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On the Heat Equation and the Index Theorem

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Dedicated to Sir William Hodge on his seventieth birthday

1. Introduction

We shall present here a new proof of the index theorem for elliptic operators based on the asymptotics of the heat equation. Since this proof has evolved, through various contributions, over a period of years, it seems appropriate to begin with a brief historical account.

The index theorem was first proved in Atiyah-Singer [5] by global topological methods notably using K-theory and cobordism. In their subsequent improved proof [1] the cobordism was eliminated but the methods remained topological. An alternative analytic approach was proposed in Atiyah-Bott [6] based on the Zeta function $\zeta(s)=\sum \lambda^{-s}$ (λ denoting eigenvalues of an operator). The idea was that the index could be expressed as the difference of two such Zeta functions, while on the other hand $\zeta(0)$ could be evaluated as an explicit integral. Zeta functions of this type had been introduced and studied for the Laplace-Beltrami operator by Minakshisundaram and Pleijel [19] many years earlier and their results were extended by Seeley [23] to the general case. The trouble with this method was that the explicit integral answer obtained for the index was extremely complicated. Notably it involved many derivatives of the coefficients of the original operator, whereas the formula obtained by topological methods could be written using two derivatives. The algebraic problems involved in this approach therefore seemed formidable.

Singer and McKean in [18] looked at this problem for the Euler-characteristic case, namely for the operator $d+d^*$: even forms $\to$ odd forms on a Riemannian manifold. Actually they used the heat equation but as is well-known (and explained in Section 4) this is quite equivalent to the Zeta-function approach. They observed that their integrand would have to be expressed in terms of the Riemannian curvature and its covariant derivatives, and in low dimensions they showed how one

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could use orthogonal symmetry to cut down the number of possible expressions. They raised the question whether, by some remarkable cancellations, the integrand α (given by the heat equation method) might not after all turn out to coincide with the (normalized) Gauss curvature K. All one knew topologically was that α and K had the same total integral and it would be a remarkable and unexpected piece of good fortune to find $\alpha = K$ locally.

It was Patodi in [21] who showed that Singer and McKean's optimism was justified and that remarkable cancellations, eliminating higher derivatives, did indeed take place. Now the Euler-characteristic is rather elementary from the topological standpoint and it was therefore highly significant when Patodi showed in [22] that his methods also extended to give the Riemann-Roch theorem for Kähler manifolds.

The one drawback to Patodi's methods was the complicated algebraic nature of the cancellation process. An alternative indirect approach was shortly afterwards discovered by Gilkey [15] who showed that the tiresome higher derivatives could be eliminated on *a priori* grounds. Roughly speaking, he showed that any integrand having the general qualitative properties of α could not involve higher derivatives – and after that it was easily identified with K. Gilkey also extended his argument to the operator on an oriented $4k$-dimensional manifold whose index gives the Hirzebruch signature. In fact, the argument turned out to be easier in this case and it leads to an elegant characterization of the Pontrjagin forms (see Section 2).

Unfortunately, despite the simplicity of Gilkey's result, the proofs were long and difficult. Part of the explanation for these difficulties was that Gilkey approached the problem from the point of view of general differential operators, rather than through Riemannian geometry, and that consequently he made no use of the tensor calculus.

The main technical contribution in our paper is to present a simple proof of Gilkey's theorem. Essentially we shall show that it follows directly from the central theorem of invariant theory for the orthogonal group and the Bianchi identities for the Riemannian curvature. This is very much in the spirit of the Singer-McKean approach. In fact, with hindsight, it seems that Singer and McKean would inevitably have been led to the Gilkey theorem had they been dealing with the signature rather than the Euler-characteristic.

After establishing the Gilkey theorem in Section 2 we proceed to generalize it in Section 3 by introducing an auxiliary vector bundle. With this generalization we can then treat the index theorem for a large class of "classical operators", including the Riemann-Roch Theorem for Kähler manifolds. This is done in Section 6 after the basic case of the Hirzebruch signature theorem has been explained in Section 5.

Having treated the classical operators by direct analysis we can now switch back to topological methods to deduce the general index theorem. The point is that the classical operators are sufficiently numerous to generate, in a certain topological sense, all elliptic operators. This is briefly explained in the final section.

In essence therefore our new proof of the index theorem follows the lines of the first proof in [5], except that cobordism has been replaced by local differential geometry. Instead of characterizing Pontrjagin numbers as global cobordism invariants we characterize Pontrjagin forms as local Riemannian invariants (of a certain type). Just as in [5] this proof has the inherent defect of not generalizing to the cases given in [3] and [4] (where, for instance, the index may be an integer mod 2 rather than a real number). Nor is it really any shorter than the proof in [1] since it uses more analysis, more differential geometry and, no less topology. On the other hand, for the classical operators associated with Riemannian structures it is more direct and more explicit: in particular, the local version of the signature theorem and its generalizations is of considerable intrinsic interest and is likely to lead to further developments.

We have written the paper in a somewhat expository style so that all the ingredients in the proof are clearly displayed. Thus in Section 4 we try to explain the Seeley formulae for $\zeta(0)$ without assuming too much expertise on the part of the reader. Also in two Appendices we prove the basic facts about Riemannian invariants – including a novel account of the First Main Theorem for orthogonal invariants: as mentioned earlier these are the essential tools in our proof of the Gilkey theorem.

Finally we should acknowledge that our whole thinking on these questions was greatly stimulated and influenced by the recent work of Gel'fand (see [14]) on Lie algebra cohomology. There are close and suggestive links between his results and ours.

For an expanded treatment of the Singer-McKean paper, following slightly different lines from our presentation, the reader may consult the book by Berger [10].

2. On Characteristic Classes of Geometric Structures

Our aim in this section will be Gilkey's characterization of the Pontrjagin classes as the only form-valued invariants of Riemannian structures satisfying a rationality and homogeneity condition. As we have found this subject to be confusing to ourselves – and, in fact, to most people – we will formulate it here in possibly greater circumspection than is our wont.

Technically, a "q-form valued invariant of Riemannian structures" is a function

$$\omega: \mathbf{R} \to A^q \tag{2.1}$$

which assigns to each manifold[1] M and Riemann structure g on M, a q-form $\omega(g)=\omega(M,g)$ on M which is "natural" or "local" in the following sense: given

$$f:\ M'\to M \tag{2.2}$$

where f is a diffeomorphism of M' onto an open submanifold of M, and a Riemannian structure g on M, then ω must satisfy the equation

$$\omega(f^*g)=f^*\omega(g). \tag{2.3}$$

According to this definition ω is then clearly locally defined and further invariant under diffeomorphisms. Technically, (2.2) and (2.3) precisely express the fact that ω be a natural transformation from the contravariant functor

$$\mathbf{R}:\ M\to \text{Riemann structures on } M$$

to the contravariant functor

$$A^q:\ M\to q\text{-forms on } M$$

over the category $\mathcal{M}$ whose objects are manifolds M, and whose morphisms are maps $f: M'\to M$ as in (2.2). An "invariant" ω of this type is called *homogeneous* of weight k if for every $\lambda\in\mathbb{R}^*$, $\lambda>0$,

$$\omega(\lambda^2 g)=\lambda^k\,\omega(g), \tag{2.4}$$

and it is called regular (rational) provided the following conditions hold:

The components of $\omega(g)$ – relative to any local coordinate system x – are given by polynomials in:

a) the components g_{ij} of g relative to x;

b) a finite number of derivatives $\dfrac{\partial^\alpha}{\partial x^\alpha}g_{ij}$ of the g_{ij}, and

c) the inverse of $\det g=\det(g_{ij})$.

1) By components relative to x we, of course, mean the classical concept of these components. Thus $g_{ij}=g(\partial/\partial x_i, \partial/\partial x_j)$ and if ω had values in, say, the one-forms, then the *components* of ω relative to x would be functions ξ_j such that $\omega(g)=\sum\xi_j\,dx^j$, and the rationality of ω would mean that

$$\xi_j=P_j\left(g_{ij};\ \frac{\partial}{\partial x^\alpha}g_{ij};\ [\det(g_{ij})]^{-1}\right)$$

where the P_j are certain *polynomials with real coefficients* in the variables indicated.

[1] All geometric objects will be assumed to be smooth, that is, C^∞.

2) To explain why the inverse of $\det(g_{ij})$ appears in this definition, note that the Riemann structures on a given vector space V are naturally identified with the homogeneous space

$$GL(n,\mathbb{R})/O(n) \qquad n=\dim V \tag{2.5}$$

which, in turn, can be identified with the orbit Q of a fixed positive definite quadratic form $g\in S_2(V^*)$.

Now $GL(n,\mathbb{R})$ and $O(n)$ are algebraic groups so that it is natural to define the regular rational functions on $GL(n,\mathbb{R})/O(n)$ as the subring of regular rational functions on $GL(n,\mathbb{R})$ – which is clearly

$$\mathbb{R}[a_{ij};(\det a_{ij})^{-1}],$$

a_{ij} being the usual coordinates in $GL(n,\mathbb{R})$ – invariant under the action of $O(n)$.

On the other hand Q carries a natural algebraic structure as one component of the non-singular symmetric matrices g_{ij}, so that its coordinate ring is naturally $\mathbb{R}[g_{ij},(\det g_{ij})^{-1}]$. Now it is a non-trivial but true fact that under the isomorphism

$$GL(n,\mathbb{R})/O(n)\simeq Q$$

these coordinate rings correspond (see Appendix I).

In short, there is a natural notion of regularity for ω at every point, and our definition simply demands that ω be i) smooth in its dependence on M and ii) regular at every point of M.

3) An intrinsic formulation of regularity takes the following form. Let $M\to T_\rho(M)$ be the functor $M\to$ Tensors on M of a given type ρ. We then write $J_m T_\rho$ for the functor $M\to m$-jets of Tensors of type ρ, and denote by

$$j_m:\ T_\rho\to J_m T_\rho$$

the natural transformation which assigns to an element in $T_\rho(M)$ its m-jet in $J_m T_\rho(M)$. With this understood ω is regular of weight k if and only if it is induced from a linear point transformation[2]

$$\overset{k}{\otimes} J_m T \xrightarrow{\ \phi\ } A^q\otimes\det{}^{2s}, \qquad \det=A^n(M), \qquad n=\dim M \tag{2.6}$$

by the formula:

$$\omega(g)=\frac{\hat\phi(g)}{\det(g)^s}$$

where $\hat\phi$ denotes the composition:

$$\mathbf{R}\xrightarrow{\ i\ } T\xrightarrow{\ \overset{k}{\otimes} j_m\ }\overset{k}{\otimes} J_m(T)\xrightarrow{\ \phi\ } A^q\otimes\det{}^{2s}. \tag{2.8}$$

[2] That means linear over the ring $\mathcal{F}(M)$ of smooth functions on M.

Here T denotes the tensors of type $(2,0)$ and i the inclusion of the positive definite forms into *all* the forms.

With these preliminaries out of the way our version of Gilkey's theorem takes the following form:

Theorem I *(Gilkey). The only regular natural transformations*

$$\omega: \mathbf{R} \to A^q$$

of weight ≥ 0, *have values in the ring* Pont(g) *generated by the Pontrjagin forms of* g, *and these have weight* 0.

This beautiful characterization of the ring Pont(g), generated by the Pontrjagin classes of a Riemannian structure actually follows quite simply from two classical results, which immediately reduce the possible ω's to certain elementary functions of the curvature R of G. The full symmetries of R then yield the rest. Indeed arguments of this type are given in [18], but somehow just fall short of the Gilkey result. For completeness sake we will, however, start from scratch and sketch in those points of the classical reduction which are not readily found in the literature.

Consider then the value of an "invariant" of our type ω, at a point $p \in M$. We may compute $\omega(g;p)$ in any coordinate system and therefore, in particular, in a geodesic coordinate system centered at p. Let us denote these coordinates by $x(p,g)$ and note that they are *unique up to an orthogonal transformation.* We next study the Taylor expansion of the components g_{ij} of g relative to $x(p,g)$ about the origin.

The first classical result which we need is the following one:

Proposition (2.11). *The Taylor expansion of* g *about* p

$$g = g^0 + g^{(1)} + g^{(2)} + \cdots$$

is given by certain universal but not unique polynomials $g^{(k)}(x,R)$ *in the* $x_i(p,g)$ *and the components relative to* $x(p,g)$ *of the curvature tensor* R^i_{jkl} *and its covariant derivatives. Explicitly this expansion starts as follows:*

$$g_{ij} = \delta_{ij} + 2/3 \sum_{\mu\nu} x^\mu x^\nu R^i_{\mu\nu j} + \cdots \tag{2.12}$$

so that

$$\det\{g_{ij}(p)\} = 1. \tag{2.13}$$

For a proof see Appendix II.

It now follows from the regularity of ω that $\omega(p,g)$ is in fact given by a certain universal polynomial $\hat{\varphi}_\omega$ in the values of the curvature and its covariant derivatives at p:

$$\omega(p,g) = \hat{\varphi}_\omega(R_p, R'_p \ldots R_p^{(k)}). \tag{2.14}$$

Next consider the behavior of this polynomial under the action of $O(n)$, the orthogonal group of dim $n = \dim M$, which mediates between the different geodesic coordinates of g centered at p. The components of R transform like the 4-th power $\overset{4}{\otimes} V$ of the standard $O(n)$ module V, and those of the k-th covariant derivative of $R^{(k)}$ as those of $\overset{4+k}{\otimes} V$. We may, therefore, interpret $\hat{\varphi}_\omega$ as a polynomial function:

$$\hat{\varphi}_\omega: W \to A^q V, \quad \text{with } W = \bigoplus_{j=0}^{k} (\otimes^{j+4} V) \tag{2.15}$$

for some k.

At this stage one might be tempted to argue that $\hat{\varphi}_\omega$ has to be an *equivariant* polynomial map relative to the action of $O(n)$, because of the invariance of ω. However, it is more correct to *average* the polynomial map $\hat{\varphi}_\omega$ over $O(n)$ thereby obtaining an equivariant map

$$\varphi_\omega: W \to A^q V \tag{2.16}$$

which, however, has to agree with $\hat{\varphi}_\omega$ on the vector $R_p(g), R'_p(g), \ldots$, etc. That is, one also has the identity:

$$\omega(p,g) = \varphi_\omega(R_p, R'_p \ldots R_p^{(k)}). \tag{2.17}$$

We argue similarly for each homogeneous component $\varphi^{(j)}_\omega$ of φ_ω so that finally $\omega(p,g)$ becomes expressed as a sum of *homogeneous* and *equivariant* polynomials in the $R_p, R'_p, \ldots$, etc. Now by complete polarization these polynomials in turn are induced from certain *equivariant and multilinear* maps

$$\varphi_\omega: \overset{N}{\otimes} V \to A^q V. \tag{2.18}$$

At this point we invoke the "Fundamental Theorem of Invariance Theory for the Orthogonal Group" as started in Hermann Weyl's beautiful book on this subject [24]. For an alternative proof see Appendix I.

Conceptually this theorem asserts that all invariants of $\overset{k}{\otimes} V$, as a $O(n)$-module, are built up out of the inner product $(\ ,\ )$ on V.

More precisely, let us call a linear map

$$\varphi: \overset{2k}{\otimes} V \to \mathbb{R}$$

an *elementary invariant* if it is of the form

$$\varphi(v_1 \otimes \cdots \otimes v_{2k}) = (v_1, v_2)(v_3, v_4)\ldots(v_{2k-1}, v_{2k}), \tag{2.19}$$

or obtained from such a φ by first performing a permutation on the v_i. Then the fundamental basis theorem for the invariants of $O(n)$ states that

Theorem. *The linear space*

$$\mathrm{Hom}_{O(n)}(\overset{k}{\otimes} V, \mathbb{R})$$

of $O(n)$-linear maps from $\overset{k}{\otimes} V$ to $\mathbb{R}$ is spanned by the elementary invariants. Thus, in particular, this space is non-trivial only for k even.

Remarks. 1) This theorem is by no means true if $O(n)$ is replaced by $SO(n)$: the "determinant" then appears as a new invariant of a quite different type, and makes it impossible directly to extend the Gilkey theorem for the category of oriented Riemann manifolds.

2) In terms of "components" this theorem asserts that *every invariant of $O(n)$ is given by a complete contraction.* That is, if $\{e_i\}$ is an orthonormal basis for V, and $\xi_{\alpha_1 \ldots \alpha_k}$ denotes the components of a point in $\overset{k}{\otimes} V$ relative to the basis $\{e_{\alpha_1} \otimes e_{\alpha_2} \otimes \cdots \otimes e_{\alpha_k}\}$, then our elementary φ of (2.19) is simply given by:

$$\varphi:\ \xi_{\alpha_1 \ldots \alpha_k} \to \xi_{\alpha_1 \alpha_1 \alpha_2 \alpha_2 \ldots \alpha_r \alpha_r} \qquad (k = 2r).$$

Here, as in the sequel, we adopt the classical convention that repeated indices are to be summed.

An easy variant of this theorem, which we leave to the reader, is now given by the following:

Corollary. *The space $\mathrm{Hom}_{O(n)}(\overset{k}{\otimes} V, \Lambda^q V)$, of equivariant maps from $\overset{k}{\otimes} V$ to $\Lambda^q V$, is non-trivial only if $k-q$ is even. Further, when $k-q=2r$, then $\mathrm{Hom}_{O(n)}(\overset{k}{\otimes} V, \Lambda^q V)$ is spanned by "elementary maps" φ which contract $2r$ indices and alternate the remaining q indices. Thus up to a permutation such a φ takes the form:*

$$\varphi:\ \xi_{\alpha_1 \ldots \alpha_k} \to \xi_{\alpha_1 \alpha_1 \alpha_2 \alpha_2 \ldots \alpha_r \alpha_r [\alpha_{2r+1}, \ldots, \alpha_k]} \tag{2.20}$$

with the bracket denoting alternation of the indices enclosed in it.

