POWER OPERATIONS IN K-THEORY

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[Received 10 January 1966]

Introduction

FOR any finite CW-complex X we can define the Grothendieck group $K(X)$. It is constructed from the set of complex vector bundles over X [see (**8**) for precise definitions]. It has many formal similarities to the cohomology of X, but there is one striking difference. Whereas cohomology is *graded*, by dimension, $K(X)$ has only a *filtration*: the subgroup $K_q(X)$ is defined as the kernel of the restriction homomorphism

$$K(X) \to K(X_{q-1}),$$

where X_{q-1} is the $(q-1)$-skeleton of X. Now $K(X)$ has a ring structure, induced by the tensor product of vector bundles, and this is compatible with the filtration, so that $K(X)$ becomes a filtered ring. There are also natural operations in $K(X)$, induced by the exterior powers, and one of the main purposes of this paper is to examine the relation between operations and filtration (Theorem 4.3).

Besides the formal analogy between $K(X)$ and cohomology there is a more precise relationship. If X has no torsion this takes a particularly simple form, namely the even-dimensional part of the integral cohomology ring

$$H^{\mathrm{ev}}(X; \mathbf{Z}) = \sum_q H^{2q}(X; \mathbf{Z})$$

is naturally isomorphic to the graded ring

$$GK(X) = \sum_q K_{2q}(X)/K_{2q-1}(X).$$

Since this isomorphism preserves the ring structures, it is natural to ask about the operations. Can we relate the operations in K-theory to the Steenrod operations in cohomology?

If we consider the way the operations arise in the two theories, we see that in both cases a key role is played by the symmetric group. It is well known [cf. (**10**)] that one way of introducing the Steenrod operations is via the cohomology of the symmetric group (and its subgroups). On the other hand, the operations on vector bundles come essentially from representations of the general linear group and the role of the symmetric group in constructing the irreducible representations of $GL(n)$ is of course classical [cf. (**11**)]. A closer examination of the two cases shows that the symmetric group enters in essentially the same way in both theories. The operations arise from the interplay of the kth power map and the action of the symmetric group S_k.

We shall develop this point of view and, following Steenrod, we shall introduce operations in K-theory corresponding to any subgroup G of S_k. Taking $k = p$ (a prime) and $G = Z_p$ to be the cyclic group of order p we find that the only non-trivial operation defined by Z_p is the Adams operation ψ^p. This shows that ψ^p is analogous to the total Steenrod power operation $\sum P^i$ and, for spaces without torsion, we obtain the precise relationship between ψ^p and the P^i (Theorem 6.5). Incidentally we give a rather simple geometrical description (2.7) of the operation ψ^p.

It is not difficult to translate Theorem 6.5 into rational cohomology by use of the Chern character, and (for spaces without torsion) we recover a theorem of Adams (**1**). In fact this paper originated in an attempt to obtain Adams's results by more direct and elementary methods.

Although the only essentially new results are concerned with the relation between operations and filtration, it seems appropriate to give a new self-contained account of the theory of operations in K-theory. We assume known the standard facts about K-theory [cf. (**8**)] and the theory of representations of *finite* groups. We do not assume anything about representations of compact Lie groups.

In § 1 we present what is relevant from the classical theory of the symmetric group and tensor products. We follow essentially an idea of Schur [see (**11**) 215], which puts the emphasis on the symmetric group S_k rather than the general linear group $GL(n)$. This seems particularly appropriate for K-theory where the dimension n is rather a nuisance (it can even be negative!). Thus we introduce a graded ring

$$R_* = \sum_k \mathrm{Hom}_{\mathbf{Z}}(R(S_k), \mathbf{Z}),$$

where $R(S_k)$ is the character ring of S_k, and we study this in considerable detail. Among the formulae we obtain, at least one (Proposition 1.9) is probably not well known. In § 2, by considering the tensor powers of a graded vector bundle, we show how to define a ring homomorphism

$$j \colon R_* \to \mathrm{Op}(K),$$

where $\mathrm{Op}(K)$ stands for the operations in K-theory. The detailed information about R_* obtained in § 1 is then applied to yield results in K-theory.

§ 3 is concerned with 'externalizing' and 'relativizing' the tensor powers defined in § 2. Then in § 4 we study the relation of operations and filtration. § 5 is devoted to the cyclic group of prime order and its related operations. In § 6 we investigate briefly our operations in connexion with the spectral sequence $H^*(X, \mathbf{Z}) \Rightarrow K^*(X)$ and obtain in particular the relation with the Steenrod powers mentioned earlier. Finally in § 7 we translate things into rational cohomology and derive Adams's result.

The general exposition is considerably simplified by introducing the functor $K_G(X)$ for a G-space X (§ 2). We establish some of its elementary properties but for a fuller treatment we refer to (**4**) and (**9**).

The key idea that one should consider the symmetric group acting on the kth power of a complex of vector bundles is due originally to Grothendieck, and there is a considerable overlap between our presentation of operations in K-theory and some of his unpublished work.

I am indebted to P. Cartier and B. Kostant for some very enlightening discussions.

1. Tensor products and the symmetric group

For any finite group G we denote by $R(G)$ the free abelian group generated by the (isomorphism classes of) irreducible complex representations of G. It is a ring with respect to the tensor product. By assigning to each irreducible representation its character we obtain an embedding of $R(G)$ in the ring of all complex-valued class functions on G. We shall frequently identify $R(G)$ with this subring and refer to it as the *character ring* of G. For any two finite groups G, H we have a natural isomorphism

$$R(G) \otimes R(H) \to R(G \times H).$$

Now let S_k be the symmetric group and let $\{V_\pi\}$ be a complete set of irreducible complex S_k-modules. Here π may be regarded as a partition of k, but no use will be made of this fact. Let E be a complex vector space, $E^{\otimes k}$ its kth tensor power. The group S_k acts on this in a natural way, and we consider the classical decomposition

$$E^{\otimes k} \cong \sum V_\pi \otimes \pi(E),$$

where $\pi(E) = \mathrm{Hom}_{S_k}(V_\pi, E^{\otimes k})$. We note in particular the two extreme cases: if V_π is the trivial one-dimensional representation, then $\pi(E)$ is the kth symmetric power $\sigma^k(E)$; if V_π is the sign representation, then $\pi(E)$ is the kth exterior power $\lambda^k(E)$. Any endomorphism T of E induces an S_k-endomorphism $T^{\otimes k}$ of $E^{\otimes k}$, and hence an endomorphism $\pi(T)$ of $\pi(E)$. Taking $T \in GL(E)$, we see that $\pi(E)$ becomes a representation space of $GL(E)$, and this is of course the classical construction for the irreducible representations of the general linear group. For our purposes, however, this is not relevant. All we are interested in are the character formulae. We therefore proceed as follows.

Let $E = \mathbf{C}^n$ and let T be the diagonal matrix $(t_1,\ldots,t_n)$. Since the eigenvalues of $T^{\otimes k}$ are all monomials of degree k in $t_1,\ldots,t_n$, it follows that, for each π, $\operatorname{Trace}\pi(T)$ is a homogeneous polynomial in $t_1,\ldots,t_n$ with integer coefficients. Moreover, $\operatorname{Trace}\pi(T) = \operatorname{Trace}(\pi(S^{-1}TS))$ for any permutation matrix S and so $\operatorname{Trace}\pi(T)$ is symmetric in $t_1,\ldots,t_n$. We define

$$\Delta_{n,k} = \operatorname{Trace}_{S_k}(T^{\otimes k}) = \sum_{\pi} \operatorname{Trace}\pi(T) \otimes [V_\pi] \in \operatorname{Sym}_k[t_1,\ldots,t_n] \otimes R(S_k),$$

where $[V_\pi] \in R(S_k)$ is the class of V_π and $\operatorname{Sym}_k[t_1,\ldots,t_n]$ denotes the symmetric polynomials of degree k. If we regard $R(S_k)$ as the character ring, then $\Delta_{n,k}$ is just the function of $t_1,\ldots,t_n$ and $g \in S_k$ given by $\operatorname{Trace}(gT^{\otimes k})$. There are a number of other ways of writing this basic element, the simplest being the following proposition:

PROPOSITION 1.1. *For any partition* $\alpha = (\alpha_1,\ldots,\alpha_r)$ *of* k *let* $\rho_\alpha \in R(S_k)$ *be the representation induced from the trivial representation of*

$$S_\alpha = S_{\alpha_1}\times S_{\alpha_2}\times\ldots\times S_{\alpha_r},$$

then

$$\Delta = \sum_{\alpha\vdash k} m_\alpha \otimes \rho_\alpha,$$

where m_α *is the monomial symmetric function generated by* $t_1^{\alpha_1}t_2^{\alpha_2}\ldots t_r^{\alpha_r}$ *and the summation is over all partitions of* k.

