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GROUP REPRESENTATIONS, λ-RINGS AND THE *J*-HOMOMORPHISM

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INTRODUCTION

THIS paper arose from a desire to apply the work of J. F. Adams 'on the groups $J(X)$' [2] to the case where X is the classifying space B_G of a finite group G. Since Adams' calculations apply only to a finite complex X, and B_G is infinite, the results could not be applied directly. Rather than quoting theorems and using limiting processes, the pure algebra has been isolated and independently reworked in such a way that it not only applies to the situation considered, but is also of general interest. This occurs in Parts I and III. The algebra used requires knowledge of special λ-rings (which arise in K-theory and elsewhere). Part I is a self-contained study of these. (A special λ-ring is a commutative ring together with operations $\{\lambda^n\}$ having the formal properties of exterior powers). Part III contains the main algebraic theorem, which readers familiar with the work of Adams [2] may recognise as essentially including a proof of his theorem '$J'(X) = J''(X)$'.

Arising from the applications of this theory, the principal theorem of the paper lies in another field of study, in the topology of group representations. Broadly speaking, a general type of problem that may be posed is this: given two representations E, F of a group G, what constraints are imposed on E and F by the existence of a given type of map between them which commutes with G-action? More precisely, if E, F are unitary (or orthogonal) representations, so that the unit spheres $S(E)$, $S(F)$ are preserved by G-action, under what conditions can there exist a G-map $\phi : S(E) \to S(F)$ where ϕ is a diffeomorphism, homeomorphism, homotopy equivalence or some other given type of map?

Some results are known, for example de Rham [17] has shown that if ϕ is a diffeomorphism, then E, F must be isomorphic representations.

In this paper, only very weak restrictions are placed on ϕ. The following theorem is proved:

If G is a p-group ($p \neq 2$) and E is irreducible, then there exists a G-map ϕ: $S(E) \to S(F)$ of degree prime to p if, and only if, F is conjugate to E.

(We recall that if G is of order N, the values of the character of a complex representation of G lie in the field $\mathbf{Q}(\omega)$ where $\mathbf{Q}$ is the field of rationals and ω is a primitive Nth root of unity. Two representations are said to be conjugate if their characters are conjugate by an element of Γ_N, the Galois group of $\mathbf{Q}(\omega)$ over $\mathbf{Q}$).

Two unitary representations E, F of an arbitrary finite group G are said to be J-equivalent if there are G-maps from $S(E)$ to $S(F)$ and from $S(F)$ to $S(E)$ both of degree prime to the order of G. This is an equivalence relation; in fact for a p-group of odd order, it will follow from a result of this paper that we only need a map in one direction to ensure equivalence. Hence for a p-group ($p \neq 2$), irreducible representations are J-equivalent if and only if they are conjugate.

More generally we will prove:

Two representations are J-equivalent if and only if their irreducible components are conjugate in pairs (by possibly different elements of Γ_N).

In Part II, explicit G-maps are constructed between conjugate representations. In Part IV algebraic invariants are constructed for representations which distinguish between those which are not J-equivalent.

The principal theorem is best described using the representation ring $R(G)$ (the free abelian group with equivalence classes of irreducible representations of G as generators). If E is a representation, denote by $[E]$ its class in $R(G)$ and if $T(G)$ is the subgroup consisting of elements $[E] - [F]$ where E and F are J-equivalent, define $J(G) = R(G)/T(G)$. ($T(G)$ is a subgroup because J-equivalence can be shown to be an additive relation.) Let $W(G)$ be the subgroup generated by the elements $[E] - [\alpha E]$ where $\alpha \in \Gamma_N$, then in standard notation $R(G)_{\Gamma_N} = R(G)/W(G)$.

The main theorem states $J(G) = R(G)_{\Gamma_N}$ for a p-group of odd order.

The remainder of this introduction consists of a brief outline of the proofs involved. In [2] Adams calculates the group $J(X)$, where X is a suitable topological space. $J(X)$ is a quotient group of $K(X)$, the 'ring of complex vector bundles' over X introduced in [6]. Adams proves that in certain favourable cases $J(X) = J''(K(X))$ where he gives a purely algebraic construction for $J''(K(X))$ as a quotient group of $K(X)$. The connecting link with this paper is the result of Atiyah [3], $\widehat{R(G)} = \mathcal{K}(B_G)$, where B_G is the classifying space of G. $\mathcal{K}(B_G) = \varprojlim K((B_G)_n)$ where $(B_G)_n$ is the n-skeleton of B_G, and $\widehat{R(G)}$ is the completion of $R(G)$ with respect to a suitable topology. If we apply the construction J'' to $\mathcal{K}(B_G) = \widehat{R(G)}$, we find $J''(\widehat{R(G)})$ is the quotient group $\widehat{R(G)}_{\Gamma_N}$. Thus we have chosen our notation to suggest the formal isomorphism $\widehat{J(G)} = J''(\widehat{R(G)}) = \widehat{R(G)}_{\Gamma_N}$. For p-groups $R(G)$ is monomorphically embedded in $\widehat{R(G)}$ and it was initially with this evidence that we embarked on a proof of $J(G) = R(G)_{\Gamma_N}$.

Recalling that $J(G) = R(G)/T(G)$, $R(G)_{\Gamma_N} = R(G)/W(G)$, we may define an epimorphism $v : R(G)_{\Gamma_N} \to J(G)$ simply by showing that $W(G) \subset T(G)$. We do this in Part II. In the Adams' programme, the analogous result has only been proved in certain favourable cases, but in the case of a large class of finite groups, including p-groups, this carries through relatively easily. It is done by constructing explicit J-equivalences between conjugate representations. If E, F are one dimensional and $F = \alpha E$ where $\alpha \in \Gamma_N$, $\alpha(\omega) = \omega^r$, then choosing complex

co-ordinates, $S(E)$ and $S(F)$ are given by complex numbers of unit modulus and a suitable G-map $S(E) \to S(F)$ is given by $z \mapsto z^r$. To find maps for irreducible representations of higher dimension we use induction from one dimensional representations of some subgroup. Since J-equivalence is additive, this shows $W(G) \subset T(G)$ and defines the epimorphism v.

It remains to show v is monomorphic. This is considerably more difficult. First we note that if $\varepsilon : R(G) \to \mathbf{Z}$ is the homomorphism induced by the dimension of a representation, then $R(G) = \mathbf{Z} \oplus I(G)$ (as abelian groups) where $I(G) = \ker \varepsilon$. It is easily seen that $W(G) \subset T(G) \subset I(G)$ and that v induces an epimorphism $\tilde{v}: I(G)_{\Gamma_N} \to \tilde{J}(G)$ where $\tilde{J}(G) = I(G)/T(G)$ Evidently $J(G) = \mathbf{Z} \oplus \tilde{J}(G)$ and it is sufficient to show that $\tilde{v}$ is a monomorphism.

Given a positive integer k, for each representation E, we define an algebraic invariant $\theta_k(E) \in R(G)$. The invariants which arise naturally satisfy $\theta_k(E \oplus E') = \theta_k(E)\theta_k(E')$. If k is prime to the order of G, then θ_k has the following important property:

Given a map $\phi: S(E) \to S(F)$ of degree r, then there exists $z \in R(G)$ where $\varepsilon(z) = r$ and $\alpha \in \Gamma_N$ such that $\theta_k(F).z = \theta_k(E).\alpha z$.

If $\widehat{I(G)}$ is the completion of $I(G)$ in a suitable topology, we show θ_k induces a map $\bar{\theta}_k: I(G) \to (1 + \widehat{I(G)})_{\Gamma_N}$ homomorphic from addition to multiplication. From the above property, it will follow that $T(G) \subset \ker \bar{\theta}_k$ and so $\bar{\theta}_k$ induces a map

$$\tilde{\theta}_k: I(G)/T(G) = \tilde{J}(G) \to (1 + \widehat{I(G)})_{\Gamma_N}.$$

For a fixed prime p, let $\Gamma = \varprojlim \Gamma_{p^n}$ be the group of units of the p-adic integers. Then for a p-group G, Γ acts in a natural way on $I(G)$ via the quotient group Γ_N where $N = p^e = |G|$. Furthermore $I(G)_{\Gamma_N} = I(G)_\Gamma$. If $p \neq 2$, Γ contains a dense cyclic subgroup with generator h where h is an integer prime to p satisfying $h^{p-1} \not\equiv 1 \bmod p^2$ and $h \bmod p$ is a generator of the multiplicative group of the field $\mathbf{Z}/p\mathbf{Z}$.

For a p-group G ($p \neq 2$), we show $\tilde{v}: I(G)_\Gamma \to \tilde{J}(G)$ is monomorphic by constructing the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccc}
 & & 0 & & & & \\
 & & \downarrow & & & & \\
 & & I(G)_\Gamma & \xrightarrow{\tilde{v}} & \tilde{J}(G) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \tilde{\theta}_h} & & \\
0 & \longrightarrow & \widehat{I(G)}_\Gamma & \xrightarrow{(\rho_h)_\Gamma} & (1 + \widehat{I(G)})_\Gamma & \longrightarrow & 0
\end{array}$$

A few words of explanation are in order. The completion of $I(G)$ may be given in several ways (of which two are $\varprojlim I(G)/I(G)^n$ and $\varprojlim I(G)/p^nI(G)$): we show these are all the same. The map i may be considered as the inclusion of $I(G)_\Gamma$ in its p-adic completion. The map $(\rho_h)_\Gamma$ is induced on co-invariants by a homomorphism $\rho_h: \widehat{I(G)} \to 1 + \widehat{I(G)}$ homomorphic from addition to multiplication.

In Part III, we introduce ρ_k for any positive integer k prime to p and show $(\rho_h)_\Gamma$ is an isomorphism. This is framed in a slightly more general context by introducing the notion of a 'p-adic γ-ring' A. This gives a more satisfying treatment and a more general result. Examples of a p-adic γ-ring include $\widehat{I(G)}$ for a finite p-group G and $Z_p \otimes \tilde{K}(X)$, where $Z_p = \varprojlim \mathbf{Z}/p^n\mathbf{Z}$ is the ring of p-adic integers and X a suitable topological space (e.g. a finite, connected CW complex). For any positive integer k prime to p, ρ_k: $A \to 1 + A$ is defined, homomorphic from addition to multiplication. There is a natural Γ-action on A and the map ρ_k commutes with this action. The main algebraic theorem of Part III states that for a p-adic γ-ring A ($p \neq 2$), ρ_h induces isomorphisms on invariants and co-invariants, $(\rho_h)^\Gamma : A^\Gamma \to (1 + A)^\Gamma$, $(\rho_h)_\Gamma : A_\Gamma \to (1 + A)_\Gamma$. The case $p = 2$ needs rather different treatment and this is also discussed.

In Part IV we consider the topological properties of θ_k and show that the induced map $\tilde{\theta}_k$ is well defined and renders the diagram commutative. The fact that $\tilde{v}$ is monomorphic (and hence v: $R(G)_\Gamma \to J(G)$ is an isomorphism) follows by a trivial diagram chasing argument.

The case of unitary representations can be applied directly to give corresponding theorems for orthogonal representations. This is done in Part V. We have not however been able to determine $J(G)$—or the corresponding group $JO(G)$ for orthogonal representations—in the case of a 2-group.

I. SPECIAL λ-RINGS

This part of the paper may be regarded as a survey of the theory of special λ-rings. It includes basic results for later use. A special λ-ring is defined in § 1; it is a commutative ring R with identity, together with a family of maps λ^n: $R \to R$ having the formal properties of exterior powers. Examples of special λ-rings include $K(X)$ for a compact space X and the representation ring $R(G)$ of a finite group G.

The free λ-ring on one generator is introduced in § 2. This is the smallest λ-ring containing an element s_1 such that $\{\lambda^n(s_1)\}_{n\geq 1}$ are algebraically independent. Using this in § 3 we show that the only natural operations on the category of special λ-rings are polynomials in the $\{\lambda^n\}$. An element x in a special λ-ring is said to be n-dimensional if $\lambda^r(x) = 0$ for $r > n$ and $\lambda^n(x) \neq 0$. We show that a natural operation is uniquely given by its action on a sum of one dimensional elements (verification principle). § § 4,5 are devoted to the discussion of certain natural operations $\{\gamma^n\}$ and $\{\psi^n\}$. The $\{\gamma^n\}$ define a filtration on a λ-ring and the $\{\psi^n\}$ are ring homomorphisms.

In § 6 we show that an n-dimensional element of a special λ-ring may be written as a sum of one dimensional elements in an extension ring (splitting principle). A certain type of natural construction τ which is only defined on finite dimensional elements is explained in § 7. It associates with each finite dimensional element $x \in R$ an element $\tau(x) \in R$ such that $\tau(x + y) = \tau(x)\tau(y)$ and is called a natural exponential map. The Bott cannibalistic class θ_k is a natural exponential map given for a one dimensional element x by $\theta_k(x) = 1 + x + \cdots + x^{k-1}$. It will be used in the proof of the main theorem in this paper.

§1. PRELIMINARIES

After Grothendieck [12], define a λ-ring to be a commutative ring R with identity and a countable set of maps $\lambda^n: R \to R$ such that for all $x, y \in R$

(1) $\lambda^0(x) = 1$

(2) $\lambda^1(x) = x$

(3) $\lambda^n(x + y) = \sum_{r=0}^{n} \lambda^r(x)\lambda^{n-r}(y)$

If t is an indeterminate, for $x \in R$ define:

(4) $\lambda_t(x) = \sum_{n \geq 0} \lambda^n(x)t^n$

then the relations (1), (3) show that λ_t is a homomorphism from the additive group of R into the multiplicative group $1 + R[[t]]^+$, of formal power series in t with constant term 1, i.e.

(5) $\lambda_t(x + y) = \lambda_t(x)\lambda_t(y)$

The relation (2) states that λ_t is a right inverse of the homomorphism $1 + \sum_{n \geq 1} x_n t^n \mapsto x_1$, in particular λ_t is a monomorphism.

The ring $\mathbf{Z}$ of integers may be given a λ-structure by defining $\lambda_t(1) = 1 + \sum m_n t^n$ where $m_1 = 1$. The 'canonical' λ-structure is given by $\lambda_t(1) = 1 + t$, then by (5) $\lambda_t(m) = (1 + t)^m$ and $\lambda^n(m) = \binom{m}{n}$.

Other examples of λ-rings are $K_G(X)$ for a compact G-space X, where G is a compact Lie group [19] (which includes the case $K(X)$ when G is trivial [4] [6] and $R(G)$, the complex representation ring of a finite group G [3]), and $K^0(A)$ for a commutative ring A with identity [8]. In these examples the λ-structure is induced by exterior powers.

A λ-homomorphism is a ring homomorphism commuting with the λ-operations; an augmented λ-ring is a λ-ring R together with a λ-homomorphism $\varepsilon: R \to \mathbf{Z}$ (where $\mathbf{Z}$ has the canonical λ-structure). If X is a G-space and $x_0 \in X$, then the inclusion $(1, \{x_0\}) \to (G, X)$ is an equivariant map which induces the augmentation $i^!: K_G(X) \to K(\{x_0\}) = \mathbf{Z}$, (it assigns to each G-bundle over X the dimension of the fibre at x_0). In particular the representation ring $R(G)$ is augmented $\varepsilon: R(G) \to \mathbf{Z}$ by the dimension of a representation. Choose a prime ideal $\mathfrak{p}$ in the ring A then the canonical map $j: A \to A_{\mathfrak{p}}$ induces the augmentation $j^!: K(A) \to K(A_{\mathfrak{p}}) = \mathbf{Z}$, [9].

The notion of a λ-ideal and λ-subring is evident and the usual elementary theorems may be proved, for example the kernel of a λ-homomorphism is a λ-ideal. In particular if R is augmented and $I = \ker \varepsilon$, then I is a λ-ideal.

If x is an element of a λ-ring and $\lambda_t(x)$ is a polynomial of degree n in t, then we say x is finite dimensional and its dimension is n. Not all elements of a λ-ring are finite dimensional but we say the ring R is finite dimensional if every element in R is a difference of such elements. The examples given are all finite dimensional and the dimension corresponds to the usual definition (fibre dimension). It is easy to show that if R is augmented and $x \in R$, then for x finite dimensional, $0 \leq \varepsilon(x) \leq \dim x$. The case of X disconnected with base point shows that in some λ-rings it is possible for $\varepsilon(x)$ to take any value in the range $0 \leq \varepsilon(x) \leq \dim x$. Since ε is a λ-homomorphism, this implies that, in general, λ-homomorphisms need not preserve dimension.