We now return to the proof of the theorem. For this purpose let us write R_α, $\alpha = (\alpha_1, \ldots, \alpha_t)$ for the components of the $(t-4)$-th covariant derivative of the curvature R_{ijkl} and define an *elementary monomial* of degree r in R to be an expression of the form

$$m(R) = \sum\nolimits_q^* R_{\alpha^1} R_{\alpha^2} \ldots R_{\alpha^r} \tag{2.21}$$

where the sum goes over *alternation of precisely q indices, and contraction of the remaining ones.* Note that this is a regular invariant of the metric since, in general coordinates, it can be expressed in terms of the R_α and the g_{ij}, while the R_α themselves are regular (tensorial) invariants of the metric.

Thus $m(R)$ describes the components of a *q-form*, and if we combine the corollary above with the decomposition of $\omega(g)$ into a sum of homogeneous equivariant polynomials φ_ω^k in R, R', etc. we obtain the following essentially classical lemma:

Lemma 1. *Every invariant ω of the type under consideration is given by a linear combination of elementary monomials $m(R)$:*

$$\omega(g) = \sum a_m\, m(R) \qquad a_m \in \mathbb{R}.$$

At this stage then, the Gilkey theorem amounts to an identification of the monomials of non-negative weight.

We start with the following quite elementary.

Lemma 2. *The weight of any R_α is 2 while the weight of an elementary monomial*

$$m(R) = \sum\nolimits_q^* R_{\alpha^1} R_{\alpha^2} \ldots R_{\alpha^r}$$

is $2r + q - \sum t_i$, where t_i is the number of indices in α^i.

Proof. Recall that g_{ij} has weight 2 and g^{ij} therefore weight -2. The classical expression for $R^i_{jkl\beta_1 \ldots \beta_t}$ is then easily seen to be of weight zero, whence $R_{ijkl,\beta_1 \ldots \beta_t} = g_{i\lambda} R^\lambda_{jkl,\beta_1 \ldots \beta_t}$ is of weight 2. A profounder reason for the weightlessness of $R^i_{jkl\beta_1 \ldots \beta_t}$ is that this tensor field is dependent only on the covariant derivative ∇_g defined by g, and ∇_g is, of course, independent of constant scalar changes in ∇: $\nabla_g = \nabla_{\lambda^2 g}$. In any case then, this yields the first assertion. To obtain the second one observe that before contraction the expression $R_{\alpha^1} R_{\alpha^2} \ldots R_{\alpha^r}$ has weight $2r$. Hence, after $(\sum t_i - q)/2$ contractions, each of which involves a raising of an index – that is, one g^{ij} – $m(R)$ ends up with a weight of

$$2r - (\{\textstyle\sum t_i - q\}/2) \times 2 = 2r + q - \sum t_i. \qquad \text{Q.E.D.}$$

In the sequel it will be best to rewrite this weight in the following form. Set $t_i = 4 + \varepsilon_i$, so that $\varepsilon = \sum \varepsilon_i$ denotes the total number of "covariant derivatives" in $m(R)$. Thus one obtains the formula:

(2.22) If $m(R) = \sum_q^* R_{\alpha^1} R_{\alpha^2} \ldots R_{\alpha^r}$, then $q = 2r + \varepsilon + \text{weight } m(R)$.

In short, for fixed r the larger the weight, the more indices have to be alternated in $m(R)$. When combined with the classical symmetries for R this bound will be seen to imply the desired result.

Recall first of all that the symmetries of the Riemannian curvature R are given by:

$$R_{i\underbrace{jkl}} = 0, \qquad R_{ij\underbrace{kl,r}} = 0 \tag{2.23}$$

and

$$R_{ijkl} = -R_{jikl}, \qquad R_{ijkl} = -R_{ijlk}. \tag{2.24}$$

Here R_{ijkl} denotes the usual components of R in any coordinate system (see [13], § 20, 23, for instance) and the bow denotes the cyclic sum over the indices indicated. For future reference we gather together the following direct consequences of these identities which we leave to the reader:

Lemma. *As a consequence of* (2.23)–(2.24), *the* R_{ijkl} *also satisfy the identities*

(2.25) $$R_{ijkl} = R_{klij},$$

(2.26) $$R_{i[jk]l} = \tfrac{1}{2} R_{il[jk]}.$$

(2.27) *Alternation of* R_{ijkl} *or* $R_{ijkl,r}$ *over any* 3 *indices yields zero.*

Because the covariant derivative on tensors commutes with the natural action of the permutation group on these tensors, these identities persist also after an arbitrary number of covariant derivatives, so that (2.27) implies:

(2.28) *Alternation of* R, *over any* 3 *amongst the first* 4 *or* 5 *indices gives* 0.

Now then consider the implications of (2.28) on an elementary monomial $m(R)$ of weight >0. For $m(R)$ not to vanish we can alternate at most 2 of the 4 first indices in any of the r factors occurring in $m(R)$. The total number of indices which can be alternated is therefore only $2r+\varepsilon$. If the weight $m(R)>0$, Eq. (2.22) shows that this condition will have to be violated at least once, whence $m(R)=0$. Thus there are no non-trivial monomials of weight >0, yielding the easier part of Gilkey's theorem.

Next consider the case of weight zero. Then, of course, $q=2r+\varepsilon$, so that in the above argument we have precisely "enough room" to alternate q indices. But to avoid zero – via the first part of (2.28) – we now have to alternate over all indices except 2 amongst the first 4 in every factor of $m(R)$. It follows that if $\varepsilon>0$ – i.e., some covariant derivative occurs amongst the indices – then $m(R)$ will have to vanish by the second part of (2.28).

At this stage then, we have the crucial fact that

(2.29) weight $m(R)=0$, $m(R)\neq 0 \Rightarrow m(R)$ contains no covariant derivatives of R.

It remains to show that all of these weightless monomials in R, lie in Pont$(\mathfrak{g})$.

Recall then that generators for this ring can be taken to be the forms

(2.30) $$\operatorname{trace} R^{(k)} = \sum R_{i_1 i_2 ..} R_{i_2 i_3 ..} \cdots R_{i_{k-1} i_k ..} R_{i_k i_1 ..}$$

where alternation is taken over all the lower indices denoted by dots. Now our surviving $m(R)$'s certainly have the property that in each factor

R_{ijkl} *precisely two* indices are involved in *contraction* and the remaining two in alternation. Hence it only remains to be seen that the *contracting indices can always be chosen to be the first two*. Let us, therefore, call $m(R)$ proper if in all of its factors the contracting indices are the first two. Proper monomials are, up to sign, clearly products of the forms trace $R^{(k)}$ and hence in Pont$(\mathfrak{g})$. Suppose now that $m(R)$ is not proper, so that say in the first factor, the contracting indices are not the first two. If they are the last two we can use (2.25) to interchange the last two with the first two, making the first factor proper. Neglecting signs and using (2.24) it remains to discuss the case when R_{ijkl} has contracting indices i and l. Then, of course, jk have to be alternated. On the other hand, by (2.26) we have the identity:

$$R_{i[jk]l} = \tfrac{1}{2} R_{il[jk]}$$

which again allows us to replace the contracting indices with the first two. Straightening out each factor at a time in this manner, we see that $m(R)$ can be put in proper form, and this then completes the proof of the Gilkey theorem.

3. A Generalization

In this section we will extend Theorem 1 to joint invariants ω, of a Riemann structure g on M and Hermitian bundle ξ over M.

Here the term "Hermitian bundle" and joint invariant will be used in the following technical sense:

Definition. By a Hermitian bundle over M, we mean a triple $\xi = (E_\xi, h_\xi, D_\xi)$ consisting of a complex vector bundle E over M, together with a Hermitian structure h_ξ on E, and a connection D_ξ on E which preserves h_ξ.

With this understood a joint invariant will be a function ω which assigns to every Riemann structure g in M, and every Hermitian bundle ξ over M, a q-form $\omega(g,\xi)\in A^q(M)$ such that if

$$f\colon M' \to M$$

is any map in the category $\mathcal{M}$, then

(3.1) $$f^*\omega(g,\xi) = \omega(f^{-1}g, f^{-1}\xi).$$

These joint invariants are now called *homogeneous* of *mixed weight* (k,l) if

(3.2) $$\omega(\lambda^2 g, \mu^2 \xi) = \lambda^k \mu^l \cdot \omega(g,\xi) \qquad \lambda, \mu > 0$$

where $\mu^2\xi$ denotes the bundle E_ξ with connection D_ξ, but with Hermitian form $\mu^2 h_\xi$:

(3.3) $$h_{\mu^2\xi} = \mu^2 h_\xi.$$

Note that D_ξ preserves the Hermitian structure $\mu^2 h_\xi$, so that $\mu^2 \xi$ is indeed a Hermitian bundle in our sense.

Finally a joint invariant is called *regular* if its local components are given by universal polynomials in the variables:

$$g_{ij},\ (\det g_{ij})^{-1},\ h_{ij},\ (\det h_{ij})^{-1},\ \Gamma^k_{ij}, \tag{3.4}$$

and their derivatives.

Here $g_{ij}\, dx^i \otimes dx^j$ describes the metric relative to local coordinates x near $p \in M$, while $h_{ij} = h_\xi(s_i, s_j)$ describes the Hermitian structure relative to a frame $s = \{s_i\}$ for E near p and Γ^i_{jk} describes the connection matrix of D_ξ relative to the frame s:

$$D_\xi\, s_i = \sum_{kj} \Gamma^i_{jk}\, dx^k \otimes s_j. \tag{3.5}$$

Now then, the generalization of Theorem I which we need is given by the following

Theorem II. *A regular joint invariant $\omega(g, \xi)$ of mixed weight (k, l) vanishes identically if $k>0$, or $l \neq 0$, while if $k=l=0$ then $\omega(g, \xi)$ has values in the ring generated by the Chern-forms of ξ and the Pontrjagin forms of g:*

$$\omega(g, \xi) = \begin{cases} 0 & \text{if } k>0 \text{ or } l \neq 0 \\ \in & \mathrm{Pont}(g) \otimes \mathrm{Chern}(\xi) \quad k=l=0. \end{cases} \tag{3.6}$$

The proof of this theorem proceeds in strict analogy to the proof of Theorem I. We start with a classical analogue of (2.11). For this purpose let us call a framing s of E near p, *synchronous* to the coordinates x, centered at p, if s is obtained from an *orthonormal frame at* p, by parallel translation along the radial lines (relative to x, of course) emanating from p. Thus given x centered at p, two synchronous frames s and s' for x are then related by unitary transformations. Note also that *because D_ξ preserves h_ξ, a synchronous frame is orthonormal.*

In this terminology the analogue of (2.11) takes the form:

Proposition (3.7). *Let Γ, K denote the components of the connection D_ξ, and the curvature of D_ξ, relative to (x, s) with s a synchronous frame to x at p.*

Then the Taylor series for Γ at p is given by certain universal (but not unique!) polynomials in K and its derivatives at p. This expansion starts with:

$$\Gamma^i_{jk} = \sum x^l\, K^i_{j,kl} + \cdots \tag{3.8}$$

so that in particular $\Gamma^i_{jk}(p) = 0$

For a proof see Appendix II.

Now if M is also equipped with a Riemannian metric g, and we choose the coordinates x in Proposition (3.7) to be *geodesic*, then (3.8) and (2.11) combine to yield the following result:

Proposition (3.9). *Suppose that in Proposition (3.7) the coordinates are chosen geodesic relative to g. Then Γ and its derivatives at p are expressible by universal polynomials in the components of K, the Riemann tensor R of g, and their covariant derivatives at p.*

Here the "covariant derivatives of K" are, of course, taken relative to the connection induced on the vector bundles $\overset{r}{\otimes} T^* \otimes E \otimes E^*$ by the Levi-Civita connection Γ_g on T and the given connection D_ξ on E.

The proposition is now clear, as these covariant derivatives are related to the ordinary derivatives by polynomial expressions in the g_{ij}, and these — by (2.11) — are in turn expressible in terms of the curvature R of g and its covariant derivatives.

With the aid of Proposition (3.9), one now argues just as before that every joint invariant of our type $\omega(g, \xi)$ must be a linear combination of elementary monomial invariants $m(R; K)$, whose value at $p \in M$ is given by:

$$m(R, K)_p = \sum_q^* R_{\alpha_1} \cdots R_{\alpha_s} \cdot K^{u_1}_{r_1 \beta_1} \cdots K^{u_t}_{r_t \beta_t} \tag{3.10}$$

where as before, the R_α and $K^u_{r\beta}$ denote the components of the appropriate covariant derivative of R and K relative to a pair (x, s) of geodesic coordinates x at p and a synchronous frame s for E. Furthermore, the sum extends over q-alternations of the Greek indices, contraction of the remaining Greek indices and contraction of all the upper Latin indices with all the lower ones.

In extending our earlier argument leading from a general ω to a linear combination of such elementary ones just one point has to be clarified. Our coordinates are now fixed up to the action of $U(m) \times O(n)$ — $U(m)$ acting on the orthonormal frame at p, $O(n)$ on the geodesic coordinates x at p. Hence to get started one needs the analogue of the invariance theorem for this group. Now the $G \times H$ invariants of $A \otimes B$ for finite dimensional G and H modules, A and B are given by the tensor products of the G invariants of A and the H invariants of B:

$$\{A \otimes B\}^{G \times H} = A^G \otimes B^H. \tag{3.11}$$

Hence we only need to understand the basic invariants for $U(m)$ acting on $\mathbb{C}$-modules.

The fundamental theorem for this case is given by:

Theorem (3.12). *Let W be the standard representation of $U(m)$ on $\mathbb{C}^m$, and denote by W^* its dual module $\mathrm{Hom}_{\mathbb{C}}(W; \mathbb{C})$. Then the $\mathbb{C}$-module,*

$$\mathrm{Hom}_{U(m)}\{\overset{r}{\otimes} W \overset{s}{\otimes} W^*; \mathbb{C}\}$$

vanishes for $r \neq s$, while when $r = s$ it is spanned by elementary contractions of the type

$$\varphi_\sigma: w_1 \otimes \cdots w_r \otimes w_1^* \otimes \cdots \otimes w_r^* \to \langle w_1, w_{\sigma(1)}^* \rangle \cdots \langle w_r, w_{\sigma(r)}^* \rangle,$$

with σ a permutation of $(1, \ldots r)$.

Equipped with this fact and (3.11) our earlier argument now extends immediately: Indeed, the components relative to (x, s) of the r-th covariant derivative of K behave as the components of $\overset{r+2}{\otimes} V \otimes (W \otimes W^*)$ which we noted $K^u_{v\beta}$. Hence, in constructing the elementary invariants the u, v and α, β indices have to be contracted separately, and all the u-indices contracted with all the v-indices.

The next step is to compute the weights of the elementary monomials (3.10).

First of all notice that $m(R, K)_p$ is independent of the metric h_ξ and hence is of weight zero in h_ξ.

Under the change $g \mapsto \lambda^2 g$, the weight of $m(R, K)$ is computed, as before, by the formula:

(3.13) $$w(m) = 2s - 2c \qquad m = m(R, K)$$

where c denotes the number of Greek contractions.

Now let ε denote the total number of covariant derivatives in m. Then a count of Greek indices yields the equation $q + 2c = 4s + 2t + \varepsilon$, or

(3.14) $$q = 2s + 2t + \varepsilon + w(m).$$

To complete the argument we now need the identities on R together with the following well-known consequence of the torsionfreeness of the Levi-Civita connection.

Bianchi Identities. The components of K satisfy the identity

(3.15) $$K^u_{v\underset{\smile}{\alpha\beta,\lambda}} = 0.$$

These relations again persist under covariant differentiation, and it follows therefore that if $K^u_{v\beta}$ involves a *covariant derivative* then alternation over *all* Greek indices yields zero.

We will use this fact together with the first identity of (2.23): $R^i_{\underset{\smile}{jkl}} = 0$, to show that the number ε_K of covariant derivatives among the K-terms must be zero. Indeed assume that $\varepsilon_K > 0$, and that $m \neq 0$. Also let ε_R denote the number of covariant derivatives in the R-terms, so that $\varepsilon = \varepsilon_R + \varepsilon_K$, and divide q correspondingly into $q_R + q_K$, with q_K the total number of Greek indices involved in alternation among the K's. Now by (3.15), $m \neq 0$ implies that

(3.16) $$q_K < 2t + \varepsilon_K.$$

On the other hand, using the first identity of (2.23) gives

(3.17) $$q_R \leqq 2s + \varepsilon_R.$$

Hence we get the *strict* inequality:

$$q < 2s + 2t + \varepsilon,$$

which is clearly incompatible with (3.14) when $\omega(m) \geqq 0$. It follows that $\varepsilon_K = 0$ and that $\omega(m) = 0$. However, now one can eliminate the ε_R by our previous argument using the second Bianchi identity $R^i_{\underset{\smile}{jkl,t}} = 0$. The same argument also shows that one must have $q_K = 2t$, so that contraction can be taken to occur only among the first two R-indices, and such expressions are then clearly in $\mathrm{Pont}(g) \otimes \mathrm{Chern}(\xi)$.

This then concludes the proof of Theorem II. Note that it has the following immediate and really quite plausible consequence:

Corollary to Theorem II. *Any regular invariant ω from Hermitian bundles ξ over M to $A^q(M)$ is given by an element of* $\mathrm{Chern}(\xi)$.

Proof. Any such an ω, when composed with the forgetful functor (concerning the Riemann structure) induces a joint regular invariant of weight 0 in g. Hence $\omega(\xi) \in \mathrm{Pont}(g) \otimes \mathrm{Chern}(\xi)$, but is clearly independent of the Riemannian structure and hence is in $\mathrm{Chern}(\xi)$.

Remark. In all the preceding discussion the Hermitian metric h_ξ plays a very minor role. In fact the same results still hold (with essentially the same proofs) if we consider simply bundles E with a linear connection. The Corollary above then takes a somewhat simpler and more natural form, asserting that the Chern forms are the only regular invariants of a "connected vector bundle".