Proof. Let E^α be the eigenspace of $T^{\otimes k}$ corresponding to the eigenvalue $t_1^{\alpha_1}t_2^{\alpha_2}\ldots t_r^{\alpha_r}$. This has as a basis the orbit under S_k of the vector

$$e_\alpha = e_1^{\otimes\alpha_1} \otimes e_2^{\otimes\alpha_2} \ldots \otimes e_r^{\otimes\alpha_r},$$

where $e_1,\ldots,e_n$ are the standard base of $\mathbf{C}^n$. Since the stabilizer of e_α is just the subgroup S_α, it follows that E^α is the induced representation ρ_α. Since S_α and S_β are conjugate if α and β are the same *partition* of k, it follows that

$$\Delta = \sum_{|\alpha|=k} t^\alpha \otimes \rho_\alpha = \sum_{\alpha\vdash k} m_\alpha \otimes \rho_\alpha,$$

where the first summation is over all *sequences* $\alpha_1, \alpha_2,\ldots$ with

$$|\alpha| = \sum \alpha_i = k.$$

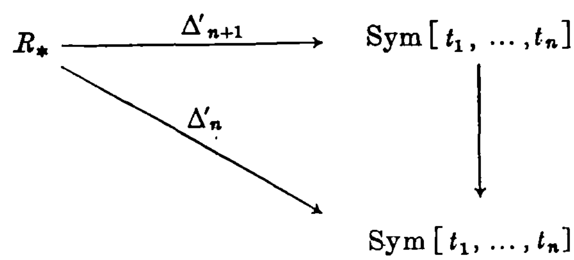
Now let us introduce the dual group

$$R_*(S_k) = \operatorname{Hom}_{\mathbf{Z}}(R(S_k), \mathbf{Z}).$$

Then $\Delta_{n,k}$ defines (and is defined by) a homomorphism

$$\Delta'_{n,k}\colon R_*(S_k) \to \operatorname{Sym}_k[t_1,\ldots,t_n].$$

From the inclusions $$S_k \times S_l \to S_{k+l}$$
we obtain homomorphisms
$$R(S_{k+l}) \to R(S_k \times S_l) \cong R(S_k) \otimes R(S_l)$$
and hence by duality
$$R_*(S_k) \otimes R_*(S_l) \to R_*(S_{k+l}).$$
Putting $R_* = \sum_{k \geqslant 0} R_*(S_k)$ we see that the above pairings turn R_* into a *commutative graded ring*. This follows from the fact, already used in Proposition 1.1, that S_α and S_β are conjugate if α and β are the same partition. Moreover, if we define
$$\Delta'_n : R_* \to \mathrm{Sym}[t_1,\ldots,t_n]$$
by $\Delta'_n = \sum \Delta'_{n,k}$, we see that Δ'_n is a *ring homomorphism*. This follows from the multiplicative property of the trace:
$$\mathrm{Trace}(g_1 g_2 T^{\otimes(k+l)}) = \mathrm{Trace}(g_1 T^{\otimes k})\mathrm{Trace}(g_2 T^{\otimes l}),$$
where $g_1 \in S_k$, $g_2 \in S_l$. Finally we observe that we have a commutative diagram



where the vertical arrow is given by putting $t_{n+1} = 0$. Hence passing to the limit we can define
$$\Delta' : R_* \to \varprojlim_n \mathrm{Sym}[t_1,\ldots,t_n].$$
Here the inverse limit is taken in the category of *graded* rings, so that
$$\varprojlim_n \mathrm{Sym}[t_1,\ldots,t_n] = \sum_{k=0} \varprojlim_n \mathrm{Sym}_k[t_1,\ldots,t_n]$$
is the direct sum (and not the direct product) of its homogeneous parts.

PROPOSITION 1.2. $\Delta' : R_* \to \varprojlim_n \mathrm{Sym}[t_1,\ldots,t_n]$ *is an isomorphism.*

Proof. Let $\sigma^k \in R_*(S_k)$ denote the homomorphism $R(S_k) \to \mathbf{Z}$ defined by
$$\sigma^k(1) = 1, \quad \sigma^k(V_\pi) = 0 \quad \text{if } V_\pi \neq 1,$$

where 1 denotes the trivial representation. Since $\pi(E)$ is the kth symmetric power of E when $V_\pi = 1$, it follows from the definition of $\Delta'_{n,k}$ that

$$\Delta'_{n,k}(\sigma^k) = h_k(t_1,\ldots,t_n)$$

is the kth homogeneous symmetric function (i.e. the coefficient of z^k in $\prod(1-zt_i)^{-1}$). Since the h_k are a polynomial basis for the symmetric functions, it follows that Δ'_n is an epimorphism for all n. Now the rank of $R(S_k)$ is equal to the number of conjugacy classes of S_k, that is the number of partitions of k, and hence is also equal to the rank of $\mathrm{Sym}_k[t_1,\ldots,t_n]$ provided that $n \geqslant k$. Hence

$$\Delta'_{n,k}: R_*(S_k) \to \mathrm{Sym}_k[t_1,\ldots,t_n]$$

is an epimorphism of free abelian groups of the same rank (for $n \geqslant k$) and hence is an isomorphism. Since

$$\mathrm{Sym}_k[t_1,\ldots,t_{n+1}] \to \mathrm{Sym}_k[t_1,\ldots,t_n]$$

is also an isomorphism for $n \geqslant k$, this completes the proof.

COROLLARY 1.3. *R_* is a polynomial ring on generators $\sigma^1, \sigma^2,\ldots$.*

Instead of using the elements $\sigma^k \in R_*(S_k)$ we could equally well have used the elements λ^k defined by

$$\lambda^k(V_\pi) = 1 \quad \text{if } V_\pi \text{ is the sign representation.}$$
$$\lambda^k(V_\pi) = 0 \quad \text{otherwise.}$$

Since $\pi(E)$ is the kth exterior power when π is the sign representation of S_k, it follows that

$$\Delta'_{n,k}(\lambda^k) = e_k(t_1,\ldots,t_n)$$

is the kth elementary symmetric function. Thus R_* is equally well a polynomial ring on generators $\lambda^1, \lambda^2,\ldots$.

COROLLARY 1.4. *Let $\Delta_{n,k} = \sum a_i \otimes b_i$ with $a_i \in \mathrm{Sym}_k[t_1,\ldots,t_n]$ and $b_i \in R(S_k)$, and suppose $n \geqslant k$. Then the a_i form a base if and only if the b_i form a base. When this is so the a_i determine the b_i and conversely, i.e. they are 'dual bases'.*

Proof. This is an immediate reinterpretation of the fact that $\Delta'_{n,k}$ is an isomorphism.

COROLLARY 1.5. *The representations ρ_α form a base for $R(S_k)$.*

Proof. Apply Corollary 1.4 to the expression for $\Delta_{n,k}$ given in Proposition 1.1. Since the m_α are a basis for the symmetric functions, it follows that the ρ_α are a basis for $R(S_k)$.

COROLLARY 1.6. *The characters of S_k take integer values on all conjugacy classes.*

Proof. The characters of all ρ_α are integer-valued and so Corollary 1.6 follows from Corollary 1.5.

Note. Corollary 1.6 can of course be deduced fairly easily from other considerations.

Let $C(S_k)$ denote the group of integer-valued class functions on S_k. By Corollary 1.6 we have a natural homomorphism

$$R(S_k) \to C(S_k).$$

This has zero kernel and finite cokernel, and the same is therefore true for the dual homomorphism

$$C_*(S_k) \to R_*(S_k).$$

The direct sum $C_* = \sum_{k \geq 0} C_*(S_k)$ has a natural ring structure, and

$$C_* \to R_*$$

is a ring homomorphism. We shall identify C_* with the image subring of R_*. From its definition, $C_*(S_k)$ is the free abelian group on the conjugacy classes of S_k. Let ψ^k denote the class of a k-cycle. Then C_* is a polynomial ring on $\psi^1, \psi^2, \ldots$. The next result identifies the subring $\Delta'(C_*)$ of symmetric functions:

PROPOSITION 1.7. $\Delta'_n(\psi^k) = m_k(t_1, \ldots, t_n) = \sum_{i=1}^{n} t_i^k$ *so that* $\Delta'(C_*)$ *is the subring generated by the power sums* m_k.

Proof. By definition we have

$$\Delta'_n(\psi^k) = \mathrm{Trace}(gT^{\otimes k}),$$

where $g \in S_k$ is a k-cycle. Now use Proposition 1.1 to evaluate this trace and we get

$$\Delta'_n \psi^k = \sum_{\alpha \vdash k} m_\alpha \rho_\alpha(g).$$

But, if $H \subset G$, any character of G induced from H is zero on all elements of G not conjugate to elements of H. Hence, taking $H = S_\alpha$, $G = S_k$, we see that $\rho_\alpha(g) = 0$ unless $\alpha = k$ (i.e. α is the single partition k). Since $\rho_k(g) = 1$, we deduce

$$\Delta'_n \psi^k = m_k,$$

as required.