We may define a λ-structure on $1 + A[[t]]^+$ for any commutative ring A with unit. We give the definition in terms of universal polynomials.

Let $\xi_1, \ldots, \xi_q, \eta_1, \ldots, \eta_r$ be indeterminates and let s_i, σ_i be the ith elementary symmetric functions in $\xi_1, \ldots, \xi_q$ and $\eta_1, \ldots, \eta_r$ respectively. Define:

(6) $P_n(s_1, \ldots, s_n; \sigma_1, \ldots, \sigma_n)$ is the coefficient of t^n in $\prod_{i,j}(1 + \xi_i \eta_j t)$

(7) $P_{n,m}(s_1, \ldots, s_{mn})$ is the coefficient of t^n in

$$\prod_{i_1 < \cdots < i_m} (1 + \xi_{i_1} \cdots \xi_{i_m} t)$$

Evidently P_n is a polynomial of weight n in the $\{s_i\}$ and also in the $\{\sigma_i\}$, $P_{n,m}$ is of weight nm in the $\{s_i\}$. In order that none of the variables involved in (6), (7) are identically zero, we must choose $r \geq n, q \geq n$ in (6) and $q \geq nm$ in (7). In any case, from the usual theory of symmetric functions [16], the identities are true for all values of q, r, m, n which are non-negative. Both P_n and $P_{n,m}$ have integer coefficients and so may be defined in any commutative ring.

Define a λ-ring structure on $1 + A[[t]]^+$ by

(8) 'addition' is multiplication of power series

(9) 'multiplication' is given by

$$(1 + \sum a_n t^n) \circ (1 + \sum b_n t^n) = 1 + \sum P_n(a_1, \ldots, a_n; b_1, \ldots, b_n) t^n$$

(10) $\Lambda^m(1 + \sum a_n t^n) = 1 + \sum P_{n,m}(a_1, \ldots, a_{mn}) t^n$

LEMMA 1.1. *$1 + A[[t]]^+$ is a λ-ring with the above structure.*

Proof. We need only show that the universal polynomials satisfy certain basic identities. For example, the fact that $\circ$ is associative is equivalent to the identity:

(11) $P_n(P_1(a_1; b_1), \ldots, P_n(a_1, \ldots, a_n; b_1, \ldots, b_n); c_1, \ldots, c_n)$
$= P_n(a_1, \ldots, a_n; P_1(b_1; c_1), \ldots, P_n(b_1, \ldots, b_n; c_1, \ldots, c_n))$

If $\xi_1, \ldots, \xi_q, \zeta_1, \ldots, \eta_r, \zeta_1, \ldots, \zeta_s$ are indeterminates where $q, r, s \geq n$ then the first n elementary symmetric functions in $\xi_1, \ldots, \xi_q$, in $\eta_1, \ldots, \eta_r$ and in $\zeta_1, \ldots, \zeta_s$ are algebraically independent ([16] § 26).

Comparing coefficients of t^n in

$$\prod (1 + \xi_i \eta_j t) \circ \prod (1 + \zeta_k t) = \prod (1 + \xi_i \eta_j \zeta_k t) = \prod (1 + \xi_i t) \circ \prod (1 + \eta_j \zeta_k t)$$

we see that (11) is indeed an identity. Similarly the other identities are satisfied. 1 is the 'zero' and $1 + t$ is the 'identity'.

Definition 1.2. A λ-ring R is said to be special if

$$\lambda_t : R \to 1 + R[[t]]^+ \text{ is a } \lambda\text{-homomorphism.}$$

In effect a special λ-ring satisfies (1)–(3) and also

(12) $\lambda_t(1) = 1 + t$ or $\lambda^n(1) = 0, n > 1$ (λ_t preserves identities)

(13) $\lambda^n(xy) = P_n(\lambda^1(x), \ldots, \lambda^n(x); \lambda^1(y), \ldots, \lambda^n(y))$

(14) $\lambda^m(\lambda^n(x)) = P_{m,n}(\lambda^1(x), \ldots, \lambda^{mn}(x))$

(12) is a special case of (14) where $n = 0$. Note that (12) ensures that the only special λ-structure on $\mathbf{Z}$ is given by the canonical structure.

Every special λ-ring R contains a λ-subring isomorphic to $\mathbf{Z}$ for if $1 \in R$ had finite additive order, then

$$1 = \lambda_t(0) = \lambda_t(m.1) = (1 + t)^m$$

and comparing coefficients of t^m we derive a contradiction. Not all λ-rings may be augmented (for example the rationals where $\lambda_t(r) = (1 + t)^r$), but if the special λ-ring R is augmented, then any element of R may be written uniquely as $x = \varepsilon(x) + (x - \varepsilon(x))$ where $\varepsilon(x) \in \mathbf{Z}$, $x - \varepsilon(x) \in I$ and so $R = \mathbf{Z} \oplus I$ considered as abelian groups. The converse is also true and so:

PROPOSITION 1.3. *R is an augmented special λ-ring if and only if there is a λ-ideal such that $R = \mathbf{Z} \oplus I$ (considered as abelian groups). $I = \ker \varepsilon$, where $\varepsilon\colon R \to \mathbf{Z}$ is the augmentation.*

THEOREM 1.4 (Grothendieck). *For any commutative ring A with identity, $1 + A[[t]]^+$ is a special λ-ring.*

Proof. We have only to verify (12)–(14) in $1 + A[[t]]^+$. Evidently $\Lambda^m(1 + at) = 1$ for $m > 1$ using the universal polynomial $P_{n,m}$ and so elements of the form $1 + at$ are one dimensional, (1 is the 'zero' in $1 + A[[t]]^+$). In particular $1 + t$ is one dimensional, which gives (12). To prove (13), (14) as in lemma 1.1 we have universal polynomial formulae to prove. But in any λ-ring which satisfies the property that the product of one dimensional elements is one dimensional, if $x = \sum x_i$, $y = \sum y_j$ are sums of one dimensional elements,

$$\lambda_t(xy) = \lambda_t(\sum x_i y_j) = \prod (1 + x_i y_j t) = \lambda_t(x) \circ \lambda_t(y)$$

and similarly $\lambda_t(\lambda^n(x)) = \Lambda^n(\lambda_t(x))$, and so x, y satisfy (13), (14).

In the λ-ring $1 + \mathbf{Z}[\xi_1, \ldots, \xi_q, \eta_1, \ldots, \eta_r][[t]]^+$, the product of one dimensional elements is one dimensional, for $(1 + at) \circ (1 + bt) = 1 + abt$. Apply the previous remark to $x = \prod (1 + \xi_i t)$, $y = \prod (1 + \eta_j t)$ and by the same argument as in lemma 1.1, (13), (14) are universally satisfied.

It is easy to see that $A \mapsto 1 + A[[t]]^+$ is a covariant functor from commutative rings with unit and ring homomorphisms to special λ-rings and λ-homomorphisms. It even preserves monomorphisms and epimorphisms. Furthermore since λ_t is a monomorphism, every special λ-ring is a λ-subring of a special λ-ring of the form $1 + A[[t]]^+$.

The λ-structure on $1 + A[[t]]^+$ may be given in a more sophisticated manner by

(15) The structure is functorial in A

(16) 'addition' is power series multiplication

(17) 'multiplication' satisfies $(1 + at) \circ (1 + bt) = 1 + abt$

(18) $\Lambda^n(1 + at) = 1$ for $n > 1$

(19) the $\{\Lambda^n\}$ satisfy (1)–(3)

The examples of λ-rings given earlier are special. We give proofs for $K_G(X)$ and $R(G)$ since they will concern us later in this paper.

THEOREM 1.5. *(i) For a compact G-space X where G is a compact Lie group, the Grothendieck group of complex G-vector bundles is a special λ-ring.*

(ii) The complex representation ring $R(G)$ of a finite group G is a special λ-ring.

Proof. Of course (*ii*) is a special case of (*i*) but we give a separate proof at a more elementary level.

(*i*) follows by the G-splitting principle of [19]. A G-vector bundle is written as a sum of one dimensional elements in some extension λ-ring, which satisfies the property that the product of one dimensional elements is one dimensional. This demonstrates that (13), (14) are satisfied by the isomorphism classes of G-vector bundles and it is trivial to extend the proof to virtual G-vector bundles.

(*ii*) follows by identifying $R(G)$ with the ring of complex characters as in [3], then $R(G)[[t]]$ is a subring of the ring of all functions $G \to \mathbf{C}[[t]]$, For any complex representation ρ of G, $\lambda_t(\rho)$ is the function given by $g \mapsto \det(1 + t\rho(g))$. $R(G)$ is special as a result of the formulae:

$$\det(1 + \rho_1(g) \otimes \rho_2(g)t) = \det(1 + \rho_1(g)t) \circ \det(1 + \rho_2(g)t)$$
$$\det(1 + \Lambda^n(\rho(g))t) = \det(\Lambda^n(1 + \rho(g)t)).$$

By restricting to the cyclic subgroup generated by g, we may assume the representations concerned are diagonal and then the above formulae are trivial.

§2. THE FREE λ-RING ON ONE GENERATOR

This is a special λ-ring U generated (as a λ-ring) by a single element s_1, such that the elements $\{\lambda^n(s_1)\}_{n\geq 1}$ are algebraically independent. U will be a λ-subring of a special λ-ring Ω in which s_1 may be written as the formal sum of an infinite number of one dimensional elements. If we write $s_n = \lambda^n(s_1)$, in effect we are setting up the formal framework in which we may factorise the power series $\lambda_t(s_1) = 1 + \sum_{n\geq 1} s_n t^n$ into an infinite product $\prod_{n\geq 1}(1 + \xi_n t)$ where $s_1 = \sum \xi_n$.

U will have the universal property that if R is a special λ-ring and $x \in R$, then there is a λ-homomorphism $u_x : U \to R$ in which $u_x(s_n) = \lambda^n(x)$.

If $\{R_r\}$ is an inverse system of special λ-rings with λ-homomorphisms $\phi_s^r : R_r \to R_s$ for $r \geq s$, then $\varprojlim_r R_r$ is a special λ-ring in a canonical manner such that the canonical homomorphisms $\phi_n : \varprojlim R_r \to R_n$ are λ-homomorphisms.

If $\Omega_r = \mathbf{Z}[\xi_1, \ldots, \xi_r]$ for $r \geq 0$ where $\lambda_t(\xi_n) = 1 + \xi_n t$ and

$$\phi_s^r(\xi_n) = \begin{cases} \xi_n & n \leq s \\ 0 & s < n \leq r \end{cases}$$

then $\Omega = \varprojlim \Omega_r$ is a special λ-ring.

We may consider Ω to be the ring of those power series in the $\{\xi_i\}$ which become polynomials in the n variables $\xi_1, \ldots, \xi_n$ when we put $\xi_r = 0$ for $r > n$. The canonical map $\phi_n : \Omega \to \Omega_n$ is given by this process and plainly $\phi_n{}^m \phi_m = \phi_n$ for $m \geq n$. If $s_n(\xi_1, \ldots, \xi_r)$ is the nth elementary symmetric function in $\xi_1, \ldots, \xi_r$, let $s_n = \varprojlim s_n(\xi_1, \ldots, \xi_r) \in \Omega$, then $\lambda^n(s_1) = s_n$. The $\{s_n\}$ are algebraically independent, for if $f(s_1, \ldots, s_n) = 0$ where f is a polynomial, then $\phi_n(f(s_1, \ldots, s_n)) = f(s_1(\xi_1, \ldots, \xi_n), \ldots, s_n(\xi_1, \ldots, \xi_n)) = 0$ and this implies f is the zero polynomial ([16] § 26). Let U be the smallest λ-subring of Ω containing s_1, then $s_n = \lambda^n(s_1) \in U$ and so $\mathbf{Z}[s_1, \ldots, s_n, \ldots]$, the ring of polynomials in the $\{s_n\}$, is contained in U. Using the axioms for a special λ-ring, we see $\mathbf{Z}[s_1, \ldots, s_n, \ldots]$ is closed under the λ-operations and so $U = \mathbf{Z}[s_1, \ldots, s_n, \ldots]$.

If R is any special λ-ring and $x \in R$, define $u_x : U \to R$ by $u_x(s_n) = \lambda^n(x)$, then using the universal polynomials P_n and $P_{n,m}$, it is easy to see that u_x is a λ-homomorphism. The image of u_x is the λ-subring generated by x (that is the smallest λ-subring containing x).

§3. NATURAL OPERATIONS ON SPECIAL λ-RINGS AND THE VERIFICATION PRINCIPLE

In this section we characterize the ring of natural operations and show a natural operation is a polynomial in the λ-operations. It is uniquely defined by its action on a (finite) sum of one dimensional elements.

Let $\underline{S}$ be the category of special λ-rings and let Op $\underline{S}$ be the ring of natural operations on $\underline{S}$, i.e. if $\mu \in$ Op $\underline{S}$, for each $A \in \underline{S}$ there is a map (not necessarily a homomorphism), $\mu_A : A \to A$ such that for a λ-homomorphism $\phi : A \to B$, $\phi\mu_A = \mu_B\phi$. Addition and multiplication of operations are calculated on values, e.g. $(\mu_A + \nu_A)(a) = \mu_A(a) + \nu_A(a)$ for $a \in A$, $\mu, \nu \in$ Op $\underline{S}$. To simplify the notation, we often omit the suffix A in μ_A.

If $\mathbf{Z}[\lambda^1, \ldots, \lambda^n, \ldots]$ is the ring of polynomials in the λ-operations, there is a well-defined homomorphism $\alpha : \mathbf{Z}[\lambda^1, \ldots, \lambda^n, \ldots] \to$ Op $\underline{S}$ defined by $\alpha(f(\lambda^1, \ldots, \lambda^n))(x) = (\lambda^1(x), \ldots, \lambda^n(x))$, where x is an element of a special λ-ring.

PROPOSITION 3.1. *α is an isomorphism, i.e. every natural operation is (uniquely) a polynomial in the λ-operations.*

Proof. $U \in \underline{S}$ so if $\alpha(f(\lambda^1, \ldots, \lambda^n)) = 0$, then $\alpha(f(\lambda^1, \ldots, \lambda^n))(s_1) = f(s_1, \ldots, s_n) = 0$ and so f is the zero polynomial, since the $\{s_i\}$ are algebraically independent. So α is a monomorphism.

If $\mu \in$ Op $\underline{S}$, then $\mu(s_1) \in U$ and $\mu(s_1) = f(s_1, \ldots, s_n)$ for some polynomial f. If $R \in \underline{S}$ and $x \in R$, by Theorem 2.1, there is a λ-homomorphism $u_x : U \to R$ where $u_x(s_n) = \lambda^n(x)$. Since μ is natural, it commutes with u_x and so

$$\begin{aligned}\mu(x) &= \mu u_x(s_1)\\ &= u_x\mu\,(s_1)\\ &= u_x f(s_1, \ldots, s_n)\\ &= f(\lambda^1(x), \ldots, \lambda^n(x))\\ &= \alpha(f(\lambda^1, \ldots, \lambda^n))(x)\end{aligned}$$

which implies $\alpha(f(\lambda^1, \ldots, \lambda^n)) = \mu$ and α is an epimorphism.

If $\mu \in$ Op $\underline{S}$ and we wish to show $\mu = f(\lambda^1, \ldots, \lambda^n)$, it is sufficient to check the identity operating on $s_1 \in U$, that is to check $\mu(s_1) = f(s_1, \ldots, s_n)$, which may be calculated in Ω.

By the universal property of an inverse limit, since the natural operations commute with the canonical maps $\phi_r : \Omega \to \Omega_r$, it is sufficient to check the formula in each $\Omega_r = Z[\xi_1, \ldots, \xi_r]$, i.e. it is sufficient to show

$$\mu(\xi_1 + \cdots + \xi_r) = f(s_1(\xi_1 \ldots \xi_r), \ldots, s_n(\xi_1, \ldots, \xi_r))$$

This is a vital property of special λ-rings which we write as:

THEOREM 3.2. (verification principle). *If $\mu \in \mathrm{Op}\ \underline{S}$, then μ is a polynomial in the λ-operations and $\mu = f(\lambda^1, \ldots, \lambda^n)$ if, and only if, the identity holds operating on a sum $\xi_1 + \cdots + \xi_r$ of one dimensional elements, for all $r \geq 0$.*

We may also consider a verification principle for more than one variable. The notion of a natural map in two variables is evident; we denote by $\mathrm{Op}_2\ \underline{S}$ the ring of such maps and so if $\nu \in \mathrm{Op}_2\ \underline{S}$ and A is a special λ-ring, there is a map $\nu_A: A \times A \to A$ such that for a λ-homomorphism $\phi: A \to B$, then $\phi\nu_A = \nu_B(\phi \times \phi)$.