4. On the Asymptotic Measures of the Heat Equation

In the previous sections we characterized certain form-valued invariants of geometric objects on a manifold. On the other hand, in the theory of a non-negative elliptic differential operator A on M one encounters a countable sequence $\mu_k(A)$ of such invariants, but taking values in the smooth measures on M, and the purpose of this section is to recall first of all, how these μ's fit into the index question and second, to explain the general algorithm which describes them in terms of the coefficients of A. This rather subtle recipe is due to Seeley [23] and generalizes an earlier formula of Minakshisundaram-Pleijel [9] treating the Laplacian of a Riemann structure on M. Of course, they dealt with these matters before the advent of the powerful Pseudo-differential operator calculus.

We start by sketching in the basic facts concerning ellipticity, in a form suitable for our later purposes, and also, hopefully, intelligible to non experts. Suppose then that E and F are bundles over M, with

$$A\colon \Gamma(E)\to\Gamma(F)$$

a differential operator of order $\leqq m$ between them. Given a local coordinate system $x=(x_1,\dots,x_n)$, over U, we write $x\cdot\xi$ for the linear function $\sum_i x_i\,\xi_i$, and consider the operator obtained from A by *conjugating* it with multiplication by the function $e^{ix\xi/\varepsilon}$, $\varepsilon>0$. From the product rule it then follows easily that this operator has an expansion of the form:

$$A_\varepsilon(U)\equiv e^{-ix\xi/\varepsilon}\circ A\circ e^{ix\xi/\varepsilon}=A_0+\varepsilon^{-1}A_1+\cdots+\varepsilon^{-m}A_m \tag{4.1}$$

where the A_i are differential operators from E to F defined over U, depending polynomially of degree i on the ξ's, and whose highest term involves no differentiation at all. Thus for all $p\in U$,

$$A_m(\xi)_p\in\operatorname{Hom}(E_p,F_p). \tag{4.2}$$

Now A is called elliptic if $A_m(\xi)$ is invertible, for all real nonzero ξ, and all coordinate charts U on M.

This notion can, of course, be expressed independently of coordinate charts, so that it is sufficient to check it at every $p\in M$ for only one system of coordinates. In fact A_m is seen to define a section

$$\sigma(A)\in\Gamma\{S_m T\otimes\operatorname{Hom}(E,F)\}, \tag{4.4}$$

where T is the tangent bundle and S_m denotes the m-th symetric power. If we interpret $\Gamma(S_m T)$ as the polynomial functions on T^* – the cotangent bundle of M – then $\sigma(A)$ may be interpreted as a polynomial section of $\operatorname{Hom}(E,F)$-lifted to T^*, and ellipticity means that on the complement of the zerosection of $T^*(M)$, $\sigma(A)$ has values in the *isomorphisms* from E to F. So interpreted $\sigma(A)$ is the *highest order symbol of* A.

In the theory of elliptic operators this nonsingularity of $\sigma(A)$ is exploited first of all in the construction of a parametrix for A, that is, an operator

$$P\colon \Gamma(F)\to\Gamma(E) \tag{4.5}$$

such that PA and AP both differ from the identity by smoothing operators S_E and S'_F respectively[3]:

$$\begin{aligned} PA&=1-S_E\\ AP&=1-S_F. \end{aligned} \tag{4.6}$$

[3] A smoothing operator S is, of course, one given by a smooth kernel K in the form

$$SU(x)=\int K_S(x,y)\,U(y)\,|dy|.$$

Indeed, the first step in the construction of such a P is to seek a P_U, which will have the desired property for sections compactly supported on the coordinate patch U, and for the construction of P_U one starts, just as a good physicist would, by "formally inverting" the operator $A_\varepsilon(U)$ near $\varepsilon=0$. That is, one inductively determines a formal power series in ε:

$$B_\varepsilon(U,x,\xi)=\sum_{k\geqq m}\varepsilon^k B_k(U,x,\xi) \tag{4.7}$$

with the B in $\Gamma\{\operatorname{Hom}(F,E)|U\}$ – for each nonzero ξ – such that

$$\left(A+\frac{1}{\varepsilon}A_1+\cdots+\frac{1}{\varepsilon^m}A_m\right)(\varepsilon^m B_m+\varepsilon^{m+1}B_{m+1}\cdots)\equiv 1. \tag{4.8}$$

Thus the equations for $B_\varepsilon\sim\{A_\varepsilon\}^{-1}$ (near $\varepsilon=0$), start off with:

$$A_m(x,\xi)\cdot B_m(x,\xi)=1, \tag{4.9}$$

whence $B_m(x,\xi)=A_m(x,\xi)^{-1}$. Thereafter:

$$A_{m-1}\circ B_m+A_m\cdot B_{m+1}=0, \tag{4.10}$$

determines $B_{m+1}(x,\xi)$ etc. Note, however, that this is a highly complicated procedure because the lower A_k's are *differential* operators. Nonetheless, these equations determine the $B_k(U)$ explicitly in terms of the coefficients of A and their derivatives. Furthermore, the B_k are clearly homogeneous of degree $-k$ in the ξ's.

In terms of the $B_k(x,\xi)=B_k(U,x,\xi)$, a local parametrix P_U is easily constructed:

Given $s\in\Gamma_c(E|U)$[4], let $\hat{s}(\xi)$ be its Fourier transform relative to a fixed trivialization of $E|U$:

$$\hat{s}(\xi)=\int_U e^{-ix\xi}\,s(x)\,dx, \tag{4.11}$$

and in terms of $\hat{s}$ define $P_U\cdot s$ by:

$$P_U\cdot s(x)=\sum_{k\geqq m}\frac{1}{(2\pi)^n}\int\varphi_k(\xi)\,e^{i(x\xi)}\,B_k(x,\xi)\,\hat{s}(\xi)\,d\xi. \tag{4.12}$$

Here $\varphi_k(\xi)$ are suitable C^∞ functions of ξ which vanish near $\xi=0$, and are 1 near ∞. These functions serve to regularize the $B_k(x,\xi)$ near $\xi=0$ and also to stagger the entries of the B_k's into the ξ-integral.

[4] $\Gamma_c(E)$ denotes sections with compact support.

The P_U's can now be combined by a partition $\{\varphi_U\}$ of unity via the equation

$$P=\sum \varphi_U P_U \psi_U \tag{4.13}$$

(where $\psi_U=1$ on supp φ_U) to yield a global "Parametrix" P for A.

Once such a P has been constructed, the theory of elliptic operators is essentially reduced to that of smoothing operators, and from that theory one deduces the following basic propositions:

Theorem (E I) (Finiteness). *Suppose that $A: \Gamma(E)\to\Gamma(F)$ is an elliptic operator over the compact manifold M, then both the kernel and cokernel of A are finite dimensional. In particular the index of A is well-defined by*

$$\text{index}(A)=\dim\ker A-\dim\operatorname{coker} A. \tag{4.14}$$

We next turn to the "Hodge theorem" in this context. Suppose then that E and F are given fixed Hermitian structures and that a volume form is chosen on M. These data define a "global" Hermitian inner product $(s,s')_E$ on $\Gamma(E)$ by:

$$(s,s')_E=\int_M h_E(s,s')\ \text{volume}. \tag{4.15}$$

Similarly, $(s,s')_F$ is defined on $\Gamma(F)$ and relative to these, A now has a formal adjoint A^*, which is again a differential operator of order $\leqq m$. By the "Laplacian $\square$ of A" one now means the operators on $\Gamma(E)$ and $\Gamma(F)$ respectively given by:

$$\square_E=A^*A,\qquad \square_F=AA^*. \tag{4.16}$$

Clearly, each of these operators is self-adjoint elliptic and $\geqq 0$. From now on we exclude the trivial case $m=0$, so $\square-\lambda$ will also be elliptic.

Now for each $\lambda\in\mathbb{R}$ we let $\Gamma_\lambda(E)$ be the eigenspace of $\square_E$ on $\Gamma(E)$ associated to λ:

$$\Gamma_\lambda(E)=\{s\in\Gamma(E)\,|\,\square_E s=\lambda s\},$$

and define $\Gamma_\lambda(F)$ similarly. With this understood one has the

Theorem (E II) (Hodge Theorem). *For all $\lambda\in\mathbb{R}$, $\Gamma_\lambda(E)$ is finite dimensional. Further $\Gamma_\lambda(E)=0$ except for a discrete set of nonnegative λ's and this countable sequence of subspaces gives an orthogonal direct sum decomposition of the Hilbert space $L_2(E)$ obtained from $\Gamma(E)$ by completion relative to $(\ ,\ )_E$. Thus:*

$$L_2(E)=\oplus\,\Gamma_\lambda(E). \tag{4.17}$$

Finally, relative to this decomposition we have the Hodge formulae:

$$\Gamma_0(E)=\operatorname{Ker} A,\qquad \Gamma_0(F)\cong\operatorname{Coker} A \tag{4.18}$$

while for every $\lambda>0$

$$A:\ \Gamma_\lambda(E)\to\Gamma_\lambda(F)\quad \textit{is an isomorphism}. \tag{4.19}$$

Note that (4.18) gives a new formula for the index of A:

$$\text{index}(A)=\dim\Gamma_0(E)-\dim\Gamma_0(F), \tag{4.20}$$

and combined with (4.19) this in turn leads to many formulas for the index of A. Indeed, let $\varphi(x)$ be *any function on $\mathbb{R}$, with $\varphi(0)=1$*. Then we have

$$\text{index}\,A=\sum_\lambda\varphi(\lambda)\dim\Gamma_\lambda(E)-\sum_\lambda\varphi(\lambda)\dim\Gamma_\lambda(F). \tag{4.21}$$

Provided only that these infinite sums converge.

Now in this context one has

Theorem (E III). (The asymptotic heat expansion.) *In the situation described above, consider the operator $\square=\square_E$; then the series*

$$h_t(\square)=\sum_\lambda e^{-t\lambda}\dim\Gamma_\lambda(E) \tag{4.22}$$

converges for every $t>0$. Furthermore, near $t=0$, $h_t(\square)$ has an asymptotic expansion of the form:

$$h_t(\square)\sim\sum_{k\geqq -n} t^{k/2m}\,U_k(\square)\qquad k\in Z, \tag{4.23}$$

where m is the order of A, and $n=\dim M$.

Finally, each $U_k(\square)$ is given as the integral over M of a certain measure $\mu_k(\square)$ on M, canonically fashioned out of the coefficients of $\square$. Thus:

$$U_k(\square)=\int_M\mu_k(\square). \tag{4.24}$$

Remarks. 1) First of all, this theorem, of course, also holds for $\square=\square_F$, and, in fact, more generally, for any nonnegative self-adjoint differential operator.

Next, observe that by (4.21), and (4.22):

$$\text{index}\,A=h_t(\square_E)-h_t(\square_F)\qquad\text{for any } t>0.$$

Hence, in conjunction with (4.23):

$$\text{index}\,A=U_0(\square_E)-U_0(\square_F)$$

and so finally by (4.24):

(4.25) $$\text{index } A = \int_M \mu_0(\square_E) - \mu_0(\square_F).$$

This is the expression for the index we have been heading for, and which we will use later on. In one sense (4.25) solves the index problem – namely it gives a formula for the index in terms of an integral explicitly made out of the coefficients of A and A^*. On the other hand, the formula for $\mu_0(\square_E)$ will be seen in a moment to be so formidable that there is little hope of interpreting these integrals directly in terms of the characteristic classes of E, M, etc. To the two older authors, the beautiful formula (4.25), therefore, seemed to be useless in this context until the recent work of our younger collaborator appeared.

2) Before describing the recipe for the μ_k a few remarks on the connection of $h_t(\square)$ with the heat equation is in order. Relative to the decomposition

$$L_2(E) = \oplus \Gamma_\lambda(E)$$

the operator $\square = \square_E$ is, of course, in diagonal form:

$$\square | \Gamma_\lambda(E) = \lambda,$$

so that $H_t = e^{-\square t}$ is a well-defined family of bounded operators acting on $L_2(E)$ and satisfying the heat equation

$$\frac{d}{dt} H_t + \square_E \cdot H_t = 0$$

with initial value $H_0 = 1$. Furthermore, at least formally:

$$h_t(\square) = \text{Trace } e^{-\square t}.$$

Hence (4.22) should be interpreted as proving that H_t is indeed of trace class for $t>0$. Actually this turns out to be a consequence of the much stronger result that for $t>0$ the abstract operator H_t is represented by a *smoothing operator* with kernel $H_t(x, y)$:

$$H_t s(x) = \int H_t(x, y)\, s(y)\, |dy|.$$

Now this being granted, the trace of H_t must exist – and is given by the usual formula for smoothing operators:

$$\text{Trace } H_t = \int_M \text{Trace } H_t(x, x)\, |dx|,$$

that is – as an integral of the measure

$$\mu_t(x) = \text{Trace } H_t(x, x)\, |dx|$$

over M.

The $\mu_t(x)$ should, therefore, be thought of as the *local* trace of H_t at x. In terms of an orthonormal base of eigenfunctions $\{\varphi_n(x)\}$ of $\square$ we have

(4.26) $$\mu_t(x) = \sum_n e^{-\lambda_n t} |\varphi_n(x)|^2 |dx|.$$

Thus $\mu_t(x)$ is a function of $\square$ only in a very abstract sense. However the asymptotic expansion (4.23) and (4.24) is a consequence of a *local asymptotic expansion*

(4.27) $$\mu_t(x) \sim \sum_{k \geq -n} t^{k/2m} \mu_k(x)$$

the μ_k being purely *local invariants* of $\square$. In fact, as will be seen later, in our cases the μ_k will depend *rationally* on the *coefficients of A and their derivatives*.

3) Our next remark is that a completely parallel development of this subject is possible, in which one replaces the series

$$h_t(\square) = \sum e^{-\lambda t} \dim \Gamma_\lambda E$$

by the "zeta function of $\square$":

$$\zeta_s(\square) = \sum \lambda^{-s} \dim \Gamma_\lambda E,$$

where we assume for the moment that $\square > 0$ so that $\lambda \neq 0$. Indeed these two are related by the formula

$$\zeta_s(\square) = \frac{1}{\Gamma(s)} \int_0^\infty t^s h_t(\square)\, dt/t$$

as follows directly from the definition of the Γ-function:

$$\Gamma(s) = \int_0^\infty t^s e^{-t}\, dt/t$$

by setting $t = \lambda t$ and then summing.

Now in this context the asymptotic expansion for $h_t(\square)$ yields that $\zeta_s(\square)$ – which at first converges only for large $\text{Re}(s)$ – actually extends to a meromorphic function in the s-plane. Moreover $\Gamma(s)\,\zeta_s(\square)$ has simple poles at the points $s_k = -k/2m$, $k \geq -n$, and its residue there is given by:

$$\text{Res } \Gamma(s)\, \zeta_s(\square) = U_k(\square).$$

In particular $\zeta_0(\square) = U_0(\square)$.

Conversely, this property of $\zeta_s(\square)$ implies the asymptotic formula for $h_t(\square)$ and, in fact, it is this property of $\zeta_s(\square)$ which Minakshisundaram and Pleijel proved originally and which Seeley extended in [23]. This was also the point of view taken in [6], relating the zeta function and the index theorem, although to deal with the zero eigenvalue one has to replace $\Box$ by $\Box + u$ (with $u > 0$).

4) Finally we should point out that, to define the local measures $\mu_k(P)$, we do not need P to non-negative self-adjoint. All that is needed is that the highest order symbol of P should have these properties [23]. We no longer have an eigenfunction expansion but $\zeta_s(P) = \text{trace}\, P^{-s}$ can still be defined and $\int \mu_k(P)$ will still give the residue of $\Gamma(s)\,\zeta_s(P)$ at $s = -k/2m$.

We now turn to Seeley's formula for the μ_k and to motivate his algorithm we will start with the following purely heuristic considerations. Let $T = \mathbb{R}^n/2\pi\,\mathbb{Z}^n$ be the n-torus defined by the lattice whose coordinates are divisible by 2π in $\mathbb{R}^n$, and consider a *constant coefficient* elliptic, self-adjoint, non-negative operator[5] A of order $2m$ acting on the C^∞ functions of T. For each integral point $\xi \in \mathbb{Z}^n$, the function $e^{ix\xi}$ is then an eigenfunction for A with eigenvalue $A(\xi)$:

$$A\, e^{ix\xi} = A(\xi) \cdot e^{ix\xi}$$

where $A(\xi) = A_0 + A_1(\xi) + \cdots + A_{2m}(\xi)$ is the zero order term of the differential operator A_ε at $\varepsilon = 1$, and therefore is a polynomial of order $2m$ in the ξ's. Further, by Fourier's theorem, the $e^{ix\xi}$ are a complete system of eigenfunctions for A. Hence

$$h_t(A) = \sum_{\xi \in \mathbb{Z}^n} e^{-tA(\xi)}.$$

Thus $h_t(A)$ is the sum of the values of the function $e^{-tA(\xi)}$ at the integer points of $\xi \in \hat{\mathbb{R}}^n$, and a little estimation shows that in its asymptotic behavior near $t=0$ this sum behaves like the corresponding integral:

$$h_t(A) \sim \int e^{-tA(\xi)}\, d\xi.$$

Now write $A(\xi) = A_m(\xi) + E(\xi)$ with E the terms of order $< 2m$ and consider the integral obtained by expanding e^{-tE}:

$$\int e^{-A_m(\xi)t} \{\textstyle\sum (-1)^\alpha t^\alpha E(\xi)^\alpha/\alpha!\}\, d\xi.$$

The term by term integration of this expression yields the desired asymptotic formula for $h_t(A)$:

$$h_t \sim \sum_{k \geq -n} t^{k\,2m}\, U_k(A).$$

The integration here is, of course, carried out in polar coordinates in ξ-space and first relative to r and it is this step which introduces the fractional and negative powers of t, as will be seen below. We still have

[5] Throughout this section A denotes an operator of the "Laplacian" type — that is, take $A = \Box_F$.

to come from the global U_k to the local measures μ_k on T. However, as these clearly should be constant in this example they must be given by:

$$\mu_k(A) = \left(\frac{1}{2\pi}\right)^n U_k(A)\,|dx|.$$

To summarize then:

$$\sum t^k \mu_k(A) \sim \left(\frac{1}{2\pi}\right)^n \int e^{-tA(\xi)}\, d\xi\, |dx|. \tag{4.28}$$

In the general case the Seeley formula is given by a similar but more delicate formula; namely, if $A: \Gamma(E) \to \Gamma(E)$ is an operator of order $2m$, and if $x = (x_1, \ldots, x_n)$ is a local coordinate of the type we are discussing, and if $x = (x_1, \ldots, x_n)$ is a local coordinate for M over U, then *properly interpreted*, the Seeley formula can be written:

$$\sum t^k \mu_k(A) \sim \left(\frac{1}{2\pi}\right)^n \lim_{\varepsilon \to 1} \int e^{-\lambda t} \{A_\varepsilon - \lambda\, \varepsilon^{-2m}\}^{-1}\, d\lambda\, d\xi\, |dx|. \tag{4.29}$$

here, as before, A_ε is the operator

$$A_\varepsilon = e^{-ix\xi/\varepsilon} A\, e^{ix\xi/\varepsilon} = A + \frac{1}{\varepsilon} A_1(\xi) + \cdots + \frac{1}{\varepsilon^{2m}} A_{2m}(\xi),$$

and the agreement is that the term in the brackets is first inverted formally near $\varepsilon = 0$, then integrated over λ along a countour enclosing the positive reals and then over ξ. In the result ε is finally set equal to 1.

