SPECTRAL ASYMMETRY AND RIEMANNIAN GEOMETRY

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1 *Main Theorems*

If A is a positive self-adjoint elliptic (linear) differential operator on a compact manifold then it has a discrete spectrum consisting of positive eigenvalues $\{\lambda\}$. In analogy with the classical Riemann zeta-function one can define, for Re(s) large,

$$\zeta_A(s) = \operatorname{Tr} A^{-s} = \sum \lambda^{-s}.$$

This has an analytic continuation to the whole s-plane as a meromorphic function of s and $s = 0$ is not a pole: moreover $\zeta_A(o)$ can be computed as an explicit integral over the manifold [9]. In this note we shall introduce a refinement of this invariant when A is no longer positive and we shall study its geometrical significance for an important class of operators (first order systems) arising from Riemannian geometry. A full exposition will be given elsewhere.

Suppose therefore that A is self-adjoint and elliptic but no longer positive. The eigenvalues are now real but can be positive or negative. We define, for Re(s) large,

$$\eta_A(s) = \sum_{\lambda \neq 0} (\operatorname{sign} \lambda)\, |\lambda|^{-s}$$

and again it is true that this extends meromorphically to the whole s-plane. A considerably more difficult result is

THEOREM 1. *$\eta_A(s)$ is finite at $s = 0$.*

The number $\eta_A(o)$—which we also write simply as $\eta(A)$—is clearly a measure of asymmetry of the spectrum of A. As a simple example take $A = i(d/dx)+t$ where $x(\text{mod } 2\pi)$ is the angular coordinate on the circle and t is a real parameter $0 < t < 1$. Then a simple computation gives $\eta(A) = 1-2t$: note that Spec A consists of $\pm n+t$ and is symmetric about 0 only if $t = \frac{1}{2}$.

For an interesting example in higher dimensions take a compact oriented Riemannian manifold X of dimension $4k-1$ and let A be the operator, acting on exterior differential forms of even degree, given by

$$A(\phi) = (-1)^{p+1} *d\phi + (-1)^p d*\phi \qquad (\deg \phi = 2p).$$

Here d is exterior differention and $*$ is the Hodge duality operator defined by the metric. It is easy to check that A is self-adjoint and elliptic. Suppose now that X is the boundary of an oriented Riemannian manifold Y which is isometric near the boundary

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to the cylinder $X \times [0, 1]$. Let $p_i(Y)$ denote the i-th Pontrjagin form of Y, that is, the differential form constructed from the Riemannian curvature, which represents the i-th Pontrjagin class. Then we have the following theorem:

THEOREM 2.

$$\operatorname{Sign}(Y) - \int_Y L_k(p(Y)) = (-1)^{k+1}\,\eta(A),$$

where $L_k(p) = L_k(p_1, p_2, \ldots, p_k)$ *is the Hirzebruch L-polynomial and* $\operatorname{sign}(Y)$ *is the signature of the quadratic form on* $H^{2k}(Y, X)$.

If X is empty Theorem 2 reduces to the Hirzebruch signature theorem for a closed $4k$-manifold. Thus we can view Theorem 2 as a differential-geometric generalization of Hirzebruch's theorem to manifolds with boundary. If we compare it with the Gauss–Bonnet theorem for manifolds with boundary we see there is an important difference. Thus in Gauss–Bonnet the boundary only contributes a term due to geodesic curvature and this vanishes when Y is a product near the boundary. *Our boundary invariant $\eta(A)$ therefore represents a new phenomenon not present in Gauss–Bonnet.* When Y is not a product near the boundary there is an additional term in the formula of Theorem 2 involving the second fundamental form of X in Y.

To understand Theorem 2 in its correct analytical context one has to re-interpret the integer $\operatorname{sign}(Y)$. To do this let $\hat{Y}$ denote the complete non-compact Riemannian manifold obtained from Y by attaching the semi-infinite cylinder $X \times [0, \infty)$ to the boundary. Then we have

THEOREM 3. *The space of square-integrable harmonic forms on $\hat{Y}$ is naturally isomorphic to the image $\hat{H}$ of $H^*(Y, X)$ in $H^*(Y)$.*

Since $\operatorname{sign}(Y)$ is in effect the signature of a non-degenerate quadratic form on $\hat{H}$, Theorem 3 leads to an interpretation for $\operatorname{sign}(Y)$ in terms of harmonic L^2-forms analogous to the situation for closed manifolds.