If f is a polynomial in $\lambda_1{}^1, \ldots, \lambda_1{}^n, \lambda_2{}^1, \ldots, \lambda_2{}^m$, f defines a map $A \times A \to A$ by $f(\lambda_1{}^1, \ldots, \lambda_1{}^n; \lambda_2{}^1, \ldots, \lambda_2{}^m)(x, y) = f(\lambda^1(x), \ldots, \lambda^n(x); \lambda^1(y), \ldots, \lambda^m(y))$. This defines a homomorphism $\beta: \mathbf{Z}[\lambda_1{}^1, \ldots, \lambda_1{}^n, \ldots; \lambda_2{}^1, \ldots, \lambda_2{}^m, \ldots] \to \mathrm{Op}_2\ \underline{S}$.

PROPOSITION 3.3. *β is an isomorphism.*

Proof. Construct a free λ-ring U_2 on two generators s_1, σ_1 analogous to U. Let $\Omega_2 = \varprojlim_{q,r} \mathbf{Z}[\xi_1, \ldots, \xi_q, \eta_1, \ldots, \eta_r]$ where the $\{\xi_i\}$ and $\{\eta_j\}$ are indeterminates and let $s_n = \varprojlim_q s_n(\xi_1, \ldots, \xi_q)$, $\sigma_n = \varprojlim \sigma_n(\eta_1, \ldots, \eta_r)$ where $s_n(\xi_1, \ldots, \xi_r)$ and $\sigma_n(\eta_1, \ldots, \eta_r)$ are the nth elementary symmetric functions. The free λ-ring U_2 on two generators is $\mathbf{Z}[s_1, \ldots, s_n, \ldots; \sigma_1, \ldots, \sigma_m, \ldots]$ where the $\{s_i\}$ and $\{\sigma_i\}$ are algebraically independent and $\lambda^n(s_1) = s_n$, $\lambda^n(\sigma_1) = \sigma_n$. U_2 satisfies the obvious universal property and 3.3 follows by analogy with 3.1.

PROPOSITION 3.4. (verification principle for two variables). *If $\mu \in \mathrm{Op}_2\ \underline{S}$, $\mu = f(\lambda_1{}^1, \ldots, \lambda_1{}^n; \lambda_2{}^1, \ldots, \lambda_2{}^m)$ if and only if the identity holds operating on a pair $\xi_1 + \cdots + \xi_q$, $\eta_1 + \cdots + \eta_r$ of sums of one dimensional elements for all $q, r \geq 0$.*

Proof. This is analogous to 3.2.

§4. THE γ-OPERATIONS

After Grothendieck, define the γ-operations on a special λ-ring by

(1) $\gamma^n(x) = \lambda^n(x + n - 1)$

If $\gamma_t(x) = 1 + \sum_{n \geq 1} \gamma^n(x)t^n$, then

(2) $\gamma_t(x) = \lambda_{t/(1-t)}(x)$, $\lambda_s(x) = \gamma_{s/(1+s)}(x)$

(3) $\gamma_t(x + y) = \gamma_t(x)\gamma_t(y)$, $\gamma^n(x + y) = \sum_{r=0}^{n} \gamma^r(x)\gamma^{n-r}(y)$

The γ-operations satisfy certain universal polynomial formulae

(4) $\gamma^n(xy) = Q_n(\gamma^1(x), \ldots, \gamma^n(x); \gamma^1(y), \ldots, \gamma^n(y))$

(5) $\gamma^n(\gamma^m(x)) = Q_{m,n}(\gamma^1(x), \ldots, \gamma^{mn}(x))$

which may be calculated by substitution into the λ-formulae.

We also have

(6) $\gamma^0(x) = 1$

(7) $\gamma^1(x) = x$

(8) $\gamma_t(m) = \left(1 + \dfrac{t}{1-t}\right)^m = (1 - t)^{-m}$

(9) $\lambda_t(x) = 1 + xt$ implies $\gamma_t(x - 1) = 1 + (x - 1)t$

(9) states that if x is one-dimensional, $x - 1$ is of γ-dimension one (or zero), more generally if x is n-dimensional, $x - n$ is of γ-dimension at most n.

Suppose R is an augmented λ-ring with augmentation $\varepsilon\colon R \to \mathbf{Z}$ and augmentation ideal $I = \ker \varepsilon$.

Define the γ-filtration by:

(10) R_n is the additive group generated by monomials $\gamma^{n_1}(a_1) \dots \gamma^{n_r}(a_r)$ where $a_i \in I$ and $\sum n_i \geq n$.

PROPOSITION 4.1. (*i*) $R_m \cdot R_n \subset R_{m+n}$
(*ii*) $R_0 = R$, $R_1 = I$
(*iii*) *R_n is a λ-ideal for $n \geq 1$.*

Proof. (*i*) is trivial and (*ii*) follows from (6), (7).

From proposition 1.3, $R = \mathbf{Z} \oplus I = \mathbf{Z} \oplus R_1$, and so R_n is certainly an ideal. To show R_n is a λ-ideal, it is sufficient to show $\lambda^r(\gamma^m(x)) \in R_m$ for $x \in I$.

$\lambda^r(\gamma^m(x)) = \lambda^r(\lambda^m(x + m - 1)) = P_{r,m}(\lambda^1(x + m - 1), \dots, \lambda^{rm}(x + m - 1))$. Now $P_{r,m}(s_1, \dots, s_{rm})$ is the coefficient of t^r in $\prod (1 + \xi_{i_1} \dots \xi_{i_m} t)$ where s_i is the ith elementary symmetric function in the $\{\xi_i\}$. Put $\xi_i = 0$ for $i \geq m$, then by inspection, $P_{r,m}(s_1, \dots, s_{m-1}, 0, 0, \dots, 0) = 0$ and so $P_{r,m}(s_1, \dots, s_{rm})$ is a sum of monomials each containing a term s_i for $i \geq m$. Thus $\lambda^r(\gamma^m(x))$ is a sum of monomials each a multiple of some $\lambda^i(x + m - 1)$ for $i \geq m$. It remains to show that $\lambda^i(x + m - 1) \in R_m$ for $i \geq m$. Put $s = i - m$, then

$$\begin{aligned}\lambda^i(x + m - 1) &= \lambda^{m+s}(x + m - 1)\\ &= \gamma^{m+s}(x + m - 1 - m - s + 1)\\ &= \gamma^{m+s}(x - s)\\ &= \sum_{r=0}^{m+s} \gamma^{m+s-r}(x)\gamma^r(-s) \text{ by (3)}\end{aligned}$$

Since $\gamma^r(-s) = 0$ for $r > s \geq 0$ by (8),

$$\lambda^i(x + m - 1) = \sum_{r=0}^{s} \gamma^{m+s-r}(x)\gamma^r(-s) \in R_m.$$

Definition 4.2. I is said to be a special γ-ring if it is a commutative ring (without identity) with operations $\{\gamma^i\}$ such that there is an augmented special λ-ring R with I as kernel of the augmentation.

If we are given a special γ-ring I, we may recover the special λ-ring R by adjoining an identity to I in the usual manner. $R = \mathbf{Z} \oplus I$ as additive abelian groups and $\varepsilon\colon R \to \mathbf{Z}$ is given by $\varepsilon(n + a) = n$ for $n \in \mathbf{Z}$, $a \in I$.

For a special γ-ring I, the γ-filtration is given as in (10), (11), I_n is the ideal generated by monomials $\gamma^{n_1}(a_1), \dots, \gamma^{n_r}(a_r)$ where $a_i \in I$, $\sum n_i \geq n$.

From Proposition 4.1, we have

PROPOSITION 4.3. (*i*) $I_m \cdot I_n \subset I_{m+n}$
(*ii*) $I_1 = I$
(*iii*) *I_n is a γ-ideal, $n \geq 1$ (i.e. I_n is closed under the γ-operations).*

If we now refer back to § § 2, 3, we have propositions for special γ-rings analogous to those for special λ-rings.

Let $W_r = \mathbf{Z}[x_1, \ldots, x_r]^+$ be the ideal of polynomials with zero constant term in the indeterminates $\{x_i\}$, where $\gamma_t(x_r) = 1 + x_r t$. Define for $r \geq s$,

$$\phi_s^r(x_n) = \begin{cases} x_n & n \leq s, \\ 0 & s < n \leq r \end{cases}$$

then $W = \varprojlim W_r$ is a special γ-ring.

If $\sigma_n(x_1, \ldots, x_r)$ is the nth elementary symmetric function in the $\{x_i\}$ and $\sigma_n = \varprojlim \sigma_n(x_1, \ldots, x_r) \in W$, then $\gamma^n(\sigma_1) = \sigma_n$. Let V be the smallest γ-subring containing σ_1, then $V = \mathbf{Z}[\sigma_1, \ldots, \sigma_n, \ldots]^+$, the ring of polynomials in the $\{\sigma_n\}$ with zero constant term. V is the free γ-ring on the generator σ_1 since the $\{\sigma_n\}$ are algebraically independent. It satisfies the following universal property:

PROPOSITION 4.4. *If I is a γ-ring and $x \in I$, there is a unique γ-homomorphism $v_x : V \to I$ in which $v_x(\sigma_n) = \gamma^n(x)$.*

We also have the following proposition analogous to 3.2:

PROPOSITION 4.5. (verification principle for special γ-rings). *Every natural operation μ on the category of special γ-rings is a polynomial in the γ-operations with zero constant term. $\mu = f(\gamma^1, \ldots, \gamma^n)$ if and only if the identity holds operating on a sum $x_1 + \cdots + x_r$ of elements of γ-dimension one, for all $r \geq 0$.*

A similar proposition holds for natural operations in more than one variable.

§5. THE ADAMS OPERATIONS

Let R be a special λ-ring, $x \in R$, define $\psi^n : R \to R$ for $n \geqslant 1$ by

$$(1)\quad \psi_{-t}(x) = t\frac{\mathrm{d}}{\mathrm{d}t}[\lambda_t(x)]/\lambda_t(x) \text{ where } \psi_t(x) = \sum_{n \geqslant 1} \psi^n(x) t^n$$

ψ^n is a nat ural operation and

$$(2)\quad \psi^n(x) = v_n(\lambda^1(x), \ldots, \lambda^n(x)) \text{ where } v_n(s_1, \ldots, s_r) = \xi_1{}^n + \cdots + \xi_r{}^n$$

is the nth Newton polynomial, s_i being the ith elementary symmetric function in the $\{\xi_j\}$.

If x is one dimensional, $\psi^n(x) = x^n$ and for a sum of one dimensional elements $a_1 + \cdots + a_m$, $\psi^n(a_1 + \cdots + a_m) = a_1{}^n + \cdots + a_m{}^n$.

It is well known in K-theory that the ψ^n are ring homomorphisms, [1]; we prove this formally by appealing to the verification principle.

PROPOSITION 5.1. *ψ^n is a λ-homomorphism.*

Proof.
$$\psi^n\Big(\sum \xi_i + \sum \eta_j\Big) = \sum \xi_i{}^n + \sum \eta_j{}^n = \psi^n\Big(\sum \xi_i\Big) + \psi^n\Big(\sum \eta_j\Big)$$
$$\psi^n\Big(\sum \xi_i \sum \eta_j\Big) = \psi^n\Big(\sum \xi_i \eta_j\Big) = \sum (\xi_i \eta_j)^n = \sum \xi_i{}^n \sum \eta_j{}^n = \psi^n\Big(\sum \xi_i\Big)\psi^n\Big(\sum \eta_j\Big)$$
$$\psi^n\Big(\lambda^m\Big(\sum \xi_i\Big)\Big) = \psi^n(s_m(\xi_1, \ldots, \xi_r)) = s_m(\xi_1{}^n, \ldots, \xi_r{}^n) = \lambda^m\Big(\sum \xi_i{}^n\Big) = \lambda^m\Big(\psi^n\Big(\sum \xi_i\Big)\Big)$$

PROPOSITION 5.2. $\psi^m\psi^n = \psi^{mn} = \psi^n\psi^m$

$$\psi^{p^r}(x) \equiv x^{p^r} \bmod p \ (p \text{ prime})$$

Proof. Immediate from the verification principle.

PROPOSITION 5.3. *If I is a special γ-ring, $x \in I_n$ then $\psi^k(x) - k^n x \in I_{n+1}$ $(n \geq 1)$.*

Proof. It is sufficient to show $\psi^k(\gamma^m(a)) - k^m\gamma^m(a) \in I_{m+1}$ for $a \in I$, since ψ^k is a γ-homomorphism.

If $\gamma_t(x_i) = 1 + x_i t$ then $\psi^k(x_i) = (1 + x_i)^k - 1$.

$\psi^k\gamma^m - k^m\gamma^m$ is a natural operation and operating on $x_1 + \cdots + x_r$,

$$\begin{aligned}\psi^k(\gamma^m(x_1 + \cdots + x_r)) - k^m\gamma^m(x_1 + \cdots + x_r) &= \psi^k(\sigma_m(x_1, \ldots, x_r)) - k^m\sigma_m(x_1, \ldots, x_r) \\ &= \sigma_m((1 + x_1)^k - 1, \ldots, (1 + x_r)^k - 1) - k^m\sigma_m(x_1, \ldots, x_r)\end{aligned}$$

This is a symmetric polynomial of degree $\geq m + 1$; by the verification principle for special γ-rings, this gives the result.

PROPOSITION 5.4. $x \in I_n \Rightarrow \psi^k(x) + (-1)^k k\lambda^k(x) \in I_{n+1}$ $(n \geq 1)$.

Proof. From the property of Newton's polynomials,

$$\psi^k(x) - \psi^{k-1}(x)\lambda^1(x) + \cdots + (-1)^{k-1}\psi^1(x)\lambda^{k-1}(x) + (-1)^k k\lambda^k(x) = 0.$$

Now for $a \in I_n$, $\psi^r(a) \in I_n$, $\lambda^r(a) \in I_n$ for $r \geq 1$.

$$\psi^k(x) + (-1)^k k\lambda^k(x) \in I_{2n} \subset I_{n+1} \text{ for } n \geq 1.$$

PROPOSITION 5.5. $x \in I_n \Rightarrow \lambda^k(x) + (-1)^k k^{n-1}x \in I_{n+1}$ $(n \geq 1)$.

Proof. For $a, b \in I_n$, $ab \in I_{2n} \subset I_{n+1}$ and so λ^n is a ring homomorphism on I_n/I_{n+1}. It is sufficient to show $\lambda^k(\gamma^{n_1}(a_1) \cdots \gamma^{n_r}(a_r)) + (-1)^k k^{n-1}\gamma^{n_1}(a_1) \cdots \gamma^{n_r}(a_r) \in I_{n+1}$ for $a_1, \ldots, a_r \in I$, $\sum n_i = n$. This is a natural operation in the r variables $a_1, \ldots, a_r$ which we denote by $\mu(a_1, \ldots, a_r)$. From 5.3, 5.4, $k\mu(a_1, \ldots, a_r) \in I_{n+1}$. There is no torsion in the free γ-ring on r generators and since μ is determined by its action in this γ-ring, we have $\mu(a_1, \ldots, a_r) \in I_{n+1}$.

Since a special γ-ring I is in particular an additive abelian group, the p-adic topology on I is well defined, with a fundamental system of neighbourhoods of zero given by $\{p^n I\}_{n \geq 0}$.

The γ-topology on I is given by the γ-filtration.

PROPOSITION 5.6. *If the γ-topology is finer than the p-adic topology on I, then writing $\psi^k(a) = \psi(k, a)$, ψ is a continuous function from $Z^+ \times I \to I$ in the p-adic topology (where Z^+ are the positive integers).*

Proof. A fundamental system of neighbourhoods of zero for the p-adic topology may also be taken as $\{p^n I + I_n\}_{n \geq 1}$. We show that given an integer $N \geq 1$, $\exists M \geq 0$ such that $p^M \mid s$ implies

$$\psi(k + s, x) - \psi(k, x) \in p^N I + I_N.$$

Suppose $x_1 + \cdots + x_r$ is a sum of elements of γ-dimension one, then

$$\begin{aligned}\psi^{k+s}(\textstyle\sum x_i) - \psi^k(\sum x_i) &= \textstyle\sum((1 + x_i)^{k+s} - 1) - \sum((1 + x_i)^k - 1) \\ &= \textstyle\sum(1 + x_i)^k((1 + x_i)^s - 1)\end{aligned}$$

Choose $M = 2N$, then if $p^M \mid s$, it is easy to see

$$\psi^{k+s}(\sum x_i) - \psi^k(\sum x_i) = p^N S_1 + S_N$$

where S_j is a symmetric function of weight $\geq j$ in the $\{x_i\}$ for $j = 1, N$. By the verification principle, $\psi^{k+s}(x) - \psi^k(x) \in p^N I + I_N$ for $x \in I$.