Thus, the inversion, λ-integration and letting ε tend to 1 replace the expression $e^{-A(\xi)t}$ in the constant coefficient case (4.28).

We will carry this program out under the assumption that $A_{2m}(\xi)$ is a multiple of the identity:

$$A_{2m}(\xi) = a \cdot 1$$

by a function $a = a(x, \xi)$. The precise expression for $A_\varepsilon(\lambda) = A_\varepsilon - \lambda\, \varepsilon^{-2m}$ is therefore:

$$A_\varepsilon(\lambda) = A + \frac{1}{\varepsilon} A^{-1} + \cdots + \frac{1}{\varepsilon^{2m}} (a - \lambda)\, 1 \tag{4.30}$$

where a is a positive function of degree $2m$ in ξ's. Inverting $A_\varepsilon(\lambda)$ formally near $\varepsilon = 0$ yields a power series:

$$B_\varepsilon(\lambda) = \varepsilon^{2m} B_{2m}(\lambda) + \varepsilon^{2m+1} B_{2m+1}(\lambda) + \cdots \tag{4.31}$$

with $B_{2m}(\lambda) = (a - \lambda)^{-1}$ and $B_{2m+1}(\lambda) = -(a - \lambda)\, A_1 \circ (a - \lambda)^{-1}$, etc. Now, and this is where the fact that A_{2m} is a multiple of 1 simplifies matters a little, the effect of A_1 on $(a - \lambda)^{-1}$ will be uniquely expressible as a finite partial fraction in $(a-\lambda)$. That is, $B_{2m+1}(\lambda)=\sum_{s=1}^{2m} B^s_{2m+1}\cdot(a-\lambda)^{-(s+1)}$, and inductively, each $B_l(\lambda)$ will have a finite expansion of the form:

$$B_l(\lambda)=\sum_s B^s_l(a-\lambda)^{-(s+1)}. \tag{4.32}$$

At this stage the integral over λ can be carried out explicitly. Indeed, over the contour in question we have:

$$\int e^{-t\lambda}(a-\lambda)^{-(s+1)}\,d\lambda=\frac{t^s}{s!}e^{-at}, \tag{4.33}$$

so that:

$$\int e^{-t\lambda}B_\varepsilon(\lambda)\,d\lambda=\sum_{s,l}\frac{\varepsilon^l}{s!}t^s B^s_l e^{-at} \qquad l\geqq 2m. \tag{4.34}$$

We turn next to the integration over ξ. For this purpose note that, as a function of $(\xi,\lambda^{\frac{1}{2m}})$, B_l is homogeneous of degree $-l$ while a is homogeneous of degree $2m$. Hence B^s_l is a polynomial of degree $2m(s+1)-l\geqq 0$:

$$\deg_\xi B^s_l=2m(s+1)-l. \tag{4.35}$$

Note also that as a function of A, the B^s_l are polynomials in the coefficients of A and their derivatives relative to the local coordinates.

The integration over ξ, is now carried out by changing to polar coordinates (r,ξ_0), with $|\xi_0|=1$ and first integrating over $r=\sqrt{\sum\xi_i^2}$. We have

$$d\xi=\frac{1}{n}r^n\cdot\frac{dr}{r}\cdot\left\{\frac{\omega}{r^n}\right\} \tag{4.36}$$

where $\omega=\sum(-1)^i\xi_i\,d\xi_1\ldots d\hat{\xi}_i\ldots d\xi_n$, so that $\omega_S=\frac{\omega}{r^n}$ restricts to the volume form on the unit sphere and is homogeneous of degree zero in the ξ's.

Hence

$$\int B^s_l(\xi)\,e^{-a(\xi)t}\,d\xi=\frac{1}{n}\int_{\xi_0}\omega_S B^s_l(\xi_0)\int_r r^{2ms-k}e^{-a(\xi_0)tr^{2m}}\,d\log r \qquad k=l-(2m+n). \tag{4.37}$$

The change of variable $\rho=a(\xi_0)tr^{2m}$, yields $r=\{\rho/at\}^{1/2m}$, and $2m\,d\log r=d\log\rho$. The inner integral therefore changes to

$$\frac{1}{2m}(at)^{-s+k/2m}\int\rho^{s-k/2m}e^{-\rho}\,d\log\rho=\frac{1}{2m}(at)^{-s+k/2m}\Gamma\left(s-\frac{k}{2m}\right) \tag{4.38}$$

substituting in (4.34) this leads to:

$$\begin{aligned}&\int e^{-t\lambda}B_\varepsilon(\lambda)\,d\lambda\,d\xi\\ &=\frac{1}{2mn}\sum\varepsilon^l t^{k/2m}\frac{\Gamma(s-k/2m)}{s!}\int_{|\xi_0|=1}B^s_l(\xi_0)\,a(\xi_0)^{-s+k/2m}\omega_S\end{aligned} \tag{4.39}$$

now passing to $\varepsilon=1$, one obtains the formula[6]:

Theorem (Seeley). *Relative to the coordinates x, on U, the asymptotic measure of A is given by $\mu_t(A)=u_t(A)|dx|$ with*

$$\begin{aligned}u_t(A)=\frac{(2\pi)^{-n}}{2mn}\sum_{\substack{k\geqq -n\\ s-\frac{k}{2m}\geqq\frac{n}{2m}}} t^{k/2m}\frac{\Gamma(s-k/2m)}{s!}\\ \cdot\int_{|\xi_0|=1}B^s_{k+2m+n}(\xi_0)\,a^{-s+k/2m}(\xi_0)\,\omega_S.\end{aligned} \tag{4.40}$$

Our concern with this formula is actually only qualitative. That is, we need to know the following corollary.

Corollary. *The function $\mu_k(A)$ defined above is homogeneous of weight $\frac{k}{2m}$ in the coefficients of A. That is*

$$\mu_k(\lambda A)=\lambda^{\frac{k}{2m}}\mu_k(A) \qquad \lambda\in R^+. \tag{4.41}$$

Furthermore, in the quadratic case, $m=1$, $\mu_k(A)$ is a polynomial in the coefficient of A relative to x, their derivatives relative to x and the function $\det^{-1}(a)$, *where* $\det a$ *is the determinant of the leading term quadratic form:*

$$a(x,\xi)=\sum a_{ij}(x)\,\xi^i\xi^j. \tag{4.42}$$

Proof. The weight property follows directly from (4.26) and (4.27). Sending A to λA, and t to $\frac{1}{\lambda}t$ in (4.26) leaves $\mu_t(x)$ invariant, hence the coefficients μ_k in (4.27) have the desired weights.

The polynomial dependence of $\mu_k(A)$ on the coefficients is a more delicate matter. Certainly (4.40) shows that each term contributing to a μ_k is of the form:

$$\Psi(A)\equiv\int_{|\xi|=1}\Psi(A;\xi)\,a_m(\xi)^{-\frac{l}{2m}}\omega_S \tag{4.43}$$

where $\Psi(A,\xi)$ is a homogenous polynomial of degree $l-n$ in ξ, and depends polynomially on the coefficients of A and their derivatives. On the

[6] Actually in [23] Seeley deals with the zeta-function rather than the heat equation but, as explained before, these are equivalent.

other hand, in its dependence on the coefficients of $a_m(\xi)$, $\Psi(A)$ is *a priori* a transcendental function. However, in the quadratic case we can rationalize the integral by the following procedure.

First, let us reformulate our problem slightly. For this purpose let $p(\xi)$ by any polynomial of degree k, in the ξ's, and in terms of it define the $n-1$ form:

$$p(a,\xi)=p(\xi)\,a(\xi)^{-\frac{k-n}{2}}\,\omega(\xi) \tag{4.44}$$

where $\omega(\xi)=\sum(-1)^i\,\xi_i\,d\xi_1\ldots d\hat{\xi}_i\ldots d\xi_n$ as before, and $a(\xi)$ is the quadratic form (4.42) which is assumed to be strictly positive. Hence $p(a,\xi)$ is regular on all of $\mathbb{R}^n-0$, so that the integral over the unit sphere

$$f(a)=\int_{S^{n-1}} p(a,\xi) \tag{4.45}$$

is a well-defined function of a, and clearly our corollary will be proved if we show that for every p the function $f(a)$ is expressed as a polynomial in the a_{ij} and $\{\det(a_{ij})\}^{-1}$. Put differently we want to show that f is in the affine coordinate ring of the space Q of positive definite quadratic forms. For this purpose recall the map

$$\pi\colon\ GL(n,\mathbb{R})\to Q$$

of the full linear group onto Q given by

$$\pi(g)=g^t\cdot g$$

which we already discussed in Section 2. By the remark made there it will be sufficient to prove a corresponding result for the pull back $\hat{f}=f\circ\pi$ of the function $f(a)$ to $GL(n,\mathbb{R})$:

$$\hat{f}(g)=\int_{S^{n-1}} p(\xi)\,\langle g\,\xi,g\,\xi\rangle^{(-k-n)/2}\,\omega(\xi), \tag{4.46}$$

with $\langle\xi,\xi\rangle$ the usual inner product on $\mathbb{R}^n$.

At this stage observe that the form $p(a,\xi)$ is closed as a form in R^n-0: in fact, $h(\xi)=p(\xi)\,a(\xi)^{-\frac{k-n}{2}}$ is homogeneous of degree $-n$, hence $dp(a,\xi)=\left(\sum_j \xi_j\frac{\partial h}{\partial\xi_j}+n\,h\right)d\xi_1\wedge\cdots\wedge d\xi_n=0$ by Euler's theorem. Thus the integral $f(a)$ depends only on the homology class of the cycle S^{n-1}. Hence we also have

$$\hat{f}(g)=\int_{\langle g\xi,g\xi\rangle=1} p(\xi)\,\langle g\,\xi,g\,\xi\rangle^{(-k-n)/2}\,\omega(\xi).$$

Now under the change of variables $\xi=g^{-1}\xi'$, $\omega(\xi)$ goes over into $(\det g)^{-1}\,\omega(\xi')$ so that we obtain the formula:

$$\hat{f}(g)=(\det g)^{-1}\int_{S^{n-1}} p(g^{-1}\,\xi)\,\omega(\xi) \tag{4.47}$$

which shows that $\hat{f}$ is in the coordinate ring $GL(n,\mathbb{R})$ as required.

5. The Hirzebruch Signature Theorem

In this section we shall combine the heat equation formula for the index explained in §4 with the Gilkey theorem of §2 and derive the Hirzebruch formula expressing the signature of a $4k$-manifold as a polynomial in the Pontrjagin classes. As is now well-known, this Hirzebruch formula is a special case of the general index formula and in the next sections we shall show how to treat the general case.

We begin by recalling the *signature operator* – this is the elliptic operator whose index is the Hirzebruch signature. So let M be a compact oriented Riemannian manifold of dimension $2l$, and let $d\colon \Omega^i\to\Omega^{i+1}$, $d^*\colon \Omega^i\to\Omega^{i-1}$ denote exterior differentiation of forms and its adjoint. Then $d+d^*$, acting on the space of all forms, is an elliptic first-order self-adjoint operator whose square is the Laplace operator Δ of the Hodge theory. We now define an involution τ on the space of all forms by $\tau(\alpha)=i^{p(p-1)+l}*\alpha$ for $\alpha\in\Omega^p$, where $*\colon \Omega^p\to\Omega^{2l-p}$ is the duality operator defined by the metric (the complicated power of i is arranged so that $\tau^2=1$: see [2, p. 575] for details). Denoting by $\Omega_\pm$ the ± 1-eigenspaces of τ, one verifies that $d+d^*$ interchanges Ω_+ and Ω_-. Finally we define the *signature operator* A to be the restriction of $d+d^*$ as an operator from Ω_+ to Ω_-, so that

$$A\colon\ \Omega_+\to\Omega_-$$

is an elliptic operator (its adjoint is then $d+d^*$ as an operator from Ω_- to Ω_+).

Ker A coincides with Ker $A^*A=(\operatorname{Ker}\Delta)\cap\Omega_+$, that is with the space H_+ of harmonic forms ω such that $\tau\omega=\omega$. Similarly Ker A^* coincides with the space H_- of harmonic forms ω such that $\tau\omega=-\omega$. Thus

$$\operatorname{index} A=\dim H_+-\dim H_-.$$

Suppose now that $l=2k$ so that $\dim M=4k$, then only Ω^{2k} gives a non-zero contribution to index A (Ω^q and Ω^{4k-q} give contributions which cancel if $q\neq 2k$) so that

$$\operatorname{index} A=\dim H_+^{2k}-\dim H_-^{2k}$$

where $H_\pm^{2k}$ are the ± 1-eigenspaces of $*$ on H^{2k} (the harmonic $2k$-forms). Since

$$\alpha\mapsto\int\alpha\wedge *\alpha$$

is positive definite for α a real $2k$-form it follows that

$$\alpha \mapsto \int \alpha \wedge \alpha$$

is positive definite on H_+^{2k} and negative definite on H_-^{2k}. Identifying harmonic forms with cohomology by the Hodge theory we see therefore that

$$\text{index } A = \text{sign}(M)$$

where sign (M) denotes the signature of the quadratic form on $H^{2k}(M; R)$ given by the cup-product:

$$x \mapsto x \cup x[M].$$

It is important to note that the operator A depends on the Riemannian metric and the *choice of orientation:* taking the opposite orientation changes τ to $-\tau$, hence interchanges Ω_+ and Ω_- and so replaces A by A^*.

Now let us apply formula (4.25) for the index of A:

$$\text{index } A = \int_M \mu_0(A^* A) - \mu_0(A A^*)$$

where

$$U_0 = \int_M \mu_0(A^* A)$$

is the constant term in the asymptotic expansion of trace e^{-tA^*A}. The measure $\mu_0(A^* A) - \mu_0(A A^*)$ depends on the orientation in a skew-symmetric fashion and hence defines a $4k$-form *independent of the orientation:* in local coordinates we associate to the measure $f(x)|dx_1\, dx_2 \ldots dx_n|$ and the orientation $(x_1, x_2, \ldots, x_n)$ the n-form $f(x)\, dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$. Thus

$$\text{index } A = \int_M \omega \tag{5.1}$$

where ω is a $4k$-form canonically associated to the Riemannian metric.

As explained in § 4 there is a purely local formula for μ_0 and hence for ω. Since $A^* A$ and $A A^*$ are just the Hodge-Laplace operator Δ (restriced to $\Omega_\pm$) their highest-order symbols are scalar matrices of the form $g^{ij}(x)\, \xi_i\, \xi_j \cdot 1$. We are therefore in a position to apply the Seeley theorem and Corollary of § 4. Thus ω is a *regular invariant of the metric* in the sense of § 2.

Let us now consider the effect of a change of scale in the metric: $g \mapsto \lambda^2 g$ with λ a positive constant. The new involution $\tilde{\tau}$ is related to the old involution τ by $\tilde{\tau}(\varphi) = \lambda^{n-2p}\, \tau(\varphi)$ for $\varphi \in \Omega^p$. If we define an automorphism ε on the space of forms by $\varepsilon(\varphi) = \lambda^p \varphi$ for $\varphi \in \Omega^p$, then ε is an isometry $\Omega \to \tilde{\Omega}$ of the old metric to the new metric and $\varepsilon\tau = \tilde{\tau}\varepsilon$. Thus ε induces isometries $\Omega_+ \to \tilde{\Omega}_+$ and $\Omega_- \to \tilde{\Omega}_-$. Moreover, since $\tilde{d} = d$, $\tilde{d}^* = \lambda^{-2} d^*$ we have

$$\varepsilon(d + d^*) = \lambda(\tilde{d} + \tilde{d}^*)\, \varepsilon$$

and so $\varepsilon A = \lambda \tilde{A} \varepsilon$ or $\tilde{A} = \lambda^{-1}(\varepsilon A \varepsilon^{-1})$. The same formula holds for the adjoints and so

$$\tilde{A}^* \tilde{A} = \lambda^{-2}(\varepsilon A^* A \varepsilon^{-1}), \qquad \tilde{A} \tilde{A}^* = \lambda^{-2} \varepsilon A A^* \varepsilon^{-1}).$$

Since the measure μ_0 is unchanged by the bundle isometry ε and is of weight zero in the operator it follows that

$$\mu_0(\tilde{A}^* \tilde{A}) = \mu_0(A^* A), \qquad \mu_0(\tilde{A} \tilde{A}^*) = \mu_0(A A^*)$$

and hence $\tilde{\omega} = \omega$, showing that ω is a regular invariant of the metric of *weight zero*. Applying the main Gilkey Theorem of § 2 we therefore deduce the following key result

Proposition. *The differential form ω appearing in (5.1) is given by a universal polynomial in the Pontrjagin forms, say*

$$\omega = f_k(p_1, p_2, \ldots, p_k) \tag{5.2}$$

where f_k is of total degree $4k$ (deg $p_i = 4i$).