COROLLARY 1.8. *Let* Q_k *be the Newton polynomial expressing the power sum* m_k *in terms of the elementary symmetric functions* $e_1, \ldots, e_k$, *i.e.*

$$m_k = Q_k(e_1, \ldots, e_k),$$

then

$$\psi_k = Q_k(\lambda^1, \ldots, \lambda^k) \in R_*.$$

Remark. Let us tensor with the rationals Q, so that we can introduce

$$\epsilon_\alpha \in R(S_k) \otimes \mathbf{Q},$$

the characteristic function of the conjugacy class defined by the partition α. Then Proposition 1.7 is essentially equivalent to the following expression [cf. (**11**) VII (7.6)] for $\Delta_{n,k}$

$$\Delta_{n,k} = \sum_{\alpha \vdash k} p_\alpha(t) \otimes \epsilon_\alpha \in \mathrm{Sym}_k[t_1,..., t_n] \otimes R(S_k) \otimes Q,$$

where p_α is the monomial in the power sums

$$p_\alpha = \prod_{i=1}^{k} (m_i)^{a_i}, \quad \alpha = 1^{a_1} 2^{a_2} \dots .$$

Since $\Delta'(\lambda_k) = e_k$, it follows that we can write $\Delta_{n,k}$ in the form

$$\Delta_{n,k} = \sum_{\alpha \vdash k} q_\alpha(t) \otimes b_\alpha,$$

where q_α is the monomial in the elementary symmetric functions

$$q_\alpha = \prod_{i=1}^{k} (e_i)^{a_i}, \quad \alpha = 1^{a_1} 2^{a_2} \dots,$$

and the b_α are certain uniquely defined elements in $R(S_k)$. We shall not attempt to find b_α in general, but the following proposition gives the 'leading coefficient' b_k.

PROPOSITION 1.9. *Let M denote the $(k-1)$-dimensional representation of S_k given by the subspace $\sum_{i=1}^{k} z_i = 0$ of the standard k-dimensional representation. Let $\Lambda^i(M)$ denote the ith exterior power of M, and put*

$$\Lambda_{-1}(M) = \sum (-1)^i \Lambda^i(M) \in R(S_k).$$

Then we have

$$\Delta_{n,k} = (-1)^{k-1} e_k(t) \otimes \Lambda_{-1}(M) + \text{composite terms},$$

where 'composite' means involving a product of at least two $e_i(t)$.

Proof. In the formula

$$\Delta_{n,k} = \sum_{\alpha \vdash k} q_\alpha(t) \otimes b_\alpha,$$

the b_α are the basis of $R(S_k)$ dual to the basis of $R_*(S_k)$ consisting of monomials in the λ^i. Thus b_k is defined by the conditions

$$\langle b_k, \lambda^k \rangle = 1,$$

$$\langle b_k, u \rangle = 0$$

if u is composite in the λ^i. Since the ψ^i are related to the λ^i by the equations of Corollary 1.8

$$\psi^k = Q_k(\lambda^1,..., \lambda^k) = (-1)^{k-1} k \lambda_k + \text{composite terms},$$

we can equally well define b_k by the conditions

$$\langle b_k, \psi^k\rangle = (-1)^{k-1}k,$$
$$\langle b_k, u\rangle = 0$$

if u is composite in the ψ^i. To prove that $b_k = (-1)^{k-1}\Lambda_{-1}(M)$, it remains therefore to check that the character $\Lambda_{-1}(M)$ vanishes on all composite classes and has value k on a k-cycle. Now, if $g \in S_k$ is composite, i.e. not a k-cycle, it has an eigenvalue 1 when acting on M; if $g = (1...r)(r+1,...s)...$ is the cycle decomposition, the fixed vector is given by

$$z_i = \frac{1}{r} \quad (1 \leqslant i \leqslant r), \qquad z_j = -\frac{1}{k-r} \quad (j > r).$$

Since $\Lambda_{-1}(M)(g) = \det(1-g_M)$, where g_M is the linear transformation of M defined by g, the existence of an eigenvalue 1 of g_M implies $\Lambda_{-1}(M)(g) = 0$. Finally take $g = (1\ 2\ ...\ k)$ and consider the k-dimensional representation $N = M \oplus 1$. Then g_N is given by the following matrix

$$g_N = \begin{pmatrix} 0 & 1 & & & \\ & & 1 & & \\ & & & \cdot & \\ & & & & \cdot \\ & & & & \cdot \quad 1 \\ 1 & & & & \end{pmatrix}$$

and so $\det(1-tg_N) = 1-t^k$. Hence

$$\det(1-tg_M) = \det(1-tg_N).(1-t)^{-1}$$
$$= \frac{1-t^k}{1-t} = 1+t+t^2+...+t^{k-1},$$

and so $$\Lambda_{-1}(M)(g) = \det(1-g_M) = k,$$

which completes the proof.

If $G \subset S_k$ is any subgroup, then we can consider the element

$$\Delta_{n,k}(G) \in \mathrm{Sym}_k[t_1,..., t_n] \otimes R(G)$$

obtained from $\Delta_{n,k}$ by the restriction $\eta: R(S_k) \to R(G)$. Similarly

$$\Delta'_{n,k}(G): R_*(G) \to \mathrm{Sym}_k[t_1,..., t_n]$$

is the composition of $\Delta'_{n,k}$ and

$$\eta_*: R_*(G) \to R_*(S_k).$$

Consider in particular the special case when $k = p$ is *prime* and $G = Z_p$ is the cyclic group of order p. The image of

$$\eta: R(S_p) \to R(Z_p)$$

is generated by the trivial representation 1 and the regular representation N of Z_p (this latter being the restriction of the standard p-dimensional representation of S_p). Hence we must have

$$\Delta_{n,p}(Z_p) = a(t) \otimes 1 + b(t) \otimes N$$

for suitable symmetric functions $a(t)$, $b(t)$. Evaluating $R(S_p)$ on the identity element we get

$$e_1^p = a + pb.$$

Evaluating on a generator of Z_p and using Proposition 1.7 we get

$$m_p = a.$$

Hence $b = \dfrac{e_1^p - m_p}{p}$ which has, of course, integer coefficients since

$$(\sum t_i)^p \equiv \sum t_i^p \mod p.$$

Thus we have established the proposition:

PROPOSITION 1.10. *Let p be a prime. Then restricting $\Delta_{n,p}$ from the symmetric group to the cyclic group we get*

$$\Delta_{n,p}(Z_p) = m_p \otimes 1 + \frac{e_1^p - m_p}{p} \otimes N,$$

where N is the regular representation of Z_p.

Let $\theta^p \in R_*(S_p)$ be the element corresponding to

$$\frac{e_1^p - m_p}{p} \in \mathrm{Sym}_p[t_1, \ldots, t_n]$$

by the isomorphism of Proposition 1.2 (for $n \geqslant p$), i.e.

$$\Delta_n' \theta^p = \frac{e_1^p - m_p}{p}.$$

Then Proposition 1.10 asserts that θ^p is that homomorphism $R(S_p) \to \mathbf{Z}$ which gives the multiplicity of the regular representation N when we restrict to Z_p. Thus, for $\rho \in R(S_p)$,

$$\eta(\rho) = \psi^p(\rho)1 + \theta^p(\rho)N, \tag{1.11}$$

where $\eta: R(S_p) \to R(Z_p)$ is the restriction.

2. Operations in K-theory

Let X be a compact Hausdorff space and let G be a finite group. We shall say that X is a *G-space* if G acts on X. Let E be a complex vector bundle over X. We shall say that E is a *G-vector bundle* over the G-space X if E is a G-space such that

(i) the projection $E \to X$ commutes with the action of G,

(ii) for each $g \in G$ the map $E_x \to E_{g(x)}$ is linear.

The Grothendieck group of all G-vector bundles over the G-space X is denoted by $K_G(X)$. Note that the action of G on X is supposed given: it is part of the structure of X. Since we can always construct an invariant metric in a G-vector bundle by averaging over G, the usual arguments show that a short exact sequence splits compatibly with G. Hence, if

$$0 \to E_1 \to E_2 \to \ldots \to E_n \to 0$$

is a long exact sequence of G-vector bundles, the Euler characteristic $\sum (-1)^i [E_i]$ is zero in $K_G(X)$. For a fuller treatment of these and other points about $K_G(X)$ we refer the reader to (4) and (9).

In this section we shall be concerned only with a trivial G-space X, i.e. $g(x) = x$ for all $x \in X$ and $g \in G$. In this case a G-vector bundle is just a vector bundle E over X with a given homomorphism

$$G \to \operatorname{Aut} E,$$

where $\operatorname{Aut} E$ is the group of vector bundle automorphisms of E. We proceed to examine such a G-vector bundle.

The subspace of E left fixed by G forms a subvector bundle E^G of E: in fact it is the image of the projection operator

$$\frac{1}{|G|} \sum_{g \in G} g,$$

and the image of any projection operator is always a sub-bundle (4). If E, F are two G-vector bundles, then the subspace of $\operatorname{Hom}(E, F)$ consisting of all $\phi_x : E_x \to F_x$ commuting with the action of G forms a subvector bundle $\operatorname{Hom}_G(E, F)$: in fact $\operatorname{Hom}_G(E, F) = (\operatorname{Hom}(E, F))^G$. In particular let V be a representation space of G, and let $\mathbf{V}$ denote the corresponding G-vector bundle $X \times V$ over X. Then, for any G-vector bundle E over X, $\operatorname{Hom}_G(\mathbf{V}, E)$ is a vector bundle, and we have a natural homomorphism $\mathbf{V} \otimes \operatorname{Hom}_G(\mathbf{V}, E) \to E$.