This shows $\psi(k, x)$ is (uniformly) continuous in the first variable in the p-adic topology: it is certainly continuous in the second variable since ψ^k is a ring homomorphism. This makes it continuous in the two variables.

§6. THE SPLITTING PRINCIPLE

The purpose of this section is to prove the following theorem:

THEOREM 6.1 (splitting principle). *If R is a special λ-ring and x is an n-dimensional element of R, then there is a special λ-ring $R' \supset R$ such that $x = x_1 + \cdots + x_n$ is the sum of n one-dimensional elements in R'. Furthermore if R is augmented by $\varepsilon\colon R \to \mathbf{Z}$ and $\varepsilon(x) = m$, then ε may be extended to R' such that* $\varepsilon(x_r)$

$$= \begin{cases} 1 & 1 \leq r \leq m \\ 0 & m < r \leq n \end{cases}$$

Theorem 6.1 is analogous to the splitting principle for a vector bundle over a compact space. As an immediate corollary, if R is finite dimensional (i.e. every element of R is the difference of finite dimensional elements), using Zorn's Lemma, R may be embedded in a λ-ring S in which every element may be decomposed into one dimensional elements. The theorem is the result of a series of lemmas.

LEMMA 6.2. *The tensor product of special λ-rings A, B is a special λ-ring in a canonical way such that the maps $A \to A \otimes B$, $B \to A \otimes B$ are λ-homomorphisms. If A, B are augmented, $A \otimes B$ is augmented in the obvious way.*

Proof. If F is a covariant functor on the category of commutative rings with identity, since $A \otimes B$ is the coproduct of A, B, there is a unique natural map $F(A) \otimes F(B) \to F(A \otimes B)$, Hence the functor $A \mapsto 1 + A[[t]]^+$ gives rise to the natural map

$$\alpha : (1 + A[[t]]^+) \otimes (1 + B[[t]])^+ \to 1 + A \otimes B[[t]]^+.$$

Given $\lambda_t'\colon A \to 1 + A[[t]]^+$, $\lambda_t''\colon B \to 1 + B[[t]]^+$, define $\lambda_t\colon A \otimes B \to 1 + A \otimes B[[t]]^+$, by $\lambda_t = \alpha(\lambda_t' \otimes \lambda_t'')$, then λ_t is a ring homomorphism such that $\lambda^1(x) = x$ and so $A \otimes B$ is a λ-ring. A calculation with the universal polynomials shows λ_t is a λ-homomorphism and so $A \otimes B$ is special.

If A, B are augmented $\varepsilon'\colon A \to \mathbf{Z}$, $\varepsilon''\colon B \to \mathbf{Z}$, then $A \otimes B$ is augmented by $\varepsilon = \varepsilon' \otimes \varepsilon''$

LEMMA 6.3. *If R is a special λ-ring and ξ is an indeterminate, then $R[\xi]$ is a special λ-ring where $\lambda_t(\xi) = 1 + \xi t$. If R is augmented, $R[\xi]$ may be augmented by $\varepsilon(\xi) = 0$ or 1.*

Proof. $R[\xi] = R \otimes \mathbf{Z}[\xi]$ where $\mathbf{Z}[\xi]$ is a special λ-ring given by $\lambda_t(\xi) = 1 + \xi t$ and augmented in two possible ways by $\varepsilon(\xi) = 0$ or 1.

LEMMA 6.4. *If S is a special λ-ring and I an ideal generated by $\{z_j\}_{j \in J}$, then I is a λ-ideal if and only if $\lambda^m(z_j) \in I$ for $m \geq 1$, $j \in J$.*

Proof. If $\lambda^m(z_j) \in I$ for $m \geq 1$, then for $a \in S$,

$$\lambda^n(az_j) = P_n(\lambda^1(a), \ldots, \lambda^n(a); \lambda^1(z_j), \ldots, \lambda^n(z_j))$$

and, since P_n is of weight n in the $\{\lambda^m(z_j)\}$, $\lambda^n(az_j) \in I$ for $n \geq 1$. Since any $z \in I$ is a finite sum $z = \sum a_r z_{j_r}$ where $a_r \in S$, $j_r \in J$, using therelation $\lambda^n(x+y) = \sum_{i=0}^{n} \lambda^{n-i}(x)\,\lambda^i(y)$, this implies $\lambda^n(z) \in I$ for any $z \in I$ and so I is a λ-ideal. The converse is trivial.

Using the notation of lemma 6.3, we have

LEMMA 6.5. *If $x \in R$, dim $x = n$, then in $R[\xi]$, the principal ideal I generated by $\xi^n - \lambda^1(x)\xi^{n-1} + \cdots + (-1)^n\lambda^n(x)$ is a λ-ideal.*

Proof. Considering $\lambda_t(x-\xi) = (1 + \lambda^1(x)t + \cdots + \lambda^n(x)t^n)(1+\xi t)^{-1}$, we see that for $r \geq 0$,

(1) $\lambda^{n+r}(x-\xi) = (-1)^{n+r}\xi^r(\xi^n - \lambda^1(x)\xi^{n-1} + \cdots + (-1)^n\lambda^n(x))$,

in particular $\lambda^n(x-\xi) = (-1)^n(\xi^n - \lambda^1(x)\xi^{n-1} + \cdots + (-1)^n\lambda^n(x))$ and so I is also generated by $\lambda^n(x-\xi)$.

To show I is a λ-ideal, from 6.4, we need only show $\lambda^m(\lambda^n(x-\xi)) \in I$, $m \geq 1$. From (1), $\lambda^{n+r}(x-\xi) \in I$, $r \geq 0$, but $\lambda^m(\lambda^n(x-\xi)) = P_{m,n}(\lambda^1(x-\xi), \ldots, \lambda^{mn}(x-\xi))$ and as was demonstrated in the proof of 4.1, $P_{m,n}(\lambda^1(x-\xi), \ldots, \lambda^{mn}(x-\xi))$ is a sum of monomials each a multiple of some $\lambda^{n+r}(x-\xi) \in I$.

LEMMA 6.6 *If $x \in R$, dim $x = n$, then there is a special λ-ring $R[x_1] \supset R$ where*

$$\dim x_1 = 1, \dim (x - x_1) = n - 1.$$

If R is augmented and $\varepsilon(x) = m$, then we may augment $R[x_1]$ by

$$\varepsilon(x_1) = \begin{cases} 1, & m > 0 \\ 0, & m = 0 \end{cases}$$

and then

$$\varepsilon(x - x_1) = \begin{cases} m-1, & m > 0 \\ 0, & m = 0 \end{cases}$$

Proof. From 6.5, since I is a λ-ideal, $R[\xi]/I$ is a special λ-ring. Trivially, if x_1 is the image of ξ in $R[\xi]/I$, $R[\xi]/I = R[x_1]$ where x_1 satisfies

$$x_1^n - \lambda^1(x)x_1^{n-1} + \cdots + (-1)^n\lambda^n(x) = 0.$$

Since $\lambda^{n+r}(x-\xi) \in I$ from (1) above, $\lambda^{n+r}(x-x_1) = 0$ in $R[x_1]$, but $\lambda^{n-1}(x-\xi) \notin I$ and so $\lambda^{n-1}(x-x_1) \neq 0$ showing $\dim(x-x_1) = n-1$.

If R is augmented by $\varepsilon\colon R \to \mathbf{Z}$, then $R[\xi]$ may be augmented either by $\varepsilon(\xi) = 1$ or by $\varepsilon(\xi) = 0$. If $\varepsilon(I) = 0$, then $R[\xi]/I$ is naturally augmented by $\varepsilon(z+I) = \varepsilon(z)$ for $z \in R[\xi]$. If $\varepsilon(x) = m$, then $\varepsilon(\lambda^r(x)) = \binom{m}{r}$. For $m > 0$, choose $\varepsilon(\xi) = 1$ then

$$\varepsilon(\xi^n - \lambda^1(x)\xi^{n-1} + \cdots + (-1)^n\lambda^n(x)) = 1 - \binom{m}{1} + \binom{m}{2} + \cdots + (-1)^m = (1-1)^m = 0$$

and this induces the augmentation on $R[x_1]$ given by $\varepsilon(x) = 1$. If $\varepsilon(x) = 0$, choose $\varepsilon(\xi) = 0$, then $\varepsilon(\xi^n - \lambda^1(x)\xi^{n-1} + \cdots + (-1)^n\lambda^n(x)) = 0$ and in this case $\varepsilon(x_1) = 0$.

This completes the proof.

By downward induction on n, theorem 6.1 is completed with $R \subset R[x_1, \ldots, x_n]$ where each x_r is one dimensional and $x = x_1 + \cdots + x_n$.

§7. NATURAL EXPONENTIAL MAPS AND THE BOTT CANNIBALISTIC CLASS θ_k

A natural exponential map τ is defined on finite dimensional elements and satisfies $\tau(x+y) = \tau(x)\tau(y)$. It is given uniquely when its value is known on one dimensional elements. We are interested in the Bott cannibalistic class θ_k which is given by $\theta_k(x) = 1 + x + \cdots + x^{k-1}$ if x is one dimensional.

If R is a special λ-ring, the set of finite dimensional elements is an additive semigroup which we denote by $P(R)$ (it is in fact a λ-semiring). A *natural exponential map* τ on the category of special λ-rings is defined to be a map $\tau_R\colon P(R) \to R$ for each special λ-ring R such that:

(1) $\tau_R(x+y) = \tau_R(x)\tau_R(y) \qquad x, y \in P(R)$

(2) If $\phi\colon R \to S$ is a λ-homomorphism then $\tau_S\phi = \tau\phi_R$.

If A is the category of augmented special λ-rings and λ-homomorphisms which commute with augmentation, then a *natural exponential map of degree k* on A is a map $\tau_A\colon P(A) \to A$, defined for each augmented special λ-ring A, such that:

(3) $\tau_A(a+b) = \tau_A(a)\tau_A(b) \qquad a, b \in P(A)$

(4) If $\phi\colon A \to B$ is a λ-homomorphism commuting with augmentation then $\tau_B\phi = \phi\tau_A$

(5) $\varepsilon(\tau_A(x)) = k^{\varepsilon(x)}$.

We will often omit the suffix in τ_R to simplify the notation.

PROPOSITION 7.1. *If τ_1, τ_2 are natural exponential maps which agree on all one dimensional elements, they agree everywhere.*

Proof. Use the splitting principle (Theorem 6.1)

We now give a definition of the Bott cannibalistic class θ_k. Let R be any special λ-ring and adjoin a one dimensional indeterminate ξ. If $\dim x = n$, the ideal $I(x)$ in $R[\xi]$ generated by $L(\xi, x) = \xi^n - \lambda^1(x)\xi^{n-1} + \cdots + (-1)^n\lambda^n(x)$ is a λ-ideal by lemma 6.5. Any element of the ideal $I(x)$ may be written uniquely as $f(\xi)L(\xi, x)$ where $f(\xi) \in R[\xi]$.

Define $\theta_k(\xi, x) \in R[\xi]$ by

(6) $\theta_k(\xi, x)L(\xi, x) = \psi^k(L(\xi, x))$

and define $\theta_k(x)$ by

(7) $\theta_k(x) = \theta_k(1, x)$.

PROPOSITION 7.2. *θ_k is a natural map of degree k such that $\theta_k(a) = 1 + a + \cdots + a^{k-1}$ for* $\dim a = 1$.

Proof. Obviously $L(\xi, x+y) = L(\xi, x)L(\xi, y)$ and so from (6), (7)

$$\theta_k(x+y) = \theta_k(x)\theta_k(y),$$

i.e. θ_k is exponential. Clearly θ_k is a natural and, if $\dim a = 1$, $L(\xi, a) = \xi - a$, $\psi^k(\xi - a) = \xi^k - a^k$, and so $\theta_k(\xi, a) = \xi^{k-1} + \xi^{k-2}a + \cdots + a^{k-1}$ giving

$$\theta_k(a) = 1 + a + \cdots + a^{k-1}.$$

If $\varepsilon(a) = 1$, $\varepsilon(\theta_k(a)) = k$ and if $\varepsilon(a) = 0$, $\varepsilon(\theta_k(a)) = 1$. Hence using the splitting principle, θ_k is of degree k.

Recalling $I(x)$ is the ideal in $R[\xi]$ generated by $L(\xi, x) = \xi^n - \lambda^1(x)\xi^{n-1} + \cdots + (-1)^n\lambda^n(x)$ we next prove:

PROPOSITION 7.3. *Let $R_x = I(x)/I(x+1)$ then R_x is a free R-module on one generator μ_x, where μ_x is the image of $L(\xi, x)$ in R_x. R_x is a special γ-ring (a special λ-ring without identity) and for $z \in R$*

(8) $\psi^k(z \cdot \mu_x) = \psi^k(z) \cdot \psi^k(\mu_x)$

(9) $\psi^k(\mu_x) = \theta_k(x)\mu_x$.

Proof. $L(\xi, x+1) = (\xi - 1)L(\xi, x)$ and so if μ_x is the image of $L(\xi, x)$ in R_x and η is the image of ξ, we see that $(\eta - 1)\mu_x = 0$ i.e. $\eta \cdot \mu_x = \mu_x$. Any element in $I(x)$ is uniquely of the form $f(\xi)L(\xi, x)$ for $f(\xi) \in R[\xi]$ and the image of $f(\xi)L(\xi, x)$ in R_x is $f(\eta)\mu_x = f(1)\mu_x$ since $\eta\mu_x = \mu_x$. So R_x is an R-module on the generator μ_x. If $a \in R$ and $a\mu_x = 0$, then $a \cdot L(\xi, x) \in I(x+1)$ and this implies $a = 0$, so R_x is a free R-module. (8) follows from the multiplicative properties of ψ^k and (9) is the image of (6) in R_x.

Proposition 7.3 may be used as the definition of θ_k and indeed this is the way it occurs in the topological context. It may be shown that (using K with 'compact supports') if $R = K_G(X)$ and $x \in R$ is the class of a G-vector bundle E over X, then $R_x = K_G(E)$. The case $X =$ point will be discussed in IV §1.

A third description of θ_k will be of use later in this paper. Suppose $x \in P(R)$ and let ζ be a primitive kth root of unity, then substituting $-\zeta^r$ for t in $\lambda_t(x)$, we get an element in $R \otimes Z[\zeta]$. Consider the product $\prod_u \lambda_{-u}(x) \in R \otimes \mathbf{Z}[\zeta]$, where the product is taken over all roots of $t^k - 1 = 0$ except 1. Identify R with its image in $R \otimes \mathbf{Z}[\zeta]$ under the canonical map $r \mapsto r \otimes 1$. We will demonstrate that $\prod \lambda_{-u}(x) \in R$ and that $\prod \lambda_{-u}(x) = \theta_k(x)$. If $\dim x = 1$, this is clear because

(10) $\prod \lambda_{-u}(x) = \prod(1 - xu) = 1 + x + \cdots + x^{k-1} \qquad (\dim x = 1).$

More generally, since

(11) $\prod \lambda_{-u}(x_1 + x_2) = \prod \lambda_{-u}(x_1) \prod \lambda_{-u}(x_2),$

if x is a sum of one dimensional elements, (10) is also true in this case. Finally by using the splitting principle and the naturality of $\prod \lambda_{-u}$, we see the result is true for arbitrary $x \in P(R)$. Thus we have proved:

PROPOSITION 7.4. $\theta_k = \prod \lambda_{-u}$ *where the product is taken over all roots of* $t^k - 1 = 0$ *except* 1.

II. J-EQUIVALENCE OF REPRESENTATIONS

In this part of the paper we define the notion of J-equivalence of representations of a finite group G. Let Γ_N be the Galois group of $\mathbf{Q}(\omega)$ over $\mathbf{Q}$ where ω is a primitive Nth root of unity and $N = |G|$. If $\alpha \in \Gamma_N$ and $\alpha(\omega) = \omega^k$, then for a (unitary) representation E of G, we show $\alpha E = \psi^k(E)$. For a certain class of groups which includes nilpotent groups, we may construct an explicit map of unit spheres $S(E) \to S(\alpha E)$ which is of degree prime to the order of G. In the notation of the introduction, this will induce the epimorphism

$$v : R(G)_{\Gamma_N} \to J(G).$$

§1. MAPS OF SPHERES

In this section we discuss the degree of a map of spheres and derive certain elementary results which are well known in algebraic topology.