Note incidentally that the measure $\mu_j(A^* A)$, being of weight $j/2$ in the operator $A^* A$ (see (4.41)), is of weight $-j$ in the metric g (with respect to the scale change $g \mapsto \lambda^2 g$).

We have now arrived, by differential-geometric methods, at the same point which Hirzebruch reached by the use of cobordism theory. From here on we simply repeat the easier part of Hirzebruch's proof which is essentially the computation of enough special cases to determine the coefficients of the universal polynomials f_k. For the sake of completeness, and because we shall need to generalize the argument in the next section, we shall review briefly Hirzebruch's argument. For more details we refer the reader to [16; Chapter II, § 6].

As basic examples we take the complex projective spaces P_{2k} and then form all products

$$M(k_1, \ldots, k_r) = P_{2k_1} \times P_{2k_2} \times \cdots \times P_{2k_r}, \qquad \sum k_i = k.$$

In each dimension $4k$ this gives us $\pi(k)$ different manifolds (where $\pi(k)$ is the number of partitions of k). Since $\pi(k)$ is also the number of coefficients of our unknown polynomial f_k the Eqs. (5.1) and (5.2) applied to the manifolds $M(k_1, \ldots, k_r)$ will yield a square system of linear equations. Provided this system is non-degenerate we can then solve uniquely for the coefficients of f_k. To check the non-degeneracy of the system it is convenient to replace the Pontrjagin classes p_j by another set S_j of multiplicative generators related to them by Newton's formulae. For a Riemannian manifold S_j is represented by the form $\left(\frac{1}{2\pi i}\right)^{2j}$. Trace R^{2j}

where R is the curvature matrix. The advantage of the S_j is that for a product we clearly have

$$S_j(M \times N) = S_j(M) + S_j(N)$$

(where we identify cohomology of M and N with their pull-backs to $M \times N$). With an appropriate ordering of the $\pi(k)$ monomials

$$S_\alpha = S_{j_1} \times S_{j_2} \times \cdots \times S_{j_r} \qquad (\sum j_i = k)$$

and the $\pi(k)$ products $M_\beta = M(k_1, \ldots, k_r)$ it then follows easily that the matrix $a_{\alpha\beta} = S_\alpha(M_\beta)$ is triangular and its diagonal entries are given by

$$a_{\alpha\alpha} = S_{j_1}(P_{2j_1}) \times \cdots \times S_{j_r}(P_{2j_r}).$$

Thus $\det(a_{\alpha\beta}) \neq 0$ *provided all the numbers* $S_j(P_{2j}) \neq 0$. But the total Pontrjagin class of the complex projective space P_{2j} is well-known to be given by

$$\sum_0^j p_i = (1+x^2)^{2j+1} \qquad (x \text{ generating } H^2(P_{2j}, \mathbb{Z}))$$

and so $S_j(P_{2j}) = (2j+1)\, x^{2j}[P_{2j}] = 2j+1 \neq 0$.

Now sign $(P_{2j}) = 1$ and the Kunneth formula for the cohomology of a product shows that sign $(M \times N) = \operatorname{sign} M \cdot \operatorname{sign} N$. Thus all the manifolds $M(k_1, \ldots, k_r)$ have signature 1. Hence, we can characterize our polynomials $f_k(p_1, \ldots, p_k)$ uniquely by the property that they give the value 1 when evaluated on the Pontrjagin classes of all the manifolds $M(k_1, \ldots, k_r)$. On the other hand the polynomials $L_k(p_1, \ldots, p_k)$ defined by the generating function

$$(5.3) \qquad \sum L_k = \prod \frac{x_j}{\tanh x_j} \qquad \text{where } \sum p_k = \prod (1+x_j^2)$$

also have this property. In fact for P_{2k} we have to compute the coefficient of x^{2k} in $\left(\frac{x}{\tanh x}\right)^{2k+1}$ and a simple residue calculation shows that this is 1. The multiplicative character of the generating function (5.3) then shows that we get the value 1 on all the products $M(k_1, \ldots, k_r)$. This completes the identification of f_k and L_k and hence the proof of the

Hirzebruch Signature Theorem. *For a compact oriented* $4k$*-manifold* M *the signature of the quadratic form on* $H^{2k}(M; R)$ *is given by*

$$\operatorname{sign}(M) = L_k(p_1, \ldots, p_k)[M].$$

As we have seen, this theorem appears, by our present methods, as the global integrated version of a local theorem which may be formulated here as follows

Local Signature Theorem. *Let* M *be a compact oriented Riemannian manifold of dimension* $4k$ *and let* $\{\varphi_n(x)\}$ *be a complete orthonormal base of eigenforms of degree* $2k$ *of the Laplace operator:* $\Delta\varphi_n = \lambda_n \varphi_n$. *Then* $\omega_t = \sum_n e^{-\lambda_n t} \varphi_n \wedge \varphi_n$ *converges for* $t > 0$ *and, as* $t \to 0$, $\omega_t \to L_k(p_1, \ldots, p_k)$ *where the* p_i *are the Pontrjagin forms.*

Proof. Letting A denote the signature operator we form the local traces $\mu_t^{\pm}(x)$ of the heat operators e^{-tA^*A} and e^{-tAA^*}. The contributions to $\mu_t^{\pm}(x)$ arising from $\Omega^q \oplus \Omega^{4k-q}$ (with $q \neq 2k$) are easily seen to cancel (since the corresponding parts of the heat operators are isomorphic). Since $\Omega_\pm \cap \Omega^{2k}$ are just the ± 1-eigenspaces of $*$ it follows that

$$\mu_t^+(x) - \mu_t^-(x) = \omega_t.$$

Hence we have an asymptotic expansion

$$\omega_t \sim \sum_{j \geq -4k} a_j t^{j/2}.$$

As before the coefficients a_j are regular invariants of the Riemannian metric and a_j is homogeneous of weight $-j$ with respect to the scale change $g \mapsto \lambda^2 g$. Hence by the Gilkey theorem $a_j = 0$ for $j < 0$ so that the constant term is the first non-vanishing term in the asymptotic expansion. Finally, for the constant term a_0, we have already shown that it coincides with $L_k(p_1, \ldots, p_k)$.

6. Other Classical Operators

Using § 3 we shall now extend the results of the previous section to the signature operator "with coefficients in an auxiliary bundle". So let M be a compact oriented Riemannian manifold of dimension $2l$ and let ξ be a Hermitian bundle over M as defined in § 3. Let Ω_ξ denote the space of differential forms with values in ξ, that is smooth sections of the vector bundle $\Lambda^*(T^*(M)) \otimes E_\xi$. Using the connection D_ξ we then define an operator d_ξ on Ω_ξ by

$$d_\xi(u \otimes v) = du \otimes v + (-1)^p u \wedge D_\xi v$$

where $u \in \Omega^p$, v is a section of E_ξ and $u \wedge D_\xi v$ denotes the element given by the pairing $\Omega \otimes \Omega_\xi \to \Omega_\xi$. Since the connection D_ξ is unitary it follows as usual that the adjoint d_ξ^* is given by

$$d_\xi^* = -* d_\xi *$$

where $*(u \otimes v) = *u \otimes v$. This shows that, if we define an involution τ on Ω_ξ as before by $\tau(\alpha) = i^{p(p-1)+l} *(\alpha)$ for $\alpha \in \Omega_\xi^p$, then $d_\xi + d_\xi^*$ anti-commutes with τ. Hence we obtain an operator A_ξ: $\Omega_\xi^+ \to \Omega_\xi^-$ where $\Omega_\xi^\pm$ denote the ± 1-eigenspaces of τ. We shall refer to A_ξ as the generalized signature operator of the bundle ξ. Note that A_ξ is independent of the metric h_ξ.

Proceeding as in § 5 we obtain a formula

$$\operatorname{index} A_\xi = \int_M \omega(g, \xi)$$

where $\omega(g, \xi)$ is a regular joint invariant of mixed weight $(0, 0)$. The weight zero in g is proved just as in § 5 while the weight zero in the Hermitian metric h_ξ is obvious because h_ξ does not enter explicitly into the definition of the operator A_ξ. Now apply Theorem II of § 3 and we deduce that $\omega(g, \xi)$ is given by a universal polynomial in the Chern-forms of ξ and the Pontrjagin forms of g

(6.1) $$\omega(g, \xi) = f(c_i(\xi), p_j(g)).$$

As a first step in identifying this polynomial we observe that we have the following additivity formula

(6.2) $$\omega(g, \xi \oplus \eta) = \omega(g, \xi) + \omega(g, \eta).$$

Here $\xi \oplus \eta$ denotes the direct sum of the two Hermitian bundles ξ, η the connection and Hermitian metric on the direct sum being defined in the obvious way. To prove (6.2) we simply note that $A_{\xi\oplus\eta} = A_\xi \oplus A_\eta$ and $\operatorname{Trace} e^{-t(P\oplus Q)} = \operatorname{Trace} e^{-tP} + \operatorname{Trace} e^{-tQ}$. We shall now show that (6.2) implies that (6.1) must be of the special form

(6.3) $$\omega(g, \xi) = \sum \operatorname{ch}_k(\xi)\, F_s(p_1(g), p_2(g), \ldots)$$

where $2k + 4s = 2l$, $\deg F_s = 4s$ and

$$\operatorname{ch} \xi = \sum_k \operatorname{ch}_k(\xi)$$

is the *Chern character* of ξ. We recall that if the total Chern class $\sum c_j(\xi)$ is factorized formally as $\prod(1 + x_i)$ then $\operatorname{ch}\xi = \sum e^{x_i}$. Thus in terms of the curvature matrix K of ξ we have[7]

$$\operatorname{ch}_k \xi = \left(\frac{1}{2\pi i}\right)^k \frac{\operatorname{Trace} K^k}{k!}.$$

Clearly the Chern character is additive: $\operatorname{ch}(\xi \oplus \eta) = \operatorname{ch}\xi + \operatorname{ch}\eta$. Moreover, any polynomial in the Chern classes can be universally expressed as a polynomial in the components of the Chern character. Hence we may rewrite (6.1) universally in the form

$$\omega(g, \xi) = f_1(\operatorname{ch}(\xi), p(g)) + f_2(\operatorname{ch}(\xi), p(g)) + \cdots + f_l(\operatorname{ch}(\xi), p(g))$$

[7] Except for the factor $k!$ and the fact that we are now dealing with complex rather than real bundles the elements ch_k are essentially the same as the S_k of § 5.

where f_r is homogeneous of degree r in the variables $\operatorname{ch}_k(\xi)$. Replacing ξ by $N\xi = \xi \oplus \xi \oplus \cdots \oplus \xi$ (N times) and using $\operatorname{ch}(N\xi) = N \operatorname{ch}\xi$ we deduce

(6.4) $$\omega(g, N\xi) = N f_1 + N^2 f_2 + \cdots + N^l f_l.$$

But, by (6.2), $\omega(g, N\xi) = N\omega(g, \xi)$. Since (6.4) holds for all positive integers N we may equate coefficients and so

$$\omega(g, \xi) = f_1(\operatorname{ch}(\xi), p(g))$$

is linear in the components of $\operatorname{ch}(\xi)$ as asserted in (6.3).

We are now reduced to identifying the polynomials F_s in (6.3) by computing sufficiently many special examples. Note incidentally that F_s depends on the dimension $n = 2l$ of our manifold so we should write it as F_s^l. In fact we shall prove

(6.5) $$F_s^l = 2^{l-2s} L_s$$

where the L_s are the Hirzebruch polynomials as in § 5. For $l = 2s$ (so that $\dim X = 4s$) (6.5) reduces to $F_s^l = L_s$ and this follows directly from the Hirzebruch signature theorem by taking ξ in (6.3) to be the trivial line-bundle.

As a basic example we shall take M to be a one-dimensional complex torus (elliptic curve) and ξ to be a holomorphic line-bundle.

In this case we shall prove

Lemma (6.6). $\operatorname{index} A_\xi = 2\int_M c_1(\xi)$.

Proof. Taking the standard metric on M arising from that on the universal covering $\mathbb{C}$ we have $*dx = dy$, $*dy = -dx$ and so

$$\tau dz = i * dz = i(dy - i dx) = dz$$
$$\tau d\bar{z} = i * d\bar{z} = i(dy + i dx) = -d\bar{z}$$
$$\tau dz \wedge d\bar{z} = -2.$$

From these it follows that we have isomorphisms

$$\sigma_+ : \Omega^0 \oplus \Omega^0 \to \Omega_+$$
$$\sigma_- : \Omega^{0,1} \oplus \Omega^{0,1} \to \Omega_-$$

given by

$$\sigma_+(f, g) = \tfrac{1}{2}(1 + \tau) f + g\, dz$$
$$\sigma_-(\lambda, \mu) = (1 - \tau)\, \mu\, dz + \lambda$$

and a commutative diagram

$$\begin{array}{ccc} \Omega^0 \oplus \Omega^0 & \xrightarrow{\bar\partial \oplus \bar\partial} & \Omega^{0,1} \oplus \Omega^{0,1} \\ \big\downarrow{\scriptstyle \sigma_+} & & \big\downarrow{\scriptstyle \sigma_-} \\ \Omega_+ & \xrightarrow{d+d^*} & \Omega_- \end{array}$$

Thus the operator A is isomorphic to two copies of $\bar\partial$. For any Hermitian structure on the holomorphic line-bundle ξ it then follows that A_ξ and $\bar\partial_\xi \oplus \bar\partial_\xi$ have (up to isomorphism) the same first-order term and hence the same index[8]. Here $\bar\partial_\xi$ denotes the natural $\bar\partial$-operator acting on sections of ξ. Now we already know, from our general discussion, that we must have a formula of the type

$$\text{index } A_\xi = \alpha \int_M c_1(\xi)$$

for some universal constant α ($\alpha = F_0^1$ in the notation of (6.3)). To prove $\alpha = 2$ it is then sufficient to choose a particular ξ with $\int_M c_1(\xi) \neq 0$. Taking ξ to be the holomorphic line-bundle determined by a single-point P of M we have $\int_M c_1(\xi) = 1$. Ker $\bar\partial_\xi$ consists of the holomorphic sections of ξ and these can be identified with the meromorphic functions φ on M having a single pole at P: the Cauchy residue formula applied to $\varphi(z)\,dz$ shows that $\operatorname{Res}_P \varphi(z)\,dz = 0$, so φ has no poles and hence is constant. Thus dim Ker $\bar\partial_\xi = 1$. Similarly, Ker $\bar\partial_\xi^*$ can be identified with the holomorphic functions having a zero at P (and no poles), hence Ker $\bar\partial_\xi^* = 0$ and so index $\bar\partial_\xi = 1$. Thus index $A_\xi = 2$ as required.

Remark. We have computed above a special case of the Riemann-Roch theorem for Riemann surfaces. Equally well we could have taken any other genus (e.g. zero) but the elliptic curve has the advantage that its tangent bundle is trivial which leads to somewhat simpler calculations.

To prove (6.5) we shall take products of the example just computed in (6.6) together with the examples of § 5. Generalizing the multiplicative property of the signature we shall need

Lemma (6.7). *Let M, N be compact oriented even-dimensional Riemannian manifolds and let ξ, η be Hermitian vector bundles over M, N respectively. Denote by $\zeta = \xi \otimes \eta$ the Hermitian vector bundle over $M \times N$ given by the tensor product with the natural induced metric and connection. Then*

$$\text{index } A_\zeta = \text{index } A_\xi \cdot \text{index } A_\eta .$$

[8] Using the stability of the index under deformation. Alternatively, a more careful calculation shows that for a suitable metric A_ξ is actually isomorphic to $\bar\partial_\xi \oplus \bar\partial_\xi$, i.e. the zero-order terms also coincide.

Proof. Direct computation shows that

$$(d_\zeta + d_\zeta^*)^2 = (d_\xi + d_\xi^*)^2 \otimes 1_\eta + 1_\xi \otimes (d_\eta + d_\eta^*)^2$$

which implies easily that

$$\operatorname{Ker}(d_\zeta + d_\zeta^*) = \operatorname{Ker}(d_\xi + d_\xi^*) \otimes \operatorname{Ker}(d_\eta + d_\eta^*).$$

Together with the fact that $\tau_\zeta = \tau_\xi \otimes \tau_\eta$ this leads at once to the required formula

$$\text{index } A_\zeta = \text{index } A_\xi \cdot \text{index } A_\eta .$$

Now consider $Y = M^k \times X$ where M is an elliptic curve and X is of dimension $4s$. Take the line-bundle $\eta = \xi^k \otimes 1$ ($\xi^k = \xi \otimes \cdots \otimes \xi$ k times) over Y and apply (6.7). We deduce

(6.8)
$$\begin{aligned} \text{index } A_\eta &= (\text{index } A_\xi)^k \cdot \text{sign } X \\ &= \Big(2 \int_M c_1(\xi)\Big)^k \cdot \text{sign } X \qquad \text{by (6.6).} \\ &= 2^k \int_{M^k} \text{ch}_k\, \xi^k \cdot \text{sign } X \end{aligned}$$

since $\text{ch}\,\xi^k = (\text{ch}\,\xi)^k = (1 + c_1(\xi))^k$. Comparing with (6.3), and cancelling the non-zero term $\int_{M^k} \text{ch}_k\, \xi^k$, we obtain

$$\int_X F_s^l(p_1, p_2, \ldots, p_s) = 2^k \text{ sign } X$$

where $k = l - 2s$. The argument in § 5, using products of complex projective spaces for X, shows that $2^{-k} F_s^l = L_s$ or $F_s^l = 2^{l-2s} L_s$, as asserted in (6.5). Thus we have established the

Generalized Hirzebruch Signature Theorem. *Let M be a compact oriented $2l$-dimensional Riemannian manifold, ξ a Hermitian bundle over M and A_ξ the generalized singature operator of ξ. Then the index of A_ξ is given by the formula*

$$\text{index } A_\xi = 2^l \cdot \text{ch}\,\xi \cdot \mathcal{L}(M)\,[M]$$

where $\mathcal{L}(M) = \prod \dfrac{x_i/2}{\tanh x_i/2}$ *and the elementary symmetric functions of the x_i^2 are replaced by the Pontrjagin classes of M.*

Remarks. 1. There is a local version of this theorem generalizing the case treated in § 5. Note that this local version continues to hold when ξ is just a connected bundle (with no metric) in view of the remark made at the end of § 3 and Remark 4 following Theorem (E III). Instead of $A_\xi^* A_\xi$

and $A_\xi A_\xi^*$ we now use the restriction of $(d_\xi+d_\xi^*)^2$ to Ω_ξ^+. These operators do not involve the metric h_ξ and their leading symbols are positive self-adjoint (with respect to any metric in ξ) because $\sigma(A_\xi)=\sigma(A)\otimes I_\xi$ where I_ξ is the identity on ξ.