Now let $\{V_\pi\}\ldots$ be a complete set of irreducible representations of G and consider the bundle homomorphism

$$\alpha : \sum_\pi \{\mathbf{V}_\pi \otimes \operatorname{Hom}_G(\mathbf{V}_\pi, E)\} \to E.$$

For each $x \in X$, α_x is an isomorphism. Hence α is an isomorphism. This establishes the following proposition:

PROPOSITION 2.1. *If X is a trivial G-space, we have a natural isomorphism*

$$K(X) \otimes R(G) \to K_G(X).$$

In particular we can apply the preceding discussion to the natural action of S_k on the k-fold tensor product $E^{\otimes k}$ of a vector bundle E. Thus we have a canonical decomposition compatible with the action of S_k

$$E^{\otimes k} \cong \sum_{\pi} \{V_\pi \otimes \mathrm{Hom}_{S_k}(V_\pi, E^{\otimes k})\}.$$

We put
$$\pi(E) = \mathrm{Hom}_{S_k}(V_\pi, E^{\otimes k}).$$

Thus π is an operation on vector bundles. In fact $\pi(E)$ is the vector bundle associated to E by the irreducible representation of $GL(n)$ ($n = \dim E$) associated to the partition π, but this fact will play no special role in what follows.

Our next step is to extend these operations on vector bundles to operations on $K(X)$. For this purpose it will be convenient to represent $K(X)$ as the quotient of a set $\mathscr{C}(X)$ by an equivalence relation (elements of $\mathscr{C}(X)$ will play the role of 'cochains'). An element of $\mathscr{C}(X)$ is a *graded* vector bundle $E = \sum_{i \in Z} E_i$, where $E_i = 0$ for all but a finite number of values of i. We have a natural surjection

$$\mathscr{C}(X) \to K(X)$$

given by taking the Euler characteristic $[E] = \sum (-1)^i [E_i]$. The equivalence relation on $\mathscr{C}(X)$ which gives $K(X)$ is clearly generated by isomorphism and the addition of *elementary* objects, i.e. one of the form $\sum P_i$ with

$$P_j = P_{j+1} \quad (\text{for some } j), \qquad P_i = 0 \quad (i \neq j, j+1).$$

Similarly for a G-space X we can represent $K_G(X)$ as a quotient of $\mathscr{C}_G(X)$, where an element of $\mathscr{C}_G(X)$ is a graded G-vector bundle.

Suppose now that $E \in \mathscr{C}(X)$ is a graded vector bundle. Then $E^{\otimes k}$ is also a graded vector bundle, the grading being defined in the usual way as the sum of the degrees of the k factors. We consider S_k as acting on $E^{\otimes k}$ by permuting factors and with the *appropriate sign change*. Thus a transposition of two terms $e_p \otimes e_q$ (where $e_p \in E_p$, $e_q \in E_q$) carries with it the sign $(-1)^{pq}$. The Euler characteristic $[E^{\otimes k}]$ of $E^{\otimes k}$ is then an element of $K_{S_k}(X)$.

PROPOSITION 2.2. *The element* $[E^{\otimes k}] \in K_{S_k}(X)$ *depends only on the element* $[E] \in K(X)$. *Thus we have an operation*:

$$\otimes k : K(X) \to K_{S_k}(X) = K(X) \otimes R(S_k).$$

Proof. We have to show that, if P is an elementary object of $\mathscr{C}(X)$, then
$$[(E \oplus P)^{\otimes k}] = [E^{\otimes k}] \in K_{S_k}(X).$$

But we have an S_k-decomposition:

$$(E \oplus P)^{\otimes k} \cong E^{\otimes k} \oplus Q.$$

We have to show therefore that $[Q] = 0$ in $K_{S_k}(X)$. To do this we regard E as a *complex* of vector bundles with all maps zero and P as a complex with the identity map $P_j \to P_{j+1}$. Then $(E \oplus P)^{\otimes k}$ is a complex of vector bundles, and S_k acts on it as a group of complex automorphisms (because of our choice of signs). The same is true for $E^{\otimes k}$ and Q. Now Q contains P as a factor, and so Q is certainly acyclic. Hence, by the remark at the beginning of this section, we have $[Q] = 0$ in $K_{S_k}(X)$ as required.

Remark. If we decompose $E^{\otimes k}$ under S_k

$$E^{\otimes k} \cong \sum_{\pi} \mathrm{V}_{\pi} \otimes \pi(E),$$

where $\pi(E) = \mathrm{Hom}_{S_k}(\mathrm{V}_{\pi}, E^{\otimes k})$, Proposition 2.2 asserts that $E \mapsto \pi(E)$ induces an operation

$$\pi: K(X) \to K(X).$$

Let $\mathrm{Op}(K)$ denote the set of all natural transformations of the functor K into itself. In other words, an element $T \in \mathrm{Op}(K)$ defines for each X a map

$$T(X): K(X) \to K(X),$$

which is natural. We define addition and multiplication in $\mathrm{Op}(K)$ by adding and multiplying values. Thus, for $a \in K(X)$,

$$(T+S)(X)(a) = T(X)(a)+S(X)a,$$
$$TS(X)(a) = T(X)a \,.\, S(X)a.$$

If we follow the operation

$$\otimes k: K(X) \to K(X) \otimes R(S_k)$$

by a homomorphism $\phi: R(S_k) \to \mathbf{Z}$ we obtain a natural map

$$T_{\phi}: K(X) \to K(X).$$

This procedure defines a map

$$j_k: R_*(S_k) \to \mathrm{Op}(K)$$

which is a group homomorphism. Extending this additively we obtain a *ring homomorphism*

$$j: R_* \to \mathrm{Op}(K).$$

We have now achieved our aim of showing how the symmetric group defines a ring of operations in K-theory. The structure of the ring R_* has moreover been completely determined in § 1. We conclude this section by examining certain particular operations and connecting up our definitions of them with those given by Grothendieck [cf (**5**); § 12] and Adams (**2**).

To avoid unwieldy formulae we shall usually omit the symbol j and just think of elements of R_* as operations. In fact it is not difficult to show that j is a monomorphism (although we do not really need this fact), so that R_* may be thought of as a subring of $\mathrm{Op}(K)$.

All the particular elements that we have described in § 1, namely $\sigma^k, \lambda^k, \psi^k, \theta^p$, can now be regarded as operations in K-theory. From the way they were defined it is clear that, if E is vector bundle, then $\lambda^k[E]$ is the class of the kth exterior power of E, and $\sigma^k(E)$ is the class of the kth symmetric power of E. A general element of $K(X)$ can always be represented in the form $[E_0]-[E_1]$, where E_0, E_1 are vector bundles. Taking $(E_0 \oplus E_1)^{\otimes k}$ as an S_k-complex and picking out the symmetric and skew-symmetric components, we find

$$\sigma^k([E_0]-[E_1]) = \sum_{j=0}^{k} (-1)^j \sigma^{k-j}[E_0]\lambda^j[E_1], \tag{1}$$

$$\lambda^k([E_0]-[E_1]) = \sum_{j=0}^{k} (-1)^j \lambda^{k-j}[E_0]\sigma^j[E_1]. \tag{2}$$

Putting formally $\lambda_u = \sum \lambda^k u^k$, $\sigma_u = \sum \sigma^k u^k$, where u is an indeterminate, and taking $E_0 = E_1$ in (1), we get

$$\sigma_u[E_1]\lambda_{-u}[E_1] = 1. \tag{3}$$

This identity could of course have been deduced from the corresponding relation between the generating functions of e_k and h_k by using the isomorphism of (1.2). Now from (2) we get

$$\begin{aligned}\lambda_u([E_0]-[E_1]) &= \lambda_u[E_0]\sigma_{-u}[E_1]\\ &= \lambda_u[E_0]\lambda_u[E_1]^{-1} \quad \text{by (3)}.\end{aligned}$$

This is the formula by which Grothendieck originally extended the λ^k from vector bundles to K. Thus our definition of the operations λ^k coincides with that of Grothendieck. Essentially the use of graded tensor products has provided us with a general procedure for extending operations which can be regarded as a generalization of the Grothendieck method for the exterior powers.†

Adams defines his operations ψ^k in terms of the Grothendieck λ^k by use of the Newton polynomials

$$\psi^k = Q_k(\lambda^1, \ldots, \lambda^k).$$

Corollary 1.8 shows that our definition of ψ^k therefore agrees with that of Adams. An important property of the ψ^k is that they are additive. We shall therefore show how to prove this directly from our definition.

PROPOSITION 2.3. *Let E, F be vector bundles, then*

$$\psi^k([E]\pm[F]) = \psi^k[E]\pm\psi^k[F].$$

† This fact was certainly known to Grothendieck.