If S^n is the n-sphere, then the nth homology group with integer coefficients $H_n(S^n)$ is a free abelian group on one generator γ. If $\phi\colon S_1^n \to S_2^n$, then ϕ induces $\phi_*\colon H_n(S_1^n) \to H_n(S_2^n)$ and the image of the generator γ_1 is a unique integer multiple of the generator γ_2. The Brouwer degree of ϕ is defined by

(1) $\phi_*(\gamma_1) = \deg \phi \,.\, \gamma_2$.

It is trivial to see that the degree of the identity is 1 and that if $\theta\colon S_2^n \to S_3^n$, then $\deg \theta\phi = \deg \theta \deg \phi$.

If we consider cohomology with integer coefficients and the induced map

$$\phi^* : H^n(S_2^n; \mathbf{Z}) \to H^n(S_1^n; \mathbf{Z}),$$

it is easily seen that $\phi^*(\delta_2) = \deg \phi \,.\, \delta_1$ where δ_r is the canonical generator of $H^n(S_r^n; \mathbf{Z})$ for $r = 1, 2$.

We may embed S^{n-1} as the standard $n-1$ sphere in real Euclidean space or S^{2n-1} in complex space $\mathbf{C}^n$. We describe the complex case as this is of interest to us. Let E be complex n-space and $S(E)$ the unit sphere in E. If $B(E)$ is the unit ball in E, $B(E)/S(E)$ is a $2n$-sphere with basepoint, which is $S(E)$ collapsed to a point. $B(E)/S(E)$ is the suspension of $S(E)$. We have the canonical suspension isomorphism

(4) $H^{2n-1}(S(E); \mathbf{Z}) \cong H^{2n}(B(E)/S(E); \mathbf{Z})$

Suppose $S(F)$ is the standard $2n-1$ sphere in the complex n-space F and $\phi\colon S(E) \to S(F)$ a continuous map. If $\xi \in E$, define $\Phi(\xi) = |\xi|\,\phi(|\xi|^{-1}\xi)$ for $\xi \neq 0$ and $\Phi(0) = 0$, then Φ induces the suspension map $\Omega\phi : B(E)/S(E) \to B(F)/S(F)$ and using the commutative diagram:

$$\begin{array}{ccc} H^{2n-1}(S(F); \mathbf{Z}) & \to & H^{2n-1}(S(E); \mathbf{Z}) \\ \wr & & \wr \\ H^{2n}(B(F), S(F); \mathbf{Z}) & \to & H^{2n}(B(E), S(E); \mathbf{Z}) \end{array} \tag{3}$$

we see that $\deg \Omega\phi = \deg \phi$.

From [6], the Chern character $\mathrm{ch} : K^*(B(E), S(E)) \to H^*(B(E), S(E); \mathbf{Q})$ is a natural transformation of graded rings where we only consider the mod 2 grading $K^* = K^0 \oplus K^1$, $H^* = H^{ev} \oplus H^{od}$.

If η is the class of the standard line bundle over $S^2 = P_1(\mathbf{C})$, then $\eta - 1$ is the canonical generator of $\tilde{K}(S^2)$ and is mapped by the Chern character onto the canonical generator of $H^2(S^2; \mathbf{Z})$. If x_0 is the basepoint of S^2,

$$K^*(S^2, x_0) = K^0(S^2, x_0) = \tilde{K}(S^2)$$

and $H^*(S^2, x_0; \mathbf{Q}) = H^2(S^2, x_0; \mathbf{Q}) = \tilde{H}^2(S^2; \mathbf{Q})$.

By inspection, the Chern character is an isomorphism $\mathrm{ch} : \tilde{K}(S^2) \to \tilde{H}^2(S^2; \mathbf{Z})$. But the tensor product of n copies of the generator of $\tilde{K}(S^2)$ is the generator of $\tilde{K}(S^{2n})$ and similarly

for cohomology. Since the Chern character is natural and preserves products, it induces the natural isomorphism:

(4) $\mathrm{ch} : \tilde{K}(S^{2n}) \to \tilde{H}^{2n}(S^{2n}; \mathbf{Z})$.

The suspension map $\Omega\phi : B(E)/S(E) \to B(F)/S(F)$ induces a commutative diagram:

$$\begin{array}{ccc} K(B(F), S(F)) & \xrightarrow{\mathrm{ch}} & H^{2n}(B(F), S(F); \mathbf{Z}) \\ \downarrow{\scriptstyle \phi^!} & & \downarrow{\scriptstyle \phi^*} \\ K(B(E), S(E)) & \xrightarrow{\mathrm{ch}} & H^{2n}(B(E), S(E); \mathbf{Z}) \end{array} \tag{5}$$

If μ_E is the canonical generator of $K(B(E), S(E))$ (given by the tensor product of n copies of $\eta - 1 \in \tilde{K}(S^2)$) then (4), (5) show $\phi^!(\mu_F) = \deg \phi \,.\, \mu_E$.

E is locally compact, then using K-theory with compact supports as in [19], we may define $K(E)$. If E^+ is the one point compactification of E,

$$K(E) = K(E^+, +) = K(B(E), S(E))$$

and we obtain:

PROPOSITION 1.1. *If ϕ: $S(E) \to S(F)$ induces $\phi^!$: $K(F) \to K(E)$, then $\phi^!(\mu_F) = \deg \phi \,.\, \mu_E$.*

Given ϕ_r: $S(E_r) \to S(F_r)$ for $r = 1, 2$, we may form the topological join:

$$\phi_1 \circ \phi_2 : S(E_1) \circ S(E_2) \to S(F_1) \circ S(F_2)$$

where $S(E_1) \circ S(E_2)$ may be identified with $S(E_1 \oplus E_2)$. We describe the 'curved topological join' because this allows the identification to respect the usual metric in complex space.

Denote by $X = X_1 \circ \cdots \circ X_n$ the curved topological join of spaces $X_1, \ldots, X_n$. This has as points formal sums $\sum t_r x_r$ where $t_r \geq 0$, $\sum {t_r}^2 = 1$, $x_r \in X_r$ and we identify $\sum t_r x_r$ and $\sum t_r x_r'$ if either $t_r = 0$ or $x_r = x_r'$ for each $r = 1, \ldots, n$.

This space is given the most coarse topology (i.e. smallest collection of open sets) such that t_r: $X \to [0, 1]$ and x^n: ${t_r}^{-1}(0, 1] \to X_r$ are continuous for $r = 1, \ldots, n$.

The usual definition of join construction is homeomorphic to the above, it is exactly the same except that the condition $\sum {t_r}^2 = 1$ is replaced by $\sum t_r = 1$. Using the curved join, if $S(E_r)$ is the unit sphere in E_r for $r = 1, 2$, we may identify $S(E_1) \circ S(E_2)$ and $S(E_1 \oplus E_2)$ since if $\xi_1 \in S(E_1)$, $\xi_2 \in S(E_2)$, then $t_1\xi_1 + t_2\xi_2 \in S(E_1 \oplus E_2)$ where ${t_1}^2 + {t_2}^2 = 1$.

It is a simple exercise to show a map ϕ: $Y \to X_1 \circ \cdots \circ X_n$ is continuous if and only if $t_r\phi$, $x_r\phi$ are continuous $r = 1, \ldots, n$. Suppose $Y = Y_1 \circ \cdots \circ Y_n$ and ϕ_r: $Y_r \to X_r$ are continuous maps for $r = 1, \ldots, n$, then define $\phi = \phi_1 \circ \cdots \circ \phi_n$: $Y \to X$ by $\phi(\sum t_r x_r) = \sum t_r \phi_r(x_r)$. By our previous remark, ϕ is continuous and so the join has the usual functorial properties.

PROPOSITION 1.2. *Given $\phi_r : S(E_r) \to S(F_r)$ $r = 1, 2$, then*

$$\deg \phi_1 \circ \phi_2 = \deg \phi_1 \deg \phi_2 .$$

Proof. The natural pairing $K(E_1) \otimes K(E_2) \to K(E_1 \oplus E_2)$ is an isomorphism and maps $\mu_{E_1} \otimes \mu_{E_2}$ onto $\mu_{E_1 \oplus E_2}$.

Using proposition 1.1, the result follows from the commutative diagram:

$$\begin{array}{ccc} K(F_1) \otimes K(F_2) & \longrightarrow & K(F_1 \oplus F_2) \\ \downarrow{\scriptstyle \phi_1{}^! \otimes \phi_2{}^!} & & \downarrow{\scriptstyle (\phi_1 \circ \phi_2)!} \\ K(E_1) \otimes K(E_2) & \longrightarrow & K(E_1 \oplus E_2) \end{array}$$

§2. *J*-EQUIVALENCE

We will be concerned with continuous maps between the unit spheres of unitary representations of a finite group G. Most of the results of this section are true for any finite group and so we frame the description in this general context. By contrast the main result of §3 will only be true for those finite groups whose irreducible representations are induced up from one dimensional representations of subgroups (these include nilpotent groups). The algebraic theorem of Part III will only be proved for p-groups.

Let E be a unitary representation of G, then E is a complex vector space with a metric given by a hermitian inner product, together with a homomorphism $\rho\colon G \to \mathrm{Aut}\ E$ (which preserves the metric). By the character of the representation we mean the complex class function given by the trace of ρ. Two representations have the same character if and only if they are isomorphic. Further facts about representations may be found in [13] or in [10].

If E is a unitary representation of G, then the unit sphere $S(E)$ of the representation space is preserved by the action of G. If F is another unitary representation, a map : $S(E) \to S(F)$ is said to be a G-map if it commutes with the action of G.

Given G-maps $\phi_r\colon S(E_r) \to S(F_r)$ for $r = 1, 2$, then ϕ_1, ϕ_2 induce $\phi\colon S(E_1 \oplus E_2) \to S(F_1 \oplus F_2)$ via the topological join. We may identify $S(E_1 \oplus E_2)$ with $S(E_1) \circ S(E_2)$ and define $\phi = \phi_1 \circ \phi_2$. Evidently ϕ is a G-map and from proposition 1.2, we have:

PROPOSITION 2.1. *Suppose $E_1, E_2; F_1, F_2$ are unitary representations of G and $\phi_r\colon S(E_r) \to S(F_r)$ is a G-map of degree k_r, $r = 1, 2$, then ϕ_1, ϕ_2 induce a G-map $\phi\colon S(E_1 \oplus E_2) \to S(F_1 \oplus F_2)$ of degree $k_1 k_2$.*

A very useful construction will be the process of inducing up a representation of a subgroup.

PROPOSITION 2.2. *Let E, F be unitary representations of the finite group H and $\phi : S(E) \to S(F)$ an H-map of degree k. If $H \leq G$ with canonical monomorphism $i\colon H \to G$, $[G : H] = m$, then ϕ induces a G-map $i_*\phi\colon S(i_*E) \to S(i_*F)$ of the spheres of the induced representations of G, which is of degree k^m.*

Proof. The induced representation space i_*E may be considered to be the complex vector space generated by the symbols (g, v) $g \in G$, $v \in E$ subject only to the relations:

(i) $(g, \lambda_1 v_1 + \lambda_2 v_2) = \lambda_1(g, v_1) + \lambda_2(g, v_2)$ $\quad g \in G, \lambda_r \in \mathbf{C}, v_r \in E, \quad r = 1, 2$

(ii) $(gh, v) = (g, hv)$ $\quad g \in G, h \in H, v \in E.$

The action of $g' \in G$ on i_*E is defined by $g'(g, v) = (g'g, v)$.

Let $H_1 = H, H_2, \ldots, H_m$ be the left cosets of H in G, then as a vector space

$i_* E = E_1 \oplus \cdots \oplus E_m$ where $E_r = \{(g, v) \mid g \in H_r,\ v \in E\}$. $S(i_* E)$ may be considered as $S(E_1) \circ \cdots \circ S(E_m)$ and given $\phi\colon S(E) \to S(F)$, define $i_* \phi\colon S(i_* E) \to S(i_* F)$ by

$$i_* \phi\left(\sum_{r=1}^{m} t_r(g_r, v_r)\right) = \sum_{r=1}^{m} t_r(g_r, \phi(v_r)) \text{ where } g_r \in H_r,\ v_r \in S(E)$$

A trivial calculation shows $i_* \phi$ commutes with G-action and using $\deg(\phi_1 \circ \phi_2) = \deg \phi_1 \deg \phi_2$, we have $\deg(i_* \phi) = (\deg \phi)^m$.

PROPOSITION 2.3. *Let E, F be unitary representations of G and $\phi\colon S(E) \to S(F)$ be a G-map of degree k. If $\alpha\colon G' \to G$, then α induces a G'-map $\alpha^* \phi\colon S(\alpha^* E) \to S(\alpha^* F)$ of degree k.*

Proof. Trivial.

We now have the necessary information to deal with the notion of J-equivalence. We would like to say two unitary representations E, F are J-equivalent if there exists a continuous G-map $\phi\colon S(E) \to S(F)$ of degree prime to the order of G. At the moment this is unsuitable because we have no reason to suppose that this relation will be symmetric. That it will be symmetric for p-groups of odd order will become apparent in Part IV; until then we give the following definition:

Definition 2.4. Two unitary representations E, F of a finite group G are said to be J-equivalent if there are G-maps $\phi\colon S(E) \to S(F)$, $\theta\colon S(F) \to S(E)$ each of degree prime to the order of G.

If E, F are J-equivalent, write $E \sim F$.

PROPOSITION 2.5.

(1) *J-equivalence is an equivalence relation.*

(2) *isomorphic representations are J-equivalent.*

(3) *J-equivalence is additive, i.e. $E_1 \sim F_1$, $E_2 \sim F_2 \Rightarrow E_1 \oplus E_2 \sim F_1 \oplus F_2$.*

(4) *if $i\colon H \to G$ is a monomorphism, E and F are unitary representations of H and $\phi\colon S(E) \to S(F)$, $\theta\colon S(F) \to S(E)$ are H-maps of degree prime to the order of G, then $i_* E \sim i_* F$. In particular, if G is a p-group and H a subgroup of G $(H \neq 1)$, then*

$$E \sim F \Rightarrow i_* E \sim i_* F.$$

(5) *if $\alpha\colon G' \to G$, E and F are unitary representations of G and $\phi\colon S(E) \to S(F)$, $\theta\colon S(F) \to S(E)$ are G-maps of degree prime to the order of G', then $\alpha^* E \sim \alpha^* F$. In particular, if G, G' are p-groups, $E \sim F \Rightarrow \alpha^* E \sim \alpha^* F$.*

Proof. (1), (2) are trivial.

(3), (4), (5) are results of propositions 2.1, 2.2, 2.3 respectively.

§3. CONJUGACY

Let G be a finite group of order N and let Γ_N be the Galois group of $\mathbf{Q}(\omega)$ over $\mathbf{Q}$ where ω is a primitive Nth root of unity. The character of a complex representation of G is a function from G to the complex numbers (it is a class function), and it is well known

that the values of the character lie in $\mathbf{Q}(\omega)$, and so we may operate on a character by an element of Γ_N. If E, F are complex representations of G, define E, F to be conjugate if their characters are conjugate by an element of Γ_N.

A representation $\rho: G \to \text{Aut } E$ may be realised over $\mathbf{Q}(\omega)$ (i.e. a basis may be chosen for E such that the matrix $\rho(g)$ has scalar entries in $\mathbf{Q}(\omega)$ for each $g \in G$). A proof may be found, for example, in [10] theorems 41.1 and 70.3. If ρ is so realised, we may apply $\alpha \in \Gamma_N$ to ρ and obtain another matrix representation $\alpha\rho$. Since a complex representation is determined up to isomorphism by its character, it is easy to see that $\rho' = \alpha\rho$ if and only if trace $\rho' = \alpha$ (trace ρ).

The notion of conjugacy of representations is evidently an equivalence relation. It is not additive since we may have $E_r = \alpha_r F_r$ $r = 1, 2$ where $\alpha_1 \neq \alpha_2$, so that $E_1 \oplus E_2$ need not be conjugate to $F_1 \oplus F_2$.

