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Eta invariants, signature defects of cusps, and values of L-functions

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Introduction

The purpose of this paper is to prove a conjecture of Hirzebruch [11] which gives a topological meaning to certain values of L-functions arising in totally real number fields. This conjecture was based on the very detailed investigation made by Hirzebruch for the case of real quadratic fields, and hinged on the fine

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structure of the cusp singularities of the Hilbert modular surfaces. It may therefore be helpful to recall the motivation and background of the conjecture.

Hirzebruch was interested in classifying these algebraic surfaces and, as a first step, it was necessary to compute their topological invariants. More precisely he wanted to find the Todd (arithmetic) genus and signature of their desingularizations. In the absence of any singularities, the Hirzebruch signature theorem would identify the signature with $\frac{1}{3}p_1$ where p_1 was the first Pontrjagin number, and this in turn could be computed by a suitable curvature integral. In fact, using the standard Poincaré metric on H this curvature integral turns out to be zero, so that the signature would be zero *if there were no singularities*. It turns out that one can actually assign a rational number to each isolated singularity of our surface which Hirzebruch calls the *signature defect*, because it measures the correction which this singularity produces in the signature theorem.

At this stage we should recall that the Hilbert modular surface attached to a real quadratic field K is obtained by dividing H^2 (where H is the upper half plane) by the discrete group $\mathrm{SL}(2, \mathcal{O})$ where $\mathcal{O}$ is the ring of integers of K. This surface is then compactified by adding a finite number of points "at infinity"—the cusps. The cusps are singular points and there are also internal singularities which are of elliptic type; that is, they arise from finite isotropy groups. The signature defect of such elliptic singularities is well understood in terms of the general G-signature theorem (which generalizes the Hirzebruch theorem to allow for finite or compact group actions G). The signature defect of the cusps is however a much more delicate affair.

Hirzebruch was able to compute the signature defect of the cusps because he found a simple and beautiful explicit resolution of these singularities. This resolution depended on the periodic continued fraction expansion of quadratic irrationals and the signature defect was then given by a simple rational formula involving the integers of this continued fraction.

If one turns from these geometric considerations to the more traditional number theory, the cusps correspond to ideal classes and to each such ideal class one can associate an L-function. These L-functions and their generalizations have been studied by Shimizu [22]. Classical methods for computing the value of these L-functions at $s = 1$ lead to the explicit formulae which (up to constant factors) coincide with the formulae for the signature defects of the cusps computed by Hirzebruch.

For totally real fields of higher degree the Hilbert modular variety is of higher dimension and the geometry of the cusp singularities is much more complicated. Explicit formulae are therefore not in general available but, based on the quadratic case, Hirzebruch conjectured that the signature defects of the cusps should still be given by values at $s = 1$ of the corresponding L-functions.

Clearly to prove this conjecture in general it is necessary to obtain a more direct connection between the L-function and the signature defect. The attempt to understand this connection was one of the main motivations leading to the results of Atiyah-Patodi-Singer [3], which extend Hirzebruch's signature theorem to the case of manifolds with boundary. Let us briefly recall the formulation of this generalized signature theorem. Let X be a $4k$-dimensional compact oriented Riemannian manifold with boundary Y and assume that, near Y, it is isometric to the product $Y \times R$. Then the signature of X is given by the formula

$$\operatorname{sign}(X) = \int_X L(p) - \eta(0).$$

Here $L(p)$ is the actual Hirzebruch L-polynomial in the Pontrjagin forms, while $\eta(0)$ is the value for $s = 0$ of an analytic function $\eta(s)$ depending only on Y (with its metric). Moreover $\eta(s)$ is defined in terms of the eigenvalues λ of a certain elliptic self-adjoint operator A on Y (up to sign A is $(*d - d*)$ on even forms) by

$$\eta(s) = \sum_{\lambda \neq 0} (\operatorname{sign} \lambda)|\lambda|^{-s}.$$

Thus $\eta(0)$ may be viewed as the differential geometric "signature defect" of the boundary, while its analytic expression is clearly analogous to that of an L-function.

With this theorem as our starting point we can now revert to Hirzebruch's conjecture taking X to be the Hilbert modular variety with the cusps chopped off (and the elliptic singularities temporarily ignored). The boundary Y of X then has one component Y_j for each cusp and this component has a fairly simple structure being a torus bundle over a torus: if $\dim K = 2k$ then the fibre torus has dimension $2k$ while the base torus has dimension $2k - 1$. One can then give Y_j a rather natural metric and consider the corresponding invariant $\eta_j(0)$.

By fairly direct although lengthy analysis we shall identify $\eta_j(0)$ with $L_j(0)$ where $L_j(s)$ is the Shimizu L-function associated to the j-th cusp. Moreover we shall show that the topological signature defect as defined by Hirzebruch and our differential geometric signature defect coincide in these cases. Essentially this means we have to compare two different connections on Y_j, namely the Riemannian connection and a flat connection (given by group invariance), and show that they yield the same value for $\eta_j(0)$. In this way we finally obtain a proof of Hirzebruch's conjecture, except that our method leads naturally to the value $L(0)$, which however coincides (up to a factor) with $L(1)$ because of the standard functional equation for L-functions.

Since the results of [3] were available ten years ago and were motivated in part by Hirzebruch's conjecture, one might ask why it has taken so long to settle the conjecture. In fact the details for the case of quadratic fields were worked out by two of us (Atiyah and Singer) around 1974, but even here the identification of $\eta(0)$ with $L(0)$ was quite lengthy. The technical difficulties in extending this to the general case were substantial. To a great extent this is because the analysis on the manifolds Y_j gets reduced to that of a linear differential operator on R^{2k-1} (the universal covering of the base of the fibration of Y_j). When $k = 1$ (the quadratic case), $R^{2k-1} = R$ and we have an ordinary differential operator, whereas in general we have a partial differential operator and the analysis becomes more involved. The length of the present paper arises mainly from these extra technical difficulties.

The paper is divided in three parts. Part I gives the precise definitions and formulates the main theorems. Part II which occupies the bulk of the paper identifies the value $L(0)$ with the eta invariant $\eta(0)$ arising from the flat connection. Part II begins with Section 5 which gives an outline of the proof with an indication of the various technical difficulties involved. Finally in Part III we show that the flat connection and the Riemannian connection have the same eta invariants so that the main result of [3] can be applied.

We understand that W. Müller has also developed a proof of Hirzebruch's conjecture, along similar but not identical lines.

1. The L-series of Shimizu

In this section we summarize some basic facts from algebraic and analytic number theory. Our main purpose is to give a precise definition of the L-functions which occur in the conjecture of Hirzebruch [11]. These L-series were studied earlier by Shimizu [22]. For more details on number theoretic background material, the reader may consult [11], [13], and [16].

Let K be a totally real algebraic number field of degree $2k$ over the rationals. There are $2k$ different embeddings of K into the reals and these embeddings will be denoted by $x \to x_j$, $j = 1, \dots, 2k$. We may assume that $x = x_1$. An element $x \in K$ is said to be totally positive when $x_j > 0$ for all j.

Suppose that M is a lattice in K. In particular, M is an additive subgroup of K which is free abelian of rank $2k$. Denote U_M^+ to be the subgroup of those units ε which are totally positive and satisfy $\varepsilon M = M$. The group U_M^+ is free abelian of rank $2k - 1$ [11, p. 200]. The symbol V will represent a subgroup in U_M^+ having finite index. If M is the lattice of all algebraic integers in K, then one may take $V = U^+$, the group of all totally positive units.

Given a pair (M, V), as above, one defines Shimizu's L-series as

$$L(M, V, s) = \sum_{\mu \in \frac{M - \{0\}}{V}} \frac{\operatorname{sign} N(\mu)}{|N(\mu)|^s}. \tag{1.1}$$

Here $N(\mu) = \mu_1\mu_2 \ldots \mu_{2k}$. Of course, the norm $N(\mu)$ does not change when one multiplies μ by a totally positive unit. The L-series (3.1) converges absolutely for $\mathrm{Re}(s) > 1$ and admits a holomorphic continuation to the entire complex plane [11, p. 230].

The rational bilinear form $\mathrm{Tr}(xy) = x_1y_1 + \cdots + x_{2k}y_{2k}$ is positive definite and non-degenerate on K [16, p. 40]. Here K is regarded as a vector space of dimension $2k$ over Q. Let M' be the dual lattice of M, with respect to Tr. Since M' is also V-invariant, the series $L(M', V, s)$ is well-defined.

The functional equation for Shimizu's L-function is easily derived using standard methods [13, pp. 254–258], [9]. Suppose that $\mathrm{Vol}(M)$ is the volume of a fundamental domain for the lattice M, with respect to the measure induced by Tr. One may define

$$H(M, V, s) = \left[\Gamma\left(\frac{s+1}{2}\right)\right]^{2k} \pi^{-k(s+1)}[\mathrm{Vol}(M)]^s L(M, V, s).$$

Then the functional equation reads

$$H(M, V, s) = (-1)^k H(M', V, 1 - s). \tag{1.2}$$

In particular, setting $s = 1$ in (1.2), one has the relation

$$L(M', V, 0) = (-1)^k \mathrm{Vol}(M) \pi^{-2k} L(M, V, 1). \tag{1.3}$$

Hirzebruch's conjecture was originally stated [11, p. 230] using the values $L(M, V, 1)$. Equation (1.3) shows that our main result, Theorem 4.1, gives an equivalent reformulation.

2. Algebraic construction of certain framed manifolds

We will now describe certain framed manifolds (X, f). In fact, X will be a solvmanifold and f the framing pushed down from a left invariant framing on a solvable group which covers X. The motivation for considering these particular manifolds arises in the work of Hirzebruch [11]. In fact, such X are obtained by slicing along the cusps of the generalized Hilbert Modular Varieties associated to totally real algebraic number fields.

Let (M, V) be a pair as in Section 1. The lattice M is mapped injectively into R^{2k} by sending $m \to (m_1, m_2, \ldots, m_{2k})$. Recall that $m \to m_i$ correspond to the different embeddings of K into the real line R. Since each $v \in V$ is a totally positive unit, one has $v_1v_2 \ldots v_{2k} = 1$. Moreover, V acts on M in this representation by componentwise multiplication. Sending $v_i \to \log v_i$ identifies V with an additive subgroup of rank $2k - 1$ in R^{2k-1}. Here R^{2k-1} is realized as a hyperplane through the origin, $\Sigma \log v_i = 0$, in R^{2k}. This allows one to extend the action of V on M to an action of R^{2k-1} on R^{2k}.

Since V acts on M, one may form the semidirect product $S(M, V)$ of the abelian groups M and V. The above remarks show that $S(M, V)$ embeds naturally as a discrete subgroup in the two term solvable Lie group $S(R^{2k}, R^{2k-1})$.

Corresponding to the exact sequences:

$$0 \to M \to S(M,V) \to V \to 0, \tag{2.1}$$
$$0 \to R^{2k} \to S(R^{2k}, R^{2k-1}) \to R^{2k-1} \to 0,$$

there is a quotient sequence of coset spaces

$$0 \to T^{2k} \to X \to T^{2k-1} \to 0 \tag{2.2}$$

where $X = S(M,V) \setminus S(R^{2k}, R^{2k-1})$. This represents the quotient space X as a $2k$-torus bundle over the $(2k-1)$-torus. Here $T^{2k} = M \setminus R^{2k}$ and $T^{2k-1} = V \setminus R^{2k-1}$.

The coset space X inherits a natural framing f on its tangent bundle, pushed down from the left invariant framing on the Lie group $S(R^{2k}, R^{2k-1})$.

3. The eta invariants of framed manifolds

In [3] a study was made of certain extensions of the Index Theorem for elliptic operators to manifolds with boundary. A general investigation was undertaken in the context of elliptic operators. Moreover, the special case of the signature operator for Riemannian manifolds was discussed in detail.

For the present work, it is necessary to examine more carefully specific elliptic operators for manifolds W with framed boundary (X, f). These operators have the same leading symbol but are different from the analogous signature operators for the underlying Riemannian structure, associated to the framing f. By applying and extending the work in [3], we will relate certain topological and spectral invariants of the framed manifolds (X, f).

The goal of the current section is to define precisely the spectral and topological invariants under consideration. The spectral invariants will be denoted by $\eta_A(0)$ and the topological invariants by $\sigma(X, f)$.

First, we proceed to describe the spectral invariants $\eta_A(0)$. Suppose that (X, f) is any framed manifold of dimension $4k-1$. The framing f defines a flat connection ∇ on the tangent bundle of X. There are induced connections on the associated bundles of exterior algebras $\Lambda^p T^*X$. Let d be the skewed covariant differential associated to this framing. This means that d is given by the composition:

$$\Gamma(\Lambda^p T^*X) \xrightarrow{\nabla} \Gamma(\Lambda^p T^*X \otimes T^*X) \to \Gamma(\Lambda^{p+1} T^*X)$$

where the second map is exterior multiplication. Denote $*$ to be the Hodge star operator obtained by regarding the framing as an orthonormal basis at each point.

The operator A defined for $\phi \in \Gamma(\Lambda^{2p}T^*X)$ by $A\phi = (-1)^{k+p+1}(*d - d^*)\phi$ is self-adjoint and elliptic acting on $\Gamma(\Lambda^{\mathrm{ev}}T^*X) = \sum_p \Gamma(\Lambda^{2p}T^*X)$, the forms of even degree. Since M is compact, A has pure point spectrum consisting of real eigenvalues λ.

One may define the eta function

$$\eta_A(s) = \sum_{\lambda \neq 0} \frac{\operatorname{sign} \lambda}{|\lambda|^s}, \qquad \operatorname{Re}(s) \gg 0, \tag{3.1}$$

where $\operatorname{sign} \lambda = \lambda/|\lambda| = \pm 1$. Standard arguments using the Mellin transform show that the η-series converges for $\operatorname{Re}(s)$, the real part of s, sufficiently large. Moreover, η has a meromorphic continuation to the entire complex plane with $\eta_A(0)$ finite [4]. The quantity $\eta_A(0)$ is our basic spectral invariant.

The topological ingredients will now be described. Again, we suppose that (X, f) is any framed manifold of dimension $4k - 1$. The Pontrjagin and Stiefel-Whitney classes of X vanish, so there is a compact oriented manifold W with $\partial W = X$.

Since X is framed, the tangent bundle of W is pulled back from a bundle over the quotient space W/X. This allows one to define the Pontrjagin classes of W as relative classes $p_i \in H^{4i}(W, X)$. Let $L_k(p_1, \ldots, p_k) \in H^{4k}(W, X)$ be the Hirzebruch L-polynomial in the relative Pontrjagin classes.

Our basic diffeomorphism invariant is

$$\sigma(X, f) = L_k(p_1, \ldots, p_k)[W, X] - \operatorname{sign} W,$$

the signature defect. Here $[W, X] \in H_{4k}(W, X)$ is the fundamental class and $\operatorname{sign} W$ is the signature of W. By applying the Hirzebruch signature theorem and the Novikov additivity of the signature [3], one sees that $\sigma(X, f)$ depends only on X and the framing f, but not on the choice of W.