Proof. Construct a graded vector bundle D with $D_0 = E, D_1 = F$ and consider $D^{\otimes k}$. The same reasoning as used in Proposition 1.1 shows that

$$[D]^{\otimes k} = \sum_{j=0}^{k} (-1)^j \operatorname{ind}_j[E^{\otimes k-j} \otimes F^{\otimes j}] \in K(X) \otimes R(S_k),$$

where $\operatorname{ind}_j : K(X) \otimes R(S_{k-j} \times S_j) \to K(X) \otimes R(S_k)$ is given by the induced representation. Here $E^{\otimes k-j}$ is an S_{k-j}-vector bundle via the standard permutation, while S_j acts on $F^{\otimes j}$ via permutation and signs. To obtain $\psi^k[D]$ we have to evaluate $R(S_k)$ on a k-cycle. As in Proposition 1.1 all terms except $j = 0$, k give zero; since the sign of a k-cycle is $(-1)^{k-1}$ we get

$$\begin{aligned}\psi^k([E]-[F]) &= \psi^k[E]+(-1)^k(-1)^{k-1}\psi^k[F] \\ &= \psi^k[E]-\psi^k[F].\end{aligned}$$

For $[E]+[F]$ the argument is similar but easier.

The multiplicative property

$$\psi^k[E \otimes F] = \psi^k[E]\psi^k[F]$$

follows at once from the isomorphism

$$(E \otimes F)^{\otimes k} \cong E^{\otimes k} \otimes F^{\otimes k}$$

and the multiplicative property of the trace.

Suppose now that we have any expansion, as in Corollary 1.4, of the basic element $\Delta_{n,k}$ in the form

$$\Delta_{n,k} = \sum a_i \otimes b_i,$$

where the $a_i \in \operatorname{Sym}_k[t_1,..., t_n]$ are a basis and the $b_i \in R(S_k)$ are therefore a dual basis (assuming $n \geqslant k$). Then, for any $x \in K(X)$, we obtain a corresponding expansion for $x^{\otimes k}$:

$$x^{\otimes k} = \alpha_i(x) \otimes b_i \in K(X) \otimes R(S_k),$$

where $\alpha_i = (\Delta')^{-1}a_i \in R_*$. This follows at once from the definition of Δ' and the way we have made R_* operate on $K(X)$.

Taking the a_i to be the monomials in the elementary symmetric functions the α_i are then the corresponding monomials in the exterior powers λ^i. Proposition 1.9 therefore gives the following proposition:†

PROPOSITION 2.4. *For any $x \in K(X)$ we have*

$$x^{\otimes k} = (-1)^{k-1}\lambda^k(x) \otimes \lambda_{-1}(M) + \text{composite terms},$$

where 'composite' means involving a product of at least two $\lambda^i(x)$ and M is the $(k-1)$-dimensional representation of S_k.

† Now that we have identified the λ^i of § 1 with the exterior powers we revert to the usual notation and write $\lambda^i(M)$ instead of $\Lambda^i(M)$, and correspondingly $\lambda_{-1}(M)$ instead of $\Lambda_{-1}(M)$.

Now let us restrict ourselves to the cyclic group Z_k. The image of $x^{\otimes k}$ in $K(X) \otimes R(Z_k)$ will be denoted by $P^k(x)$ and called the *cyclic kth power*. In the particular case when $k = p$ (a prime), (1.11) leads to the following proposition:

PROPOSITION 2.5. *Let p be a prime and let $x \in K(X)$. Then the cyclic pth power $P^p(x)$ is given by the formula*

$$P^p(x) = \psi^p(x) \otimes 1 + \theta^p(x) \otimes N \in K(X) \otimes R(Z_p),$$

where N is the regular representation of Z_p.

Now ψ^p and θ^p correspond, under the isomorphism

$$\Delta' : R_* \to \varprojlim_n \operatorname{Sym}[t_1,..., t_n],$$

to the polynomials $\sum t_i^p$ and $\dfrac{(\sum t_i)^p - \sum t_i^p}{p}$ respectively. Hence they are related by the formula

$$\psi^p = (\psi^1)^p - p\theta^p,$$

so that, for any $x \in K(X)$, we have

$$\psi^p(x) = x^p - p\theta^p(x).$$

Substituting this in (2.5) we get the formula

$$P^p(x) = x^p \otimes 1 + \theta^p(x) \otimes (N - p). \tag{2.6}$$

This is a better way of writing (2.5) since it corresponds to the decomposition

$$R(Z_p) = \mathbf{Z} \oplus I(Z_p),$$

where $I(Z_p)$ is the augmentation ideal. Thus

$$\theta^p(x) \otimes (N - p) \in K(X) \otimes I(Z_p)$$

represents the difference between the pth cyclic power $P^p(x)$ and the 'ordinary' pth power $x^p \otimes 1$.

Proposition 2.5 leads to a simple geometrical description for $\psi^p[V]$, where V is a vector bundle. Let T be the automorphism of $V^{\otimes p}$ which permutes the factors cyclically and V_j be the eigenspace of T corresponding to the eigenvalue $\exp(2\pi ij/p)$. Then

$$\psi^p[V] = [V_0] - [V_1]. \tag{2.7}$$

In fact from Proposition 2.5 we see that

$$[V_0] = \psi^p[V] + \theta^p[V],$$
$$[V_j] = \theta^p[V] \quad (j = 1,..., p-1).$$

3. External tensor powers

For a further study of the properties of the operation $\otimes k$ it is necessary both to 'relativize' it and to 'externalize' it.

First consider the relative group $K_G(X, Y)$, where X is a G-space, Y a sub G-space. As with the absolute case we can consider $K_G(X, Y)$ as the quotient of a set $\mathscr{C}_G(X, Y)$ by an equivalence relation. An object E of $\mathscr{C}_G(X, Y)$ is a G-complex of vector bundles over X acyclic over Y, i.e. E consists of G-vector bundles E_i (with $E_i = 0$ for all but a finite number) and homomorphisms

$$\to E_i \xrightarrow{d} E_{i+1} \xrightarrow{d}$$

commuting with the action of G, so that $d^2 = 0$ and over each point of Y the sequence is exact. An elementary object P is one in which $P_i = 0$ $(i \neq j,\ j+1)$, $P_j = P_{j+1}$, and $d: P_j \to P_{j+1}$ is the identity. The equivalence relation imposed on $\mathscr{C}_G(X, Y)$ is that generated by isomorphism and addition (direct sum) of elementary objects. Then, if $E \in \mathscr{C}_G(X, Y)$, its equivalence class $[E] \in K_G(X, Y)$. For the details we refer to (**4**). For the analogous results in the case when there is no group, i.e. for the definition of $K(X, Y)$ as a quotient of $\mathscr{C}(X, Y)$, we refer to (**7**) [Part II].

Consider next the *external* tensor power. If E is a vector bundle over X, we define $E^{\boxtimes k}$ to be the vector bundle over the Cartesian product X^k (k factors of X) whose fibre at the point $(x_1 \times x_2 \times \ldots \times x_k)$ is $E_{x_1} \otimes E_{x_2} \otimes \ldots \otimes E_{x_k}$. Thus $E^{\boxtimes k}$ is an S_k-vector bundle over the S_k-space X^k, the symmetric group S_k acting in the usual way on X^k by permuting the factors. Clearly, if

$$d: X \to X^k$$

is the diagonal map, we have a natural S_k-isomorphism

$$d^*(E^{\boxtimes k}) \cong E^{\otimes k}. \tag{3.1}$$

If E is a complex of vector bundles over X, then we can define in an obvious way $E^{\boxtimes k}$, which will be a complex of vector bundles over X^k. Moreover $E^{\boxtimes k}$ will be an S_k-complex of vector bundles, X^k being an S_k-space as above. If E is acyclic over $Y \subset X$, then $E^{\boxtimes k}$ will be acyclic over the subspace of X consisting of points $(x_1 \times x_2 \times \ldots \times x_k)$ with $x_i \in Y$ for at least one value of i. We denote this subspace by $X^{k-1}Y$ and we write $(X, Y)^k$ for the pair $(X^k, X^{k-1}Y)$. Thus we have defined an operation

$$\boxtimes k: \mathscr{C}(X, Y) \to \mathscr{C}_{S_k}(X, Y)^k.$$

The proof of (2.2) generalizes at once to this situation and establishes

PROPOSITION 3.2. *The operation* $E \mapsto E^{\boxtimes k}$ *induces an operation*

$$\boxtimes k : K(X, Y) \to K_{S_k}(X, Y)^k.$$

COROLLARY 3.3. *If* x *is in the kernel of* $K(X) \to K(Y)$, *then* $x^{\boxtimes k}$ *is in the kernel of*

$$K_{S_k}(X^k) \to K_{S_k}(X^{k-1}Y).$$

Proof. This follows at once from (3.2) and the naturality of the operation $\boxtimes\, k$.

From (3.1) we obtain the commutative diagram

$$\begin{array}{ccc} K(X) & \xrightarrow{\ \boxtimes k\ } & K_{S_k}(X^k) \\ & \searrow{\scriptstyle \otimes k} & \downarrow{\scriptstyle d^*} \\ & & K_{S_k}(X) \end{array} \qquad (3.4)$$

4. Operations and filtrations

From now we assume that the spaces $X, Y, \ldots$ are *finite CW-complexes*. Then $K(X)$ is filtered by the subgroups $K_q(X)$ defined by

$$K_q(X) = \operatorname{Ker}\{K(X) \to K(X_{q-1})\},$$

where X_{q-1} denotes the $(q-1)$-skeleton of X. Thus $K_0(X) = K(X)$ and $K_n(X) = 0$ if $\dim X < n$. Moreover, as shown in (8), we have

$$K_{2q}(X) = K_{2q-1}(X)$$

for all q. Since any map $Y \to X$ is homotopic to a cellular map, it follows that the filtration is natural.