2. In § 5 the results were restricted to manifolds with dimensions divisible by 4. Here all even-dimensional manifolds are included. This is because we have used a complex vector bundle ξ and this has Chern classes in all even dimensions.

As we shall show in the next section, the Generalized Signature Theorem implies, via topological arguments, the general index theorem and hence in particular we obtain other classical special cases such as the Riemann-Roch theorem. However, there is some interest in deriving these special cases directly by methods similar to those used for the signature theorem. Where applicable this will of course lead to appropriate local versions of the theorems concerned and these would not be obtainable by global topological arguments. We shall therefore treat briefly the Dirac and $\bar\partial$-operators by the heat equation method, and we begin with the former.

Let us recall[9] that M is said to be a *spin-manifold* if the structure group of its tangent bundle can be lifted from $SO(n)$ to Spin(n). The necessary and sufficient condition for this is the vanishing of the second Stiefel-Whitney class $\omega_2(M)$: for example $P_n(\mathbb{C})$ is a spin-manifold if and only if n is odd. A choice of lifting to Spin(n) defines a definite spin-structure and, up to isomorphism, the different spin-structures are in one-one correspondence with the elements of $H^1(M; Z_2)$. Thus the spin-structure is essentially unique if M is simply-connected. Now Spin(n) has a basic representation space S (the spin space): if $n=2l$ this is the direct sum of two irreducible representations S^+ and S^- each of dimension 2^{l-1}. Associated to the spin-structure of M there are then two corresponding vector bundles E^+ and E^- whose cross-sections are the spinor-fields on M – analogous to tensor-fields. The *Dirac operator* is then a first order self-adjoint elliptic differential operator acting on the sections of $E=E^+\oplus E^-$ and defined as the composition

$$\Gamma(E)\xrightarrow{D}(E\otimes T^*)\xrightarrow{C}\Gamma(E)$$

where D is induced by the Riemannian connection of M and C is Clifford multiplication (induced by a linear map of Spin(n)-modules: $S\otimes R^n\to S$). Note the analogy with exterior differentiation of forms which may be defined similarly but using exterior multiplication of forms in lieu of Clifford multiplication. The Dirac operator switches E^+ and E^- (because

[9] For further details on spin-manifolds and the Dirac operator see [2, § 5].

C does) and so is of the form $\begin{pmatrix}0 & B\\ B^* & 0\end{pmatrix}$ where $B: \Gamma(E^+)\to\Gamma(E^-)$ is elliptic. The index of B is called the spinor-index of M.

The operator B, like the signature operator A of § 5, has the following basic properties

(1) B is canonically defined by the Riemannian metric, the orientation and the spin-structure

(2) reversing the orientation takes B into B^*

(3) the metric scale change $g\mapsto\lambda^2 g$ takes B into $\lambda^{-1}B$.

Properties (1) and (2) are clear from what has been said. To verify (3) we define an isometry $T^*\to\tilde T^*$ by $\xi\mapsto\lambda\xi$ where T^* and $\tilde T^*$ denote the cotangent bundle of M with the old and new metrics respectively. Then we get an induced isomorphism $\alpha: E\to\tilde E$ which preserves connections. Since Clifford multiplication is bilinear it follows that $\tilde B=\lambda^{-1}\alpha B\alpha^{-1}$ so that, modulo the isometry α, $\tilde B$ coincides with $\lambda^{-1}B$.

The only significant difference between the operators B and A is that B depends on the spin-structure. However, locally the spin-structure is unique and so the measures $\mu_j(B^*B)$ and $\mu_j(BB^*)$ depend only on the metric and orientation. Proceeding now exactly as in § 5 we deduce that the spinor-index is given by some universal polynomial in the Pontrjagin classes. Thus the spinor-index is zero if dim M is not divisible by 4 and if dim $M=4s$ then

$$\text{spinor-index}(M)=h_s(p_1,\dots,p_s)[M].$$

It remains to identify the polynomials h_s and to show in fact that they coincide with the $\hat A$-polynomials defined by Hirzebruch (see [2, § 5]). This could be done by computing explicitly the spinor-index of sufficiently many examples. For instance we could take products of quarternionic projective spaces[10] $P_n(\mathbb{H})$ $(n\geq 2)$ and (in dimension 4) a quartic surface in $P_3(\mathbb{C})$: these are all spin-manifolds and have linearly independent Pontrjagin numbers [9, § 2.3]. Instead we shall show how to deduce the equality $h_s=\hat A_s$ by using the generalized signature theorem. To do this, we first extend B to an operator B_ξ where ξ is an auxiliary Hermitian bundle (just as A was extended to A_ξ). As before we deduce a formula of the type

$$\text{index } B_\xi=\sum \mathrm{ch}_k(\xi)\cdot H_s^l(p_1,\dots,p_s)[M] \qquad (6.9)$$

where $2k+4s=2l=\dim M$ and $\deg H_s^l=4s$. Moreover the formal properties of Clifford multiplication yield a multiplicative formula for B_ξ analogous to (6.7) and the torus example of (6.6) gives index $B_\xi=\int_M c_1(\xi)$.

[10] Recall that the complex projective spaces $P_{2k}(\mathbb{C})$, which we used earlier, are not spin-manifolds.

Arguing as before we then deduce

$$H_s^l = h_s. \tag{6.10}$$

To use the generalized signature theorem we now appeal to the following

Lemma (6.11). index $B_\xi =$ index A *where ξ is the total spin bundle E of M with the metric and connection induced by the Riemannian structure of M.*

Proof. There is a natural bundle isomorphism $E \otimes E \cong \Lambda^*(T^*)$ induced by the isomorphism $S \otimes S \cong \Lambda^*(R^n)$ of Spin (n)-modules, and this takes $\Gamma(E^\pm \otimes E)$ into $\Omega_\pm$. It is a simple matter of linear algebra to check that the symbols of A and B_ξ coincide[11] via the above isomorphisms and hence index $A =$ index B_ξ.

Comparing (6.9) and (6.10) with the generalized signature theorem we therefore deduce

$$\begin{aligned} 2^l \prod \frac{x_i/2}{\tanh x_i/2}[M] &= \operatorname{ch} E \sum h_s[M] \\ &= \prod (e^{x_i/2} + e^{-x_i/2}) \sum h_s[M] \end{aligned} \tag{6.12}$$

(using the formula for the character of the Spin representation). Now assume as inductive hypothesis that $h_s = \hat{A}_s$ for $s < k$ where

$$\sum \hat{A}_s = \prod \frac{x_i}{e^{x_i/2} - e^{-x_i/2}}.$$

Taking M to have dimension $2l = 4k$ and comparing (6.12) with the obvious expansion

$$2^l \prod_{i=1}^{l} \frac{x_i/2}{\tanh x_i/2}[M] = \prod_{i=1}^{l} (e^{x_i/2} + e^{-x_i/2}) \sum_{s=0}^{k} \hat{A}_s[M]$$

we then deduce

$$\hat{A}_k(p_1, \ldots, p_k)[M] = h_k(p_1, \ldots, p_k)[M].$$

Since this holds for all spin-manifolds of dimension $4k$ it follows that $\hat{A}_k = h_k$ establishing the induction and completing the identification of the Spinor-index.

All this can be extended from spin-manifolds to spin^c-manifolds, namely those manifolds M for which $\omega_2(M)$ is the reduction modulo 2 of an integral cohomology class $c_1 \in H^2(M; Z)$. The structure group of $T(M)$ can now be lifted to the complex Spinor group Spin$^c(n)$ – defined as the quotient of Spin $(n) \times U(1)$ by the group of order 2 generated by $(\varepsilon, -1)$ where ε generates the kernel of Spin$(n) \to SO(n)$ (see [8] for more

[11] In fact a more careful computation shows that $A = B_\xi$.

details). Note that the map $(u, v) \mapsto v^2$ ($u \in$ Spin(k), $v \in U(1)$) induces a homomorphism Spin$^c(n) \to U(1)$ and hence we get a complex-line-bundle L (with metric) associated to the spin^c-structure. In fact, the class c_1 is the first Chern class of L. If we give L a Hermitian structure ξ as defined in § 3 (i.e. we choose a unitary connection for L) then we can define a Dirac operator on M acting on the Spin-bundle E (associated to the spin representation of Spin$^c(n)$). If $n = 2l$ this is again of the form $\begin{pmatrix} 0 & B^c \\ (B^c)^* & 0 \end{pmatrix}$ where $B^c: \Gamma(E^+) \to \Gamma(E^-)$ is a self-adjoint elliptic operator.

If M is a spin-manifold (so $\omega_2(M) = 0$) then it has an obvious spin^c-structure with ξ trivial but it also has others. Thus if η is any Hermitian line-bundle we have a spin^c-structure on M with $\xi = \eta^2$, and so $c_1(\xi) = 2c_1(\eta)$ as differential forms. In this case the operator B^c coincides with B_η. Since M is always locally a spin-manifold, it follows that B^c is always locally of the form B_η. Now the heat equation method gives as before a local formula for index B^c and, since $B^c = B_\eta$ locally (with $\eta^2 = \xi$), the previous formula for the spinor index with auxiliary bundle leads at once to the formula

$$\text{index } B^c = e^{c_1(\xi)/2} \sum \hat{A}_s[M]. \tag{6.13}$$

A complex manifold with Hermitian metric has a natural spin^c-structure arising from the diagram

$$\begin{array}{ccc} & & \mathrm{Spin}^c(2n) \\ & \nearrow & \downarrow \\ U(n) & \longrightarrow & SO(2n) \times U(1) \end{array}$$

where the horizontal arrow is given by the inclusion into $SO(2n)$ and the determinant map to $U(1)$ [8, § 5]. Here c_1 is the usual first Chern class of the complex tangent bundle and so the right side of (6.13) is just the Todd genus

$$T_n(c_1, \ldots, c_n)[M]$$

where the Todd polynomials are defined by

$$\sum T_k = \prod \frac{x_i}{1 - e^{-x_i}}$$

and c_j is the j-th elementary symmetric function in the x_i. The operator B^c of the spin^c-structure is closely related to the $\bar\partial$-operator of the complex structure. More precisely, if we consider the $\bar\partial$-complex

$$0 \to \Omega^{0,0} \xrightarrow{\bar\partial} \Omega^{0,1} \xrightarrow{\bar\partial} \cdots \xrightarrow{\bar\partial} \Omega^{0,n} \to 0 \tag{6.14}$$

and form the corresponding elliptic operator

$$\bar{\partial}+\bar{\partial}^*: \sum \Omega^{0,2p} \to \sum \Omega^{0,2p+1} \tag{6.15}$$

as in [2], this always has the same symbol as B^c. If, moreover the metric on M is a Kähler metric, then the operator (6.15) coincides[12] with B^c (for details see [17]). On the other hand, as explained in [2], the index of the operator (6.15) is equal to the Euler-characteristic of the global sections of the $\bar{\partial}$-complex (6.14) and this in turn is equal to the Euler-characteristic of the sheaf O of germs of holomorphic functions (the arithmetic genus). Thus (6.13) yields the Todd formula

$$\sum_{p=0}^{n} (-1)^p \dim H^p(M,O) = T_n(c_1,\dots,c_n)[M]$$

for any compact complex manifold and the stronger local version for a Kähler manifold.

More generally, the full Hirzebruch Riemann-Roch Theorem for a holomorphic vector bundle V follows in the same way by taking the Dirac operator with an auxiliary Hermitian structure ξ on V. Again for a Kähler manifold we get the local version provided (see [17]) we take the unique unitary connection on V which is compatible with the complex structure – that is $D_\xi v = \sum \omega_i \otimes v_i$ where the v_i are a local holomorphic basis of V and the ω_i are forms of type (1, 0). This local version of the Riemann-Roch Theorem was treated directly, by the heat equation method, in [22]. In our present treatment, it followed as a by-product of the Riemannian case.

At this stage we should perhaps say a few words about the most classical case of all, namely the Gauss-Bonnet theorem which is concerned with the ordinary Euler characteristic. This appears as the index of the operator

$$d+d^*: \sum \Omega^{2p} \to \sum \Omega^{2p+1} \tag{6.16}$$

and a direct treatment by the heat equation was given in [21]. Unlike the signature operator the operator (6.16) does not depend on the orientation of M and this means that the local contribution for its index given by the heat equation is naturally a measure and not an n-form. To identify it directly by invariance theory methods is therefore more difficult than before: we need more information than simply its behaviour under change of scale. Gilkey has shown how to do this, and in fact he treated this case first. Here we shall show how to deduce the Gauss-Bonnet theorem from our previous results, in the same spirit as we derived the Riemann-Roch Theorem.

[12] Up to a constant factor $\sqrt{2}$.

The heat equation gives us an integral expression for the Euler-characteristic

$$\chi(M) = \int_M \mu$$

where μ is some measure depending on the metric. Our problem, which is a purely local one, is to identify μ with the Euler form ($(2\pi)^{-l} K$, where[13] $\dim M = 2l$ and K is the Gauss curvature). Now as observed in the proof of (6.11) we have $A = B_\xi$, where A is the signature operator, B is (half) the Dirac operator and ξ is the total spin bundle E with its natural Hermitian structure. In fact this equation is compatible with the decomposition $E = E^+ \oplus E^-$ and the decomposition into even and odd forms. Thus we have $A^+ = B_{\xi^+}$, $A^- = B_{\xi^-}$ where $A^+: \Omega_+^{\mathrm{ev}} \to \Omega_-^{\mathrm{odd}}$ and $A^-: \Omega_+^{\mathrm{odd}} \to \Omega_-^{\mathrm{ev}}$. Since A^*A and AA^* are just the Laplace operator on Ω_+ and Ω_- it follows that our measure μ is given by

$$\begin{aligned}\mu &= \mu_0(\Delta^{\mathrm{ev}}) - \mu_0(\Delta^{\mathrm{odd}})\\ &= [\mu_0(\Delta_+^{\mathrm{ev}}) - \mu_0(\Delta_-^{\mathrm{odd}})] - [\mu_0(\Delta_+^{\mathrm{odd}}) - \mu_0(\Delta_-^{\mathrm{ev}})]\\ &= [\mu_0(A^{+*}A^+) - \mu_0(A^+A^{+*})] - [\mu_0(A^{-*}A^-) - \mu_0(A^-A^{-*})]\\ &= \alpha^+ - \alpha^- \quad \text{say.}\end{aligned}$$

Now the local version of the generalized Spinor-index formula (6.9) using ξ^+ and ξ^- as auxiliary Hermitian bundles gives

$$\begin{aligned}\alpha^+ &= \sum \mathrm{ch}_k E^+ \hat{A}_s\\ \alpha^- &= \sum \mathrm{ch}_k E^- \hat{A}_s\end{aligned} \qquad (k+2s=l).$$

Now using the character formulae for the half-spin representations S^+ and S^- we have

$$\mathrm{ch}\, S^+ - \mathrm{ch}\, S^- = \prod_{i=1}^{l} (e^{x_i/2} - e^{-x_i/2}) = \prod_{i=1}^{l} x_i + \text{higher terms}$$

where $x_1, \dots, x$ are coordinates mod 2π for the maximal torus of $SO(2l)$. Applied to the curvature matrix the term $\prod x_i$ gives (by definition) the Euler form e. Thus

$$\mathrm{ch}\, E^+ - \mathrm{ch}\, E^- = e$$

and so

$$\alpha = \alpha^+ - \alpha^- = e \cdot \hat{A}_0 = e$$

as required.

In conclusion we should point out that there are a number of other routes to the results of this section, that is to calculating the precise polynomials which arise for the classical operators. For instance we

[13] χ and μ are trivially zero if dim M is odd.

could have started with Spin^c and the complex structure cases, bypassing the signature. We would then have had to prove that the arithmetic genus of complex projective space is equal to 1, and use Hirzebruch's original characterization of the Todd polynomials. An alternative and enlightening approach to all cases would be to combine the results of [6] and [11]. The point is that we have sufficiently many examples arising amongst homogeneous spaces of compact lie groups and these admit large groups of isometries – in particular one-parameter groups with isolated fixed-points. The Lefschetz formula of [6] (which can be proved essentially by a simpler use of the heat equation than employed here) gives a formula in terms of fixed-points for the index-character (as a function on the group). Evaluating this at the identity and using the differential-geometric residue-type result of [11] this formula gets converted into one involving Pontrjagin numbers[14]. One advantage of this approach is that the various polynomials (L, T, $\hat{A}$ etc.) appear very naturally and do not have to be known in advance. Moreover, no explicit calculations of an index would need to be made.

7. The General Index Theorem

In the original proof of the index theorem [5] and [20] the generalized signature theorem was established by the use of cobordism and the general index theorem was then deduced by the use of K-theory. Now that we have established the generalized signature theorem by differential-geometric methods we can proceed as before to the general index theorem. We shall recall briefly how this is done.

If $P: \Gamma(E) \to \Gamma(F)$ is an elliptic differential operator of order k on M its highest order symbol gives a vector bundle isomorphism $\sigma(P): \pi^* E \to \pi^* F$ where $\pi: S(M) \to M$ is the projection of the unit cotangent bundle of M. Now let $B(M)$ denote the unit ball bundle in T^*M, so that $B(M)$ is a $2n$-manifold with boundary $S(M)$. Taking two copies $B(M)^+$ and $B(M)^-$ we glue them along their common boundary to form a closed $2n$-manifold

$$\Sigma(M) = B(M)^+ \cup_{S(M)} B(M)^-.$$

Denoting by $\pi^\pm: B(M)^\pm \to M$ the projections we form the vector bundle

$$V(\sigma) = \pi^{+*}(E) \cup_{\sigma(P)} \pi^{-*}(F)$$

in which $\sigma(P)$ is used to identify E and F along $S(M)$. This vector bundle $V(\sigma)$ contains the topological information that is needed to compute index P. In fact, index P is given by

[14] The results of [11] would also need to be extended to include auxiliary bundles.