4. Relationship between L-series and eta invariants

The basic definitions and background material have been summarized above in Sections 1–3. Given this preparation, we may now state our main result which identifies the value at $s = 0$ of the Shimizu L-function with a signature defect:

THEOREM 4.1. *Let (M, V) be a pair as described in Section 1, and (X, f) the associated framed manifold of Section 2. Then*

$$L(M', V, 0) = \eta_A(0) = \sigma(X, f).$$

The proof of Theorem 4.1 breaks into two distinct parts both in terms of the concepts and techniques involved. For the purposes of exposition, it is suitable to divide our main result into two separate theorems.

Our first goal will be to relate L-functions to the analytic eta invariant:

THEOREM 4.2.

$$L(M', V, 0) = \eta_A(0).$$

The proof of Theorem 4.2 will be presented in Part II.

In Part III, the proof of Theorem 4.1 will be completed relating the analytic eta invariant to the signature defect.

THEOREM 4.3.

$$\eta_A(0) = \sigma(X, f).$$

Theorem 4.3 is essentially an application of the main result of [3] although some extra work is needed to deal with connections having non-zero torsion. Theorem 4.2 involves a direct comparison of the two analytic functions $L(s)$ and $\eta(s)$. There are a number of algebraic and analytic complications which require detailed treatment, so that Part II is in fact the longest and most substantial part of the paper.

In particular, Theorem 4.1 implies the rationality of $L(M', V, 0)$. This is a known result [21].

5. Outline of the proof of Theorem 4.2

This Part II of the paper is devoted to the proof of Theorem 4.2. Since the details are quite lengthy, it seems sensible first to give some overview of the general attack.

In preparation, we need to recall a known result concerning the eta invariant for deformations of operators. Let $A(u)$ be a one-parameter family of first order self-adjoint elliptic operators acting on sections of a vector bundle over a compact manifold X. Denote $\eta(u, s)$ to be the value at s of the analytic continuation of the eta series (1.1) associated to $A(u)$. To compute the dependence of $\eta(u, s)$ on u, one has [4]:

PROPOSITION 5.1. *Suppose that zero is not an eigenvalue of $A(u)$ for $u_1 \leq u \leq u_2$. Then*

$$\eta(u_2, s) - \eta(u_1, s) = -s\left[\Gamma\left(\frac{s+1}{2}\right)\right]^{-1} \int_{u_1}^{u_2} \int_0^{\infty} t^{(s-1)/2} \operatorname{Tr}(\dot{A} e^{-tA^2})\, dt\, du$$

where

$$\dot{A} = \frac{d}{du} A.$$

Now let us return to the setting of Part I. We proceed to outline the proof of Theorem 4.2. Let (M, V) be as in Section 1. If (X, f) is the associated framed manifold, then the operator A: $\Gamma(\Lambda^{ev}T^*X) \to \Gamma(\Lambda^{ev}T^*X)$ was defined in Section 3.

Recall that X is a T^{2k} torus bundle over the torus T^{2k-1}. The first step is to unwind A in a Fourier series along the fibers of this bundle. Let $(M' - 0)/V$ be the collection of orbits of V in $M' - 0$. As in Section 1, M' denotes the dual lattice of M. For each $\mu \in (M' - 0)/V$, one unwinds a piece of A: $L^2(X, \Lambda^{ev}T^*X) \to L^2(X, \Lambda^{ev}T^*X)$ into an operator A_μ: $L^2(R^{2k-1}, \Lambda^{ev}R^{4k-1}) \to L^2(R^{2k-1}, \Lambda^{ev}R^{4k-1})$.

After unwinding along the fibers, one has a decomposition, in the sense of unitary equivalence:

(5.2) $$A = A_0 \oplus \sum_{\mu \in (M'-0)/V} A_\mu .$$

Here A_0: $L^2(T^{2k-1}, \Lambda^{ev}R^{4k-1}) \to L^2(T^{2k-1}, \Lambda^{ev}R^{4k-1})$ corresponds to forms which are constant along the fibers.

The detailed description of unwinding along the fibers is given in Section 7. In general, one hopes that the A_μ will be easier to analyze than A. For motivation, note that A acts on the multiply connected space X for which $\pi_1(X) = S(M, V)$. However, each A_μ is defined over the simply connected Euclidean space R^{4k-1}. Nevertheless, serious new difficulties arise from the noncompactness of R^{4k-1}. Recall that our original manifold X was compact.

From (5.2), one has the formula:

$$\eta_A(s) = \eta_0(s) + \sum_{\mu \in (M'-0)/V} \eta_\mu(s) \qquad \operatorname{Re}(s) \gg 0,$$

valid for $\operatorname{Re}(s)$ sufficiently large. Here $\eta_0(s)$ is the eta function of A_0 and $\eta_\mu(s)$ is the eta function of A_μ.

It will be shown in Lemma 8.1 that $\eta_0(s) = 0$. Therefore, we actually have

(5.3) $$\eta_A(s) = \sum_{\mu \in (M'-0)/V} \eta_\mu(s) \qquad \operatorname{Re}(s) \gg 0.$$

Already, one notices the formal similarity between (5.3) and the definition of the series $L(M', V, s)$, given by (1.1). However, both (1.1) and (5.3) apply only for the real part of s sufficiently large. Theorem 4.2 concerns the analytic continuations of these series to $s = 0$.

To proceed further, we need to investigate the dependence, upon μ, of the spectrum of A_μ. In fact, one has, up to unitary equivalence:

(5.4) $$A_\mu = \operatorname{sign} N(\mu)|N(\mu)|^{1/2k} B_h$$

where $h = |N(\mu)|^{-1/2k}$. Here $N(\mu)$ is the norm of μ as defined in Section 1. The operator B_h: $L^2(R^{2k-1}, \Lambda^{\mathrm{ev}}R^{4k-1}) \to L^2(R^{2k-1}, \Lambda^{\mathrm{ev}}R^{4k-1})$ is given by Lemma 8.3. In particular, $\eta_\mu(s)$ depends only on $N(\mu)$.

The proof of (5.4) relies upon computations of the local formulae for A and A_μ. Detailed calculations are presented in Sections 6 and 8. These explicit formulas are also employed later in Sections 10, 11, and 12.

Let $\eta(h, s)$ denote the eta function of B_h. From (5.3) and (5.4), one immediately has the equality:

$$\eta_A(s) = \sum_{\mu \in (M'-0)/V} \operatorname{sign} N(\mu)|N(\mu)|^{-s/2k}\eta(h(\mu), s) \tag{5.5}$$

where $h(\mu) = |N(\mu)|^{-1/2k}$.

We would now like to apply Proposition 5.1, in order to understand the change of $\eta(h, s)$ under deformation of h. A slight technical difficulty arises here since it is not clear that $\dim \operatorname{Ker}(B_h) = 0$ throughout the desired deformation.

This problem of spectral flow across zero is easily overcome. After scaling, we may assume that $0 < h(\mu) < \varepsilon$ for all $h(\mu)$, $\mu \in (M' - 0)/V$. Here ε is chosen so that Corollary 8.6 applies. This scaling will not change $L(M', V, 0)$ as is obvious from the definition (1.1). The invariance of $\eta_A(0)$ under scaling is a consequence of Theorem 4.3 (whose proof in Part III is logically independent of Theorem 4.2).

After scaling, we may apply Proposition 5.1 to deduce:

$$\eta(h_1, 0) = \eta(h_2, 0) \tag{5.6}$$

if $0 < h_1, h_2 < \varepsilon$. The equality (5.6) is shown in Proposition 11.12. The proof of (5.6) requires significant technical work which will be described later in this section.

Let α be chosen so that $0 < h(\mu) < \alpha < \varepsilon$, for all $h(\mu)$, $\mu \in (M' - 0)/V$. The equality (5.6) motivates the following rewriting of (5.5):

$$\begin{aligned} \eta_A(s) &= \eta(\alpha, s)L(M', V, s/2k) \\ &\quad + \sum_{\mu \in (M'-0)/V} \operatorname{sign} N(\mu)|N(\mu)|^{-s/2k}[\eta(h(\mu), s) - \eta(\alpha, s)]. \end{aligned} \tag{5.7}$$

Formula (5.7) is valid for $\operatorname{Re}(s) \gg 0$, as is immediate from the definition (1.1) of $L(M', V, s)$.

The equality (5.6) leads one to speculate that the analytic continuation of the last summand in (5.7) vanishes at $s = 0$. Of course, this is not at all obvious.

Set

$$\gamma(s) = \sum_{\mu \in (M'-0)/V} \operatorname{sign} N(\mu)|N(\mu)|^{-s/2k}[\eta(h(\mu), s) - \eta(\alpha, s)]. \tag{5.8}$$

A crucial point in the proof of Theorem 4.2 is to show that $\gamma(0) = 0$. We now describe the work involved.

Using Proposition 5.1, we may write

$$\gamma(s) = s\left[\Gamma\left(\frac{s+1}{2}\right)\right]^{-1} \sum_{\mu \in (M'-0)/V} \operatorname{sign} N(\mu)|N(\mu)|^{-s/2k} \times \int_{h(\mu)}^{\alpha} \int_0^{\infty} t^{(s-1)/2} \mathfrak{L}(t, h)\, dt\, dh. \tag{5.9}$$

Here $\mathfrak{L}(t, h) = \operatorname{Tr}(\dot{B}_h e^{-tB_h^2})$ where $\dot{B}_h = dB_h/dh$. Although B_h does not act over a compact manifold, Proposition 5.1 still applies, by the rewinding argument of Section 9.

Our key technical result is to describe the asymptotic behavior of $\mathfrak{L}(t, h)$, for small values of h. In fact, one has

$$\mathfrak{L}(t, h) \sim b_0(t) + b_1(t)h + b_2(t)h^2 + \cdots \tag{5.10}$$

with $b_j(t)$ bounded for $0 < t < \infty$ and satisfying $b_j(t) = O(e^{-c_j t})$, as $t \to \infty$. Here $c_j > 0$ is a positive constant.

The proof of (5.10) is quite involved. In Section 9, we show that $\mathfrak{L}(t, h)$ has an asymptotic expansion in powers of h. However, a priori, there may be some negative powers of h appearing. In Section 11, we use the Feynman-Kac formula to show that, for the special operator B_h under consideration, the singular terms in h will vanish. The argument of Section 11 relies essentially on the algebraic cancellation lemmas proved in Section 10.

Let us pause briefly to note that (5.6) is an easy consequence of Proposition 5.1 and (5.10). Details are given in Proposition 11.12.

We now return to the analytic continuation of $\gamma(s)$. Using (5.10), we see that $\mathfrak{L}(t, h)$ is integrable down to $h = 0$. Consequently, for $\operatorname{Re}(s)$ sufficiently large,

$$\gamma(s) = -s\left[\Gamma\left(\frac{s+1}{2}\right)\right]^{-1} \sum_{\mu \in (M'-0)/V} \operatorname{sign} N(\mu)|N(u)|^{-s/2k} \times \int_0^{h(\mu)} \int_0^{\infty} t^{(s-1)/2} \mathfrak{L}(t, h)\, dt\, dh + R(s).$$

Here

$$R(s) = s\left[\Gamma\left(\frac{s+1}{2}\right)\right]^{-1}\left[\int_0^{\alpha}\int_0^{\infty} t^{(s-1)/2}\mathfrak{L}(t, h)\, dt\, dh\right] L(M', V, s/2k).$$

Now $L(M', V, s)$ is holomorphic in s [11, p. 230], and in particular $L(M', V, 0)$ is finite. Moreover, by (5.10) $\int_0^{\alpha}\int_0^{\infty} t^{(s-1)/2}\mathfrak{L}(t, h)\, dt\, dh$ converges down to $s = 0$.

Consequently, for the continuation of $R(s)$, to $s = 0$:

$$R(0) = O[\Gamma(\tfrac{1}{2})]^{-1}\left[\int_0^\alpha \int_0^\infty t^{-1/2}\mathcal{L}(t, h)\, dt\, dh\right] L(M', V, 0) = 0.$$

Using (5.10) again, we may write

$$\gamma(s) = -s\left[\Gamma\left(\frac{s+1}{2}\right)\right]^{-1} \sum_{j=0}^{2k} \frac{1}{j+1}\left[\int_0^\infty t^{(s-1)/2} b_j(t)\, dt\right]$$

$$\times L(M', V, s/2k + (j+1)/2k) - s\left[\Gamma\left(\frac{s+1}{2}\right)\right]^{-1}$$

$$\times \sum_{\mu \in (M'-0)/V} \operatorname{sign} N(\mu)|N(\mu)|^{-1-(s+1)/2k} G(h(\mu)) + R(s)$$

where $G(h)$ is bounded as $h \to 0$. Recall that $h(\mu) = |N(\mu)|^{-1/2k}$.

However, the series defining Shimizu's L-function:

$$L(M', V, s) = \sum_{\mu \in (M'-0)/V} \operatorname{sign} N(\mu)|N(\mu)|^{-s}$$

converges absolutely for $\mathrm{Re}(s) > 1$, as noted in Section 1.

If we set

$$R_1(s) = -s\left[\Gamma\left(\frac{s+1}{2}\right)\right]^{-1}$$

$$\times \sum_{\mu \in (M'-0)/V} \operatorname{sign} N(\mu)|N(\mu)|^{-1-(s+1)/2k} G(h(\mu)) + R(s),$$

then it follows that

$$R_1(0) = -O[\Gamma(\tfrac{1}{2})]^{-1} \sum_{\mu \in (M'-0)/V} \operatorname{sign} N(\mu)|N(\mu)|^{-1-1/2k} G(h(\mu)) + R(0) = 0.$$

Finally, one has

$$\gamma(0) = -O[\Gamma(\tfrac{1}{2})]^{-1} \sum_{j=0}^{2k} \frac{1}{j+1}\left[\int_0^\infty t^{-1/2} b_j(t)\, dt\right] \tag{5.11}$$

$$\times L(M', V, (j+1)/2k) + R_1(0) = 0.$$

Note that $b_j(t)$ is bounded to $t = 0$.

We are almost finished with the proof of Theorem 4.3. Returning to formula (5.7), we see that it may be rewritten as

$$\eta_A(s) = \eta(\alpha, s)L(M', V, s/2k) + \gamma(s). \tag{5.12}$$

This is just the definition of $\gamma(s)$ given in (5.8).

Evaluating (5.12) at $s = 0$, one has via (5.11):

$$\eta_A(0) = \eta(\alpha, 0)L(M', V, 0).$$

Since $0 < \alpha < \varepsilon$, it follows from (5.6) that the value $\eta(\alpha, 0)$ is independent of the choice for α. By the formula for B_α given in Lemma 8.3, $\eta(\alpha, 0)$ depends only upon k. Recall that $2k$ is the degree of K over the rationals; see Section 1.

At this point, we know that

$$\eta_A(0) = c_k L(M', V, 0) \tag{5.13}$$

where $c_k = \eta(\alpha, 0)$.

In Theorem 12.7, the constant c_k will be computed. In fact, $c_k = 1$. Thus

$$\eta_A(0) = L(M', V, 0).$$

The outline for the proof of Theorem 4.2 is now complete. Details will be supplied in the remaining sections of Part II.

6. Computing the operator A

An operator A was defined for framed manifolds (X, f) in Section 3, using the skewed covariant differential associated to the framing. The purpose of this section is to give explicit formulas for A when X is one of the algebraic manifolds constructed in Section 2. Thus, $X = S(M, V) \setminus S(R^{2k}, R^{2k-1})$. Recall that the framing f on X descends from a left invariant framing on the solvable group $S(R^{2k}, R^{2k-1})$.