In [8] it is shown that $K(X)$ is a *filtered ring*, i.e. that $K_p K_q \subset K_{p+q}$. In particular it follows that

$$x \in K_q(X) \Rightarrow x^k \in K_{kq}(X).$$

We propose to generalize this result to the tensor power $\otimes k$.

We start by recalling (5) that, for any finite group, there is a natural homomorphism

$$\alpha : R(G) \to K(B_G),$$

where B_G is the classifying space of G. This homomorphism arises as follows. Let A be the universal covering of B_G and V be any G-module. Then $A \times_G V$ is a vector bundle over B_G. The construction $V \mapsto A \times_G V$ induces the homomorphism

$$\alpha: R(G) \to K(B_G).$$

This construction can be generalized as follows. Let X be a G-space and denote by X_G the space $A \times_G X$. If V is a G-vector bundle over X, then

$$V_G = A \times_G V$$

is a vector bundle over X_G. The construction $V \mapsto V_G$ then induces a homomorphism

$$\alpha_X: K_G(X) \to K(X_G).$$

A couple of remarks are needed here. In the first place there is a clash of notation concerning B_G. To fit in with our general notation we should agree that 'B' is a point space. Secondly X_G, like B_G, is not a finite complex. Now B_G can be taken as an infinite complex in which the q-skeleton $B_{G,q}$ is finite for each q, and $K(B_G)$ can be defined by

$$K(B_G) = \varprojlim_q K(B_{G,q}).$$

If we suppose that G acts cellularly on X, then we can put $X_{G,q} = A_q \times_G X$, where A_q is the universal covering of $B_{G,q}$ and $X_{G,q}$ will be a finite complex. We then define

$$K(X_G) = \varprojlim K(X_{G,q}).$$

In fact, as will become apparent, there is no need for us to proceed to the limit. All our results will essentially be concerned with finite skeletons. We have introduced the infinite spaces B_G, X_G because it is a little tidier than always dealing with finite approximations.

Applying the above to the group S_k and the spaces X (trivial action) and X^k (permutation action) we obtain a commutative diagram

$$\begin{array}{ccc} K_{S_k}(X^k) & \xrightarrow{\alpha_{X^k}} & K(X^k_{S_k}) \\ \downarrow d^* & & \downarrow d^* \\ K_{S_k}(X) & \xrightarrow{\alpha_X} & K(X_{S_k}) \\ \| & & \| \\ K(X) \otimes R(S_k) & \longrightarrow & K(X \times B_{S_k}), \end{array} \tag{4.1}$$

where d^* is induced by the diagonal map $d: X \to X^k$.

PROPOSITION 4.2. *Let $x \in K_q(X)$, then*

$$\alpha_{X^k}(x^{\boxtimes k}) \in K_{kq}(X^k_{S_k}).$$

Proof. By hypothesis x is in the kernel of

$$K(X) \to K(X_{q-1}).$$

Hence applying (3.3) with $Y = X_{q-1}$ we deduce that $x^{\boxtimes k}$ is in the kernel of ρ in the following diagram

$$\begin{array}{ccc} K_{S_k}(X^k) & \xrightarrow{\alpha_{X^k}} & K(X^k_{S_k}) \\ \downarrow{\scriptstyle\rho} & & \downarrow \\ K_{S_k}(X^{k-1}X_{q-1}) & \longrightarrow & K((X^{k-1}X_{q-1})_{S_k}) \end{array}$$

The required result now follows from this diagram, provided that we verify that

$$(X^k_{S_k})_{kq-1} \subset (X^{k-1}X_{q-1})_{S_k}.$$

But any cell σ of the $(kq-1)$-skeleton of $X^k_{S_k} = X^k \times_{S_k} A$ arises from a product of k cells of X and a cell of A. Hence at least one of the cells of X occurring must have dimension less than q, and so σ is contained in

$$(X^{k-1}X_{q-1})_{S_k} = X^{k-1}X_{q-1} \times_{S_k} A,$$

as required.

Since the filtration in K is natural, Proposition 4.2 together with the diagram (4.1) and Corollary 3.3 gives our main result:

THEOREM 4.3. *Let* $\otimes k : K(X) \to K(X) \otimes R(S_k)$ *be the tensor power operation, and let*

$$\alpha : K(X) \otimes R(S_k) \to K(X \times B_{S_k})$$

be the natural homomorphism. Then

$$x \in K_q(X) \Rightarrow \alpha(x^{\otimes k}) \in K_{kq}(X \times B_{S_k}).$$

COROLLARY 4.4. *Let* $\dim X \leqslant n$ *and let* $x \in K_q(X)$. *Then the image of* $x^{\otimes k}$ *in* $K(X) \otimes K(B_{S_k, kq-n-1})$ *is zero.*

Proof. By Theorem 4.3 $x^{\otimes k}$ has zero image in $K(X \times B_{S_k, kq-n-1})$. But for any two spaces A, B the map

$$K(A) \otimes K(B) \to K(A \times B)$$

is injective (6). Hence $x^{\otimes k}$ gives zero in $K(X) \otimes K(B_{S_k, kq-n-1})$ as required.

Remark. Theorem 4.3 suggests that for any finite group G and G-space X we should define a filtration on $K_G(X)$ by putting

$$K_G(X)_q = \alpha_X^{-1} K_q(X \times B_G).$$

With this notation Theorem 4.3 would read simply

$$x \in K_q(X) \Rightarrow x^{\otimes k} \in K_{S_k}(X)_{kq}.$$

To exploit Theorem 4.3 we really need to know the filtration on $K(B_{S_k})$ as is shown by the following theorem:

THEOREM 4.5. *Assume that $K(X)$ is torsion-free and let* $\dim X \leqslant n$. *Let $x \in K_q(X)$ and assume that all products $\lambda^i(x)\lambda^j(x)$ with $i, j > 0$, $i+j \leqslant k$ vanish. Then $\lambda^k(x)$ is divisible by the least integer m for which*

$$m\alpha\lambda_{-1}(M) \in K_{kq-n}(B_{S_k}),$$

M being as in Proposition 2.4. *In particular this holds in the stable range* $n < 2q$.

Proof. The hypotheses and Proposition 2.4 imply that

$$x^{\otimes k} = (-1)^{k-1}\lambda^k(x) \otimes \lambda_{-1}(M) \in K(X) \otimes R(S_k).$$

Let $A = K(B_{S_k})/K_{kq-n}(B_{S_k})$, so that A is a subgroup of $K(B_{S_k,kq-n-1})$. From Corollary 4.4 and the fact that $K(X)$ is free it follows that the image of $x^{\otimes k}$ in $K(X) \otimes A$ must be zero. Hence $\lambda^k(x)$ must be divisible by the order of the image of $\lambda_{-1}(M)$ in A, i.e. by the least integer m for which

$$m\alpha\lambda_{-1}(M) \in K_{kq-n}(B_{S_k}).$$

Remark. In the proof of Proposition 1.9 we saw that the character of $\lambda_{-1}(M)$ vanishes on all composite cycles of S_k. Thus, if k is not a prime-power, the character of $\lambda_{-1}(M)$ vanishes on all elements of S_k of prime-power order and so by (5) [(6.10)] $\lambda_{-1}(M)$ is in the kernel of the homomorphism

$$R(S_k) \to \widehat{R(S_k)}.$$

Hence $\alpha\lambda_{-1}(M) = 0$ and so Theorem 4.5 becomes vacuous. *Thus Theorem* 4.5 *is of interest only when k is a prime-power.*

In order to obtain explicit results it is necessary to restrict from S_k to the cyclic group Z_k. In this case the calculations are simple. First we need the lemma:

LEMMA 4.6. *Let $Y = B_{Z_k}$, then*

$$K(Y_{2q-1}) \cong R(Z_k)/I(Z_k)^q.$$

Proof. Since Y has no odd integer cohomology, it follows that $K^1(Y, Y_{2q-1}) = 0$, and so from the exact sequence of this pair we deduce

$$K(Y_{2q-1}) \cong K(Y)/K_{2q}(Y).$$

But we know [(5) (8.1)] that

$$K(Y) \cong \widehat{R(Z_k)},$$

and $K_{2q}(Y)$ is the ideal generated by $I(Z_k)^q$. Hence

$$K(Y)/K_{2q}(Y) \cong R(Z_k)/I(Z_k)^q,$$

and the lemma is established.

Remark. The results quoted from (5) are quite simple, and we could easily have applied the calculations used there directly to Y_{2q-1}.

Combining Corollary 4.4 and Lemma 4.6 we deduce the proposition:

PROPOSITION 4.7. *Let* $\dim X \leqslant 2m$ *and let* $x \in K_{2q}(X)$. *Then the kth cyclic power* $P^k(x) \in K(X) \otimes R(Z_k)$ *is in the image of* $K(X) \otimes I(Z_k)^{kq-m}$.