Index Theorem. *Index* $P = \{\mathrm{ch}\, V(\sigma)\, \pi^* \vartheta(M)\}\, [\Sigma(M)]$ *where* $\vartheta(M)$ *denotes the total Todd class of* $T(M) \otimes_R \mathbb{C}$, $\pi = \pi_\Sigma$ *is the projection* $\Sigma(M) \to M$ *and* $\Sigma(M)$ *has its natural orientation*[15].

Routine calculations show that this formula, when applied to the classical operators of § 5 and 6 reduces precisely to the index formulae obtained there. To prove the general index theorem we therefore want to show that any operator is equivalent in some sense to a classical operator.

First, by using pseudo-differential operators, we can allow our symbols σ to be arbitrary smooth bundle isomorphisms over $S(M)$. This shows that the index depends only on the vector bundle $V(\sigma)$ over $\Sigma(M)$, and this bundle can be arbitrary. Moreover if $V = \pi^* W$ for some W on M then the index is zero (this corresponds to the case of an operator not involving differentiation i.e. multiplication by an invertible matrix function). Thus $V(\sigma(P)) \mapsto \mathrm{index}\, P$ induces a homomorphism $K(\Sigma(M)) \to Z$ which is zero on the image of $K(M)$: here $K(M)$ denotes as usual the Grothendieck group generated by complex vector bundles on M. Using the basic results of K-theory it can now be proved that, if M is oriented and of even-dimension, then the elements of $K(\Sigma(M))$ defined by generalized signature operators generate this group modulo the image of $K(M)$ and 2-torsion. This is then enough to prove the index theorem for the even orientable case. The odd-dimensional case can be reduced to the even case by multiplying by a circle. To treat a non-orientable manifold M we pass to the oriented double cover $\tilde{M}$. The existence of a local formula (from the heat equation) proves that index $\tilde{P} = 2$ index P where $\tilde{P}$ is the lift to $\tilde{M}$ of the elliptic operator P. Since the cohomology formula also gets multiplied by 2 the index theorem for M follows from that for $\tilde{M}$.

We shall now add a few words of explanation on why the symbols of the generalized signature operators generate the group $K(\Sigma(M))$ in the sense stated above. Suppose first we consider the local problem in which M is replaced by a small neighborhood of a point. Then $K(\Sigma(M))$ gets replaced by $K(S^{2n})$, the Grothendieck group of vector bundles on the $2n$-sphere and $K(M)$ gets replaced by K (point) so that $\pi^* K(M)$ is given by the trivial bundles on S^{2n}. The fundamental theorem of K-theory asserts that $K(S^{2n})$ modulo trivial bundles is an infinite cyclic group. This is just a reformulation of the periodicity theorem which states that the homotopy groups $\pi_{2n-1}(GL(N, C))$ (for N large) are periodic in n with period 2 and hence (by considering the case $n=1$) are infinite cyclic. The passage from $\pi_{2n-1}(GL(N, C))$ to $K(S^{2n})$ is given by our construction $\sigma \mapsto V(\sigma)$. Now it is a remarkable fact that

[15] If $x_1, \ldots, x_n$ are local coordinates on M, $\xi_i = dx_i$ corresponding coordinates for the fibre of $B(M)^+$, the orientation of $B(M)^+$ (and so of $\Sigma(M)$) is given by the (ordered) coordinates $x_1, \xi_1, \ldots, x_n, \xi_n$.

the symbol of the operator $\bar{\partial}+\bar{\partial}^*: \Sigma\Omega^{0,2p} \to \Sigma\Omega^{0,2p+1}$ *in* $\mathbb{C}^n$ *gives precisely a generator of* $\pi_{2n-1}(GL(N, C))$ (here $N=2^{n-1}$). This makes it plausible that, on a complex manifold, the extension of this operator to auxiliary vector bundles should provide generators for $K(\Sigma(M))$ mod the image of $K(M)$. In fact, these can be deduced from the local statement by standard techniques of algebraic topology. Alternatively one can give proofs of the local theorem which extend automatically to the global case. For a Spin^c-manifold the same results hold using the Dirac operator, but for a general oriented manifold we have to use the signature operator and locally its symbol gives 2^n times the generator of $\pi_{2n-1}(GL(N, C))$ (see, for example, the explicit calculation for $n=1$ in the proof of Lemma (6.6)). For a fuller account of the periodicity theorem and its ramifications the reader may consult [12] and the bibliography given there.

Appendix I

Invariant Theory for the Orthogonal Group

Let k denote any field of characteristic zero, $\bar{k}$ its algebraic closure. We shall begin by recalling a few elementary facts about the orthogonal group over k and $\bar{k}$. In the first place the special orthogonal group $SO(n, \bar{k})$ is *connected* (in the Zariski topology). This may be proved as follows:

(i) $SO(n, \bar{k})$ acts transitively on the quadric Q in $P_{n-1}(\bar{k})$ (representing null vectors) and the map $SO(n, \bar{k}) \to Q$ given by choosing a base-point in Q is open.

(ii) Over $\bar{k}$ we may choose coordinates so that our quadratic form is $x_1 x_n + \sum_{n>i,j\geq 2} a_{ij} x_i x_j$ and our base-point is $(1, 0, 0 \dots 0)$. Let A denote the matrix (a_{ij}), the isotropy group then consists of matrices of the form

$$\begin{pmatrix} a & b & c \\ 0 & T & d \\ 0 & 0 & a^{-1} \end{pmatrix}$$

where $a \in \bar{k}$ is non-zero, $TAT^t = A$, $2ac + bAb^t = 0$, $bAT^t + ad^t = 0$ and hence is parametrized by (T, a, d) with $a \neq 0$ and $T \in SO(n-2, \bar{k})$.

(iii) Assuming inductively the connectedness of $SO(r, \bar{k})$ for $r<n$ we see that the isotropy group in (ii) is connected. Combined with (i) this yields the connectedness of $SO(n, \bar{k})$. Finally the induction starts trivially with $n=0, 1$.

Next we recall the Cayley map

$$S \mapsto C(S) = (1+S)(1-S)^{-1}$$

which transforms the skew-symmetric matrix into the (special) orthogonal matrix $C(S)$. Its image is the open set where $\det(1+C) \neq 0$. Since $SO(n)$ is connected it follows that $SO(n)$ is birationally equivalent (over k) to affine space and hence has sufficiently many points with coordinates in k: that is, for any extension K of k, a rational function on $SO(n, K)$ which vanishes on $SO(n, k)$ vanishes identically. Multiplying $C(S)$ by the matrix

$$J = \begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

extends this result to the whole of $O(n)$.

With these preliminaries out of the way we turn to the invariant theory. Let $\mathcal{M}(n, k)$, $\mathcal{S}\mathcal{M}(n, k)$ denote the $n \times n$ matrices and the symmetric $n \times n$ matrices over k. Then we shall prove the following key lemma:

Lemma. *Let* $f: \mathcal{M}(n, k) \to k$ *be a polynomial such that* $f(BA) = f(A)$ *for all* $B \in O(n, k)$. *Then there is a polynomial* $F: \mathcal{S}\mathcal{M}(n, k) \to k$ *such that* $f(A) = F(A^t A)$.

Proof. Observe first that f will retain its orthogonal invariance under any field extension K of k. In fact $(A, B) \mapsto f(BA) - f(A)$ is a polynomial function on $\mathcal{M}(n, K) \times O(n, K)$ vanishing on $\mathcal{M}(n, k) \times O(n, k)$. By the remarks made earlier this implies its vanishing identically. Now let K be the field $k(p_{ij})$ where $p_{ij} = p_{ji}$ are indeterminates representing the generic symmetric matrix P. Let L be the splitting field over K of the equation $\det(P - \lambda^2) = 0$. Thus in L we can construct all the eigenvalues $\lambda_1, \dots, \lambda_n$ of P and their square roots $\sqrt{\lambda_i}$. Note that the λ_i are distinct and so we can explicitly construct a symmetric matrix

$$Q = \sum_{i=1}^{n} \sqrt{\lambda_i} \prod_{j \neq i} \frac{P - \lambda_j}{\lambda_i - \lambda_j}$$

which is a square root of P: $Q^2 = P$. If σ is any element of the Galois group of L over K we also have $(\sigma Q)^2 = P$. Since Q and σQ are symmetric it follows that $B = \sigma Q Q^{-1}$ is orthogonal and so

$$\sigma f(Q) = f(\sigma Q) = f(BQ) = f(Q).$$

This holds for all σ and hence $f(Q) \in K = k(P)$, that is $f(Q) = \varphi(P)$ with φ a rational function. If we specialize P to have coefficients in k then this equation will continue to make sense provided we assume

i) P has distinct eigenvalues,

ii) denominator of $\varphi(P) \neq 0$.

If, in addition, we take P non-singular then every A with $A^t A = P$ lies in the $O(n, k)$-orbit of Q and hence the equation

$$f(A) = \varphi(A^t A)$$

holds for a Zariski open set of $A \in \mathcal{M}(n, k)$. It therefore holds for all $A \in \mathcal{M}(n, k)$. Moreover φ must be a polynomial F because if $\varphi = F/H$ in irreducible form, then choosing A such that $H(A^t A) = 0$, $F(A^t A) \neq 0$ gives $F(A^t A) = H(A^t A) f(A) = 0$ a contradiction.

Corollary. *Let $f: GL(n, k) \to k$ be a regular function invariant under $O(n, k)$, then $f(A) = F(A^t A)$ for some regular function F on the space of non-degenerate symmetric matrices.*

Proof. $f(A) = h(A)(\det A)^{-2N}$ for some N with $h(A)$ an invariant polynomial. Hence by the lemma

$$f(A) = H(A^t A)(\det(A^t A))^{-N}$$

for some polynomial H.

Remarks. 1. For $k = \mathbb{C}$ the holomorphic counterpart of this corollary follows at once from the existence of a local section of the map $A \mapsto A^t A$. Such a local section is given by the implicit function theorem or more explicitly by choosing the symmetric matrices near $A = 1$ (complementary to the Lie algebra of $O(n, C)$) and then translating by $GL(n, C)$. In the algebraic situation a Zariski-local section does not exist and instead we have to pass to a finite covering before a section can be constructed – hence the Galois theory.

2. Our lemma is just the special case of the main theorem for orthogonal invariants of n vectors in n-space. In fact, if we denote the columns of A by $a_1, \ldots, a_n$ then the (i, j)-th entry in $A^t A$ is just the scalar product $a_i \cdot a_j$.

We propose now to show how our lemma together with the main theorem for invariants of $GL(n)$ leads to the main theorem for $O(n)$. The point of this reduction is that the main theorem for $GL(n)$ has a variety of proofs, some quite simple and direct, whereas for $O(n)$ the only proof given in [24] involves the use of the mysterious Capelli identity.

Let us begin by recalling the theorem for the linear group[16]. It asserts that all linear maps

$$\overset{r}{\otimes} V \overset{s}{\otimes} V^* \to \mathbb{C}$$

[16] The following proof is extracted from [24, Chapter III].

invariant under $GL(V)$ are spanned by the complete contractions in which we pair off the V and V^* factors (this implies $r = s$). Since $V \otimes V^* \cong \operatorname{End}(V)$ an equivalent formulation is that the only endomorphisms of $\overset{r}{\otimes} V$ which commute with the action of $GL(V)$ are linear combinations of permutations. More formally, if we denote by $\mathcal{A}, \mathcal{B}$ respectively the images in $\operatorname{End}(\overset{r}{\otimes} V)$ of the group algebras of $GL(V)$ and of the symmetric group, the theorem asserts that $\mathcal{B}$ is equal to the commutator algebra $\mathcal{A}'$ of $\mathcal{A}$. Now the basic results on representations of finite groups assert that $\mathcal{B}$ is a direct sum of full matrix algebras and is equal to its double commutator: $\mathcal{B} = \mathcal{B}''$. Hence $\mathcal{A}' = \mathcal{B}$ is equivalent to $\mathcal{A} = \mathcal{B}'$. But $\mathcal{B}'$ is just the r-th symmetric power $S^r(\operatorname{End} V) \subset \overset{r}{\otimes} \operatorname{End} V$ and $\mathcal{A}$ is the subspace spanned by the diagonal elements $A \otimes A \otimes \cdots \otimes A$ $(A \in \operatorname{End} V)$. The equality $\mathcal{A} = \mathcal{B}'$ expresses the familiar fact that a symmetric multilinear function is uniquely determined by the associated polynomial. We see, therefore, that the main theorem for $GL(n)$ is an easy consequence of the double commutator theorem.

We return now to the main theorem for $O(n)$. Let $V = k^n$ and put $W = \overset{r}{\otimes} V$. We want to identify all the $O(n, k)$-invariant linear maps $\varphi: W \to k$. Given φ define $f: (\operatorname{End} V) \times W \to k$ by $f(A, \omega) = \varphi(A\omega)$ (where $A\omega$ denotes the action of $\operatorname{End} V$ on W). Since φ is $O(n)$-invariant we have

$$f(BA, \omega) = f(BA\omega) = \varphi(A\omega) = f(A, \omega) \qquad \text{for } B \in O(n, k).$$

Now we may regard f as a polynomial map

$$\operatorname{End} V \to W^*.$$

Applying the lemma to each component of this map it follows that

$$f(A, \omega) = F(A^t A, \omega)$$

for some polynomial F (linear in the variable ω). But from its definition we see that

$$f(AR^{-1}, R\omega) = f(A, \omega) \qquad \text{for } R \in GL(V)$$

and so

$$F((R^{-1})^t P R^{-1}, R\omega) = F(P, \omega),$$

where P here denotes a symmetric matrix. In other words F is a polynomial map

$$S^2(V^*) \times W \to k$$

invariant under $GL(V)$ and linear on W. We are now in a position to apply the main theorem for $GL(V)$. We deduce that $r = 2s$ is even. F is of degree s on $S^2(V^*)$ and is a linear combination of complete contractions. Explicitly F is a combination of the functions

$$(P, \omega) \mapsto (u_1^i\, p_{ij}\, v_1^j)\,(u_2^k\, p_{kl}\, v_2^l) \dots$$

where ω is the tensor product of $u_1, v_1, u_2, v_2 \dots u_s, v_s$ in some order (and u_1^i denotes the components of $u_1 \in V$). Evaluating F on the unit matrix $p_{ij} = \delta_{ij}$ we deduce that $\varphi(\omega) = f(1, \omega) = F(1, \omega)$ is a linear combination of complete contractions. This completes the proof of the main theorem for $O(n)$.

Appendix II

The Proof of Propositions 2.11 and 3.7

For the sake of completeness we append here a proof of the above differential geometric propositions. We start with 3.7 because it is the simpler of the two.

Our task is therefore to compute, up to arbitrary high order, the components Γ of a connection D on E at a point $p \in M$ in terms of the corresponding information for the curvature K of D at p, provided these components are computed relative to a synchronous frame s for E at p. Now the formulas expressing the synchronism of s with the coordinates x centered at p are as follows:

Let θ_j^i be the connection form, for D relative to s, that is, near p:

$$D s_i = \theta_j^i s_j.\text{[17]}$$

Also let $\mathscr{R}$ denote the radial vector field in the x coordinates:

$$\mathscr{R} = x^i \frac{\partial}{\partial x^i}.$$

Then the synchronism of s with x is simply expressed by the condition that the inner product of θ_j^i with $\mathscr{R}$ vanish:

(a 1) $$\iota(\mathscr{R})\, \theta_j^i \equiv 0.$$

Now the x-components of Γ are given by

(a 2) $$\theta_j^i = \Gamma_{jk}^i\, dx^k,$$

and those of the curvature K by

(a 3) $$d\theta_j^i - \theta_k^i \wedge \theta_j^k = K_{jkl}^i\, dx^k \wedge dx^l$$

with K_{jkl}^i skew in k and l.

[17] We continue to use the convention that repeated indices are summed.

To relate them, recall that the Lie derivative along $\mathscr{R}$, — again to be denoted by $\mathscr{R}$ — acts on the algebra of forms via the relation

(a 4) $$\mathscr{R} = \iota(\mathscr{R})\, d + d\, \iota(\mathscr{R}).$$

Hence, by (a 1) and (a 4) we have

(a 5) $$\begin{aligned}\mathscr{R}\, \theta_j^i &= \iota(\mathscr{R})\, d\theta_j^i = \iota(\mathscr{R})\, (d\theta_j^i - \theta_k^i \wedge \theta_j^k)\\ &= 2 x^k K_{jkl}^i\, dx^l.\end{aligned}$$

Next, applying the derivation $\mathscr{R}$ to (a 2) one finds the relation:

$$\mathscr{R}\, \theta_j^i = (\mathscr{R}\Gamma_{jk}^i)\, dx^k + \Gamma_{jk}^i\, dx^k,$$

so that equating coefficients we obtain:

(a 6) $$\mathscr{R}\Gamma_{jk}^i + \Gamma_{jk}^i = 2 x^l K_{jlk}^i.$$

This relation now immediately yields Proposition (3.7). Explicitly let us write $\hat{\Gamma}$ and $\hat{K}$ etc., for the formal Taylor series relative to x about p of the function indicated, and $\hat{\Gamma}[n]$, $\hat{K}[n]$ etc. for the term of homogeneity n in this expansion. Then by Euler's formula $\mathscr{R}$ preserves these components and multiplies $\hat{\Gamma}[n]$ by n. Hence (a 6) gives rise to the explicit formula:

$$(n+1)\, \hat{\Gamma}_{jk}^i[n] = 2 x^l \hat{K}_{jlk}^i[n-1],$$

for the Taylor series of Γ in terms of the Taylor series for K. Q.E.D.