There are counterparts of Theorem 2 for all the other classical operators associated to a Riemannian metric†. For simplicity we shall describe only the case of the Dirac operator. Let X, Y be as above and assume further that Y (hence also X) is a spin-manifold. Let D denote the Dirac operator of X, acting on the spinor fields. This is a self-adjoint elliptic operator hence $\eta(D)$ is defined. Then we have

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† There are interesting cases whenever $\dim X$ is odd. It is not necessary for $\dim X$ to be of the form $4k-1$: when $\dim X = 4k+1$ the operator iA is self-adjoint and elliptic; it generalizes $i\,d/dx$ in the circle example given earlier.

THEOREM 4.

$$\text{Spin}\,(\hat{Y})+h^+ - \int_{\hat{Y}} \hat{A}(p(Y)) = \tfrac{1}{2}(h-\eta(D)).$$

Here $\hat{A}$ is the Hirzebruch $\hat{A}$-polynomial and h, h^+, Spin$(\hat{Y})$ are integers defined as follows. First $h = \dim H$ where H is the space of harmonic spinors on X (the null space of D^2 or D). Next Spin$(\hat{Y})$ is the " L^2-spinor index ", namely it is $p-q$ where p and q are respectively the dimensions of the square-integrable harmonic $+$ and $-$ spinors on $\hat{Y}$: recall that the spin-bundle of $\hat{Y}$ breaks up into two parts S^+ and S^- corresponding to the $\frac{1}{2}$-spin representations of Spin$(4k)$. Finally, identifying S^+ on the cylinder with the spin bundle of X, we define $h^+ = \dim H^+$ where $H^+ \subset H$ consists of limiting values of harmonic sections of S^+. More precisely $\phi \in H^+$ if there is a harmonic section ψ of S^+ such that $\psi - \phi \in L^2$ (in the cylinder).

The minor formal differences between Theorems 2 and 4 are easily accounted for. For forms the analogues of h^+ and $h/2$ cancel because of Poincaré duality and the factor $\frac{1}{2}$ multiplying $\eta(D)$ disappears because the analogue of D consists of two copies of A (one for even forms and one for odd forms).

2. *Generalizations*

If T is a linear transformation commuting with our self-adjoint operator we can define the function

$$\eta_{A,T}(s) = \sum_{\lambda \neq 0} (\text{sign}\,\lambda)\ \text{Trace}\ T_\lambda\,|\lambda|^{-s}$$

where the summation is taken over distinct eigenvalues and T_λ is the transformation induced by T on the λ-eigenspace. In the Riemannian cases such transformations arise naturally from the (compact) group G of isometries (preserving orientation and spin structure where relevant). Putting $s = 0$ this leads to invariants $\eta(A, g)$ for $g \in G$ and Theorems 2 and 4 have natural generalizations in which the integral over Y is replaced by an integral over Y^g—the fixed-point set of g. If g has no fixed-points on X then Y^g is closed and the integrals become *independent of the metric*. The generalization of Theorem 2 then identifies $\eta(A, g)$ with the invariant for fixed-point free actions defined in [3 : §6]. An important example of this theorem is given by taking X to be a sphere and g to be a rotation. In this case the result can be verified directly by computing the eigenvalues and comparing the resulting value of $\eta(A, g)$ with the explicit formula given in [3]: this was done by D. B. Ray.

A closely related generalization arises if we extend our operator A to a local coefficient system given by a unitary representation χ of the fundamental group of X. We then obtain invariants $\eta(A,\chi)$, and $\eta(A,\chi) - \dim\chi\,.\,\eta(A)$ is *independent of the metric.* If $\pi_1(X)$ is finite this invariant is essentially equivalent—via Fourier transforms—to the invariant $\eta(A, g)$ for the universal cover of X(with $g \in \pi_1(X)$). However if $\pi_1(X)$ is infinite we have a new situation not covered by the other point of view. In this way the invariant of [3] is extended to infinite fundamental groups.

The preceding remarks show that there is a strong analogy between our η-invariants and the analytic torsion of Ray–Singer [7][8]. While this analogy played a role in suggesting the form of Theorem 2 we are far from having a unified point of view embracing [7] [8] and our present results.