If y_i, $i = 1, \ldots, 2k - 1$ and x_j, $j = 1, \ldots, 2k$ are suitable coordinates on R^{2k-1} and R^{2k}, then a left invariant framing on the cotangent bundle of $S(R^{2k}, R^{2k-1})$ is spanned by dy_i, $e^{-y_j} dx_j$, [11]. Here one defines $y_{2k} = -y_1 - y_2 - \cdots - y_{2k-1}$. This left invariant framing descends to a framing f on T^*X. The y_i correspond to the base and the x_j correspond to the fiber in the fiber bundle (2.2) with X as total space.

Let $A\colon \Gamma(\Lambda^{\mathrm{ev}} T^*X) \to \Gamma(\Lambda^{\mathrm{ev}} T^*X)$ be the differential operator constructed in Section 1 for any framed manifold X. Denote $C = A^2$. Then C preserves the parity of forms. The operator C is analogous to the Laplace-Beltrami operator. However, C is constructed using the skewed covariant differential of the flat connection corresponding to f. The same construction using the standard exterior derivative would give the Laplace-Beltrami operator on forms.

The framing f induces framings on the associated bundles of exterior algebras. By taking components with respect to these framings, one may regard A and C as operators acting on functions with values in a fixed vector space $\mathfrak{M}$. Here $\mathfrak{M}$ is identified with the fiber of $\Lambda^{\mathrm{ev}} T^*X$, over any point in X.

The differential operator A is self-adjoint. From the defining formula $A = \pm(*d - d*)$, one finds that it is of the form

$$A = \sum F_j \frac{\partial}{\partial y_j} - \sqrt{-1} \sum E_m e^{y_m} \frac{\partial}{\partial x_m} \tag{6.2}$$

where F_j, E_m: $\mathfrak{M} \to \mathfrak{M}$ are endomorphisms independent of both x and y. The representation (6.2) of A follows alternatively from the group invariance of A. Notice that $\{\partial/\partial y_j, e^{y_m}\partial/\partial x_m\}$ form a basis for the Lie algebra of $S(M, V)$.

By computing the symbol of A and noting that A^2 has the same symbol as the laplacian, one deduces

Proposition 6.4. *If (X, f) is a framed manifold associated to a pair (M, V) as in Section 4, then the basic operator A is given by:*

$$A = \sum F_j \frac{\partial}{\partial y_j} - \sqrt{-1} \sum E_m e^{y_m} \frac{\partial}{\partial x_m}$$

where F_j, E_m: $\mathfrak{M} \to \mathfrak{M}$ are endomorphisms independent of both x and y. Moreover, $F_j^2 = -1$, $E_m^2 = 1$, and distinct pairs from the collection $\{F_j, E_m\}$ will anticommute.

7. Unwinding along the fibers

In this section, we introduce an idea which is of basic importance. That is, we decompose the operator A using a Fourier series along the fibers. Much more general formulations of this method are well known in the study of representation theory for compact solvmanifolds, [7]. However, because we are studying a quite specific solvmanifold X, it seems desirable to give a self-contained and elementary treatment.

Recall that our manifold X is a T^{2k} torus bundle over the $2k - 1$ torus T^{2k-1}. The fibering is given as in (2.2). One may unwind this bundle to a trivial bundle over the universal cover R^{2k-1} of T^{2k-1}.

If one expands in a Fourier series along the fibers, there is a corresponding unwinding of A. The effect is to decompose A, up to unitary equivalence, as a direct sum:

$$A = A_0 + \sum_{\mu \in (M'-0)/V} A_\mu .$$

Here, each A_μ, $\mu \neq 0$, acts on sections of a trivial bundle over the simply connected Euclidean space R^{2k-1}. The advantages of expanding in a Fourier series along the fibers are twofold. First, one replaces differentiation operators along the fibers by simpler multiplication operators. Secondly, the A_μ act over the simply connected space R^{2k-1}. Thus, one avoids problems associated to the fundamental group of X. Of course, A acts on sections of a trivial bundle over X. Recall that $\pi_1(X)$ is the solvable group $S(M, V)$.

Unfortunately, new difficulties are introduced by the noncompactness of R^{2k-1}. Our original manifold X is compact. In particular, certain analytic results for elliptic operators, which are routine for compact manifolds, will require more careful discussion in the noncompact setting.

Let us now give a detailed description of the unwinding of A along the fibers.

Recall that X is the bundle associated to the action of $V = \pi_1(T^{2k-1})$ on $T^{2k} = M \setminus R^{2k}$. In the coordinate system of Section 2, this is given by componentwise multiplication. Each vector-valued function ϕ on X may be expanded in a Fourier series along the fibers T^{2k}, of the fibering (2.2). Thus, one has in local coordinates

$$\phi(x, y) = \sum_{\mu \in M'} \phi_\mu(y) e^{i\mu \cdot x}$$

where M' is the dual lattice to M. Here the y's are the base coordinates and x's are the fiber coordinates, as in Section 6.

However, since (2.2) is not a trivial bundle, $\phi_\mu(y)$, $\mu \neq 0$, will not be periodic in y. To obtain a global Fourier decomposition, one must unwrap the bundle (2.2) to a trivial bundle over the universal cover R^{2k-1} of T^{2k-1}. Let $(M' - 0)/V$ denote the orbits of V in $M' - 0$. One has an isomorphism of $\mathfrak{M}$-valued L^2-spaces:

$$L^2(X, \mathfrak{M}) \to L^2(T^{2k-1}, \mathfrak{M}) \oplus \sum_{(M'-0)/V} L^2(R^{2k-1}, \mathfrak{M}), \tag{7.1}$$

obtained by sending $\phi \to \phi_0 + \Sigma_{(M'-0)/V} \phi_\mu$, with $\mu \in (M' - 0)/V$. Here $\mathfrak{M}$ is the vector space of Section 6. In fact, $\mathfrak{M}$ may be identified with $\Lambda^{\mathrm{ev}} T^* X$.

The identification (7.1) sends differentiations $\partial/\partial x_j$ along the fibers into multiplication operators $\sqrt{-1}\,\mu_j$. Thus, one has a corresponding decomposition of A as

$$A \to A_0 \oplus \sum_{(M'-0)/V} A_\mu. \tag{7.2}$$

Referring to Proposition 6.4, and replacing $\partial/\partial x_j$ by $\sqrt{-1}\,\mu_j$, one deduces the basic lemma:

LEMMA 7.3. *The operators A_0, A_μ are of the form:*

$$A_0 = \sum F_j \frac{\partial}{\partial y_j},$$

$$A_\mu = \sum F_j \frac{\partial}{\partial y_j} + \sum \mu_m E_m e^{y_m},$$

where F_j, E_m: $\mathfrak{M} \to \mathfrak{M}$ are endomorphisms independent of y. The summation in j runs from 1 to $2k - 1$, while the summation in m runs from 1 to $2k$. Moreover,

$F_j^2 = -1$, $E_m^2 = 1$, *and distinct pairs of the collection* $\{E_m, F_j\}$ *will anticommute.*

The operator $C = A^2$ also unwinds as

$$C \to C_0 \oplus \sum_{(M'-0)/V} C_\mu. \tag{7.4}$$

By squaring the formulas given in Lemma 7.3, we may deduce:

LEMMA 7.5. *The operator* C_0 *is just*

$$C_0 = -\sum \frac{\partial^2}{\partial y_l^2}.$$

For $\mu \neq 0$, the operator C_μ is given by

$$C_\mu = -\sum \frac{\partial^2}{\partial y_l^2} + V$$

where the endomorphism V depends upon μ.

We may write $V = V_1 + V_2$, with

$$V_1 = \sum_{m=1}^{2k} \mu_m^2 e^{2y_m}$$

and

$$V_2 = \sum_{m=1}^{2k-1} \mu_m F_m E_m e^{y_m} - \mu_{2k}\left(\sum_{j=1}^{2k-1} F_j\right) E_{2k} e^{y_{2k}}.$$

The unusual form of V_2 arises from the fact that y_{2k} is by definition $-y_1 - y_2 - \cdots - y_{2k-1}$.

8. The operators A_μ

Our present purpose is to investigate the spectrum of A_μ. In particular, we will show that, up to unitary equivalence, A_μ depends only upon $N(\mu)$. Here $N(\mu)$ is the norm of μ, as defined in Section 1.

Recall that A_μ is the piece, of our basic operator A, which corresponds to the orbit of μ in M'/V. Since A has pure point spectrum, so does each A_μ. We will employ the explicit formulas for the A_μ given in Lemma 7.3.

If $\mu = 0$, one has

LEMMA 8.1. *The operator* A_0: $L^2(T^{2k-1}, \mathfrak{M}) \to L^2(T^{2k-1}, \mathfrak{M})$ *is unitarily equivalent to* $-A_0$. *Consequently,* λ *and* $-\lambda$ *appear with equal multiplicity in the spectrum of* A_0, *for any real number* λ.

Proof. Let E_m be as in Proposition 6.4. Then, since A is self-adjoint, one deduces $E_m^* = E_m$. Here $1 \le m \le 2k$ is arbitrary. However, $E_m^2 = 1$, so that E_m must also be unitary, $E_m E_m^* = 1$.

By Lemma 7.3, one has $A_0 = \Sigma F_j \partial/\partial y_j$, where each F_j anticommutes with E_m. So $E_m A_0 E_m = -A_0$, for any m, which establishes the required unitary equivalence.

For the rest of this section, we assume that $\mu \neq 0$. Recall that A_μ: $L^2(R^{2k-1}, \mathfrak{M}) \to L^2(R^{2k-1}, \mathfrak{M})$ is of the form

$$A_\mu = \sum F_j \frac{\partial}{\partial y_j} + \sum \mu_m e^{y_m} E_m$$

where F_j, E_m: $\mathfrak{M} \to \mathfrak{M}$ are endomorphisms independent of y.

Denote $N(\mu) = \mu_1 \mu_2 \ldots \mu_{2k}$, as in Section 1, for the norm of μ. To begin, one has

LEMMA 8.2. *The operator A_μ is unitarily equivalent to*

$$A'_\mu = \operatorname{sign} N(\mu)\left[\sum F_j \frac{\partial}{\partial y_j} + \sum |\mu_m| e^{y_m} E_m\right]$$

where the vertical bars denote absolute value.

Proof. For each l, the endomorphism E_l: $\mathfrak{M} \to \mathfrak{M}$ induces a unitary equivalence, E_l: $L^2(R^{2k-1}, \mathfrak{M}) \to L^2(R^{2k-1}, \mathfrak{M})$, as in the proof of Lemma 8.1.

Moreover, using Lemma 7.3,

$$E_l A_\mu E_l = -\left[\sum F_j \frac{\partial}{\partial y_j} + \sum_{m \neq l} \mu_m e^{y_m} E_m - \mu_l e^{y_l} E_l\right].$$

The lemma follows from successive conjugation by each E_l for which the corresponding $\mu_l < 0$.

One may make the change of variables $y_k \to y_k - \log(|\mu_k|/|N(\mu)|^{1/2k})$. This preserves the relation $y_{2k} = -y_1 - y_2 - \cdots - y_{2k-1}$. Moreover, A'_μ is transformed into the unitarily equivalent operator

$$A''_\mu = \operatorname{sign} N(\mu)|N(\mu)|^{1/2k}\left[|N(\mu)|^{-1/2k} \sum F_j \frac{\partial}{\partial y_j} + \sum e^{y_m} E_m\right].$$

Combining these observations with Lemma 8.2, one has

LEMMA 8.3. *The operator A_μ is identified up to unitary equivalence, denoted $\sim$, as*

$$A_\mu \sim \operatorname{sign} N(\mu)|N(\mu)|^{1/2k} B_h$$

where $h = |N(\mu)|^{-1/2k}$, *and*

$$B_h = h \sum F_j \frac{\partial}{\partial y_j} + \sum e^{y_m} E_m.$$

Here we assume $\mu \neq 0$. The index j is summed from 1 to $2k - 1$ and the index m is summed from 1 to $2k$.

Now let $h > 0$ be arbitrary and put $D_h = B_h^2$.

For the special value $h = |N(\mu)|^{1/2k}$, given in Lemma 8.3, one clearly has

$$C_\mu = A_\mu^2 \sim h^{-2} B_h^2 = h^{-2} D_h. \tag{8.4}$$

By squaring the formula defining B_h, in Lemma 8.3, we easily deduce

LEMMA 8.5. *For any* $h > 0$, *the operator* $h^{-2} D_h$ *may be written as*

$$h^{-2} D_h = -\sum \frac{\partial^2}{\partial y_j^2} + W_h,$$

where $W_h = h^{-2} W_1 + h^{-1} W_2$, *with* W_1, W_2 *endomorphisms independent of* h. *Specifically,*

$$W_1 = \sum_{m=1}^{2k} e^{2y_m},$$

$$W_2 = \sum_{m=1}^{2k-1} F_m E_m e^{y_m} - \left(\sum_{j=1}^{2k-1} F_j \right) E_{2k} e^{y_{2k}}.$$

We pause to record:

COROLLARY 8.6. *For* h *sufficiently small, say* $0 < h < \varepsilon$, *zero does not appear in the spectrum of* B_h.

Proof. By examining the formulas in Lemma 8.5, we see that $W_h > 0$, for any y, if h is sufficiently small. So D_h will be positive definite. Since $D_h = B_h^2$, the corollary follows.

9. Existence of the asymptotic expansion of the heat kernel

The next three sections are devoted to establishing the basic asymptotic expansion (5.10) of our outline. In the present section, we show the existence of such asymptotic expansions in powers of h. However, a priori, some negative powers of h may appear. In Sections 10 and 11, we prove vanishing of the singular terms. This general approach is reminiscent of the heat equation proof of the Index Theorem [2]. However, new complications arise since we are working with operators over the noncompact Euclidean space R^{2k-1}.

Let B_h be given by Lemma 8.3. We define

$$\mathfrak{L}(t, h) = \operatorname{Tr}\left(\dot{B}_h e^{-tB_h^2}\right).$$

Since B_h may be realized as being unwound from an operator over a compact manifold, it is easy to see that the trace is well-defined.

From the explicit formula given for B_h in Lemma 8.3, it follows that

$$\mathfrak{L}(t, h) = \sum \operatorname{Tr}\left(F_j \frac{\partial}{\partial y_j} e^{-tD_h}\right) \tag{9.1}$$

where $D_h = B_h^2$.

For this section, we will not need the special form of the endomorphisms F_j. Fix some endomorphism U: $\mathfrak{M} \to \mathfrak{M}$ with U having constant coefficients, independent of y. Denote, for any given $1 \leq i \leq 2k - 1$:

$$\mathfrak{R}(t, h) = \operatorname{Tr}\left(U \frac{\partial}{\partial y_i} e^{-tD_h}\right). \tag{9.2}$$

To demonstrate existence of an asymptotic expansion for $\mathfrak{R}(t, h)$, as $h \downarrow 0$, we will realize the D_h as being unwound from a family of operators over a compact solvmanifold. This allows one to apply the standard elliptic theory for compact manifolds. Thus, as far as existence of an expansion is concerned, one can avoid difficulties stemming from the noncompactness of R^{2k-1} and the rapid growth of the potential W, as given in Lemma 8.5.