A similar argument yields the more familiar Proposition (2.11). Here the bundle in question is the tangent bundle of M, and the connection the unique torsion free connection which preserves the given Riemann structure g.

Now our problem is to express the Taylor-components at p of g, in terms of those of the curvature — which we now denote by R — provided a canonical coordinate system centered at p is used.

Let then x be such a coordinate system, and let s_i be the orthonormal frame obtained from $\left.\frac{\partial}{\partial x^i}\right|_p$ by parallel transport along the radial geodesics through p. The dual frame to $\{s_i\}$ is therefore a frame of 1-forms $\{\theta^i\}$ well defined on M near p.

As before the connection form of g relative to $\{s_i\}$ will be denoted by θ_j^i and the radial field by $\mathscr{R}$.

The geometric assumptions now translate into the following formulae:

(a 7) $$\iota(\mathscr{R})\, \theta^i = x^i \qquad \iota(\mathscr{R})\, \theta_j^i = 0 \qquad g_{ij}\, dx^i \otimes dx^j = \theta^\alpha \otimes \theta^\alpha.$$

The first two follow from the fact that the radial lines are geodesics and that the frame s is parallel along these. The last expresses the orthonormality of the frame s. We now introduce the functions a^i_j relating the frames $\{\theta^i\}$ and dx^i:

$$\theta^i = a^i_j\, dx^j.$$

In terms of these we clearly have

$$g_{ij} = a^\alpha_i a^\alpha_j$$

so that it is sufficient to determine the a's in terms of the R's.

To this end we again apply the derivation $\mathscr{R}$ but this time twice. We will also need the radial function r, defined by

$$r^2 = x^i x^i.$$

Finally observe that then the expressions x^i/r, or dx^i/r are homogeneous of degree zero in r, and hence annihilated by $\mathscr{R}$.

Applying $\mathscr{R}$ to θ^i, yields

(a 8)
$$\begin{aligned}\mathscr{R}\,\theta^i &= \iota(\mathscr{R})\, d\theta^i + d\,\iota(\mathscr{R})\,\theta^i\\ &= \iota(\mathscr{R})\, d\theta^i + dx^i.\end{aligned}$$

On the other hand, the torsion-freeness of the connection is expressed by:

(a 9)
$$d\theta^i = \theta^i_j \wedge \theta^j.$$

Hence (a 8) goes over into

$$\begin{aligned}\mathscr{R}\,\theta^i &= \iota(\mathscr{R})\,\theta^i_j \wedge \theta^j + dx^i\\ &= -\theta^i_j x^j + dx^i.\end{aligned}$$

We next apply the operator $r\cdot\mathscr{R}\cdot 1/r$, to obtain

$$\begin{aligned}r\cdot\mathscr{R}\cdot\frac{1}{r}\cdot\mathscr{R}\,\theta^i &= -\iota(\mathscr{R})\, d\theta^i_j x^j\\ &= -2R^i_{jkl}\, x^j x^k\, dx^l.\end{aligned}$$

On the other hand the left-hand side can be computed in terms of the a's yielding

$$\mathscr{R}\,\theta^i = (\mathscr{R}\,a^i_j + a^i_j)\, dx^j$$

$$r\cdot\mathscr{R}\cdot\frac{1}{r}\cdot\mathscr{R}\,\theta^i = (\mathscr{R}^2 a^i_j + \mathscr{R}\,a^i_j)\, dx^j.$$

Hence equating coefficients one obtains the relation

(a 10)
$$(\mathscr{R}^2 + \mathscr{R})\, a^i_l = -2R^i_{jkl}\, x^j x^k.$$

Finally, applying the Taylor series construction, this translates into:

(a 11)
$$(n^2+n)\,\hat{a}^i_l[n] = -2x^j x^k \hat{R}^i_{jkl}[n-2],$$

which explicitly yields $\hat{a}^i_l[n]$ in terms of $\hat{R}$ for all $n>0$. For $\hat{a}^i_j[0]$ we of course have the identity δ^i_j by construction, so this relation explicitly gives the dependence of a on R — and hence also implicitly that of g on R. Q.E.D.

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A Flatness Criterion in Grothendieck Categories

Friedrich Ulmer (Zürich)

Let $\mathfrak{A}$ be an abelian category in which filtered direct limits are exact, i.e. a Grothendieck AB 5) category [4]. Let $U \in \mathfrak{A}$ be an object, $\Lambda = [U, U]$ its endomorphism ring and denote with $\mathfrak{Mod}_\Lambda$ the category of right Λ-modules. For each $X \in \mathfrak{A}$ the ring Λ acts on the group $[U, X]$ of all morphisms $U \to X$ by means of composition, and the lifted hom-functor $[U, -]: \mathfrak{A} \to \mathfrak{Mod}_\Lambda$ has a left adjoint $\otimes_\Lambda U: \mathfrak{Mod}_\Lambda \to \mathfrak{A}$. Let $n \geqq 1$ and $U_i = U$ for $i = 1, 2, \ldots, n$. The aim of this note is to show that $\otimes_\Lambda U: \mathfrak{Mod}_\Lambda \to \mathfrak{A}$ is an exact functor iff the kernel of every morphism $\bigoplus_{i=1}^{n} U_i \to U$ is an epimorphic image of a suitable direct sum of copies of U. This generalizes the theorem of Gabriel-Popescu [3] asserting the exactness of $\otimes_\Lambda U$ provided U is a generator in $\mathfrak{A}$. (In this case, the evaluation morphism

$$[U, X] \otimes_\Lambda X \xrightarrow{\cong} X$$

is an isomorphism for each $X \in \mathfrak{A}$ and thus the canonical functor $\mathfrak{A} \xrightarrow{\cong} \mathfrak{Mod}_\Lambda/\ker \otimes_\Lambda U$ is an equivalence.) The flatness criterion given here – an outgrowth of [11] which took its final form in a seminar held at the ETH in the summer semester 1971 – cannot be obtained by adapting the localization techniques of Gabriel-Popescu [3] or the method of Takeuchi [10]. For if $U \in \mathfrak{A}$ is flat over its endomorphism ring but not a generator, then the full subcategory of all $X \in \mathfrak{A}$ with the property

$$[U, X] \otimes_\Lambda X \xrightarrow{\cong} X$$

plays the rôle of $\mathfrak{A}$, and this subcategory need not be a Grothendieck AB 5) category. A counter example is given by any torsion free abelian group U which is complete in the $\mathbb{Z}$-adic topology (cf. example b)). If $\mathfrak{A}$ is a category of modules over a ring R, then the above criterion coincides with that of Cartan-Eilenberg [2], p. 123 (resp. Bourbaki [1], Chapt. I, § 2, No. 11, Cor. 1, Prop. 13) and it also implies directly that of Lazard [5], i.e. $U \in \mathfrak{A}$ is flat over Λ iff it is a filtered direct limit of finitely generated free Λ-modules. By a slight modification one can obtain Lazard's and Cartan-Eilenberg's characterization also for U as an R-module. Some examples and counter examples which illustrate the criterion can be found at the end.

Inventiones math. 28, 277–280 (1975)

Andrew Ranicki

Errata to the Paper:

On the Heat Equation and the Index Theorem

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The joint paper of the above title which appeared in Inventiones math. **19**, 279–330 (1973), though correct in principle, contained some technical errors which we shall here explain and rectify. Our thanks are due to D. Epstein, Y. Colin de Verdière and A. Vasquez whose computations and queries alerted us to our errors.

1. The Notion of Regularity

The main error occurs on page 306 where it is implicitly assumed that the coefficients of the two operators A^*A and AA^* (associated to the signature operator A) are polynomial functions in the g_{ij}, their derivatives and $(\det g)^{-1}$. As we shall show later this is not quite true – the coefficients also involve $\sqrt{\det g}$ and the inverses of the principal minors of the matrix g_{ij}. Thus the form ω in (5.1) is *not* a regular invariant of the metric in the sense of § 2, and so the Gilkey Theorem as formulated on p. 284 does not apply.

To correct this we shall widen the notion of regularity (so as to include, in particular, the form ω above) and then check that our proof of Gilkey's Theorem still holds in this wider context.

In § 2 regularity was only defined for *invariants* of a Riemann structure g (i.e. satisfying the naturality or invariance property (2.3)). It will perhaps make for greater clarity if we introduce our new notion of regularity for *any* function of g, independently of the invariance property. We shall say that $f(g)$ is a regular function of g if, in any coordinate system, we have

$$f(g)(x)=\sum_\alpha a_\alpha(x, g(x))\, m_\alpha \qquad \text{(finite sum)}$$

where $a_\alpha(x, y)$ are C^∞ functions and m_α denotes a monomial in the partial derivatives of $g(x)$. Here $g(x)$ stands of course for the classical components $g_{ij}(x)$ relative to the basis dx^i given by the coordinates $(x_1, \ldots, x_n)$. Clearly regularity is a local property and it has only to be checked in one coordinate system. The essential difference between this definition and that of p. 282–284 is that we now allow C^∞ dependence on g and do not insist on polynomial dependence on g and g^{-1}. Another less significant difference is that we now allow the coefficients a_α to depend also on x. If f is both regular *and* invariant then this dependence is illusory – in fact translation invariance alone shows the a_α must be independent of x. For a differential form regularity is defined in terms of regularity of its components relative to the usual basis $dx^{i_1}\wedge\cdots\wedge dx^{i_p}$.

2. Proof of the Gilkey Theorem

The proof of the Gilkey Theorem given in §2 is a "point-wise" proof using geodesic coordinates. For this reason the C^∞ dependence on g introduced in our new definition of regularity is quite innocuous. In more detail let $\omega(g)$ be a regular form-valued invariant of g in our new sense. Then in $\mathbb{R}^n$ each component $\omega_\beta(g)$ is given by an expression:

$$\omega_\beta(g)[x] = \sum a_\alpha^\beta(x, g(x))\, m_\alpha$$

which we call the universal polynomial of ω. To evaluate $\omega(g)$ at a point p of a given Riemannian manifold M we choose a geodesic coordinate system centered at p and interpret it as a map of an ε-ball about 0 in $\mathbb{R}^n$

$$f\colon \mathbb{R}^n_\varepsilon \to M$$

sending 0 to p. The invariance property of ω implies that $f^*(\omega(g)_p) = \omega(f^*g)_0$. Applying our universal polynomial to the right-hand side now yields

$$f^*(\omega_\beta(g)_p) = \sum_\alpha a_\alpha^\beta(0, f^*g|_0)\, m_\alpha(f^*g)_0$$

and as the coordinates are geodesic for f^*g at 0 we see that $f^*g|_0$ is the unit matrix. Furthermore we may now apply Proposition (2.1) to $f^*(g)$ so that the monomials $m_\alpha(f^*g)_0$ are given by polynomials in the curvature of f^*g and its covariant derivatives (at 0). In this way we arrive at formula (2.17) and the proof now proceeds as before.

3. The Signature Operator

The operators A^*A and AA^* of p. 306 are just the restrictions of the Hodge Laplacian Δ to the subspaces $\Omega_\pm$ of Ω (the space of all forms). Certainly Δ, relative to the usual basis of the $dx^{k_1} \wedge \cdots \wedge dx^{k_r}$, has coefficients which are polynomial in g, its derivatives and $(\det g)^{-1}$. However this basis is not compatible with the decomposition $\Omega = \Omega_+ \oplus \Omega_-$ (determined by the eigenspaces of $*$). This difficulty is a consequence of the fact that $\Lambda^k(T^*)$ is associated to the principal tangent bundle via a representation of $GL(n, \mathbb{R})$, whereas $\Omega_\pm$ are associated only to the principal $SO(n)$-bundle (determined by g) via a representation of $SO(n)$ which is *not the restriction of a $GL(n, \mathbb{R})$ representation.*

The upshot is that we have to resort to an *adhoc framing* of $\Omega_\pm$, which can be constructed as follows. Let $\phi^1, \ldots, \phi^n$ be an orthonormal frame of T^* obtained from the $dx^1, \ldots, dx^n$ by applying the Gramm-Schmidt procedure. In terms of these ϕ's the $*$ operator and the corresponding τ operator defined by

$$\tau\alpha = i^{p(p-1)+l} * \alpha \qquad \text{for } \alpha \in \Omega^p,$$

is especially simple. Indeed if $\phi^K = \phi^{k_1} \wedge \cdots \wedge \phi^{k_r}$ is an exterior monomial then

$$\tau\phi^K = \sigma(K) \cdot \phi^{L}$$

where L denotes the complementary monomial and $\sigma(K)$ is ± 1, when $n/2 = l$ is even, and $\pm i$ when l is odd. It follows that if Φ_n denotes the subspace of Ω generated by $\phi^1, \ldots, \phi^{n-1}$ then $\tau\Phi_n \subset \phi^n \wedge \Phi_n$, so that in particular

We may therefore frame Ω_+, with the forms $(\phi^K + \tau\phi^K)/\sqrt{2} = \phi^K_+$ where ϕ^K does not involve ϕ^n, and similarly frame Ω_- by $\phi^K_- = (\phi^K - \tau\phi^K)/\sqrt{2}$. Furthermore the ϕ^K_+, ϕ^K_- together give rise to an orthonormal framing of Ω.

Now the ϕ's are related to the dx's by a triangular matrix

$$\phi = T\,dx$$

whose coefficients are C^∞ functions of the g_{ij} but are not just polynomials in the g_{ij} and $\det g^{-1}$. Indeed here square roots of $\det g$ and inverses of principal minors of g will appear.

In any case relative to the frame ϕ^K_+, ϕ^K_- the operators d and d^* will have regular coefficients (in our new sense) and therefore $(d + d^*)^2$ also. But in this frame the operators A^*A and AA^* just correspond to the "diagonal" parts of this matrix operator and hence still have regular coefficients. Moreover their leading terms are (in any base) the scalar operator

$$-\sum g^{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

Since regular functions of g are closed under multiplication and under differentiation by C^∞ vector fields the Seeley formula (4.40) applied to AA^* and A^*A shows that their heat expansion coefficients are regular functions of g. The form ω appearing in (5.1) is therefore a regular invariant of g and so we can apply the Gilkey Theorem and proceed as before.

4. Other Operators

For the generalized signature operators A_ξ of §6 the argument is quite analogous. The definition of regularity in §3 is widened in a similar manner by allowing polynomials in the variables in (3.4) to have coefficients depending on g and (for functions not necessarily invariant) on x. The generalized Gilkey Theorem (Theorem II on p. 290) is still true and can be applied to the form $\omega(g, \xi)$ in (6.1) as before.

The Dirac operator B on p. 314–315 presents essentially the same features as the signature operator. To write its coefficients out explicitly we must first choose an orthonormal base of the two Spin bundles E^+ and E^-. Since the Spin representations are not representations of $GL(n, \mathbb{R})$ a local coordinate system on M does not automatically give rise to such a base, so we must again use the Gramm-Schmidt process to orthogonalize the dx^i. The coefficients of B are then regular functions of g as before while the leading terms in BB^* and B^*B are scalar and given by $-\sum g^{ij} \dfrac{d^2}{\partial x_i \partial x_j}$.

5. Corollary on Page 303

A second error occurs in the last part of the Corollary on p. 303. The statement that in the quadratic case $\mu_k(A)$ is a polynomial is incorrect and should be modified by replacing $\mu_k(A)$ with $(\det a)^{\frac{1}{2}} \mu_k(A)$. The error crept in through a wrong sign on p. 305 where, after the change of variable $\xi = g^{-1}\xi'$, we wrote down

$$\hat{f}(g) = (\det g)^{-1} \int p(g^{-1}\xi)\,\omega(\xi)$$

instead of the correct expression:

$$\hat{f}(g) = |(\det g)|^{-1} \int_{S^{n-1}} p(g^{-1}\xi)\,\omega(\xi)$$

It follows that it is $|\det g|\,\hat{f}(g)$ which is in the coordinate ring of $GL(n, \mathbb{R})$ rather than $\hat{f}(g)$.

This alteration does not now affect the rest of the paper in view of our widened definition of regularity. Incidentally the necessity for the factor $(\det a)^{\frac{1}{2}}$ is at once seen by considering the case $A = d + d^*: \Omega^{\text{even}} \to \Omega^{\text{odd}}$ which leads to the Gauss-Bonnet formula. For the signature operator the square root is eventually cancelled out by another factor $(\det a)^{\frac{1}{2}}$ occurring in the coefficients of A (as observed in (3) above), which explains why the Hirzebruch formula for the signature is rational in the g_{ij}.

6. Appendix II

There is an unfortunate confusion of notation on p. 328 which affects the precise recurrence formula (a 11) but does not vitiate the main conclusion. Precisely the lower line of the formula

$$\begin{aligned} r \cdot \mathscr{R} \cdot \frac{1}{r} \cdot \mathscr{R}\theta^i &= -i(\mathscr{R})\,d\theta^i_j\,x^j \\ &= -2R^i_{jkl}\,x^j x^k\,dx^l \end{aligned}$$

is wrong if R^i_{jkl} is to have its standard meaning, that is, if the curvature matrix of g relative to the frame $\partial/\partial x^i$ is to be given by

$$\tfrac{1}{2} R^i_{jkl}\,dx^k\,dx^l.$$

The correct expression is obtained by replacing the $-2R^i_{jkl}$ of our paper with

$$a^i_\alpha\, b^\beta_j\, R^\alpha_{\beta kl}.$$

where b is the inverse matrix to a. Indeed the a's and b's correct for the switch of frames: $\theta^i \to dx^i$, while the minus sign corrects for the switch $dx^i \to \frac{\partial}{\partial x^i}$ and the 2 is cancelled by 1/2 above.

Correcting (a 10) and (a 11) correspondingly we obtain the recursion (a 11)

$$(n^2+n)\,\hat{a}^i_l[n] = x^j x^k \{\widehat{R^\alpha_{\beta jkl}}\, a^i_\alpha\, b^\beta_j\}[n-2]$$

which still serves to determine the $\hat{a}$'s in terms of the $\hat{R}$'s.

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Examples

Bombieri, E., Giusti, E.: Harnack's inequality for elliptic differential equations on minimal surfaces. Inventiones math. **15**, 24–46 (1971).

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