witt group'. The following two propositions describe examples which show that 'torsion-free Witt group' does not imply 'pythagorean' either.

Proposition 6.4. *Let $F = \mathbb{R}((x))((y))((z))((w))$ be the iterated Laurent series field in the unknowns x, y, z, w over the field of real numbers $\mathbb{R}$. Consider the quaternion algebras $D_1 = (x, y)_F$ and $D_2 = (z, w)_F$ and the biquaternion algebra $D = D_1 \otimes_F D_2$. For $\ell = 1, 2$, let $\{1, i_\ell, j_\ell, k_\ell\}$ be the usual F-basis for D_ℓ and let σ_ℓ be the orthogonal involution on D_ℓ that sends i_ℓ to $-i_\ell$ and that fixes the other basis elements. Let $\sigma = \sigma_1 \otimes \sigma_2$ be the resulting orthogonal involution on D. Then:*
(1) *D is a division algebra.*
(2) $W_t(D,\sigma) = 0$.
(3) (D,σ) *is not pythagorean.*

Proof. (1) Let v be the standard (x, y, z, w)-adic valuation on F (see for instance [W, §3]). Note that F is Henselian with respect to v. An application of Springer's theorem shows that the Albert form $\langle x, y, -xy, -z, -w, zw\rangle$ of D is anisotropic (we obtain six residue forms of dimension 1 over $\mathbb{R}$, that are necessarily isotropic). Hence D is a division algebra, cf. [Lam, Chap. III, Thm. 4.8].

(2) Since F is Henselian, the valuation v extends uniquely to a valuation on D (see [Mo, Thm. 2]), which we also denote by v. We now claim that the residue division

algebra $\overline{D}$ is isomorphic to $\mathbb{R}$. The proof of this claim goes as follows: Since $\mathrm{char}(\mathbb{R}) = 0$, the division algebras D_1 and D_2 are tame (in the sense of [JW, §6]). By [JW, Corollary 6.7] we have $\Gamma_D \subseteq \Gamma_{D_1} + \Gamma_{D_2}$ (this sum takes place in the divisible closure of Γ_F). We first compute Γ_{D_1}. Since $i_1^2 = x$ and $v(x) = (1,0,0,0)$ we have $v(i_1) = (1/2,0,0,0)$. Similarly $v(j_1) = (0,1/2,0,0)$ and $v(k_1) = (1/2,1/2,0,0)$. Let γ_1 be the quaternion conjugation on D_1. Since v extends uniquely from F to D_1 and $v \circ \gamma_1$ is a valuation on D_1 we have $v(a) = v \circ \gamma_1(a)$ for every $a \in D_1$. In particular $v(\gamma_1(a)a) = 2v(a)$. If we write $a = a_0 + a_1 i_1 + a_2 j_1 + a_3 k_1$ we obtain $\gamma_1(a)a = a_0^2 - xa_1^2 - ya_2^2 + xya_3^2$. Since the four terms in this sum have different valuation we get

$$\begin{aligned} v(a) &= \frac{1}{2}\min\{\varepsilon_0 0, \varepsilon_1 v(x), \varepsilon_2 v(y), \varepsilon_3 v(xy)\} \\ &= \frac{1}{2}\min\{\varepsilon_0 0, \varepsilon_1(1,0,0,0), \varepsilon_2(0,1,0,0), \varepsilon_3(1,1,0,0)\}, \end{aligned}$$

where $\varepsilon_i = 0$ if $a_i = 0$, and 1 otherwise (for $i = 0,\dots,3$; this is to account for the presense or absence of a_i).

This yields $\Gamma_{D_1} = \frac{1}{2}\mathbb{Z}\times\frac{1}{2}\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$. A similar argument shows that $\Gamma_{D_2} = \mathbb{Z}\times\mathbb{Z}\times\frac{1}{2}\mathbb{Z}\times\frac{1}{2}\mathbb{Z}$. Since $\Gamma_D \subseteq \Gamma_{D_1} + \Gamma_{D_2}$ we get $\Gamma_D = \frac{1}{2}(\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z})$. In particular $[\Gamma_D : \Gamma_F] = 16$ and by Draxl's "Ostrowski Theorem" (see [JW, Equation 1.2]), we obtain $[\overline{D} : \overline{F}] = 1$, i.e. $\overline{D} = \overline{F} = \mathbb{R}$. This proves the claim.

Using now that $W_t(\mathbb{R}) = 0$ and also $W_{-1}(\mathbb{R}) = 0$, [Lar, Theorem 3.7] implies that $W_t(D,\sigma) = 0$.

(3) Consider the sum of two hermitian squares

$$\begin{aligned} a &= \sigma(j_1 \otimes j_2 + 1 \otimes 1)(j_1 \otimes j_2 + 1 \otimes 1) + \sigma(i_1 \otimes j_2)(i_1 \otimes j_2) \\ &= (j_1 \otimes j_2 + 1 \otimes 1)^2 - (i_1 \otimes j_2)^2 \end{aligned}$$

in (D,σ). We will show that a is not a hermitian square in (D,σ) by means of a signature computation. Let $P \in X_F$ be the ordering for which $x,y,z,w >_P 0$ and let F_P be the real closure of F at P. Then

$$D \otimes_F F_P \cong M_4(F_P).$$

If a were a hermitian square, then the hermitian forms $\langle a\rangle_\sigma$ and $\langle 1\rangle_\sigma$ over (D,σ) would be isometric. We will shortly see, however, that $\mathrm{sign}_P^\star\langle a\rangle_\sigma = \pm 4$, while $\mathrm{sign}_P^\star\langle 1\rangle_\sigma = 0$. Thus the forms are not isometric and so a is not a hermitian square.