Choose some compact solvmanifold X and some $h_0 = h(\mu_0)$, so that D_{h_0} arises from a piece of the operator C: $L^2(X, \mathfrak{M}) \to L^2(X, \mathfrak{M})$. This means that $h_0 = |N(\mu_0)|^{-1/2k}$, for some $\mu_0 \in M'$. Moreover, as in (8.4), C_{μ_0} is unitarily equivalent to $h_0^{-2} D_{h_0}$. Let π: $L^2(X, \mathfrak{M}) \to L^2(X, \mathfrak{M})$ denote the orthogonal projection onto those forms whose Fourier series along the fiber has non-zero entries only for μ in the orbit of μ_0. Then $\pi C \pi \sim C_{\mu_0} \sim h_0^{-2} D_{h_0}$, where $\sim$ denotes unitary equivalence.

Set $h = h_0 \gamma$. To investigate the behavior of (9.2) as $\gamma \downarrow 0$, we will imbed C into a family of operators C_γ and let $\gamma \downarrow 0$. The C_γ will be chosen to guarantee that $\pi C_\gamma \pi \sim h_0^{-2} D_h$. This reduces one to studying the family of elliptic operators C_γ over the compact manifold X. Standard methods may then be applied.

In the notation of (6.1), set

$$C_\gamma = -\left[\gamma^2 \sum \frac{\partial^2}{\partial y_l^2} + \sum e^{2y_m} \frac{\partial^2}{\partial x_m^2}\right] + \gamma R. \tag{9.3}$$

Here R is the first order part of C.

Unwinding C_γ in a Fourier series along the fibers, corresponding to the orbit of μ_0, one obtains an operator $C_{\mu_0, \gamma}$. Following through the steps leading to

Lemma 7.5, we see that

$$C_{\mu_0,\gamma} = -\gamma^2 \sum \frac{\partial^2}{\partial y_l^2} + V_1 + \gamma V_2$$

where V_1, V_2 are as in Lemma 7.5, with $\mu = \mu_0$.

Continuing the identification process, with the additional parameter γ, we find that $C_{\mu_0,\gamma}$ is unitarily equivalent to

$$h_0^{-2} D_h = -\gamma^2 \sum \frac{\partial^2}{\partial y_l^2} + h_0^{-2} W_1 + \gamma h_0^{-1} W_2$$

where $h = h_0 \gamma$ and W_1, W_2 are given by Lemma 8.5.

Thus D_h is unitarily equivalent to $\pi h_0^2 C_\gamma \pi$. Consequently,

$$\mathfrak{R}(t, h) = \operatorname{Tr}\left(\pi U \frac{\partial}{\partial y_i} e^{-t h_0^2 C_\gamma} \pi \right) \tag{9.4}$$

with $h = h_0 \gamma$.

The main result of this section is:

THEOREM 9.5. *Let $\mathfrak{R}(t, h)$ be given by* (9.2). *As $h \downarrow 0$, one has an asymptotic expansion:*

$$\mathfrak{R}(t, h) \sim h^{-2k} t^{-4k-2} \big(a_0(t) + a_1(t) h t^{1/2} + a_2(t) h^2 t + \cdots \big).$$

Moreover $a_j(t)$ is bounded as $t \downarrow 0$ and has at most polynomial growth as $t \uparrow \infty$.

If $\mathfrak{L}(t, h)$ is defined by (9.1), *then $\mathfrak{L}(t, h)$ has an asymptotic expansion of the same type.*

Once the expansion for $\mathfrak{R}(t, h)$ is known, the analogous expansion for $\mathfrak{L}(t, h)$ follows immediately from (9.1). Thus, it suffices to establish the result for $\mathfrak{R}(t, h)$.

To prepare for the proof of Theorem 9.5, we derive:

LEMMA 9.6. *Let π: $L^2(X, \mathfrak{M}) \to L^2(X, \mathfrak{M})$ be the orthogonal projection onto those sections of $\mathfrak{M}$ whose Fourier series along the fibers contains non-zero entries only from the orbit of $\mu_0 \in M'$.*

Then, for $\phi \in C^\infty(X, \mathfrak{M})$, supported near any given point in X, one has the local representation

$$\pi\phi = D(\omega \# \phi).$$

Here ω is a continuous function and $\#$ denotes convolution. Moreover, D is a differential operator of order $2k + 2$, acting along the fibers.

Proof (Lemma 9.6). Let $|\mu|^2 = \mu_1^2 + \mu_2^2 + \cdots + \mu_{2k}^2$ for $\mu \in M'$ and denote vol to be the volume of a fundamental region in R^{2k}, for the dual lattice M'.

For $x, \bar{x} \in T^{2k}$, the fiber, set

$$\omega(x, \bar{x}) = \frac{1}{\text{vol}} \sum_{\mu \in V\mu_0} \frac{e^{i\mu(x-\bar{x})}}{\|\mu\|^{2k+2}}$$

where $V\mu_0$ is the V-orbit of μ_0 in M'. Note that the right-hand side converges absolutely, even if one sums over all $\mu \in M'$.

Let $P = -\Sigma\partial^2/\partial x_i^2$. Then

$$P_x^{k+1}\omega(x, \bar{x}) = \frac{1}{\text{vol}} \sum_{\mu \in V\mu_0} e^{i\mu(x-\bar{x})} = \pi(x, \bar{x})$$

in the sense of distributions. Here $\pi(x, \bar{x})$ is the distribution kernel of π. This gives the lemma with $D = P^{k+1}$.

We now proceed with the proof of Theorem 9.5:

Proof (*Theorem* 9.5). We will employ (9.4) and investigate the asymptotic behavior of the right-hand side as $\gamma \downarrow 0$.

One may cover X by a finite number of coordinate charts Ω, which are compatible with the fibering of X. Thus $\Omega = \Omega_y \times \Omega_x$, with $\Omega_y \subset R^{2k-1}$ and $\Omega_x \subset R^{2k}$. Let $\psi_\alpha(y)$ be a partition of unity subordinate to Ω.

In the notation of (9.3), we let $E_\alpha(t)$ be a fundamental solution for $\partial/\partial t - \Sigma\partial^2/\partial y_l^2$ and $F_\alpha(t)$ a fundamental solution for $\partial/\partial t - \Sigma e^{2y_m}\partial^2/\partial x_m^2$. For concreteness, we assume that E_α, F_α are defined on some neighborhood of the support of ψ_α and that E_α satisfies Dirichlet boundary conditions.

Let C_γ be given by (9.3). As a parametrix for the fundamental solution $K(t)$ of $\partial/\partial t + C_\gamma$, we employ

$$G(t, (x, y), (\bar{x}, \bar{y})) = \sum_\alpha \psi_\alpha(y) E_\alpha(t\gamma^2, (x, y), (\bar{x}, \bar{y})) \times F_\alpha(t, (x, y), (\bar{x}, \bar{y}))\psi_\alpha(\bar{y}).$$

Standard methods [1], [14] give $K(t)$ as an infinite sum with $G(t)$ as leading term.

Analyzing this construction for $K(t)$ in the usual way, we find that $K(t)$ has an asymptotic expansion as $\gamma \downarrow 0$:

$$K(t, (x, y), (\bar{x}, \bar{y})) \sim (t\gamma^2)^{-(2k-1)/2} \exp\left(\frac{-|y-\bar{y}|^2}{4t\gamma^2}\right) \times \sum_\alpha \psi_\alpha(y) F_\alpha(t, (x, y), (\bar{x}, \bar{y}))\psi_\alpha(\bar{y}) \times \sum_{j=0}^{\infty} b_j((x, y), (\bar{x}, \bar{y}), t)\gamma^j t^{j/2}$$

where b_j are smooth and bounded, along with their derivatives. This expansion is valid for $((x, y), (\bar{x}, \bar{y}))$ in a neighborhood of the diagonal.

Applying Lemma 9.6 and integrating, we find that

$$\operatorname{Tr}\left(\pi U \frac{\partial}{\partial y_i} e^{-th_0^2 C_\gamma} \pi\right) = \operatorname{Tr}\left(\pi U \frac{\partial}{\partial y_i} K\left(th_0^2\right) \pi\right)$$

has an asymptotic expansion as $\gamma \downarrow 0$:

$$\operatorname{Tr}\left(\pi U \frac{\partial}{\partial y_i} e^{-th_0^2 C_\gamma} \pi\right) \sim \gamma^{-2k}\left(c_0(t) + c_1(t)\gamma t^{1/2} + c_2(t)\gamma^2 t + \cdots\right)$$

with $c_j(t) = O(t^{-4k-2})$, $0 < t \leq 1$, and $c_j(t)$ bounded as $t \to \infty$.

Since $h = h_0 \gamma$, with h_0 fixed, Theorem 9.5 follows by reference to formula (9.4).

10. Vanishing of algebraic traces

Theorem 9.5 shows the existence of an asymptotic expansion for $\mathcal{L}(t, h)$, in powers of h. A priori, the coefficients of negative powers of h may be non-zero. Our goal in Sections 10 and 11 is to show that these singular terms actually vanish. Although different in detail, our vanishing theorems are reminiscent of the paper of V. K. Patodi [17].

Sections 10 and 11 represent a key step in the entire proof of Theorem 4.2. At the conclusion of these two sections, we will have established the basic formula (5.10) from our outline.

The present section is purely algebraic in nature. We show the vanishing of certain traces using the fact that $\{F_j, \sqrt{-1}\, E_m\}$, of Lemma 7.3, define a representation of a $4k - 1$ dimensional Clifford algebra, [2]. That is, $F_j^2 = -1$, $E_m^2 = 1$, and distinct pairs from the collection $\{F_j, E_m\}$ will anticommute. The methods employed in this section are standard. We include proofs of Lemmas 10.1 and 10.2 for completeness.

LEMMA 10.1. *Consider a product* $\prod E_{i_s} \prod F_{j_r}$, *where* $i_1 < i_2 < \cdots$, *and* $j_1 < j_2 < \cdots$. *Then*

$$\operatorname{Tr}\left(\prod E_{i_s} \prod F_{j_r}\right) = 0$$

if at least one but not all $4k - 1$ *of the* E_i*'s and* F_j*'s appear in the product expression.*

Proof. Set $\mathfrak{D} = \prod E_{i_s} \prod F_{j_r}$. Let $l < 4k - 1$ be the total number of E_i's and F_j's appearing in $\mathfrak{D}$.

If l is even, then $\mathfrak{D}$ anticommutes with any E_i or F_j which appears in $\mathfrak{D}$. If l is odd, then $\mathfrak{D}$ anticommutes with any E_i or F_j which does not appear in $\mathfrak{D}$.

In either case, $\mathfrak{D}$ anticommutes with an invertible matrix M, so that

$$\operatorname{Tr}(\mathfrak{D}) = \operatorname{Tr}(M\mathfrak{D}M^{-1}) = -\operatorname{Tr}(\mathfrak{D})$$

and thus $\operatorname{Tr}(\mathfrak{D}) = 0$.

LEMMA 10.2. *The product $E_{2k}\prod_{i=1}^{2k-1}E_iF_i$ is the $2^{2n} \times 2^{2n}$ identity matrix, where $n = 2k - 1$. Consequently,*

$$\operatorname{Tr}\left(E_{2k}\prod_{i=1}^{2k-1}E_iF_i\right) = 2^{2n}.$$

Proof. $\mathfrak{D} = E_{2k}\prod_{i=1}^{2k-1}E_iF_i$ commutes with all the E_j's and F_j's; hence $\mathfrak{D}$ also commutes with the $SO(4k-1)$ action on $\mathfrak{M} = \Lambda^{ev}T^*X$.

By the representation theory of $SO(4k-1)$, [6, p. 258], each Λ^{2p} is a distinct irreducible of $SO(4k-1)$. Thus $\mathfrak{D} = \pm 1$ on every Λ^{2p}, where the sign $\pm$ may depend upon p. Here Λ^{2p} is the $2p'$-th exterior power of the standard representation.

Computing out the explicit formulas for the E_j's and F_j's and using that $\mathfrak{D}$ commutes with all these endomorphisms, one verifies that $\mathfrak{D} = \pm 1$ on $\mathfrak{M} = \Lambda^{ev}T^*X$.

A tedious chase through the orientation conventions of [3] and [11] verifies that $\mathfrak{D} = 1$, the identify matrix.

Of course, one may also prove Lemmas 10.1 and 10.2 by doing a direct calculation. This seems less enlightening than the method above.

The following result will be crucial to our estimates of heat kernels:

LEMMA 10.3. *Suppose that W_2 is the endomorphism of Lemma 8.5. Let $y_1, y_2, \ldots, y_l$ be points in R^{2k-1}. Then,*

a) *For any $1 \leq i \leq 2k - 1$ and $0 \leq l < 2k$, one has*

$$\operatorname{Tr}(F_iW_2(y_1)W_2(y_2)\cdots W_2(y_l)) = 0$$

and

b) *For any $1 \leq i \leq 2k$ and $0 \leq l < 2k - 1$, one has*

$$\operatorname{Tr}(E_iW_2(y_1)W_2(y_2)\cdots W_2(y_l)) = 0.$$

Lemma 10.3 follows immediately from Lemma 10.1 and the explicit expression for W_2, given in Lemma 8.5.

11. Estimate of the heat kernel

In this section, the proof of the basic expansion for $\mathfrak{L}(t, h)$ is completed. That is, formula (5.10) of our outline is established. We will work over the noncompact Euclidean space R^{2k-1} and use the Feynman-Kac representation of the heat kernel.

There is a formal procedure due to Kac [12] for obtaining asymptotic expansions directly from the Feynman-Kac formula. It appears that his method cannot be rigorously justified in our case. This is due to the extremely rapid growth at infinity of the derivatives of the potential W_h. Recall that W_h is the zero'th order term in our basic operator $h^{-2}D_h$ of Lemma 8.5.

Fortunately, we have already shown the existence of an expansion for $\mathcal{L}(t, h)$ in Section 9. The essential idea was to utilize the fact that D_h was unwound from an elliptic operator on a compact manifold, by the method of Section 7. The technique of Section 9 will be called the rewinding argument.

Note that the expansion of Theorem 9.5, for $\mathcal{L}(t, h)$, contains some coefficients of negative powers of h. Our present purpose is to show the vanishing of these singular terms. It is very important that the basic expansion (5.10), for $\mathcal{L}(t, h)$, should not contain negative powers of h. The essential point is to obtain an upper bound on $\mathcal{L}(t, h)$, i.e., $|\mathcal{L}(t, h)| \leq F(t)$, for suitable $F(t)$.

We establish the required upper bound on $\mathcal{L}(t, h)$ by a delicate argument using the Feynman-Kac formula. For this, the special nature of our potential is crucial. In particular, the algebraic lemmas of Section 10 will be needed.

Let $D_h = B_h^2$ be the second order differential operator described in Lemma 8.5. Recall that D_h is of the form

$$D_h = h^2(\Delta + W_h)$$

where $\Delta = -\sum_{i=1}^{2k-1}\partial^2/\partial y_i^2$ is the standard Laplacian, acting on $\mathfrak{M}$-valued functions.

The zero'th order term W_h is written as

$$W_h = h^{-2}W_1 + h^{-1}W_2$$

Moreover, W_1 and W_2 are endomorphisms, of the trivial vector bundle $\mathfrak{M}$, which are independent of $h > 0$.