We recall that the complex representation ring $R(G)$ is the free abelian group generated by the isomorphism classes of irreducible complex (or unitary) representations of G. If $\xi_1, \ldots, \xi_m$ are the classes of irreducible representations of G, every element of $R(G)$ is uniquely of the form $\sum n_r \xi_r$ where $n_r \in \mathbf{Z}$. The representations of G are given by $n_r \geq 0$, $r = 1, \ldots, m$.

$R(G)$ is isomorphic to the character ring of G. This is the subring of class functions from the set of conjugacy classes of G to the complex numbers generated by the complex characters. $R(G)$ is a special λ-ring (see I.1.5) and in particular the Adams operators are defined on $R(G)$. If ξ_r is one dimensional $\psi^k(\xi_r) = \xi_r{}^k$. We may also operate on $R(G)$ by elements of Γ_N, since the elements of $R(G)$ may be considered as difference characters.

PROPOSITION 3.1. *If χ is a complex (difference) character, then so is $\psi^r(\chi)$ and $[\psi^r(\chi)](g) = \chi(g^r)$. If $\alpha \in \Gamma_N$ and $\alpha(\omega) = \omega^k$ (where k is prime to the order of the group) then $\alpha\chi = \psi^k(\chi)$.*

Proof. It is sufficient to consider χ as the character of a matrix representation ρ. Restricting to the cyclic group C generated by g, $\rho(g)$ is equivalent to a diagonal matrix $\text{diag}(\omega_1, \ldots, \omega_s)$ where ω_i is a qth root of unity and q is the order of g. Then $\chi(g) = \sum \omega_i$ and $\psi^r(\chi)(g) = \sum \omega_i{}^r = \chi(g^r)$. Evidently if $\alpha(\omega) = \omega^k$, $[\alpha\chi](g) = \chi(g^k) = [\psi^k(\chi)](g)$.

COROLLARY 3.2. *The Adams operators are periodic of period dividing the order of the group i.e. $\psi^{N+r} = \psi^r$ where $N = |G|$. In particular if ε: $R(G) \to \mathbf{Z}$ is the augmentation induced by the dimension of the representations, then $\psi^N = \varepsilon$.*

It is trivial Galois theory that the elements of $R(G)$ invariant under Γ_N consist precisely of those (virtual) representations with rational characters. (It is also well known [10], that a complex representation E with rational character need not be a rational representation, but that $m(E)$. E is rational where the integer $m(E)$ is the Schur index). This gives a complete description of the subring of invariants $R(G)^{\Gamma_N}$.

This paper may be considered as an attempt to give a topological description of the quotient ring of coinvariants $R(G)_{\Gamma_N} = R(G)/W(G)$ where $W(G)$ is the subgroup of $R(G)$ generated by elements $\{x - \alpha x\}$, $x \in R(G)$, $\alpha \in \Gamma_N$.

Algebraically $R(G)_{\Gamma_N}$ is easily described. If the representation E is irreducible, then trivially so is the conjugate representation αE for $\alpha \in \Gamma_N$. Let us order the classes of irreducible representations $\xi_1, \ldots, \xi_s, \ldots, \xi_m$ such that no two of $\xi_1, \ldots, \xi_s$ are conjugate and yet their conjugates exhaust $\xi_1, \ldots, \xi_m$. Then $W(G)$ is generated by elements of the form $\{\xi_i - \alpha\xi_i\}$ where $1 \leq i \leq s$, $\alpha \in \Gamma_N$ and we have:

PROPOSITION 3.3. *$R(G)_{\Gamma_N}$ is the free abelian group on the generators*

$$\{\xi_i + W(G)\}, \qquad 1 \leq i \leq s.$$

§4. $J(G)$ AND THE EPIMORPHISM $\nu : R(G)_{\Gamma_N} \to J(G)$

Let G be any finite group, let $T(G) \subset R(G)$ be the additive subgroup generated by elements $[E] - [F]$ where E, F are J-equivalent unitary representations and $[E]$ denotes the isomorphism class of E. Define $J(G) = R(G)/T(G)$.

Remark. At this stage there is no reason to suppose that $[E] - [F] \in T(G)$ implies E and F are J-equivalent, since $[E] - [F] = [E \oplus C] - [F \oplus C]$. There may exist suitable maps $\phi : S(E \oplus C) \to S(F \oplus C)$ and $\theta : S(F \oplus C) \to S(E \oplus C)$ which are not of the form $\phi = \phi_1 \circ \phi_2$, $\theta = \theta_1 \circ \theta_2$, since ϕ and θ are only asked to be continuous G-maps, they are not expected to be linear. It will be a result of the main theorem for p-groups ($p \neq 2$) proved in this paper that $[E] - [F] \in T(G) \Rightarrow E$, F are J-equivalent.

In order that an epimorphism ν: $R(G)_{\Gamma_N} \to J(G)$ be defined, it is sufficient to show that $W(G) \subset T(G)$. (Recall that $W(G)$ is the subgroup of $R(G)$ generated by elements $\{x - \alpha x\}$, $x \in R(G)$, $\alpha \in \Gamma_N$).

Since J-equivalence is additive (2.5(3)), it is sufficient to show that E is J-equivalent to αE where E is irreducible.

LEMMA 4.1. *If F is a one dimensional unitary representation of a finite-group G and $\alpha \in \Gamma_N$, then F is J-equivalent to αF.*

Proof. Suppose $\alpha(\omega) = \omega^k$ (where k is prime to $N = |G|$). Since F is one dimensional, we may choose a complex co-ordinate z for the representation space so that $S(F)$ is given by the set of complex numbers of unit modulus. If χ is the character of F, then the action of $g \in G$ on F may be written as $z \mapsto \chi(g) \,.\, z = \zeta z$ where ζ is an Nth root of unity. We may also consider αF to have the same underlying space with the action of $g \in G$ on αF given by $z \mapsto \alpha\chi(g) \,.\, z = \zeta^k z$. Define ϕ: $S(F) \to S(\alpha F)$ by $\phi(z) = z^k$; it is a G-map of degree k. A map θ: $S(\alpha F) \to S(F)$ may be defined in a similar manner.

In [10] an M-group is defined to be a finite group G such that every irreducible representation of G is induced from a one dimensional representation of some subgroup. In particular, by [10] theorem 52.1, a nilpotent group is an M-group.

PROPOSITION 4.2. *If E is an irreducible unitary representation of an M-group G and $\alpha \in \Gamma_N$, then E is J-equivalent to αE.*

Proof. There is a subgroup $H \leq G$ and a one dimensional representation F of H such that $i_* F = E$ (where $i : H \to G$ is the canonical injection). The Galois group Γ_N acts also on

$R(H)$ and its action commutes with induction, i.e. $i_*(\alpha F) = \alpha(i_* F) = \alpha E$. By lemma 4.1, F and αF are J-equivalent so we can choose maps ϕ: $S(F) \to S(\alpha F)$, θ: $S(\alpha F) \to S(F)$ of degree prime to the order of G. Then by proposition 2.5(4), $i_* F$ and $i_* \alpha F$ are J-equivalent.

Since J-equivalence is additive (2.5(3)), we may use proposition 4.2 to construct explicit J-equivalences between conjugate representations of M-groups. So we see that $W(G) \subset T(G)$ and we may define v: $R(G)_{\Gamma_N} \to J(G)$ by $v(x + W(G)) = x + T(G)$ for $x \in R(G)$. v is obviously an epimorphism.

Let ε: $R(G) \to \mathbf{Z}$ be the augmentation induced by the dimension of a representation and let $I(G) = \ker \varepsilon$, then $R(G) = \mathbf{Z} \oplus I(G)$ as abelian groups.

If E and F are J-equivalent, then there is a map $S(E) \to S(F)$ of degree prime to the order of the group, and for the degree to be defined, the dimensions of E and F must be equal. This shows $T(G) \subset I(G)$.

Define $\tilde{J}(G) = I(G)/T(G)$

then $J(G) = \mathbf{Z} \oplus \tilde{J}(G)$ as abelian groups.

It is also easy to see that $W(G) \subset I(G)$ and that, in the usual notation,

$$I(G)_{\Gamma_N} = I(G)/W(G)$$

and $R(G)_{\Gamma_N} = \mathbf{Z} \oplus I(G)_{\Gamma_N}$

This implies:

PROPOSITION 4.3. *If G is an M-group (in particular if G is a p-group), there is a canonical epimorphism v: $R(G)_{\Gamma_N} \to J(G)$ and this induces an epimorphism $\tilde{v}$: $I(G)_{\Gamma_N} \to \tilde{J}(G)$.*

III. THE ALGEBRAIC THEOREM

The notion of a p-adic γ-ring is defined here. It is made sufficiently general so that it includes $Z_p \otimes I(G)$ where $I(G)$ is the augmentation ideal for a p-group G and also $Z_p \otimes \tilde{K}(X)$ for a finite, connected CW complex X. The algebraic theorem is given in terms of a p-adic γ-ring and so includes both cases.

§1 is devoted to properties of $I(G)$ for later use. In §2 a p-adic γ-ring is defined. The Adams operations $\{\psi^k\}$, when acting on a p-adic γ-ring are shown to be continuous in the integer k (for the p-adic topology), and so induce operations $\{\psi^\alpha\}$ where α may be any p-adic integer. If Γ is the multiplicative group of units in the p-adic integers, $\alpha \in \Gamma$ acts via the operation ψ^α.

In §3 we introduce the operation ρ_k for a p-adic γ-ring A, as a homomorphism from the additive group of A to the multiplicative group $1 + A$. If $p \neq 2$, Γ has a dense cyclic subgroup generated by h, where $(h, p) = 1$, $h^{p-1} \not\equiv 1 \bmod p^2$, and $h \bmod p$ is a generator of the multiplicative group of the field $\mathbf{Z}/p\mathbf{Z}$. In §4 we prove the main algebraic theorem of the paper, that ρ_h induces an isomorphism on invariants and coinvariants $A^\Gamma \cong (1 + A)^\Gamma$, $A_\Gamma \cong (1 + A)_\Gamma$. The case $p = 2$ is discussed, and by introducing the notion of orientability for a p-adic γ-ring, a corresponding theorem can be proved in this case.

Both theorems use an induction argument. The corresponding part of Adams work [2] utilises a filtration on $K(X)$ given topologically by using the skeletons of X. Here the induction uses a natural algebraic filtration, $A = A(1) \supset A(2) \supset \cdots$, given in terms of the γ-operations.

Another point of note is that in [2], the result is found by using the properties of Bernoulli numbers, here they are avoided.

The algebraic theorem holds good for $\widehat{I(G)} = Z_p \otimes I(G)$ where G is a p-group ($p \neq 2$). The group Γ acts on $\widehat{I(G)}$ via the finite quotient group Γ_N where $N = p^e = |G|$ and so the theorem implies $\rho_h : \widehat{I(G)}_{\Gamma_N} \to (1 + \widehat{I(G)})_{\Gamma_N}$ is an isomorphism.

In §5 we discuss the connection between the natural exponential map θ_k of I.§7 and ρ_k.

§1. THE TOPOLOGY OF I(G)

We recall that $I(G)$ is a free abelian group on the generators $\xi_2 - \varepsilon(\xi_2), \ldots, \xi_m - \varepsilon(\xi_m)$ where $\xi_1 = 1, \xi_2, \ldots, \xi_m$ are the isomorphism classes of irreducible representations of G. $I(G)$ may be given a topology in three ways, with neighbourhoods of zero being given by:

(1) $I(G) \supset pI(G) \supset \cdots \supset p^n I(G) \supset \cdots$ p-adic topology
(2) $I(G) \supset I(G)^2 \supset \cdots \supset I(G)^n \supset \cdots$ $I(G)$-adic topology
(3) $I(G) = I(G)_1 \supset I(G)_2 \supset \cdots \supset I(G)_n \supset \cdots$ γ-topology (1.4(11))

PROPOSITION 1.1. *For a p-group G, the topologies* (1), (2), (3) *are equivalent.*

Proof. It is shown in [3] corollary 12.3 that the topologies (2) and (3) coincide. (It is an easy exercise to show this result is true replacing $I(G)$ by any γ-ring I which has a finite number of generators, each of finite γ-dimension). We now show (1),(2) determine the same topology. From [3] proposition 6.13, $|G| \cdot I(G)^n \subset I(G)^{n+1}$ for any finite group, and so if $|G| = p^e$, $p^e I(G)^n \subset I(G)^{n+1}$. It remains to show a power of $I(G)$ lies in $pI(G)$. It is sufficient to show for a representation ξ, that $(\xi - \varepsilon(\xi))^m \in pI(G)$, for some m. Take $m = p^e = |G|$, then

$$(\xi - \varepsilon(\xi))^{p^e} \equiv \xi^{p^e} - \varepsilon(\xi)^{p^e} \bmod pR(G).$$

Since
$$\xi^{p^e} \equiv \psi^{p^e}(\xi) \bmod pR(G) \qquad \text{(I.5.2)}$$

and
$$\psi^{p^e}(\xi) = \varepsilon(\xi) \qquad \text{(II.3.2)}$$

We have
$$(\xi - \varepsilon(\xi))^{p^e} \equiv \varepsilon(\xi) - \varepsilon(\xi)^{p^e} \bmod pR(G)$$
$$\equiv 0 \bmod pR(G).$$

Since $(\xi - \varepsilon(\xi))^{p^e} \in I(G)$, it lies in $pI(G)$ and this completes the proof.

§2. p-ADIC γ-RINGS

We recall some facts about the completion of an abelian group. If $H \supset H_1 \supset \cdots \supset H_n \supset \cdots$ is a filtration of subgroups on an abelian group H, the filtration topology on H is given by fundamental neighbourhoods of zero $\{H_n\}_{n \geq 1}$. It is Hausdorff if and only if

$\cap H_n = 0$. The completion of H in the filtration topology is $\hat{H} = \varprojlim H/H_n$. If the topology is Hausdorff, the canonical map $H \to \hat{H}$ is a monomorphism and we may identify H as a dense subset of $\hat{H}$. In this case H is complete if, and only if, $H = \hat{H}$, or alternatively if, and only if, every Cauchy sequence converges to a limit.

The filtration $\{p^n H\}_{n \geq 0}$ gives rise to the p-adic topology. Let $Z_p = \varprojlim \mathbf{Z}/p^n\mathbf{Z}$ (the p-adic integers) then the p-adic completion $\hat{H}$ is a Z_p-module. If B is a finitely generated abelian group, the completion of B is a finitely generated Z_p-module, $\hat{B} = Z_p \otimes B$. The quotients $\{B/p^n B\}$ are finite and so compact in the discrete (p-adic!) topology. Thus for finitely generated abelian groups, p-adic completion is an exact functor, since inverse limit is exact for compact groups [11]. The p-adic completion of a finitely generated abelian group is compact in the p-adic topology. It is to be remembered that subgroups and quotient groups of finitely generated abelian groups are finitely generated. This implies that $\hat{B}/\hat{C} = (B/C)^{\wedge}$, if C is a subgroup of B.

We may give Z_p a special λ-ring structure by $\lambda_t(\alpha) = (1 + t)^{\alpha}$. Since the positive integers Z^+ are dense in Z_p, if $\alpha \in Z_p$, there is a sequence $\{\alpha_n\}$ in Z^+ which converges to α. It is easily seen that $\binom{\alpha_n}{r} \to \binom{\alpha}{r}$ and so $\lambda^r(\alpha) = \lim \lambda^r(\alpha_n)$. If R is a special λ-ring, then so is $Z_p \otimes R$ by I.6.2. The λ-structure on $Z_p \otimes R$ may be described in terms of that on R. Since $\lambda^k(\alpha_n x) = P_k(\lambda^1(\alpha_n), \ldots, \lambda^k(\alpha_n); \lambda^1(x), \ldots, \lambda^k(x))$, where P_k is a polynomial of weight k in the $\lambda^r(\alpha_n) = (r!^{\alpha_n})$, $r = 1, \ldots, k$, we see that $\lim \lambda^k(\alpha_n x) = \lambda^k(\lim \alpha_n x) = \lambda^k(\alpha x)$. Thus $\lambda_t(\alpha x) = \lim \lambda_t(\alpha_n x) = \lim[\lambda_t(x)^{\alpha_n}] = \lambda_t(x)^{\alpha}$ for $\alpha \in Z_p$. Also $\lim \gamma^k(\alpha_n x) = \gamma^k(\alpha x)$ and $\gamma_t(\alpha x) = \gamma_t(x)^{\alpha}$. Trivially $\psi^k(\alpha x) = \alpha \psi^k(x)$.