Theorems 2 and 4 show that our η-invariants are closely related to the invariants studied by Chern and Simons [4]. However there are significant differences and these can be exploited. Thus suppose X has a given framing f (that is, a trivialization of its tangent bundle). Then, by the Pontrjagin–Thom construction, (X, f) determines an element in the stable homotopy group of spheres in dimension $4k-1$. In particular one can associate to (X, f) its Adams e-invariant, which identifies a finite cyclic summand in π^S_{4k-1}. It is reasonable to ask for an explicit way of calculating $e(X,f)$, and this is provided by the following:†

THEOREM 5.

$$e(X,f) = \tfrac{1}{2}(\eta(D)-h) - \int_X \hat{A}(f) \qquad \begin{array}{ll} \bmod Z & (k \text{ even}) \\ \bmod 2Z & (k \text{ odd}). \end{array}$$

Here D, h are associated to the metric and spin structure defined by f as in Theorem 4, and $\hat{A}(f)$ is a $(4k-1)$-form on X canonically defined by the metric, the framing and the $\hat{A}$-polynomial. More precisely $\hat{A}(f) = f^*(\omega)$ where f is now viewed as a section of the principal orthogonal bundle Q and ω is the form on Q defined by the metric connection and $\hat{A}$ in the manner of Chern-Simons [4]. Universally ω is characterized by the requirement that $d\omega = \hat{A}(p)$.

The right-hand side of Theorem 5 is a real number given by analysis on X: in fact it has to be *rational*, since $e \in Q/Z$. Such rationality properties of the η-invariant are connected with rational values of some L-functions in number theory. The connection comes about because certain manifolds X of the form G/Γ with G, Γ solvable, G a Lie group and Γ discrete, arise naturally in number theory and have natural framings [5]. In fact the present work was, to a considerable extent, motivated by the attempt to understand some of the results in [5]. The details of all this will be explained elsewhere.

3. *Outline of proof*

Since Theorems 2 and 4 generalize certain cases of the index theorem on closed manifolds it is natural to try and formulate them as index theorems for elliptic operators with boundary condition. The difficulty is that the operators on Y which are involved are known not to admit any local elliptic boundary conditions of the conventional kind. There are non-vanishing topological obstructions to this as explained in [6; Appendix I]. The way round this difficulty is to consider more general boundary conditions of a *global* character. More precisely (for the case of spinors

† In the notation of [1] $e = e_c$ for k even and $\tfrac{1}{2}e = e_R$ for k odd.

say) we define the boundary operator B acting on a section ϕ of S^+ on Y by: $B\phi = P(\phi|X)$ where P is the spectral projection of the Dirac operator D on X corresponding to eigenvalues $\lambda \geqslant 0$. Such boundary operators present no serious analytical problems and have been considered by various authors although not from our point of view in relation to index problems.

Once we have set the problem up as an index problem for an elliptic operator with boundary condition, standard analytical methods on the lines of **[10]** show that our index can be evaluated in terms of a contribution from the interior and a contribution from the boundary. The first of these is locally the same as the contribution for a closed manifold and, for the classical Riemannian operators, have been determined in **[2]** where they are identified with the appropriate polynomial in the Pontrjagin forms. The boundary contribution is computed by a direct and elementary calculation in terms of the eigenvalues of D and gives the right hand side of Theorem 4. The index of our problem (for spinors) contributes the first two terms in Theorem 4. The reason why $L^2(Y)$ enters is that the harmonic spinors satisfying our boundary condition are essentially those which decay exponentially in the cylinder.

The Gauss–Bonnet formula is much simpler than Theorem 2 or 4 precisely because the operator associated to the Euler characteristic (namely $d+d^*$: even forms $\rightarrow$ odd forms) does admit a local elliptic boundary condition: the obstruction vanishes in this case.

Finally we shall comment on Theorem 1. Although, for convenience, this was stated first we do not know of a direct analytic proof. In fact Theorem 1 can be deduced as a Corollary of Theorem 4 (and its analogues) by topological methods rather in the way the general index theorem was derived in **[2]** from the classical special cases of Riemannian geometry. An essentially equivalent method is to try and treat the general case of $\eta(A)$ by a boundary value problem on the lines of the Riemannian cases. However topological methods are needed to show that a suitable boundary value problem can be set up.

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