Specifically, one has, according to Lemma 8.5:

(11.1)
$$W_1 = \sum_{m=1}^{2k} e^{2y_m},$$
$$W_2 = \sum_{m=1}^{2k-1} F_m E_m e^{y_m} - \left(\sum_{j=1}^{2k-1} F_j\right) E_{2k} e^{y_{2k}}$$

where $y_{2k} = -y_1 - y_2 - \cdots - y_{2k-1}$.

Our goal is to obtain an upper bound for $\mathcal{L}(t, h) = \mathrm{Tr}(\dot{B}_h e^{-tD_h})$, for h sufficiently small. Note that the right-hand side is of trace class, by the rewinding argument.

Since D_h has matrix-valued terms, the Feynman-Kac representation of $\exp(-tD_h)$ requires the solution of stochastic integral equations [5]. In particular, one does not obtain a simple exponential term appearing, as in the case of scalar potentials.

To avoid these probabilistic technicalities, we attack the problem in two steps. First, by use of Duhamel's principle and the algebraic cancellation lemmas of Section 10, a suitable upper bound is established in terms of heat kernels for scalar potentials. We then employ the Feynman-Kac formula for operators acting on functions.

Let P_1 and P_2 be second order elliptic operators acting on sections of a vector bundle over a compact manifold. Suppose that P_1 and P_2 both have leading order symbol given by the metric tensor. Thus $P_1 - P_2$ is a first order operator.

Duhamel's principle [14] states:

$$\begin{aligned} e^{-P_2} - e^{-P_1} &= \int_0^1 e^{-sP_1}(P_1 - P_2)e^{-(1-s)P_2}\,ds \\ &= e^{-P_1}(P_1 - P_2) \# e^{-P_2}. \end{aligned} \tag{11.2}$$

Here # is an abbreviation for the convolution product appearing on the line above.

By iterating Duhamel's principle l times, we find that:

$$\begin{aligned} e^{-P_2} = \sum_{j=0}^{l-1} e^{-P_1}(P_1 - P_2) \# e^{-P_1}(P_1 - P_2) \# \cdots \\ \# e^{-P_1}(P_1 - P_2) \# e^{-P_1} + e^{-P_1}(P_1 - P_2) \\ \# e^{-P_1}(P_1 - P_2) \# \cdots \# e^{-P_1}(P_1 - P_2) \# e^{-P_2}. \end{aligned} \tag{11.3}$$

Here l is any integer greater than or equal to one.

Let $\bar{D}_h$ denote the operator D_h with the matrix-valued potential, hW_2, removed. Thus $\bar{D}_h = h^2(\Delta + h^{-2}W_1)$ is a scalar operator times the identity matrix. Note that $\|W_2\| \ll W_1$ at infinity, so one expects $\bar{D}_h$ to be the dominant part of D_h.

We apply (11.3) with $P_1 = t\bar{D}_h$ and $P_2 = tD_h$. Although these operators act over the noncompact space R^{2k-1}, the use of (11.3) is justified by the rewinding argument, as in Section 9. In particular, D_h and $\bar{D}_h$ may be simultaneously unwound from operators over a compact manifold.

Since $D_h - \bar{D}_h = hW_2$, substitution in (11.3) gives:

$$\begin{aligned} e^{-tD_h} = \sum_{j=0}^{l-1} (-th)^j e^{-t\bar{D}_h}W_2 \# e^{-t\bar{D}_h}W_2 \# \cdots \\ \# e^{-t\bar{D}_h}W_2 \# e^{-t\bar{D}_h} + (-th)^l e^{-t\bar{D}_h}W_2 \\ \# e^{-t\bar{D}_h}W_2 \# \cdots \# e^{-t\bar{D}_h}W_2 \# e^{-tD_h}. \end{aligned} \tag{11.4}$$

Recall that $\mathfrak{L}(t, h) = \mathrm{Tr}(\dot{B}_h e^{-tD_h})$. From (9.1), one obtains:

$$\mathfrak{L}(t, h) = \sum_{i=1}^{2k-1} \mathfrak{L}_i(t, h) \tag{11.5}$$

with

$$\mathfrak{L}_i(t, h) = \mathrm{Tr}\left(F_i \frac{\partial}{\partial y_i} e^{-tD_h}\right).$$

Our object is to obtain an upper bound for $\mathfrak{L}(t, h)$. Of course, it suffices to give an upper bound for each $\mathfrak{L}_i(t, h)$, $1 \leq i \leq 2k - 1$.

Let i be fixed. Using (11.4) and the definition of $\mathfrak{L}_i(t, h)$, we derive:

(11.6)

$$\mathfrak{L}_i(t, h) = \sum_{j=0}^{l-1} (-th)^j \mathrm{Tr}\left(F_i \frac{\partial}{\partial y_i} e^{-t\bar{D}_h} W_2 \# e^{-t\bar{D}_h} W_2 \# \cdots \# e^{-t\bar{D}_h} W_2 \# e^{-t\bar{D}_h}\right)$$
$$+ (-th)^l \mathrm{Tr}\left(F_i \frac{\partial}{\partial y_i} e^{-t\bar{D}_h} W_2 \# e^{-t\bar{D}_h} W_2 \# \cdots \# e^{-t\bar{D}_h} W_2 \# e^{-tD_h}\right).$$

Here $l \geq 1$ is an integer.

Now set $l = 2k$ and apply the algebraic cancellation Lemma 10.3a, to write:

(11.7)

$$\mathfrak{L}_i(t, h) = (-th)^{2k} \mathrm{Tr}\left(F_i \frac{\partial}{\partial y_i} e^{-t\bar{D}_h} W_2 \# e^{-t\bar{D}_h} W_2 \# \cdots \# e^{-t\bar{D}_h} W_2 \# e^{-tD_h}\right).$$

Note that $\exp(-t\bar{D}_h)$ is a scalar operator and therefore it does not disturb the vanishing of traces given by Lemma 10.3.

It is important to realize the essential use of algebraic arguments to eliminate the lower order terms in h, which occur in (11.6). For this, we employed very specific information about the symbol of our basic operator B_h. Recall that, by definition, one has $D_h = B_h^2$.

It will be straightforward to obtain an upper bound, for the right-hand side of (11.7), in terms of scalar operators. Thus, the first step in estimating $\mathfrak{L}_i(t, h)$ is essentially complete. We have circumvented the difficulties arising from the matrix-valued potential W_2.

To proceed further, we will employ the Feynman-Kac representation of the heat kernel $\exp(-t\bar{D}_h)$. Note that this operator acts on functions. Thus, by the results of [12] and [18], we have:

$$\begin{aligned}(11.8)\qquad \exp(-t\bar{D}_h)(y,\xi) &= (4\pi t h^2)^{-n/2}\exp\left(\frac{-|y-\xi|^2}{4th^2}\right),\\ &E\left[\exp\left(-\int_0^{th^2} h^{-2}W_1(\xi + x(\tau))\,d\tau \,|\, x(th^2) = y - \xi\right]\right..\end{aligned}$$

Here, E denotes the conditional expectation for Wiener measure with respect to paths starting at the origin, $x(0) = 0$, and reaching $y - \xi$ at parameter value th^2, $x(th^2) = y - \xi$. Moreover, $n = 2k - 1$ denotes the dimension of the underlying Euclidean space.

Also, if $\Delta = -\Sigma\partial^2/\partial y_i^2$ denotes the usual Laplacian, then there is a standard formula:

$$(11.9)\qquad \exp(-th^2\Delta)(y,\xi) = (4\pi th^2)^{-n/2}\exp\left(\frac{-|y-\xi|^2}{4th^2}\right).$$

This may be derived by Fourier transforms.

Using (11.7) and the Feynman-Kac formula, we will establish:

Proposition 11.10. *Let $\mathfrak{L}_i(t, h)$ be defined as in* (11.5). *Here* $1 \le i \le 2k - 1$ *is arbitrary. Then, one has the estimate*:

$$|\mathfrak{L}_i(t,h)| \le C_1 e^{-C_2 t}$$

for h sufficiently small and $0 < t < \infty$. The symbols C_1 and C_2 denote positive constants, independent of h.

Proof. i) The first step is to obtain an upper bound, for $\mathfrak{L}_i(t, h)$, involving only scalar operators. We use the expression for $\mathfrak{L}_i(t, h)$, given in (11.7), as our starting point.

By the rewinding argument and the analogous fact for compact manifolds [14]:

$$\begin{aligned}\mathfrak{L}_i(t,h) = (-th)^{2k}\int_{-\infty}^{\infty}\mathrm{Tr}\Big(F_i\frac{\partial}{\partial y_i}e^{-t\bar{D}_h}W_2 \# e^{-t\bar{D}_h}W_2 \# \cdots\\ \# e^{-t\bar{D}_h}W_2 \# e^{-tD_h}\Big)(\xi,\xi)\,d\xi.\end{aligned}$$

Here Tr denotes the pointwise trace of the kernel for the convolution product, restricted to the diagonal. The integral runs over R^{2k-1}. Moreover, the convolution # appears $2k$ times.

Clearly, since F_i has constant entries:

$$|\mathfrak{L}_i(t, h)| \leq C_3(th)^{2k} \int_{-\infty}^{\infty} \left\| \frac{\partial}{\partial y_i} e^{-t\bar{D}_h} W_2 \# e^{-t\bar{D}_h} W_2 \# \cdots \# e^{-t\bar{D}_h} W_2 \# e^{-tD_h} \right\| (\xi, \xi)\, d\xi.$$

From (11.1), one has $W_1 + hW_2 \geq C_4$, for h sufficiently small. Here $C_4 > 0$ is a positive constant. Since $D_h = h^2\Delta + W_1 + hW_2$, the Trotter product formula [23, p. 297] yields

$$\|e^{-tD_h}\|(y, \xi) \leq e^{-C_4 t} e^{-th^2\Delta}(y, \xi)$$

with $\Delta = -\Sigma \partial^2/\partial y_i^2$, the standard Laplacian on R^{2k-1}.

By substitution, one has

$$(11.10a) \quad |\mathfrak{L}_i(t, h)| \leq C_3 e^{-C_4 t}(th)^{2k} \int_{-\infty}^{\infty} \left\| \frac{\partial}{\partial y_i} e^{-t\bar{D}_h} W_2 \# e^{-t\bar{D}_h} W_2 \# \cdots \# e^{-t\bar{D}_h} W_2 \right\| \# e^{-th^2\Delta}(\xi, \xi)\, d\xi.$$

This gives an upper bound, for $\mathfrak{L}_i(t, h)$, involving only scalar potentials.

ii) To continue, we need a representation for the kernel $\partial/\partial y_i \exp(-t\bar{D}_h)(y, \xi)$. By Duhamel's principle (11.2), with $P_1 = th^2\Delta$ and $P_2 = t\bar{D}_h$, one may write:

$$e^{-t\bar{D}_h}(y, \xi) = e^{-th^2\Delta}(y, \xi) - e^{-th^2\Delta} tW_1 \# e^{-t\bar{D}_h}(y, \xi).$$

Here we used the definition $\bar{D}_h = h^2\Delta + W_1$. It follows that

$$(11.10b) \quad \frac{\partial}{\partial y_i} e^{-t\bar{D}_h}(y, \xi) = \frac{\partial}{\partial y_i} e^{-th^2\Delta}(y, \xi) - \frac{\partial}{\partial y_i} e^{-th^2\Delta} tW_1 \# e^{-t\bar{D}_h}(y, \xi).$$

This last formula is useful, since the Euclidean heat kernel, $\exp(-th^2\Delta)(y, \xi)$, is given explicitly by (11.9). From calculus, one deduces:

$$(11.10c) \quad \left| \frac{\partial}{\partial y_i} e^{-th^2\Delta}(y, \xi) \right| \leq C_5 (th^2)^{-1/2} \exp\left(\frac{-th^2\Delta}{2} \right)(y, \xi).$$

Substituting (11.10b) and (11.10c) into (11.10a) gives:

$$|\mathfrak{L}_i(t, h)| \leq |\mathfrak{D}_1| + |\mathfrak{D}_2|$$

where

$$\mathfrak{D}_1 = C_6 e^{-C_4 t}(th)^{2k}(th^2)^{-1/2} \int_{-\infty}^{\infty} e^{-th^2\Delta/2} \#\| W_2 e^{-t\bar{D}_h} W_2 \# \cdots \# e^{-t\bar{D}_h} W_2 \| \# e^{-th^2\Delta}(\xi, \xi)\, d\xi.$$

The convolution # appears $2k$ times in $\mathfrak{D}_1$. Moreover,

$$\mathfrak{D}_2 = C_7 e^{-C_4 t}(th)^{2k}(th^2)^{-1/2} t \times \int_{-\infty}^{\infty} e^{-th^2\Delta/2} \# \| W_1 e^{-t\bar{D}_h} W_2 \# e^{-t\bar{D}_h} W_2 \# \cdots \# e^{-t\bar{D}_h} W_2 \| \# e^{-th^2\Delta}(\xi, \xi)\, d\xi.$$

Here, the convolution # appears $2k + 1$ times.

iii) The same procedure will be used to estimate the terms $\mathfrak{D}_1$ and $\mathfrak{D}_2$. We begin with $\mathfrak{D}_1$.

From the explicit formulas (11.1), one has

$$\| W_2(y_1) - W_2(y_2) \| \leq C_8 W_1^{1/2}(y_1) \| y_1 - y_2 \|$$

for y_1 and y_2 in a uniform neighborhood of the diagonal, i.e., for $\|y_1 - y_2\|$ uniformly bounded. Note that $\|W_2\| = O(W_1^{1/2})$.

Using this estimate, and the rapid decay of the heat kernel (11.8) outside the diagonal, we may deduce:

$$|\mathfrak{D}_1| \leq C_9 e^{-C_{10} t}(th)^{2k}(th^2)^{-1/2} \int_{-\infty}^{\infty} e^{-th^2\Delta/2} \# e^{-t\bar{D}_h} \# \cdots \# e^{-t\bar{D}_h} \# e^{-th^2\Delta}(\xi, \xi) |W_1(\xi)|^k\, d\xi.$$

Since the heat kernels appearing are positive and scalar valued, we have omitted the norm symbol.

By the semigroup property of $\exp(-t\bar{D}_h)$:

$$|\mathfrak{D}_1| \leq C_{11} e^{-C_{10} t}(th)^{2k}(th^2)^{-1/2} \int_{-\infty}^{\infty} e^{-th^2\Delta/2} \# e^{-t\bar{D}_h} \# e^{-th^2\Delta}(\xi, \xi) |W_1(\xi)|^k\, d\xi.$$

Using the explicit formulas (11.8) and (11.9), one has

$$|\mathfrak{D}_1| \leq C_{12} e^{-C_{13} t}(th)^{2k}(th^2)^{-1/2} \int_{-\infty}^{\infty} e^{-t\bar{D}_h}(\xi, \xi) |W_1(\xi)|^k\, d\xi.$$

This last integral may be estimated from above by (11.8) and the method of Ray [18, p. 317] to give:

$$|\mathfrak{D}_1| \leq C_{14} e^{-C_{15} t}(th)^{2k}(th^2)^{-1/2}(th^2)^{-(2k-1)/2} \int_{-\infty}^{\infty} e^{-tW_1(\xi)} |W_1(\xi)|^k\, d\xi.$$

Recall that our Euclidean space has dimension $2k - 1$.