If B is a special γ-ring, then, by definition, there is a special augmented λ-ring R such that $B = \ker \varepsilon$ where ε is the augmentation. Tensoring the exact sequence $0 \to B \to R \to \mathbf{Z} \to 0$ with Z_p, then $0 \to Z_p \otimes B \to Z_p \otimes R \to Z_p \to 0$ is exact since $\mathrm{Tor}_1(Z_p, \mathbf{Z}) = 0$. Evidently $Z_p \otimes B$ is a λ-ideal and is thus a special γ-ring.

Definition 2.1. If A is a γ-ring, it is said to be a p-adic γ-ring if it is the completion of some γ-ring B, $A = Z_p \otimes B$ where

(1) B is finitely generated as an abelian group.

(2) the γ-topology on B is finer than the p-adic topology.

Examples. (1) $\widehat{I(G)}$ for a p-group G (proposition 1.1)

(2) $Z_p \otimes \tilde{K}(X)$ for a finite, connected, CW complex X.

In this case, note that $\tilde{K}(X)_n = 0$ for large n and so the γ-topology is discrete.

(3) pZ_p where $\psi^k(x) = \begin{cases} x \text{ if } (p, k) = 1 \\ 0 \text{ if } p \mid k \end{cases}$

This example is vital in the proof of the main algebraic theorem. We may calculate the $\{\lambda^k\}$ using I.5(1) which gives

$$\psi_t(x) = -t \frac{\mathrm{d}}{\mathrm{d}t} [\log \lambda_{-t}(x)].$$

Hence
$$-t\frac{d}{dt}[\log \lambda_{-t}(x)] = \left[\sum_1^\infty t^k - \sum_1^\infty t^{pk}\right]x$$
$$= \left[-t\frac{d}{dt}\log(1-t) + \frac{t}{p}\frac{d}{dt}\log(1-t^p)\right]x$$

and so
$$\lambda_{-t}(p) = \frac{(1-t)^p}{1-t^p}$$

i.e.
$$\lambda_t(p) = \frac{(1+t)^p}{1-(-t)^p}.$$

This gives
$$\gamma_t(p) = \lambda_{t/(1-t)}(p) = [(1-t)^p - (-t)^p]^{-1}$$

and so
$$\gamma_t(-p) = (1-t)^p - (-t)^p.$$

To see pZ_p is a p-adic γ-ring with this structure, we observe that a λ-ring with no torsion is special if and only if the $\{\psi^k\}$ are ring homomorphisms such that $\psi^{kr} = \psi^k\psi^r$. These conditions are obviously satisfied in this case. Alternatively, $pZ_p = Z_p \otimes p\mathbf{Z}$ and it may be verified that the λ-structure is special from the following alternative description:

Let C be the cyclic group of order p with generator g; let χ be an irreducible character such that $\chi(g)$ is a primitive pth root of unity. The invariant subgroup of $I(C)$ under the action of the Galois group Γ_p is generated by $\mu = 1 + \chi + \cdots + \chi^{p-1} - p$. The map
$$\theta: I(C)^{\Gamma_p} \to p\mathbf{Z}$$
defined by evaluation on the generator g is an isomorphism which gives the required γ-ring structure on $p\mathbf{Z}$.

Since $\gamma_t(-p) = (1-t)^p - (-t)^p$, either $\gamma^r(-p) = 0$ or $p \mid \gamma^r(-p)$, but $\gamma^{p-1}(-p) = (-1)^{p-1}p$ and so the lowest power of p attainable in B_n is in $[\gamma^{p-1}(-p)]^m$ where
$$(m-1)(p-1) < n \leq m(p-1).$$
i.e. $B_{(m-1)(p-1)+1} = B_{(m-1)(p-1)+2} = \cdots = B_{m(p-1)} = p^m\mathbf{Z}$. The γ-topology and p-adic topology coincide and $B_n = B_{n+1}$ unless $n \equiv 0 \bmod(p-1)$, in which case B_n/B_{n+1} is the cyclic group of order p.

(4) $Z_p \otimes F$ where F is finitely generated abelian group written additively, the product of two elements is defined to be zero and $\lambda^k(x) = (-1)^{k-1}k^{n-1}x$ for some fixed integer n. It is an interesting calculation to show $\gamma_t(x) = 1 + xf_n(t)$ where $f_n(t)$ satisfies the recurrence relations $f_1(t) = t, f_{n+1}(t) = t(1-t)f_n'(t)$ and so $\gamma^m(x) = 0, m > n$ and the γ-topology is discrete. Since the operations $\{\lambda^k\}$, $\{\gamma^k\}$ are given by multiplication by constants, this is called a scalar γ-ring. A topological instance is given by $F = \tilde{K}(S^{2n})$.

We have already remarked that the Adams operations are p-adically continuous on A. If we temporarily write $\psi^k(a) = \psi(k, a)$ for $k \in Z^+$, $a \in A$, then $\psi: Z^+ \times B \to B$ is p-adically continuous (I.5.6). But Z^+ is dense in Z_p, B is dense in A and A is complete, so by continuity, ψ extends to a continuous map $\psi: Z_p \times A \to A$. Thus we have:

LEMMA 2.2. *The domain of the Adams operations $\{\psi^k\}$ in the variable k extends by*

continuity, to give continuous operations $\psi^\alpha: A \to A$ $(\alpha \in Z_p)$ *on a p-adic* γ*-ring* A. *If* $\beta \in Z_p$, $\psi^\alpha \psi^\beta = \psi^{\alpha\beta}$.

Let Γ be the group of units of Z_p, then if $\alpha \in \Gamma$, α acts on A via the operation ψ^α. By lemma 2.2, this defines a (p-adically) continuous Γ-action on A and makes A into a compact topological Γ-module.

Of course it may happen that the Adams operators are actually periodic on B, i.e. $\psi^{p^e+k} = \psi^k$ for some integer e, as is the case on $I(G)$ where G is a p-group of order p^e. In this case Γ acts on both B and $Z_p \otimes B$ through the finite quotient Γ_{p^e}. (It is easily seen that $\Gamma = \varprojlim_n \Gamma_{p^n}$ where Γ_m is the Galois group of $\mathbf{Q}(\omega)$ over $\mathbf{Q}$ and ω is a primitive mth root of unity). Since B is finitely generated, tensoring with Z_p is exact, and so commutes with the formation of invariants and coinvariants. So we have:

PROPOSITION 2.3. *If* B *is as in definition* 2.1 *and the Adams operators* $\{\psi^k\}$ *are periodic in* k *of period* p^e, *then* Γ *acts on* B *and* $\hat{B} = Z_p \otimes B$ *via the quotient group* Γ_{p^e}. *Furthermore* $(\hat{B})^\Gamma = (B^\Gamma)\hat{}$, $(\hat{B})_\Gamma = (B_\Gamma)\hat{}$.

If B satisfies the requirements of proposition 2.3, we may simplify the notation for invariants by writing $\hat{B}^\Gamma$, and for coinvariants by writing $\hat{B}_\Gamma$. In the case of $B = I(G)$ for a p-group G, we have:

COROLLARY 2.4. *For a p-group* G *of order* $N = p^e$,

$$I(G)_\Gamma = I(G)_{\Gamma_N}, \quad \widehat{I(G)_\Gamma} = \widehat{I(G)}_{\Gamma_N}.$$

Suppose A is a p-adic γ-ring and $A = \hat{B} = Z_p \otimes B$ as usual. Define $A(n)\hat{} = (B_n)\hat{} = Z_p \otimes B_n$, the closure of the nth ideal in the γ-filtration on B. Since $\lim \gamma^k(\alpha_n x) = \gamma^k(\alpha x)$ where $\{\alpha_n\}$ is a sequence of integers with limit $\alpha \in Z_p$, we see $A_n \subset A(n)$. Evidently $(A_n)\hat{} = A(n)$, but since A_n need not be closed in the p-adic topology, we do not necessarily have $A_n = A(n)$. From the definition of a p-adic γ-ring, the filtration topology $\{A(n)\}_{n\geq 1}$ is finer than the p-adic topology on A. Since A is complete in the p-adic topology, a fortiori it is complete (Hausdorff) in the $\{A(n)\}$ topology and so $A = \varprojlim A/A(n)$. In this discussion, we have proved:

PROPOSITION 2.5. *If* A *is a p-adic* γ *ring and* $A(n) = (A_n)\hat{}$, *the filtration topology* $\{A(n)\}_{n\geq 1}$ *is finer than the p-adic topology on* A. *Also* $A = \varprojlim A/A(n)$.

Note. With two possible topologies available, if we refer to a topology without qualification, we mean the p-adic topology. Likewise, a continuous map will mean p-adically continuous.

Next we observe that

$$A(n)/A(n+1) = (B_n)\hat{}/(B_{n+1})\hat{} = (B_n/B_{n+1})\hat{} = Z_p \otimes (B_n/B_{n+1})$$

Thus $A(n)/A(n+1)$ is a p-adic γ-ring with the structure of Example (4) above (this follows from I.5.5). The λ-operations acting on B_n/B_{n+1} are given by $\lambda^k(x) = (-1)^{k-1}k^{n-1}x$. Since $\lambda^k(\alpha x) = \lim \lambda^k(\alpha_n x)$ where $\{\alpha_n\}$ is a sequence of integers with limit $\alpha \in Z_p$, λ^k acts by the same formula on $A(n)/A(n+1)$. Thus we have:

PROPOSITION 2.6. *$A(n)/A(n+1)$ is a scalar p-adic γ-ring.*

(i) $\lambda^k(a) = (-1)^{k-1}k^{n-1}a$ *for* $a \in A(n)/A(n+1)$

(ii) $\gamma^k(a) = c(k,n)a$ *where* $c(k,n) \in \mathbf{Z}$ *depends only on* k, n.

(iii) $\psi^k(a) = k^n a$.

(iv) the Γ*-action is defined by* $\psi^\alpha(a) = \alpha^n a$ *for* $\alpha \in \Gamma$.

§3. THE OPERATION ρ_k

If A is a p-adic γ-ring, a series $\sum_{r\geq 1} a_r$, with $a_r \in A(r)$, converges in the p-adic topology since it converges in the filtration topology $\{A(n)\}_{n\geq 1}$ which is finer. This shows that the set $1 + A$ is a multiplicative group. It is a compact, topological group, with fundamental neighbourhoods of 1 given by $\{1 + p^n A\}_{n\geq 0}$, or equivalently $\{1 + p^n A + A(n)\}_{n\geq 1}$, since the filtration $\{A(n)\}$ is finer than $\{p^n A\}$.

Let $a \in A$, $\alpha \in Z_p$, then define $\gamma_\alpha(a) = 1 + \sum \alpha^r \gamma^r(a) \in 1 + A$. For fixed α, γ_α is a homomorphism from the additive group of A to the multiplicative group $1 + A$. It is evident that γ_α is p-adically continuous using the alternative system of neighbourhoods on $1 + A$ (given any N, choose n such that $\gamma^k(p^n A) \subset p^N A$ for $k = 1, \ldots, N-1$, then $\gamma_\alpha(p^n A) \subset 1 + p^N A + A(N)$).

More generally, if $\alpha \in S$ where S is any finitely generated Z_p-algebra, then $\gamma_\alpha(a) \in 1 + S \otimes_{Z_p} A$. In particular, let $(k,p) = 1$ and let $S = Z_p[t]/\Phi_k(t)$ where Φ_k is the kth cyclotomic polynomial, then $S = Z_p[\zeta]$, where ζ is (in the algebraic closure of Q_p) a primitive root of $t^k - 1 = 0$. Note that Φ_k is irreducible over Q_p, since $(k,p) = 1$: this follows by showing Φ_k is irreducible over $\mathbf{Z}/p\mathbf{Z}$ and a fortiori, irreducible over Q_p. If we consider the product $\prod(1-u)$ over all roots $\{u\}$ of $t^k - 1 = 0$ except 1, we find $\prod(1-u) = k$. This shows $\dfrac{u}{u-1} \in Z_p[\zeta]$ and so $\gamma_{u/(u-1)}(a) \in 1 + Z_p[\zeta] \otimes_{Z_p} A$.

Define $\rho_k(a) = \prod \gamma_{u/(u-1)}(a)$ where the product is taken over all roots of $t^k - 1 = 0$ except 1.

Since $Z_p[\zeta]$ is a free Z_p-module containing Z_p as a direct summand, we may consider $A = Z_p \otimes_{Z_p} A$ as a direct summand of $Z_p[\zeta] \otimes_{Z_p} A$. We demonstrate that $\rho_k(a) \in 1 + A$.

If $\gamma_t(a) = 1 + at$, then

$$\rho_k(a) = \prod\left(1 + \frac{ua}{u-1}\right) = \frac{\prod(1 - u(a+1))}{\prod(1-u)} = k^{-1}(1 + (a+1) + \cdots + (a+1)^{k+1}).$$

If V is the free γ-ring on the generator σ_1 (I§4), then, using a limiting argument, $\rho_k(\sigma_1) \in 1 + V$; hence by the universal property of V(I.4.4), for arbitrary $a \in A$, $\rho_k(a) \in 1 + A$.

PROPOSITION 3.1. *If A is a p-adic γ-ring, it is an additive compact topological Γ-module, $1 + A$ is a multiplicative compact topological Γ-module, and*

$\rho_k: A \to 1 + A$ is a (p-adically) continuous Γ-module homomorphism.

Proof. In §2 we have already seen that A is a compact topological Γ-module. For the same reasons, so is $1 + A$. Since $\rho_k(p^n a) = [\rho_k(a)]^{p^n}$, ρ_k is p-adically continuous. Since ψ^α commutes with the γ-operations, it follows that ρ_k is a Γ-module homomorphism.

§4. THE MAIN ALGEBRAIC THEOREM

For most of this section we insist $p \neq 2$. If p is odd, the group of units of Z_p is monogenic (contains a dense cyclic subgroup), whereas the group of units of Z_2 is not. Specifically for $p \neq 2$, Γ is topologically generated by the integer h where $(h, p) = 1$, $h^{p-1} \not\equiv 1 \bmod p^2$ and $h \bmod p$ is a generator of the multiplicative group of the field $\mathbf{Z}/p\mathbf{Z}$; for $p = 2$, Γ is the direct product of the monogenic group with generator 3 and the group $\{\pm 1\}$. In the proof of theorem 4.1, we will use the fact that Γ is monogenic and so we must omit $p = 2$. A refinement which includes $p = 2$ will be considered in theorem 4.5.

Until then we assume $p \neq 2$, so Γ is monogenic, with generator η, and η acts on a p-adic γ-ring via the Adams operation ψ^h, where h is the integer defined above. The action of Γ on any topological Γ-module M is uniquely defined by the action of η, since the powers of η are dense in Γ. The map $(1 - \eta)$ defined by $(1 - \eta)x = x - \eta x$ is a continuous homomorphism. If M is Hausdorff, $\ker(1 - \eta)$ is closed and $\ker(1 - \eta) = M^\Gamma$, the invariant submodule. Moreover if M is compact, the image of $(1 - \eta)$ is compact and hence closed. But $\operatorname{im}(1 - \eta)$ contains all elements of the form $(1 - \eta^r)x$ and so contains the closure of such elements. Since the powers of η are dense in Γ, we see $\operatorname{im}(1 - \eta) = (1 - \Gamma)M$ and so $\operatorname{coker}(1 - \eta) = M_\Gamma$, the coinvariant quotient module. We note that if A is a p-adic γ-ring, in particular it is compact and so the above remarks apply.

Any continuous homomorphism $f: M_1 \to M_2$ of topological Γ-modules induces homomorphism f^Γ and f_Γ on invariants and coinvariants.

THEOREM 4.1. *For any p-adic γ-ring A where $p \neq 2$, $\rho_h : A \to 1 + A$ induces isomorphisms $(\rho_h)^\Gamma$ and $(\rho_h)_\Gamma$ where h is a generator of Γ.*

This is the main algebraic theorem and the proof will proceed using induction on the filtration $\{A(n)\}_{n \geq 1}$.

LEMMA 4.2. *If $0 \to X \to Z \to Y \to 0$ is an exact sequence of p-adic γ-rings and theorem 4.1 is true for X, Y, then it is true for Z.*

Proof. If M is a compact topological Γ module, then M^Γ and M_Γ are the homology groups of the complex $0 \to M \xrightarrow{1-\eta} M \to 0$. In particular this is true if M is a p-adic γ-ring. The short exact sequence $0 \to X \to Z \to Y \to 0$ gives rise to the exact homology sequence $0 \to X^\Gamma \to Z^\Gamma \to Y^\Gamma \to X_\Gamma \to Z_\Gamma \to Y_\Gamma \to 0$. (ker-coker sequence). ρ_k induces a homomorphism of such sequences:

$$\begin{array}{ccccccccccccc}
0 & \longrightarrow & X^\Gamma & \longrightarrow & Z^\Gamma & \longrightarrow & Y^\Gamma & \longrightarrow & X_\Gamma & \longrightarrow & Z_\Gamma & \longrightarrow & Y_\Gamma & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & (1+X)^\Gamma & \to & (1+Z)^\Gamma & \to & (1+Y)^\Gamma & \to & (1+X)_\Gamma & \to & (1+Z)_\Gamma & \to & (1+Y)_\Gamma & \to & 0
\end{array}$$

Two applications of the Five Lemma give the required result.