The case when $k = p$, a prime, is of particular interest because Z_p is then the p-Sylow subgroup of S_p. This means that, as far as p-primary results go, nothing is lost on passing from S_p to Z_p. In the next section therefore we shall study this case in detail.

5. The prime cyclic case

LEMMA 5.1. *Let* $\rho \in R(Z_p)$ *denote the canonical one-dimensional representation of* Z_p,

$$N = \sum_{i=0}^{p-1} \rho^i$$

the regular representation and $\eta = \rho - 1$.

Then in $\widehat{R(Z_p)}$ *we have*

$$p^k(N-p) = (-1)^k \eta^{(k+1)(p-1)} + \text{higher terms}.$$

Proof. Since $\rho^p = 1$, we have $(1+\eta)^p = 1$. Thus $\eta^p = -p\eta\epsilon$, where $\epsilon \equiv 1 \bmod \eta$ and so is a unit in $\hat{R}$. Hence

$$(-p)\eta \sim \eta^p, \tag{1}$$

where we write $a \sim b$ if $a = \epsilon b$ with $\epsilon \equiv 1 \bmod \eta$. Now the identity

$$\sum_{i=0}^{p-1} (1+t)^i = \frac{(1+t)^p - 1}{t} \equiv p + t^{p-1} \quad \bmod pt$$

with t replaced by η shows that

$$\begin{aligned} N-p &\equiv \eta^{p-1} \quad \bmod p\eta \\ &\equiv \eta^{p-1} \quad \bmod \eta^p \quad \text{by (1)}. \end{aligned}$$

Hence we have $$(N-p) \sim \eta^{p-1}. \tag{2}$$

From (1) we have $$(-p)^k \eta \sim \eta^{k(p-1)}\eta,$$

and so $$(-p)^k \eta^{p-1} \sim \eta^{(k+1)(p-1)}. \tag{3}$$

The lemma now follows from (2) and (3).

COROLLARY 5.2. *The order of the image of* $(N-p)$ *in* $R(Z_p)/I(Z_p)^n$ *is* p^k *where k is the least integer such that* $k+1 \geqslant \dfrac{n}{p-1}$.

Proof. $I(Z_p)$ is the ideal (η).

We can now state the explicit result for the prime case:

THEOREM 5.3. *Suppose that* $\dim X \leqslant 2(q+t)$ *with* $t < q(p-1)$ *and let* $x \in K_{2q}(X)$. *Then* $\theta^p(x)$ *is divisible by* p^{q-r-1}, *where*

$$r = \left[\frac{t}{p-1}\right].$$

Proof. Since $\dim X < 2qp$, we have $x^p = 0$. Hence by Proposition 2.5 we have

$$P^p(x) = \theta^p(x) \otimes (N-p) \in K(X) \otimes R(Z_p).$$

By Proposition 4.7 it follows that $\theta^p(x)$ is divisible by the order of the image of $(N-p)$ in $R(Z_p)/I(Z_p)^n$, where

$$n = pq-q-t.$$

From Theorem 5.3 it follows that $\theta^p(x)$ is divisible by p^k, where k is the least integer for which

$$(k+1) \geqslant q - \frac{t}{p-1},$$

namely
$$k = q - \left[\frac{t}{p-1}\right] - 1.$$

COROLLARY 5.4. *Let the hypotheses be the same as in Theorem* 5.3. *Then* $\psi^p(x)$ *is divisible by* p^{q-r}, *where* $r = \left[\frac{t}{p-1}\right]$.

Proof. ψ^p and θ^p are related by the formula

$$\psi^p(x) = x^p - p\theta^p(x).$$

Since $x^p = 0$ in our case, we have

$$\psi^p(x) = -p\theta^p(x),$$

and so the result follows at once from Corollary 5.2.

Remark. Taking $t = 0$ we find that $\psi^p(x)$ is divisible by p^q on the sphere S^{2q}. Note that this result was not fed in explicitly anywhere. It is of course a consequence of the periodicity theorem, and the computation we have used for $K(B_{Z_p})$ naturally depended on the periodicity theorem.

The preceding results take a rather interesting form if X has no torsion. First we need a lemma:

LEMMA 5.5. *Suppose that* X *has no torsion* (*i.e.* $H^*(X, \mathbf{Z})$ *has no torsion*) *and let* $x \in K(X)$. *Suppose that the image of* x *in* $K(X_q)$ *is divisible by* d. *Then* x *is divisible by* d modulo $K_{q+1}(X)$, *i.e.*

$$x = dy + z, \quad y \in K(X), \quad z \in K_{q+1}(X).$$

Proof. Let A, B denote the image and cokernel of

$$j^*: K(X) \to K(X_q).$$

From the exact sequence of the pair (X, X_q) we see that B is isomorphic to a subgroup of $K^1(X, X_q)$. But, since X is torsion-free, so is X/X_q. Hence $K^1(X, X_q)$ is free and therefore also B. Hence, if $a \in A$ is divisible by d in $K(X_q)$, it is also divisible by d in A. Taking $a = j^*(x)$ therefore we have

$$j^*(x) = dj^*(y) \quad \text{for some } y \in K(X),$$

and so $x = dy+z$, for some $z \in \operatorname{Ker} j^* = K_{q+1}(X)$.

Using this lemma we now show how Corollary 5.4 leads to the following proposition:

PROPOSITION 5.6. *Suppose that X has no torsion and let $x \in K_{2q}(X)$. Then there exist elements*

$$x_i \in K_{2q+2i(p-1)}(X) \quad (i = 0, 1, \ldots, q)$$

such that

$$\psi^p(x) = \sum_{i=0}^{q} p^{q-i} x_i,$$

Moreover we can choose $x_q = x^p$.

Proof. By Theorem 5.3 the restriction of $\psi^p(x)$ to the $2(q+t)$-skeleton, with $t = i(p-1)-1$, is divisible by p^{q-i+1}. By Corollary 5.4 it follows that $\psi^p(x)$ is divisible by p^{q-i+1} modulo $K_{2q+2i(p-1)}(X)$. The required result now follows by induction on i. Since $\psi^p(x) \equiv x^p \bmod p$ and $x^p \in K_{2pq}(X)$, it follows that x^p is a choice for x_q.

The elements x_i occurring in Lemma 5.6 are not uniquely defined by x. If, however, we pass to the associated graded group $GK^*(X)$ and then reduce mod p, we see that the element

$$\bar{x}_i \in G^{2q+2i(p-1)}K(X) \otimes Z_p$$

defined by x_i is uniquely determined from the relation

$$\psi^p x = \sum_{i=0}^{q} p^{q-i} x_i.$$

If we multiply x by p or add to it anything in $K_{2(q+1)}(X)$, we see from Lemma 5.5 that $\bar{x}_i$ is unchanged. Hence $\bar{x}_i$ depends only on

$$\bar{x} \in G^{2q}K(X) \otimes Z_p.$$

Now we recall [(8) § 2] that, since X has no torsion, we have an isomorphism of graded rings

$$H^*(X, \mathbf{Z}) \cong GK^*(X),$$

and hence

$$H^{2q}(X, Z_p) \cong G^{2q}K(X) \otimes Z_p.$$

By this isomorphism the operation $\bar{x} \to \bar{x}_i$ must correspond to some cohomology operation. In the next section we shall show that this is precisely the Steenrod power P_p^i.

6. Relation with cohomology operations

In the proof of Proposition 4.2 we verified that there was an inclusion

$$j:(X^k, X^k_{2kq-1}) \to (X, X_{2q-1})^k.$$

Hence we can consider the map

$$K(X, X_{2q-1}) \to K(X^k_{S_k}, (X^k_{S_k})_{2kq-1})$$

given by $x \mapsto \alpha j^* x^{\boxtimes k}$. If we follow this by a cellular approximation to the diagonal map $X_{S_k} \to X^k_{S_k}$, we obtain a map

$$\mu: K(X, X_{2q-1}) \to K(X_{S_k}, (X_{S_k})_{2kq-1}).$$

From its definition this is compatible with the operation

$$x \mapsto d^*\alpha x^{\boxtimes k} = \alpha x^{\otimes k}$$

for the absolute groups, i.e. we have a commutative diagram

$$\begin{array}{ccc} K(X, X_{2q-1}) & \longrightarrow & K(X_{S_k}, (X_{S_k})_{2kq-1}) \\ \downarrow & & \downarrow \\ K(X) & \longrightarrow & K(X_{S_k}) \end{array} \tag{6.1}$$

On the other hand, by restricting X to X_{2q} and X_{S_k} to $(X_{S_k})_{2kq}$ we obtain another commutative diagram

$$\begin{array}{ccc} K(X, X_{2q-1}) & \xrightarrow{\mu} & K(X_{S_k}, (X_{S_k})_{2kq-1}) \\ \downarrow & & \downarrow \\ K(X_{2q}, X_{2q-1}) & & K((X_{S_k})_{2kq}, (X_{S_k})_{2kq-1}) \\ \| & & \| \\ C^{2q}(X) & \xrightarrow{\nu} & C^{2kq}(X_{S_k}) \end{array} \tag{6.2}$$

where ν is the map of cochains given by

$$\nu(c) = d^*[(c \otimes c \otimes \ldots \otimes c) \otimes_\Gamma 1]. \tag{6.3}$$

Here we have made the identification

$$C^*(X^k_{S_k}) = (C^*(X) \otimes_Z \ldots \otimes_Z C^*(X)) \otimes_\Gamma C^*(A),$$

where $A \to B_{S_k}$ is the universal S_k-bundle and Γ is the integral group ring of S_k, and similarly we identify

$$C^*(X_{S_k}) = C^*(X) \otimes_\Gamma C^*(A).$$

The commutativity of Diagram 6.2 depends of course on the fact that the isomorphism

$$K(X_{2q}, X_{2q-1}) \cong C^{2q}(X)$$

is compatible with (external) products.