Finally, calculus and the definition of $W_1(\xi)$, given by (11.1), may be applied to give:

$$|\mathfrak{D}_1| \leq C_{16} e^{-C_{15} t}.$$

iv) The term $\mathcal{D}_2$ is estimated by the same method as for $\mathcal{D}_1$. By repeating the steps in iii) with small modification, we obtain:

$$|\mathcal{D}_2| \leq C_{17}e^{-C_{18}t}(th)^{2k}(th^2)^{-1/2}t(th^2)^{-(2k-1)/2}\int_{-\infty}^{\infty} e^{-tW_1(\xi)}|W_1(\xi)|^{k+1}\,d\xi.$$

Calculus and the definition of $W_1(\xi)$ give:

$$|\mathcal{D}_2| \leq C_{19}e^{-C_{20}t}.$$

v) Since $|\mathcal{L}_i(t, h)| \leq |\mathcal{D}_1| + |\mathcal{D}_2|$, as shown above, Theorem 11.10 follows from the estimates for $\mathcal{D}_1$, $\mathcal{D}_2$ given in Part iii), iv) of the proof.

The proof of Proposition 11.10 is complete.

By combining 9.5 and 11.10, we deduce:

PROPOSITION 11.11. *Let* $\mathcal{L}(t, h) = \mathrm{Tr}(\dot{B}_h e^{-tD_h})$, *where* B_h *and* D_h *are as in Section* 8. *Then, as* $h \downarrow 0$, *there is an asymptotic expansion*:

$$\mathcal{L}(t, h) \sim b_0(t) + b_1(t)h + b_2(t)h^2 + \cdots$$

with $b_j(t)$ *bounded, for* $0 < t < \infty$, *and satisfying* $b_j(t) = O(e^{-C_j t})$, *as* $t \to \infty$. *Here* $C_j > 0$ *is a positive constant, for* $j \geq 0$.

Proof. One has $\mathcal{L}(t, h) = \Sigma\mathcal{L}_i(t, h)$, where $\mathcal{L}_i(t, h)$ are as in (11.5). Therefore, it suffices to show the existence of such expansions for each $\mathcal{L}_i$, $1 \leq i \leq 2k - 1$.

For the remainder of the proof, we fix a value of i. Suppose that, for some $l \geq 0$, we have

$$\mathcal{L}_i(t, h) \sim h^{-l}(\beta_{-l}(t) + \beta_{-l+1}(t)h + \cdots)$$

for some functions $\beta_m(t)$. Notice that Theorem 9.5 gives such an expansion with $l = 2k$. One has, by 11.10,

$$\beta_{-l}(t) = \lim_{h\to 0} h^l\mathcal{L}_i(t, h) = 0$$

since $l \geq 0$. It follows, by a finite induction, that

$$\mathcal{L}_i(t, h) \sim \beta_0(t) + \beta_1(t)h + \beta_2(t)h^2 + \cdots$$

as required.

A similar induction argument, starting from 9.5, and using 11.10 at each step, gives the desired estimates for the $\beta_m(t)$.

Now let $\eta(h, s)$ be the eta series (3.1) for our basic operator B_h:

$$\eta(h, s) = \sum_{\lambda\neq 0} \frac{\operatorname{sign}\lambda}{|\lambda|^s} \qquad \mathrm{Re}(s) \gg 0.$$

Here B_h is as given in Lemma 8.3.

The following result can be deduced from 11.10:

PROPOSITION 11.12. *Let $0 < h_1, h_2 < \varepsilon$, where ε is sufficiently small. Then $\eta(h_1, 0) = \eta(h_2, 0)$. In other words, the analytic continuations of $\eta(h_1, s)$ and $\eta(h_2, s)$ agree at $s = 0$.*

Proof. If ε is suitably chosen, then, by 8.6, zero does not occur in the spectrum of B_h, $0 < h < \varepsilon$. Thus, one may apply 5.1 to write:

$$\eta(h_2, s) - \eta(h_1, s) = -s\left[\Gamma\left(\frac{s+1}{2}\right)\right]^{-1}\int_{h_1}^{h_2}\int_0^\infty t^{(s-1)/2}\mathfrak{L}(t, h)\,dt\,dh$$

for $0 < h_1, h_2 < \varepsilon$ and $\operatorname{Re}(s) \gg 0$. By 11.10, the integral converges down to $s = 0$, so that

$$\eta(h_2, 0) - \eta(h_1, 0) = -O\left[\Gamma\left(\tfrac{1}{2}\right)\right]^{-1}\int_{h_1}^{h_2}\int_0^\infty t^{-1/2}\mathfrak{L}(t, h)\,dt\,dh = 0.$$

This proves Proposition 11.12.

12. Normalization

The proof of Theorem 4.2 is almost complete. We have given the technical details needed to establish formula (5.13) of our outline. Thus, $\eta(0) = c_k L(M', V, 0)$, for some constant c_k, depending only on k. It remains to evaluate c_k.

Let B_h be the basic operator of Lemma 8.3. According to our outline of Section 5, and the subsequent details in Sections 6 to 11, c_k is the value of the eta series of B_h at $s = 0$, for h sufficiently small. Proposition 11.12 shows that, as long as h is small enough, this value is independent of h.

We define, for $\beta > 0$ sufficiently large,

$$P_1 = \sum F_j\frac{\partial}{\partial y_j} + \beta\sum e^{y_l}E_l.$$

Clearly, by the definition of B_h given in Lemma 8.3, one has $P_1 = \beta B_h$, where one imposes the relation $\beta = h^{-1}$. It follows from formula (3.1) that the eta series of P_1 and B_h continue to the same value at $s = 0$.

Suppose $\eta_1(s)$ denotes the eta series of P_1. Clearly, $c_k = \eta_1(0)$, as long as β is sufficiently large. Therefore, we do not indicate the dependence of $\eta_1(s)$ upon β.

One may define an operator P_0: $L^2(R^{2k-1}, \mathfrak{M}) \to L^2(R^{2k-1}, \mathfrak{M})$ via the formula:

$$P_0 = \sum F_j\frac{\partial}{\partial y_j} + \frac{\beta}{2}(e^{y_1} + e^{y_{2k}})(E_1 + E_{2k}) + \beta\sum_{1<l<2k} e^{y_l}E_l.$$

Since $c_1 P_1^2 \le P_0^2 \le c_2 P_1^2$, the minimax principle implies that P_0 has pure point spectrum. Moreover, the eta series $\eta_0(s)$ of P_0 is absolutely convergent for $\mathrm{Re}(s)$ large.

We record the easy

LEMMA 12.1. *For all s, one has $\eta_0(s) = 0$.*

Proof. Let us conjugate P_0 successively by the unitary matrices $\sqrt{-1}\,F_1, \sqrt{-1}\,F_2, \ldots, \sqrt{-1}\,F_{2k-1}, 1/\sqrt{2}\,(E_1 + E_{2k}), E_2, \ldots, E_{2k-1}$. We find that P_0 is unitarily equivalent to $-P_0$. Lemma 12.1 now follows from the definition (3.1) of $\eta_0(s)$.

To compute the constant $c_k = \eta_1(0)$, we deform linearly from P_0 to P_1. Set $P_u = uP_1 + (1-u)P_0$, so that

$$(12.2)\quad P_u = \sum F_j \frac{\partial}{\partial y_j} + \frac{\beta}{2}\left[(1+u)e^{y_1} + (1-u)e^{y_{2k}}\right]E_l + \frac{\beta}{2}\left[(1+u)e^{y_{2k}} + (1-u)e^{y_1}\right]E_{2k} + \beta \sum_{1<l<2k} e^{y_l} E_l.$$

Clearly, one has

$$(12.3)\qquad \dot{P}_u = \frac{d}{du}P_u = P_1 - P_0 = \frac{\beta}{2}\left(e^{y_1} - e^{y_{2k}}\right)\left(E_1 - E_{2k}\right).$$

Define $Q_u = P_u^2$. A computation using (12.2) shows that one has:

LEMMA 12.4. *The operator Q_u is of the form:*

$$Q_u = -\sum \frac{\partial^2}{\partial y_j^2} + \beta^2 W_1 + \beta W_2$$

where

$$W_1(y) = \tfrac{1}{2}(u^2+1)\left(e^{2y_1} + e^{2y_{2k}}\right) + (1-u^2)e^{y_1+y_{2k}} + \sum_{1<l<2k} e^{2y_l}$$

and

$$W_2(y) = F_1\left[\left(\frac{1+u}{2}\right)E_1 + \left(\frac{1-u}{2}\right)E_{2k}\right]e^{y_1} - \sum F_j\left[\left(\frac{1+u}{2}\right)E_{2k} + \left(\frac{1-u}{2}\right)E_1\right]e^{y_{2k}} + \sum_{1<l<2k} e^{y_l}F_l E_l.$$

From this we can deduce:

LEMMA 12.5. *For β sufficiently large,* $\mathrm{Ker}(P_u) = 0$, *for all* $0 \le u \le 1$.

Proof. When β is large, $\beta^2 W_1 + \beta W_2$ is positive definite for all u and y. Thus $\mathrm{Ker}(Q_u) = 0$. Since $Q_u = P_u^2$, the lemma follows.

The primary technical step, in computing $\eta_1(0)$, is:

LEMMA 12.6. *As $t \downarrow 0$, one has*

$$\mathrm{Tr}\left(\dot{P}_u e^{-tQ_u}\right) = \frac{-2}{\sqrt{\pi}}\frac{1}{u^2+1}t^{-1/2} + O(1)$$

where the error estimate is independent of u. Here $0 < u \le 1$.

Proof. The rewinding argument of Section 9 shows that $\mathrm{Tr}(\dot{P}_u e^{-tQ_u})$ has an asymptotic expansion as $t \downarrow 0$, with coefficients depending on u. A priori, a finite number of the coefficients may be singular as u decreases to zero. However, we will employ the Feynman-Kac formula to show bounded dependence upon u, and to calculate the first non-vanishing term in t. This is the same type of method as used in Sections 9, 10, and 11. Therefore, for more detail on technical points, the reader may refer back to these sections.

Let $\bar{Q}_u$ be defined as $\bar{Q}_u = \Delta + \beta^2 W_1$, where $\Delta = -\Sigma\partial^2/\partial y_i^2$ is the standard Laplacian on R^n, for $n = 2k - 1$. Thus $\bar{Q}_u$ is the dominant scalar part of Q_u. By Lemma 12.4, we see that $Q_u - \bar{Q}_u = \beta W_2$. Note that $\|W_2\| \ll W_1$ at infinity.

We apply Duhamel's principle (11.3) with $P_1 = \bar{Q}_u$ and $P_2 = Q_u$. Substitution in (11.3) yields

$$e^{-tQ_u} = \sum_{j=0}^{l-1}(-\beta t)^j e^{-t\bar{Q}_u}W_2 \# e^{-t\bar{Q}_u}W_2 \# \cdots \# e^{-t\bar{Q}_u}W_2 \# e^{-t\bar{Q}_u}W_2 \# e^{-t\bar{Q}_u}$$
$$+ (-\beta t)^l e^{-t\bar{Q}_u}W_2 \# \cdots \# e^{-t\bar{Q}_u}W_2 \# e^{-tQ_u}$$

for any integer $l \ge 1$. It follows that

$$\mathrm{Tr}\left(\dot{P}_u e^{-tQ_u}\right) = \sum_{j=0}^{l-1}(-\beta t)^j \mathrm{Tr}\left(\dot{P}_u e^{-t\bar{Q}_u}W_2 \# e^{-t\bar{Q}_u}W_2 \# \cdots \# e^{-t\bar{Q}_u}\right)$$
$$+ (-\beta t)^l \mathrm{Tr}\left(\dot{P}_u e^{-t\bar{Q}_u}W_2 \# e^{-t\bar{Q}_u}W_2 \# \cdots \# e^{-t\bar{Q}_u}W_2 \# e^{-tQ_u}\right).$$

Now set $l = 2k$. By using formula (12.3) for $\dot{P}_u$ and the algebraic trace Lemma 10.3b), we may write:

$$\mathrm{Tr}\left(\dot{P}_u e^{-tQ_u}\right) = \mathfrak{D}_1 + \mathfrak{D}_2. \tag{12.6a}$$

Here, with $n = 2k - 1$,

$$\mathfrak{D}_1 = (-\beta t)^n \mathrm{Tr}\left(\dot{P}_u e^{-t\bar{Q}_u}W_2 \# e^{-t\bar{Q}_u}W_2 \# \cdots \# e^{-t\bar{Q}_u}W_2 \# e^{-t\bar{Q}_u}\right).$$

The convolution # appears $2k - 1$ times. Moreover,

$$\mathfrak{D}_2 = (-\beta t)^{n+1}\mathrm{Tr}\left(\dot{P}_u e^{-t\bar{Q}_u}W_2 \# e^{-t\bar{Q}_u}W_2 \# \cdots \# e^{-t\bar{Q}_u}W_2 \# e^{-tQ_u}\right).$$

The convolution # appears $2k$ times in the definition of $\mathfrak{D}_2$.

By applying the method of Theorems 11.10, 11.11, with small modifications, we deduce:

$$\mathfrak{D}_1 = (-\beta t)^n\mathrm{Tr}\left(\dot{P}_u W_2^{2k-1}e^{-t\bar{Q}_u} \# e^{-t\bar{Q}_u} \# \cdots \# e^{-t\bar{Q}_u}\right) + O(1)$$

and $\mathfrak{D}_2 = O(1)$. These error estimates are uniform in $0 \le u \le 1$.

Thus, by (12.6a), one has:

$$\mathrm{Tr}\left(\dot{P}_u e^{-tQ_u}\right) = (-\beta t)^n\mathrm{Tr}\left(\dot{P}_u W_2^{2k-1}e^{-t\bar{Q}_u} \# \cdots \# e^{-t\bar{Q}_u}\right) + O(1).$$

The convolution # appears n times.

By the definition (11.2) of the convolution and the semigroup property of $\exp(-t\bar{Q}_u)$, we have:

$$\mathrm{Tr}\left(\dot{P}_u e^{-tQ_u}\right) = -(\beta t)^n[(2k-1)!]^{-1}\mathrm{Tr}\left(\dot{P}_u W_2^{2k-1}e^{-t\bar{Q}_u}\right) + O(1).$$

The rewinding argument shows that we may write:

$$\mathrm{Tr}\left(\dot{P}_u e^{-t\bar{Q}_u}\right) = -(\beta t)^n[(2k-1)!]^{-1}\int_{-\infty}^{\infty}\mathrm{Tr}\left(\dot{P}_u W_2^{2k-1}\right)(\xi)e^{-t\bar{Q}_u}(\xi,\xi) + O(1).$$

The lead term estimate of D. B. Ray [18] now gives:

$$\mathrm{Tr}\left(\dot{P}_u e^{-tQ_u}\right) = (4\pi t)^{-n/2}\left(-\frac{1}{n!}\right)(\beta t)^n\int_{-\infty}^{\infty} e^{-t\beta^2 W_1(y)}\mathrm{Tr}\left(\dot{P}_u W_2^n\right)dy + O(1)$$

Using Lemmas 10.1, 10.2 and calculus, we find:

$$\mathrm{Tr}\left(\dot{P}_u e^{-tQ_u}\right) = (4\pi)^{-n/2}(\beta^2 t)^{n/2}(\mathfrak{D}_1 + \mathfrak{D}_2)2^{2n} + O(1).$$

Here

$$\mathfrak{D}_1 = -\frac{\beta}{2}\int_{-\infty}^{\infty} e^{-t\beta^2 W_1(y)}e^{2y_1+y_2+\cdots+y_{2k-1}}dy,$$

$$\mathfrak{D}_2 = -\frac{\beta}{2}\int_{-\infty}^{\infty} e^{-t\beta^2 W_1(y)}e^{-2y_1-y_2-\cdots-y_{2k-1}}dy.$$

The change of variables $\bar{y}_1 = -y_1 - y_2 - \cdots - y_{2k-1}$, $\bar{y}_j = y_j$, $j > 1$, gives $\mathfrak{D}_1 = \mathfrak{D}_2$. Moreover, by employing elementary calculus methods to estimate $\mathfrak{D}_1$, one finds

$$\mathrm{Tr}\left(\dot{P}_u e^{-tQ_u}\right) = \frac{-2}{\sqrt{\pi}}\frac{1}{u^2+1}t^{-1/2} + O(1).$$

This proves Lemma 12.6.