In order to compute $\mathrm{sign}_P^\star\langle a\rangle_\sigma$ we follow the method of Example 3.4. The algebra D is generated by the basic tensors $i_1 \otimes 1$, $j_1 \otimes 1$, $1 \otimes i_2$ and $1 \otimes j_2$. We extend scalars to the real closure of F at P, $D \longrightarrow D \otimes_F F_P$, and then apply the splitting isomorphism

$$\xi_P : (D \otimes_F F_P, \sigma \otimes \mathrm{id}_{F_P}) \xrightarrow{\sim} (M_4(F_P), \mathrm{ad}_{\varphi_P})$$

induced by the algebra isomorphisms

$$\begin{aligned} \eta_\ell : D_\ell \otimes_F F_P &\xrightarrow{\sim} M_2(F_P) \\ i_\ell \otimes 1 &\longmapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ j_\ell \otimes 1 &\longmapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \end{aligned}$$

for $\ell = 1, 2$. A straightforward computation shows that

$$\xi_P(a \otimes 1) = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix}.$$

Since $\sigma \otimes \mathrm{id}_{F_P}$ is an orthogonal involution, φ_P is a quadratic form over F_P. Let Φ_P denote the Gram matrix of φ_P. Since ξ_P is an isomorphism of algebras with involution we have that

$$\xi_P \circ (\sigma \otimes \mathrm{id}_{F_P}) = \mathrm{ad}_{\varphi_P} \circ \xi_P,$$

from which it follows (by easy, but tedious computations) that we may take

$$\Phi_P = \pm \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Since $\Phi_P = \Phi_P^{-1}$ we have

$$\Phi_P^{-1} \xi_P(a \otimes 1) = \pm \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 2 \end{bmatrix},$$

from which it follows that $\psi_P \simeq \pm\langle 1, 1, 1, 1 \rangle$. We conclude that $\mathrm{sign}_P^\star \langle a \rangle_\sigma = \pm 4$.

On the other hand, since φ_P is clearly hyperbolic, it follows from Lemma 3.6 that $\mathrm{sign}_P^\star \langle 1 \rangle_\sigma = \mathrm{sign}_P \varphi_P = 0$. ■

Proposition 6.5. *We use the same notation as in the previous proposition, except that we let $\sigma = \sigma_1 \otimes \gamma_2$, where γ_2 denotes quaternion conjugation on D_2, so that σ is a symplectic involution on D. Then:*

(1) *D is a division algebra.*
(2) *$W_t(D, \sigma) = 0$.*
(3) *(D, σ) is not pythagorean.*

Proof. (1) & (2): identical to the proof of Proposition 6.4(1) & (2).

(3) The proof is similar to the proof of Proposition 6.4(3). We explain the main differences. Consider the sum of three hermitian squares

$$\begin{aligned} a &= \sigma(j_1 \otimes 1 + 1 \otimes 1)(j_1 \otimes 1 + 1 \otimes 1) + 2\sigma(i_1 \otimes i_2)(i_1 \otimes i_2) \\ &= (j_1 \otimes 1 + 1 \otimes 1)^2 + 2(i_1 \otimes i_2)^2 \end{aligned}$$

in (D, σ). Let $P \in X_F$ be the ordering for which $x, y >_P 0$ and $z, w <_P 0$. Then

$$D \otimes_F F_P = (D_1 \otimes_F D_2) \otimes_F F_P \cong M_2(F_P) \otimes_{F_P} \mathbb{H}_P \cong M_2(\mathbb{H}_P).$$

Let η_1 be as before and let η_2 be the isomorphism $D_2 \otimes_F F_P \xrightarrow{\sim} \mathbb{H}_P$ defined by letting $\eta_2(i_2 \otimes 1) = i$ and $\eta_2(j_2 \otimes 1) = j$. Let ξ_P be the induced isomorphism

$$(D \otimes_F F_P, \sigma \otimes \mathrm{id}_{F_P}) \xrightarrow{\sim} (M_2(\mathbb{H}_P), \mathrm{ad}_{\varphi_P}).$$

This time the form φ_P is hermitian over $(\mathbb{H}_P, -)$ and a computation shows that we may take $\Phi_P = \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Another computation shows that $\xi_P(a \otimes 1) = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$. Hence

$$\Phi_P^{-1} \xi_P(a \otimes 1) = \pm \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

from which it follows that $\mathrm{sign}_P^\star \langle a \rangle_\sigma = \pm 2$. Since again $\mathrm{sign}_P^\star \langle 1 \rangle_\sigma = 0$ it follows that a cannot be hermitian square. ∎

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