The map ν defined by (6.3) induces a map of cohomology (denoted also by ν)

$$\nu: H^{2q}(X, \mathbf{Z}) \to H^{2kq}(X_{S_k}, \mathbf{Z}).$$

The diagrams (6.1) and (6.2) then establish the following

Proposition 6.4. *Let $x \in K_{2q}(X)$ be represented by $a \in H^{2q}(X, \mathbf{Z})$ in the spectral sequence $H^*(X, \mathbf{Z}) \Rightarrow K^*(X)$. Then $\alpha(x^{\otimes k}) \in K_{2kq}(X_{S_k})$ is represented by $\nu(a) \in H^{2kq}(X_{S_k}, \mathbf{Z})$ in the spectral sequence*

$$H^*(X_{S_k}, \mathbf{Z}) \Rightarrow K^*(X_{S_k}),$$

where ν is induced by the formula (6.3).

Remarks. (1) It seems plausible that one could in fact define a tensor-power operation mapping the spectral sequence of X into the spectral sequence of X_{S_k}. Proposition 6.4 concerns itself only with the extreme members E_2 and E_∞ (and only for even dimensions).

(2) The map ν is essentially the parent of all the Steenrod operations, while $x \longmapsto x^{\otimes k}$ is the parent of all the operations in K-theory introduced in § 2. Proposition 6.4 contains therefore, in principle, all the relations between operations in the two theories. We proceed to make this explicit in the simplest case:

Theorem 6.5. *Suppose that X has no torsion so that we may identify $H^*(X, Z_p)$ with $GK^*(X) \otimes Z_p$. If $x \in K_{2q}(X)$ we denote the corresponding element of $H^{2q}(X, Z_p)$ by $\bar{x}$. Let*

$$\psi^p x = \sum_{i=0}^{q} p^{q-i} x_i$$

be the decomposition of $\psi^p x$ given by (5.6). *Then we have*

$$\bar{x}_i = P_p^i(\bar{x}),$$

where $$P_p^i: H^{2q}(X, Z_p) \to H^{2q+2i(p-1)}(X, Z_p)$$

is the Steenrod power (for $p = 2$ we put $P^i = Sq^{2i}$).

Proof. By Proposition 6.4 the map

$$P: K(X) \to K(X) \otimes R(Z_p)$$

induces $$\bar{P}: H^*(X, Z_p) \to H^*(X, Z_p) \otimes H^*(Z_p, Z_p), \quad (1)$$

where $\bar{P}$ is ν reduced mod p. Now by (2.6) and Lemma 5.5 (choosing $x_q = x^p$) we have the following expression for $P(x)$,

$$P(x) = x_q \otimes 1 - \sum_{i=0}^{q-1} x_i \otimes p^{q-i-1}(N-p). \tag{2}$$

By definition of the Steenrod powers [(**10**) 112] we have

$$\bar{P}(\bar{x}) = \sum_{i=0}^{q} (-1)^{q-i} P^i(\bar{x}) \otimes \eta^{(q-i)(p-1)},$$

where η is the canonical generator of $H^2(Z_p; Z_p)$.

Comparing (1) and (2) and using Lemma 5.1 we have the result.

Remark. Proposition 6.5, together with the kind of calculations made in (**3**), leads to a very simple proof of the non-existence of elements of Hopf invariant 1 mod p (including the case $p = 2$).

7. Relation with Chern characters

If the space X has no torsion, it is possible to replace the operations ψ^k by the Chern character

$$\mathrm{ch}: K^*(X) \to H^*(X; \mathbf{Q}).$$

In fact ch is a monomorphism and ψ^k can be computed from the formulae

$$\mathrm{ch}\, x = \sum_q \mathrm{ch}_q(x), \qquad x \in K(X),\ \mathrm{ch}_q(x) \in H^{2q}(X; \mathbf{Q})$$

$$\mathrm{ch}\, \psi^k x = \sum_q k^q \mathrm{ch}_q(x).$$

Conversely one can define $H^*(X; \mathbf{Q})$ and ch purely in terms of the ψ^k (**3**). It is reasonable therefore to try to express Theorems 5.6 and 6.5 in terms of Chern characters. We shall see that we recover the results of Adams (**1**), at least for spaces without torsion.

If X is without torsion, we identify $H^*(X; \mathbf{Z})$ with its image in $H^*(X; \mathbf{Q})$. If $a \in H^*(X; \mathbf{Q})$, we can write $a = b/d$ for $b \in H^*(X; \mathbf{Z})$ and some integer d. If d can be chosen prime to p, we shall say that a is *p-integral.*

THEOREM 7.1. *Let X be a space without torsion, $x \in K_{2q}(X)$ and p a prime. Then*

$$p^t \mathrm{ch}_{q+n}(x)$$

is p-integral, where $t = \left[\dfrac{n}{p-1}\right]$.

Proof. We proceed by induction on n. For $n = 0$ (and all q) the result is a consequence of the periodicity theorem (**8**). We suppose therefore that $n > 0$ and the result established for all $r \leqslant n-1$. By Proposition 5.6 we have

$$\psi^p x = \sum_{i=0}^{q} p^{q-i} x_i, \quad x_i \in K_{2q+2i(p-1)}(X),$$

and so

$$\operatorname{ch} \psi^p x = \sum_{i=0}^{q} p^{q-i} \operatorname{ch} x_i.$$

Taking components in dimension $2(q+n)$ we get

$$p^{q+n} \operatorname{ch}_{q+n}(x) = \sum_{i=0}^{t} p^{q-i} \operatorname{ch}_{q+n}(x_i), \quad t = \left[\frac{n}{p-1}\right]. \tag{1}$$

In particular, for $n = 0$, we have

$$\operatorname{ch}_q(x) = \operatorname{ch}_q(x_0). \tag{2}$$

Since X has no torsion, this implies that

$$y = x_0 - x \in K_{2q+2}(X).$$

Replacing x_0 by $x+y$ in (1) and multiplying by p^{t-q} we get

$$p^t(p^n-1)\operatorname{ch}_{q+n}(x) = p^t \operatorname{ch}_{q+n} y + \sum_{i=1}^{t} p^{t-i} \operatorname{ch}_{q+n}(x_i). \tag{3}$$

But by the inductive hypothesis (with q replaced by $q+1$ and $q+i(p-1)$ $(i \geqslant 1)$) we see that all terms on the right-hand side of (3) are p-integral. Hence $p^t \operatorname{ch}_{q+n}(x)$ is p-integral and so the induction is established.

For any $x \in K_{2q}(X)$ we denote by $\bar{x} \in H^{2q}(X, Z_p)$ the corresponding element obtained from the isomorphism

$$G^{2q}K(X) \otimes Z_p \cong H^{2q}(X; Z_p).$$

Now, by Theorem 7.1, $p^t \operatorname{ch}_{q+t(p-1)} x$ is p-integral. We may therefore reduce it mod p and obtain an element of $H^{2q+2t(p-1)}(X; Z_p)$. It follows from Theorem 7.1 that this depends only on $\bar{x}$. We denote it therefore by $T^t(\bar{x})$, so that T^t is an operation

$$H^{2q}(X; Z_p) \to H^{2q+2t(p-1)}(X; Z_p).$$

We now identify this operation.

THEOREM 7.2. *The operation* $\sum_{i \geqslant 0} T^i$ *is the inverse of the 'total' Steenrod power* $\sum_{i \geqslant 0} P^i$,

i.e.

$$(\sum T^i) \circ (\sum P^i) = \text{identity}.$$

Proof. As in Theorem 7.1 we have

$$\psi^p x = \sum_{i=0}^{q} p^{q-i} x_i.$$

Now in equation (1) above take $n = t(p-1)$ and multiply by p^{t-q}. Then reducing mod p we get

$$0 = \sum_{i=0}^{t} T^{t-i}(\bar{x}_i) \quad (t > 0),$$

$$\bar{x} = T^0(\bar{x}_0).$$

But by Theorem 6.5 we have $\bar{x}_i = P^i\bar{x}$, and so we deduce

$$0 = \Big(\sum_{i=0}^{t} T^{t-i}P^i\Big)\bar{x}, \quad \bar{x} = T^0P^0\bar{x}.$$

In other words, the composition

$$(\sum T^i) \circ (\sum P^i)$$

is the identity operator as required.

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