The main result of the present section is

PROPOSITION 12.7. *The eta invariant of P_1, at $s = 0$, is the number one,*

$$\eta_1(0) = 1.$$

Thus, the universal constant c_k, appearing in formula (5.13), *is one,*

$$c_k = 1.$$

Proof. We observed at the beginning of this section that $c_k = \eta_1(0)$. Thus, it suffices to find the value of $\eta_1(0)$. Using Lemma 12.5, one may apply Proposition 5.1 to write:

$$\eta_1(s) = \frac{-s}{\Gamma\left(\frac{s+1}{2}\right)} \int_0^1 \int_0^\infty t^{(s-1)/2} \operatorname{Tr}\left(\dot{P}_u e^{-tQ_u}\right) dt\, du \tag{12.8}$$

for $\operatorname{Re}(s)$ sufficiently large. One needs to justify using Proposition 5.1 in this noncompact setting. However, when $\operatorname{Re}(s)$ is large, only elementary arguments based on the minimax principle are required.

We now use Lemma 12.6 to analytically continue the right-hand side of (12.8). This gives

$$\eta_1(0) = \frac{-2}{\sqrt{\pi}} \int_0^1 \frac{-2}{\sqrt{\pi}} \frac{du}{u^2 + 1} = 1$$

With the evaluation of the constant c_k, the proof of Theorem 4.2 is complete.

13. Preparatory discussion

The goal of Part III of the paper is to prove Theorem 4.3. Recall that this theorem identifies the signature defect and the analytic eta invariant for the special framed manifolds constructed algebraically in Section 2.

Theorem 4.3 is essentially a long corollary to the general result of [3] on certain elliptic boundary value problems. In this section, we describe what can be deduced immediately from [3] for the eta invariants of a general framed manifold. The eta invariants under consideration are those of the flat connection associated to the framing.

The eta invariants for the Levi-Civita connection of a Riemannian metric were studied very thoroughly in [3]. For our special framed manifolds, of Section 2, we can control the deformation from the Levi-Civita connection, of the metric induced by the framing, to the flat connection, given by the framing. This leads to a proof of Theorem 4.3. Details are given in Sections 14 and 15.

In the remainder of this section, we discuss the eta invariant theorem for a general framed manifold X^{4k-1}, with framing f. Our eta invariants will be those of the flat connection for the given framing.

Let W^{4k} be a compact Riemannian manifold with a given connection on TW preserving the metric. Suppose that $X^{4k-1} = \partial W$ is our framed manifold with framing f. Assume that, in a product neighborhood of the boundary X, the metric and connection coincide with the product metric and product flat connection induced by the framing f. Of course, this means that the connection on TW will, in general, have non-zero torsion and thus cannot coincide with the Levi-Civita connection.

Note that such W can always be found for a given (X, f). This is a consequence of cobordism theory and partition of unity arguments. We assume that W is chosen and fixed throughout the discussion.

We may define the analytic eta invariant $\eta_A(0)$, as in Section 3, by using the skewed covariant differential of the framing. The main result of [3] now specializes to give:

THEOREM 13.1. *We may write* $\eta_A(0) = \int_W \mathfrak{D} - l$. *Here* l *is an integer. The integrand* $\mathfrak{D}$ *is locally defined. Moreover,* $\mathfrak{D}$ *is invariant under scaling of the metric on* W.

Proof. Let $d\colon \Lambda^p W \to \Lambda^{p+1} W$ be the skewed covariant differential obtained from the connection on TW. Suppose δ is the adjoint of d. Clearly, $d + \delta$ is a first order elliptic operator, acting on the bundle of differential forms.

Define an involution τ by $\tau\phi = i^{p(p-1)+2k} * \phi$, for $\phi \in \dot{\Lambda}^p W$, where $*$ is the Hodge star operator. Since $\tau^2 = 1$, we have a decomposition of the bundle of differential forms $\Lambda = \Lambda^+ \oplus \Lambda^-$, into the ± 1 eigenspaces of τ. It may be verified that $d + \delta$ interchanges Λ^+ and Λ^-.

To obtain Theorem 13.1, one applies the general result of [3] to the modified signature complex:

$$\Lambda^+ W \xrightarrow{d+\delta} \Lambda^- W.$$

Here, the suitable non-local boundary conditions are imposed. It follows routinely that $\eta_A(0) = \int_W \mathfrak{D} - l$. Also, l is an integer and $\mathfrak{D}$ is locally defined.

To complete the proof, we need only show that $\mathfrak{D}$ is invariant under scaling of the metric. Suppose that ϕ is an eigenfunction of $\Delta = (d + \delta)^2$ with eigenvalue λ. Define Ψ by $\Psi_p = c^p \phi_p$, where ϕ_p is the p-form component of ϕ. Then Ψ is an eigenfunction of the Laplacian $\bar{\Delta}$, for the scaled metric $\bar{g} = c^2 g$, with eigenvalue $\bar{\lambda} = c^{-2}\lambda$. The scale invariance of $\mathfrak{D}$ now follows from the standard formula:

$$K(t, x, y) = \sum_{\lambda} e^{-t\lambda} \phi_\lambda(x) \otimes \phi_\lambda(y).$$

Here the sum runs over the eigenfunctions ϕ_λ of Δ.

For the remainder of this section, we will investigate the nature of the integrand $\mathfrak{D}$.

To identify $\mathfrak{D}$ more precisely, one may employ the invariance theory methods of [1]. In that work, only the torsion-free Levi-Civita connection was considered. However, a routine extension of the argument in [1] shows that $\mathfrak{D}$ is an $O(4k)$ invariant polynomial map, in the components of the torsion T and curvature R and their covariant derivatives for the extended connections, with values in $4k$ forms. For more details on how the torsion T enters, the reader is referred to Section 1 of [8].

If T is identically zero, then $\mathfrak{D}$ depends only on the curvature of the Levi-Civita connection. The fact that $\mathfrak{D}$ is invariant under scaling of the metric is then sufficient to identify $\mathfrak{D}$ with the L-polynomial of Hirzebruch [1].

Unfortunately, when $T \neq 0$, there are many more scale invariant maps in the components of T and its covariant derivatives. In fact, the proof of $\mathfrak{D} = L_k$, for $T = 0$, in [1, pp. 286–289] relied heavily on symmetries of the curvature tensor. Such arguments fail when the torsion is non-vanishing.

To summarize, we may state:

LEMMA 13.2. *The integrand $\mathfrak{D}$ appearing in Theorem 13.1 is an $O(4k)$ invariant polynomial in the components of the curvature tensor R and torsion tensor T and their covariant derivatives, for the metric preserving connection on T^*W, with values in $4k$-forms. Moreover, $\mathfrak{D}$ is invariant under scaling of the metric.*

If T and all its covariant derivatives vanish, at some point, then $\mathfrak{D} = L_k(\Omega)$ is the Hirzebruch L-polynomial at that point. Here Ω denotes the curvature form.

To gain more control of the situation, we will restrict the connection on T^*W to be of a special type. Note that Theorem 13.1 applies to any connection which coincides with the flat connection on a product neighborhood of the boundary and preserves the extended metric.

First, recall the following [10, p. 48]:

LEMMA 13.3. *Let M be a complete Riemannian manifold. Suppose that there is an alternating bilinear map T: $TM \times TM \to TM$. Then there is a unique connection ∇ on TM such that*:

(i) *∇ preserves the Riemannian metric.*

(ii) *∇ has torsion tensor T.*

Suppose now that T_0 denotes the torsion tensor of the flat connection given by the framing on X. We regard T_0 as a tensor over $X \times I$ by lifting from the projection $X \times I \to X$. Here I is the unit interval.

Let $0 \leq y \leq 1$ be the standard coordinate on I. Choose a function $f \in C^\infty[0, 1]$ satisfying:

$$f(y) = \begin{cases} 1 & |y| < \varepsilon \\ 0 & |y| > 1 - \varepsilon \end{cases}$$

for some $0 < \varepsilon < 1/2$. Denote T to be the tensor $f(y)T_0$.

Considering $X \times I$ as a sufficiently small product neighborhood of $X = \partial W \subset W$, we may extend the metric given by the framing on X to a metric g on W, so that g is a product metric on $X \times I$. Similarly, T may be extended to W by setting $T = 0$ on $W - X \times [0, 1 - \varepsilon]$. Let ∇ be the unique connection, given in Lemma 13.3, which preserves the metric g and has torsion tensor T.

It follows from 13.3 that on $X \times I$ the curvature and torsion tensors and their covariant derivatives, for ∇, depend polynomially upon the derivatives of f and the torsion tensor T_0 and its covariant derivatives with respect to the flat connection on X, after we raise and lower indices via g.

Recall that an invariant polynomial map P has weight k [1, p. 282] if $P \to \lambda^k P$ under scaling of the metric $g \to \lambda^2 g$. Of course, when $k \geq 0$, we say that P has non-negative weight.

By consolidating the above discussion, we may deduce:

PROPOSITION 13.5. *Let the connection ∇ on W be chosen as described above. Then the local integrand $\mathfrak{D}$ appearing in Theorem* 13.1 *satisfies*:

i) *On $W - X \times [0, 1 - \varepsilon]$, $\mathfrak{D} = L_k(\Omega)$, the Hirzebruch L-polynomial applied to the curvature form Ω.*

ii) *On $X \times [0, 1 - \varepsilon]$,*

$$\mathfrak{D} = \sum a_i(f) P_i(T_0).$$

Here $\mathfrak{D}$ is regarded as an $O(4k)$ invariant polynomial with values in $4k$-forms.

Here $a_i(f)$ is a polynomial in f and the derivatives of f with values in $\Lambda^1(I)$. Moreover, each $P_i(T_0)$ is an $O(4k - 1)$ invariant polynomial, in the components of T_0 and its covariant derivatives with respect to the flat connection, having values in the $4k - 1$ forms on X. Moreover, each P_i has non-negative weight.

Proof. It is important to see that each P_i has non-negative weight. For this, observe that the derivatives of f have weight zero and $a_i(f)$ is obtained, from these derivatives, by contractions with g^{ij}, actually g^{nn}, where $n = 4k$. Therefore, each $a_i(f)$ has non-positive weight. Since $\mathfrak{D}$ has weight zero, the P_i must have non-negative weight.

It should be emphasized that invariance theory methods do not allow one to conclude $P_i(T_0) = 0$ in general.

We close this section with some general facts concerning polynomial maps of non-negative weight in the components of a general torsion tensor T. Recall that T is of weight zero when considered as a tensor of type $(1, 2)$, in indicial notation T^i_{jk}. Specifically, the definition is

$$T(Y, Z) = \nabla_Y Z - \nabla_Z Y - [Y, Z] \tag{13.6}$$

for vector fields Y, Z. Formula (13.6) is unaffected by scaling of the metric.

Lowering the first index gives a tensor of weight two:

$$T_{ijk} = g_{is} T^s_{jk}.$$

In general, the only symmetry of the torsion tensor is

$$T_{ijk} = -T_{ikj}. \tag{13.7}$$

This fact follows from (13.6) and Lemma 13.3.

If α is a multi-index with length $l = |\alpha| \geq 3$, then one denotes:

$$T_\alpha = T_{\alpha_1 \alpha_2 \alpha_3,\, \alpha_4 \alpha_5 \cdots \alpha_l}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_l)$. The indices $\alpha_4, \alpha_5, \ldots, \alpha_l$ refer to covariant derivatives of T.

Every invariant polynomial is a finite linear combination of elementary monomials. By the definition of [1, p. 286], an elementary monomial $m(T)$, in T, is given by:

$$m(T) = \sum_q^{*} T_{\alpha^1} T_{\alpha^2} \cdots T_{\alpha^r}. \tag{13.8}$$

Alternation runs over precisely q indices and the remaining indices are contracted. Moreover, α^i are multi-indices.

By analogy with Lemma 2 of [1, p. 287], it is easy to derive:

PROPOSITION 13.9. *The elementary monomial* $m(T)$ *given in* (13.8) *has weight* $2r + q - \Sigma|\alpha^i|$. *Here* $|\alpha^i|$ *denotes the length of the multi-index* α^i.

Proof. Each T_{α^i} has weight two before any contractions are applied. The total number of contractions is $[\Sigma|\alpha^i| - q]/2$. Moreover, each contraction decreases the weight by two. Thus, $m(T)$ has weight

$$2r - 2\left[\sum|\alpha^i| - q\right]/2 = 2r + q - \sum|\alpha^i|.$$

14. The integer part

In Sections 14 and 15, we study the eta invariant for the flat connection on the special framed manifolds $X = S(M, V)\backslash S(R^{2k}, R^{2k-1})$. Here, the flat connection comes from a left invariant framing on the solvable group $S(R^{2k}, R^{2k-1})$. The goal is to identify the integrand $\mathfrak{D}$ and integer l appearing in Theorem 13.1. The method in both cases is by a deformation argument from the flat connection to the Levi-Civita connection of a left invariant metric.

In this section, we show that the integer l appearing in Theorem 13.1 is $\operatorname{sign} W$, the signature of W. For the operator associated to the Levi-Civita connection, of any Riemannian metric, this is proved in [3]. In general, such integers will jump under deformation of elliptic operators, even if the leading symbol of the operator is preserved. This is due to the spectral flow of eigenvalues across the origin [4]. However, for the special case under consideration, it will be shown that l is invariant under deformation from the flat connection, of the left invariant framing, to the Levi-Civita connection, of the associated left invariant metric.

Let $X = S(M, V)\backslash S(R^{2k}, R^{2k-1})$ be one of the framed manifolds constructed in Section 2. Formula (6.2) shows that the eta invariant operator of the flat connection is given by:

$$A_0 = \sum F_j f_j - \sqrt{-1} \sum E_m e_m. \tag{14.1}$$

Here $f_j = \partial/\partial y_j$ and $e_m = e^{y_m}\partial/\partial x_m$ are a basis for the Lie algebra of the solvable group $S(R^{2k}, R^{2k-1})$. Moreover, F_j, E_m: $\mathfrak{M} \to \mathfrak{M}$ are endomorphisms of the fixed vector space $\mathfrak{M} = \Lambda^{ev}\mathfrak{S}$, where $\mathfrak{S}$ denotes the Lie algebra of the solvable group $S(R^{2k}, R^{2k-1})$.

We may decompose the Hilbert space of square integrable $\mathfrak{M}$-valued functions on X as an orthogonal direct sum:

$$L^2(X, \mathfrak{M}) = H \oplus H^{\perp}. \tag{14.2}$$

Here H is the space of constant $\mathfrak{M}$-valued functions. Thus H is finite dimensional and isomorphic to $\mathfrak{M}$. Moreover, $H^{\perp}$ is the orthogonal complement of H in $L^2(X, \mathfrak{M})$.

The relevance of the decomposition (14.2) is suggested by:

LEMMA 14.3. *Given a positive constant $c > 0$, then by a uniform scaling of the lattices M and V, we may assure*:

i) $\operatorname{Ker} A_0 = H$,

ii) $A_0^2 - c > 0$ *on* $H^{\perp}$.

Here A_0, as in (14.1), *is the operator associated to the flat connection.*

Proof. This follows from the formulas in Lemma 7.3. For a related observation, see Corollary 8.6.

Now let A_1 be the eta operator associated to the Levi-Civita connection of the metric induced by the framing. Thus, A_1 is constructed as in Section 3, but with the Levi-Civita connection instead of the flat connection.

One may write:

$$A_1 = A_0 + G \tag{14.4}$$

where $G\colon \mathcal{M} \to \mathcal{M}$ is a constant endomorphism. Formula (14.4) follows from the fact that A_0 and A_1 are first order operators with the same leading symbol and descend from left invariant operators on the Lie group $S(R^{2k}, R^{2k-1})$.

Let us deform linearly from A_1 to A_0. We set

$$A_t = tA_1 + (1-t)A_0 = A_0 + tG \qquad \text{for } 0 \leq t \leq 1. \tag{14.5}$$

LEMMA 14.6. *Assume that the lattices M and V have been suitably scaled. Then* $\operatorname{Ker} A_t \subset H$, *for all* $0 \leq t \leq 1$. *Consequently,*

$$\operatorname{Ker} A_t = \begin{cases} H & t = 0 \\ \operatorname{Ker} G & t > 0. \end{cases}$$

Here Ker G *denotes the kernel of G as an endomorphism of the finite dimensional vector space H.*

Proof. This follows from Lemma 14.3 and formula (14.5). One just chooses $c > \|G\|^2$.

Now suppose that η_t is the eta invariant of A_t. That is, η_t is the value of the eta function (3.1) at $s = 0$. Since each A_t preserves the decomposition (14.2), one has

$$\eta_t = \eta_t(H) + \eta_t(H^\perp)$$

where η_t is the eta invariant of A_t restricted to H. Recall that H is finite dimensional, so there is no problem regarding analytic continuation of the eta series restricted to H.

We study each piece of η_t separately, to determine the dependence upon t:

LEMMA 14.7. *If M, V are scaled as in Lemma* 14.3, *then* $\eta_t(H^\perp)$ *is a continuous function of t.*

Proof. According to Lemma 14.6, we have $H^\perp \cap \operatorname{Ker}(A_t) = 0$ for all $0 \leq t \leq 1$. Since discontinuities arise only via the spectral flow across the origin, the result follows (see Proposition 5.1).

Also, one has

LEMMA 14.8. *For all* $0 \leq t \leq 1$, *one has* $\eta_t(H) = 0$.

Proof. If (y_i, x_j) are the coordinates chosen at the beginning of Section 6, then the map $y_i \to y_i$, $x_1 \to -x_1$, $x_j \to x_j$, $j \geq 2$, defines an orientation reversing local diffeomorphism near the origin.

This map induces an isometry $i_H \colon H \to H$ which anticommutes with the action of A_1. Since G is exactly the restriction of A_1 to H, i_H anticommutes with G. However, A_t restricts to H as tG, so i_H and A_t will anticommute, as endomorphisms of H, for $t > 0$. Lemma 14.8 follows.

Combining Lemmas 14.7 and 14.8, one deduces:

LEMMA 14.9. *Under the hypothesis of Lemma* 14.3, *the eta invariant* η_t *of* A_t *acting on* $L^2(X, \mathfrak{M})$ *is a continuous function of* t, $0 \leq t \leq 1$.

We can now establish:

THEOREM 14.10. *If* $X = S(M, V) \setminus S(R^{2k}, R^{2k-1})$, *with the lattices* M, V *suitably uniformly scaled, then the integer* l *appearing in Theorem* 13.1 *is the signature of* W, $l = \operatorname{sign} W$.

Proof. Using the general theorem of [3, p. 57] on elliptic boundary value problems, we find that, for the operator A_t:

$$l_t = \int_W \mathfrak{D}_t - \eta_t$$

where l_t is an integer. Moreover, the integrand $\mathfrak{D}_t$ is a continuous function of t. By Lemma 14.9, η_t is also continuous for the special case under consideration. Therefore l_t is a constant, since it is a continuous integer-valued function. In particular, $l = l_0 = l_1$. However, A_1 is the operator associated to the Levi-Civita connection of a Riemannian metric and from [3, p. 66], we know that $l_1 = \operatorname{sign} W$. This proves Theorem 14.10.

The technical devices of scaling the lattices M and V, employed in this section, will not disturb the proof of Theorem 4.3. To see this, let A be the operator associated to the flat connection. If $\lambda \in \operatorname{Spec} A$, then, under the uniform scaling of Theorem 14.10, $\lambda \to b^2\lambda$. Here b^2 is a positive constant, independent of λ. By the definition (3.1) of the analytic eta invariant, $\eta_A(s) \to b^{-2s}\eta_A(s)$. Thus $\eta_A(0)$ is independent of the uniform scaling. Similarly, $\sigma(X, f)$ is a diffeomorphism invariant and therefore unchanged under scaling.

5. Identifying the integrand

Let $X = S(M, V) \setminus S(R^{2k}, R^{2k-1})$ be one of the algebraically defined framed manifolds appearing in Section 2. Suppose that $X = \partial W$ and that the flat connection on TX, given by the framing, has been extended to a connection on TW, by the construction of Section 13.

The purpose of the present section is to show that, under these circumstances, the integrand $\mathfrak{D}$, of Theorem 13.1, is the Hirzebruch L-polynomial. That is, $\mathfrak{D} = L_k(\Omega)$. Here Ω is the curvature form of the extended connection on TW. Our work in the current section is purely local and therefore does not depend upon the discrete group $S(M, V)$.

In Proposition 13.5, we observed that $\mathfrak{D} = L_k(\Omega)$ outside a collar neighborhood of the boundary X. Moreover, on this collar, $\mathfrak{D}$ is an invariant polynomial of specified type. In particular, $\mathfrak{D}$ is a sum of invariant polynomials of positive weight in the torsion tensor T of the flat connection on TX. Our notation differs slightly from Proposition 13.5, where T_0 denoted the torsion tensor of the flat connection on TX.

Now, T has special symmetries, since it is defined by the Lie bracket on the solvable group $S(R^{2k}, R^{2k-1})$. By exploiting these extra symmetries in T, we will show that all invariant polynomials of positive weight must vanish.

Thus, $\mathfrak{D} = 0$ on the collar neighborhood of $X = \partial W$. Also $L_k(\Omega) = 0$, on the collar, since the metric is a product near the boundary. Combined with Proposition 13.5, this gives $\mathfrak{D} = L_k(\Omega)$ everywhere on W.

Let us turn to the detailed proof of the above assertions.

Suppose that $\mathfrak{S}$ is the Lie algebra of $S(R^{2k}, R^{2k-1})$. For vector fields $Y, Z \in \mathfrak{S}$, it follows from formula (13.6) that the torsion is given by:

$$T(Y, Z) = -[Y, Z]. \tag{15.1}$$

Here, we are working with the flat connection which defines elements of $\mathfrak{S}$ to be parallel vector fields.

We begin by recording:

LEMMA 15.2. *The torsion tensor T of the flat connection associated to $\mathfrak{S}$ is parallel. This means that all covariant derivatives of T vanish identically.*

Proof. This follows immediately from (15.1) and would hold for any Lie group. It does not require the special structure of the group $S(R^{2k}, R^{2k-1})$.

Lemma 15.2 shows that, in studying the elementary monomials $m(T)$ of (13.8), one may assume $|\alpha^i| = 3$, for all i. Therefore, Proposition 13.9 gives that weight $(m(T)) = 2r + q - 3r = q - r$.

We are concerned with the case $q = 4k - 1 = \dim X$ and weight $(m(T)) \geq 0$. Therefore, one has $r \leq 4k - 1$ for each elementary monomial under consideration.

Let us consider the special structure of the Lie algebra of $S(M, V)$. In Section 6, a basis $f_i = \partial/\partial y_i$, $1 \leq i \leq 2k - 1$, and $e_{j+2k-1} = e^{y_j}\partial/\partial x_j$, $1 \leq j \leq 2k$, was given for the Lie algebra $\mathfrak{S}$. Recall that by definition $y_{2k} = -y_1 - \cdots - y_{2k-1}$.

LEMMA 15.3. *In the frame field f_i, e_γ, the only non-vanishing components of T are*:

(i) $T^{\gamma}_{\gamma i} = -T^{\gamma}_{i\gamma} = 1$, *where* $\gamma - i = 2k - 1$. *Moreover*, $1 \leq i \leq 2k - 1$ *and* $2k \leq \gamma \leq 4k - 2$.

(ii) $T^{\gamma}_{\gamma i} = -T^{\gamma}_{i\gamma} = -1$, *where* $\gamma = 4k - 1$ *and* $1 \leq i \leq 2k - 1$ *is arbitrary.*

Proof. This follows by explicit computation using (15.1) and the local coordinate expressions for f_i and e_γ.

Since we have chosen an orthonormal frame field, $g_{ij} = \delta_{ij}$, Lemma 15.3 also holds when the first index in T is lowered.

A useful consequence of Lemma 15.3 is the identity:

$$T_{\underset{\smile}{ijk}} = 0. \tag{15.4}$$

Here, the bow denotes alternation.

The main technical result of this section is:

PROPOSITION 15.5. *Let T be the torsion tensor of the flat connection on the $4k - 1$ dimensional manifold $X = S(M,V) \setminus S(R^{2k}, R^{2k-1})$. Then any $O(4k - 1)$ invariant polynomial map of non-negative weight, in the components of T and its covariant derivatives, with values in $4k - 1$-forms, must vanish identically.*

Proof. By the usual invariant theory of $O(4k - 1)$, [1, p. 286], it suffices to establish this result for the elementary monomials $m(T)$ of (13.8). We assume that a non-zero $m(T)$ exists and reason by contradiction.

For convenience, we let the Greek indices γ refer to the vector fields e_γ, $2k \leq \gamma \leq 4k - 1$. Latin indices $1 \leq j, l, m \leq 4k - 1$ will refer either to the e_j's or the f_j's.

As a consequence of Proposition 13.9 and Lemma 15.2, we may assume that

$$m(T) = \sum_q^* T_{j_1 l_1 m_1} T_{j_2 l_2 m_2} \cdots T_{j_r l_r m_r}$$

where $q = 4k - 1$ and $r \leq q$. This was already observed.

By Lemma 15.3, we may write

$$m(T) = \sum_q^* T_{\gamma_1 l_1 m_1} T_{\gamma_2 l_2 m_2} \cdots T_{\gamma_r l_r m_r}$$

since it is only necessary to sum over non-zero terms.

Since $q = 4k - 1$, we must alternate over at least one of the second or third indices in $2k - 1$ of the T's. This is because each γ_i is restricted to the range $2k \leq \gamma_i \leq 4k - 1$.

Moreover, by the basic symmetry (13.7), if either the second or third index is alternated in any T, we may assume that both indices are alternated. Otherwise, $m(T)$ factors through $4k$-forms and must vanish since $\dim X = 4k - 1$.

By reordering the terms, one may write

$$m(T) = \sum_q^{*} T_{\gamma_1 l_1 m_1} T_{\gamma_2 l_2 m_2} \cdots T_{\gamma_n l_n m_n} T_{\gamma_{2k} l_{2k} m_{2k}} \cdots T_{\gamma_r l_r m_r} \tag{15.6}$$

where $n = 2k - 1$. The bow denotes alternation in all indices involved.

Since $q = 4k - 1 = 2(2k - 1) + 1$, there is just one more alternation in addition to those indicated explicitly in (15.6). By (15.4), we may alternate at most two indices in any given copy of T. Consequently, an index must be alternated in some $T_{\gamma_s l_s m_s}$, for $s \geq 2k$. Finally, only a first index, some γ_s, can be alternated, since otherwise the map factors through $4k$-forms.

Again, by reordering the terms, we may assume that

$$m(T) = \sum_q^{*} T_{\gamma_1 l_1 m_1} T_{\gamma_2 l_2 m_2} \cdots T_{\gamma_n l_n m_n} T_{\gamma_{2k} l_{2k} m_{2k}} \cdots T_{\gamma_r l_r m_r} \tag{15.7}$$

with $n = 2k - 1$. All $q = 4k - 1$ alternations have been indicated, so the remaining indices must be contracted.

By Lemma 15.3, one may suppose that $\gamma_i = l_i$ or $\gamma_i = m_i$, $1 \leq i \leq r$, for each term giving a non-zero contribution to the sum (15.7). Therefore, $\gamma_1, \gamma_2, \ldots, \gamma_{2k-1}, \gamma_{2k}$ must have different values.

Each of the γ_i's for $1 \leq i \leq 2k - 1$ must be contracted with an index in some $T_{\gamma_s l_s m_s}$, $s > 2k$. This follows from Lemma 15.3 and the fact that $\gamma_1, \gamma_2, \ldots, \gamma_{2k}$ are distinct. By Lemma 15.3, no two γ_i's can be contracted with indices in the same copy of T. This restriction forces $r \geq 2k + 2k - 1 = 4k - 1$.

Pursuing this argument one step further, observe that either $\gamma_{2k} = l_{2k}$ or $\gamma_{2k} = m_{2k}$. In each case l_{2k}, respectively m_{2k}, must be contracted with an index in yet another copy of T. This yields $r \geq 2k + 2k = 4k$. However, we have already shown that the condition of non-negative weight requires $r \leq 4k - 1$. This fact was noted at the beginning of our proof. Since the inequalities $r \leq 4k - 1$ and $r \geq 4k$ cannot both hold, we have obtained a contradiction.

Thus $m(T) = 0$. The proof of Proposition 15.5 is complete.

By combining the results of Proposition 13.5 and Proposition 15.5, we may deduce:

Proposition 15.8. *Let the connection ∇ on TW be chosen as described in Proposition 13.5 and let $X = S(M, V) \setminus S(R^{2k}, R^{2k-1})$ be one of the manifolds* *considered above. Then the integrand* $\mathfrak{D}$, *appearing in Theorem* 13.1, *is given by* $\mathfrak{D} = L_k(\Omega)$, *the Hirzebruch L-polynomial in the curvature form* Ω *of* ∇.

Finally, Theorem 4.3 follows from (13.1) and (15.8) because, by the theory of relative characteristic classes, one has

$$\int_W \mathfrak{D} = L_k(p_1, \ldots, p_k)[W, X]$$

where $p_i \in H^{4i}(W, X)$ are the relative Pontrjagin classes associated to the framing of X.

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