Proof of Theorem 4.1. For each $n \geq 1$, we have an exact sequence:

$$0 \to A(n)/A(n+1) \to A/A(n+1) \to A/A(n) \to 0.$$

Since $A = \varprojlim A/A(n)$ (proposition 2.5), by induction on n we need only prove two facts:

(*i*) the theorem holds for $A(n)/A(n+1)$, $n \geq 1$ (recall $A = A(1)$)

(*ii*) $\varprojlim$ commutes with $^{\Gamma}$ and $_{\Gamma}$ on $\{A/A(n)\}$ and on $\{(1+A)/(1+A(n))\}$

Now (*ii*) is certainly true because $\varprojlim$ is an exact functor for compact groups ([11] chapter VIII 5.6), and the groups concerned are compact by propositions 2.6 and 3.1.

It remains to prove (*i*). We recall the structure of $A(n)/A(n+1)$ given in proposition 2.6. Since $\psi^h(a) = h^n a$ for $a \in A(n)/A(n+1)$, then $(1-\eta)a = (1-h^n)a$. If $n \not\equiv 0 \bmod(p-1)$, then $p \nmid 1 - h^n$, implying $1 - h^n$ has an inverse in Z_p. So $1 - \eta$ is an isomorphism and the invariants and coinvariants are zero. In this case the theorem is trivially true.

We need therefore only consider $n \equiv 0 \bmod(p-1)$. We will in fact show ρ_h is itself an isomorphism in this case (and so induces isomorphisms on invariants and coinvariants.) Since $\gamma^k(x) = c(k,n)x$ where $c(k,n) \in \mathbf{Z}$, $\rho_h(x) = 1 + \mathrm{d}(h,n)x$ where $\mathrm{d}(h,n) \in Z_p$ and depends only on h, n. Now multiplication by $\mathrm{d}(h,n)$ on $A(n)/A(n+1)$ is an isomorphism if, and only if, $p \nmid \mathrm{d}(h,n)$. We need not compute $\mathrm{d}(h,n)$ directly, we need only demonstrate the existence of a p-adic γ-ring, such that $A(n)/A(n+1) \neq 0$ for $n \equiv 0 \bmod(p-1)$, and such that ρ_h is an isomorphism on $A(n)/A(n+1)$, $n \equiv 0 \bmod(p-1)$. The most economical example of such a ring is given in example 3 of §2.

$$A = pZ_p \text{ where } \gamma_t(-p) = (1-t)^p - (-t)^p,\ \psi^k = \begin{cases} \text{identity} & (k,p) = 1 \\ 0 & p \mid k. \end{cases}$$

In this case $A(r(p-1)) = p^r Z_p$, $A(r(p-1)+1) = p^{r+1}Z_p$ and so $A(n)/A(n+1) = 0$ if $n \not\equiv 0 \bmod(p-1)$ and $A(n)/A(n+1)$ is the cyclic group of order p if $n \equiv 0 \bmod(p-1)$. If $n = r(p-1)$, a generator for $A(n)/A(n+1)$ is the image of p^r.

$$\rho_h(-p) = \prod_u \left[\left(1 - \frac{u}{u-1}\right)^p - \left(\frac{u}{1-u}\right)^p \right] \text{ where the product is taken over all roots of}$$

$t^h - 1 = 0$ except 1, and so

$$\rho_h(p) = \frac{\prod(1-u)^p}{\prod(1-u^p)} = h^{p-1} = 1 + \frac{h^{p-1}-1}{p} \cdot p.$$

If $m = \dfrac{h^{p-1}-1}{p}$, then m is an integer prime to p since $h^{p-1} \not\equiv 1 \bmod p^2$. For $p \neq 2$, by induction on r, we have $(1+mp)^{p^{r-1}} \equiv 1 + mp^r \bmod p^{r+1}$, thus $\rho_h(p^r) = \rho_h(p)^{p^{r-1}} \equiv 1 + mp^r \bmod p^{r+1}$ and so in the γ-ring $A(n)/A(n+1)$ where $n \equiv 0 \bmod(p-1)$, we have $\rho_h(a) = 1 + ma$. Since $(m,p) = 1$ and $A(n)/A(n+1)$ is the cyclic group of order p, ρ_h is an isomorphism.

Thus A is the required example to complete the theorem.

COROLLARY 4.3. $(\rho_h)_\Gamma : \widehat{I(G)}_\Gamma \to (1 + \widehat{I(G)})_\Gamma$ *is an isomorphism for a p-group G of odd order.*

Further insight into the structure of p-adic γ-rings and into the proof of Theorem 4.1 is obtained by recalling that the group Γ of p-adic units decomposes naturally as a direct product

$$\Gamma \cong U \times (\mathbf{Z}/p\mathbf{Z})^*.$$

Here U is the subgroup of Γ consisting of elements congruent to 1 mod p, and $(\mathbf{Z}/p\mathbf{Z})^*$ is the multiplicative group of the finite field $\mathbf{Z}/p\mathbf{Z}$. To get this decomposition we have to split the exact sequence

$$1 \to U \to \Gamma \to (\mathbf{Z}/p\mathbf{Z})^* \to 1,$$

that is we have to lift from $\mathbf{Z}/p\mathbf{Z}$ to Z_p the roots of the equation

$$x^{p-1} - 1 = 0$$

This lifting is possible by Hensel's Lemma.

Let Γ_0 be the subgroup of Γ obtained in this way: it is isomorphic to $(\mathbf{Z}/p\mathbf{Z})^*$ and hence cyclic of order $p-1$. Now if C is cyclic of order $p-1$ generated by T, the group algebra $Z_p[C]$ has the elements

$$\rho_i = \prod_{j \neq i} \frac{T - \alpha^j}{\alpha^i - \alpha^j} \qquad i = 0, 1, \ldots, p-2$$

where α generates Γ_0, as a maximal set of orthogonal idempotents. Hence any $Z_p[C]$-module M has a canonical decomposition

$$M = \oplus M_i \qquad M_i = \rho_i M$$

so that, on M_i, T acts as multiplication by α^i. Applying this with $M = A$ (p-adic γ-ring) and $C = \Gamma_0$ we see that we have a direct sum decomposition

$$A = \oplus A_i$$

where A_i is the sub Z_p-module of A on which ψ^α acts as multiplication by α^i: in particular A_0 is acted on trivially by Γ_0. Since Γ_0 acts on A by ring automorphisms it follows that $A_i A_j \subset A_{i+j}$. Since U commutes with Γ_0 each A_i is stable under U. Thus A has the structure of a graded U-ring (graded by $\mathbf{Z}/(p-1)\mathbf{Z}$). This grading of A is related in quite a simple way to its γ-filtration $\{A(n)\}$. If we put $A_i(n) = A_i \cap A(n)$ for the induced filtration on A_i then (2.6) (*iii*) shows that

$$A_i(n) = A_i(n+1) \text{ if } n \not\equiv i \bmod (p-1).$$

Thus the filtration of A_i goes in jumps of $(p-1)$.

When we come to consider the invariants and coinvariants of Γ the above decomposition of A clearly implies

$$A^\Gamma \cong {A_0}^U \quad A_\Gamma \cong (A_0)_U$$

Thus Theorem (4.1) is true for A if and only if it is true for A_0. On the other hand our proof of (4.1) in fact gives the stronger result:

PROPOSITION 4.4. *For the p-adic γ-ring A_0 ($p \neq 2$) the homomorphism $\rho_h : A_0 \to 1 + A_0$ is an isomorphism.*

The case $p = 2$. If the action of Γ is such that the elements $1, -1 \in \Gamma$ have the same action, then Γ acts via the quotient group $\Gamma' = \Gamma/\{\pm 1\}$. Since Γ' is monogenic for $p = 2$, (with generator the image of 3) it might be hoped that in this case the proof would go through for a 2-adic γ-ring. These hopes are shown to be ill-founded by considering the example at the end

of the proof of theorem 4.1 for $p = 2$. In this case, $\psi^{-1} =$ identity and so Γ acts via the quotient group Γ'. We see that $\rho_k(2) = k$, so we must choose $k \equiv 3 \bmod 4$ to ensure ρ_k is an isomorphism on $A(1)/A(2)$. But in this case $\rho_k(4) = k^2 \equiv 1 \bmod 8$ and ρ_k is not an isomorphism on $A(2)/A(3)$.

More subtle refinements will be needed for the prime 2, which correspond to considering real K-theory instead of complex K-theory. We give a brief outline of this situation. If A is any γ-ring, A is said to be *oriented* if $\gamma_t(a) = \gamma_{1-t}(a)$ for all $a \in A$. (Suppose, when we adjoin a unit to A in the standard way to give a special augmented λ-ring R, that R is finite dimensional. Then, in this case, A is oriented if, and only if, every finite-dimensional element $x \in R$ satisfies $\lambda^r(x) = \lambda^{n-r}(x)$ where $\dim x = n$.) Equating coefficients in $\sum \gamma^r(a)t^r = \sum \gamma^r(a)(1-t)^r$, we find

(1) $1 = 1 + \sum_{r \geq 1} \gamma^r$ i.e. $\gamma^1 = -\sum_{r \geq 2} \gamma^r$ and $A(1) = A(2)$

(2) $\gamma^1 = -\sum_{r \geq 1} r\gamma^r$

and in general,

(3) $\gamma^k = (-1)^k \gamma^k + (-1)^{k+1}(k+1)\gamma^{k+1} +$ higher terms in the γ-filtration.

From (3), we find $A(2n-1) = A(2n)$ for $n \geq 1$.
[Notice the significant difference between this and the filtration of the real case in [2]. There the filtration $F_1 \supset F_2 \supset \cdots \supset F_n \supset \cdots$ satisfies $F_n = F_{n+1}$ for $n \equiv 3, 5, 6, 7 \bmod 8$.]

We shall now require the 'completion' of the verification principle for those γ-rings which are complete in the γ-topology. In this case, a natural operation is a power series in the $\{\gamma^r\}$, and an identity holds if, and only if, it holds operating on a sum of elements of γ-dimension one. (c.f. I.4.5). If $a = x_1 + \cdots + x_n$ is a sum of elements of γ-dimension one,

$$-\frac{\sum_{r \geq 1} r\gamma^r(a)t^{r-1}}{1 + \sum_{r \geq 1} \gamma^r(a)t^r} = -\frac{d}{dt} \log \gamma_t(a) = -\sum_{s=1}^{n} \frac{x_s}{1 + x_s t}.$$

Put $t = 1$, then $\dfrac{-\sum r\gamma^r(a)}{1 + \sum \gamma^r(a)} = \sum \{(1 + x_s)^{-1} - 1\} = \psi^{-1}(\sum x_s) = \psi^{-1}(a)$. [$\psi^{-1}$ is defined because $-1 \in Z_p$].

From the identities (1), (2), $\psi^{-1} =$ identity, when acting on an orientable p-adic γ-ring. Thus the Γ-action on an orientable p-adic γ-ring factors through the quotient $\Gamma' = \Gamma/\{\pm 1\}$, which is monogenic for all primes, including 2.

For $p \neq 2$, define $\sigma_k(a) = \rho_k(\frac{1}{2}a) = (\rho_k(a))^{1/2}$ (2 has an inverse in Z_p),

For $p = 2$, define $\sigma_3(a) = \gamma_{\zeta/(\zeta-1)}(a)$ where $\zeta^2 + \zeta + 1 = 0$.

If A is an orientable 2-adic γ-ring, we have

$$\sigma_3(a) = \gamma_{\zeta/(\zeta-1)}(a) = \gamma_{(1-\zeta[\zeta-1]^{-1})}(a) = \gamma_{1/(1-\zeta)}(a) = \gamma_{\bar{\zeta}/(\bar{\zeta}-1)}(a)$$

where $\bar{\zeta} = 1/\zeta = -1 - \zeta$ is the conjugate of ζ under the Galois group of $Q_2(\zeta)$ over Q_2. From this it follows that $\sigma_3(a) \in 1 + A$: in fact if

$$\sigma_3(a) = 1 + \alpha + \beta\zeta \quad \alpha, \beta \in A,$$

then $$1 + \alpha + \beta\zeta = 1 + \alpha + \beta(-1 - \zeta)$$

and so $$\beta + 2\beta\zeta = 0$$

which implies $\beta = 0$ and hence $\sigma_3(a) \in 1 + A$.

We can now prove:

THEOREM 4.5. *If A is an orientable p-adic γ-ring, $\sigma_h : A \to 1 + A$ induces isomorphisms $(\sigma_h)^{\Gamma'}$ and $(\sigma_h)_{\Gamma'}$ on invariants and coinvariants where p is any prime and h is a generator of $\Gamma' = \Gamma/\{\pm 1\}$.*

The proof is by the same technique as in theorem 4.1. The only significant difference is the calculation at the end. The example of a p-adic γ-ring is $2pZ_p$, where

$$\gamma_t(-2p) = [(1 - t)^p - (-t)^p]^2.$$

Evidently this γ-ring is orientable.

For $p \neq 2$, $\sigma_h(2p) = h^{p-1}$. Multiplication by 2 is an isomorphism; since ρ_h is an isomorphism, so is σ_h.

For $p = 2$, $\sigma_3(4) = \left[\left(1 - \left(\frac{\zeta}{\zeta - 1}\right)\right)^2 - \left(-\frac{\zeta}{\zeta - 1}\right)^2\right]^{-2} = -3.$

We need only show σ_3 is an isomorphism, on $A(2n + 1)/A(2n)$ since $A(2n - 1) = A(2n)$ for any orientable γ-ring. But if $A = 4Z_2$, $A(2n - 1) = A(2n) = 4^n Z_2$. By induction on r,

$$(1 - 4)^{4^{r-1}} \equiv 1 + 4^r \bmod 4^{r+1}, \; r \geq 2,$$

giving $$\sigma_3(4) = 1 - 4,$$

$$\sigma_3(4^r) \equiv 1 + 4^r \bmod 4^{r+1}, \; r \geq 2,$$

and so σ_3 is an isomorphism on A(2n + 1)/A(2n).

§5. THE CONNECTION BETWEEN θ_k AND ρ_k

Recall that θ_k and ρ_k are defined by:

(1) $\theta_k(x) = \prod \lambda_{-u}(x)$ for finite dimensional x in a special λ-ring R.

(2) $\rho_k(a) = \prod \gamma_{u/(u-1)}(a)$ for $a \in A$ (p-adic γ-ring)

where in both cases the product is taken over all roots of $t^k - 1 = 0$ except 1.

Since $\lambda_s = \gamma_{s/(1+s)}$ (I.4.(2)), in some suitable formal setting θ_k and ρ_k will agree. In this section, we describe the required situation.

Let R be an augmented special λ-ring augmented by ε: $R \to \mathbf{Z}$, where $B = \ker \varepsilon$. We assume that R satisfies the following:

(3) R is finitely generated as an abelian group by $x_1 = 1, x_2, \ldots, x_m$ which are finite-dimensional.

(4) $\varepsilon(x_r) = \dim x_r$ for $r = 1, \ldots, m$.

(5) The γ-topology on B is finer than the p-adic topology.

Note that (4) implies $\varepsilon(x) = \dim x$ when x is finite dimensional. In this case

$$\gamma_t(x - \varepsilon(x)) = \lambda_{t/(1-t)}(x)\left(1 + \frac{t}{1-t}\right)^{-\varepsilon(x)} = \lambda_{t/(1-t)}(x)(1-t)^{\varepsilon(x)}.$$

This shows $\gamma_t(x - \varepsilon(x))$ is a polynomial in t of degree $\leq \dim x$, and so the γ-dimension of $x - \varepsilon(x)$ is not greater than the dimension of x. Since B is generated as an abelian group by $x_2 - \varepsilon(x_2), \ldots, x_m - \varepsilon(x_m)$, it is straightforward to see that the B-adic topology coincides with the γ-topology on B. Of greater importance is the fact that $A = Z_p \otimes B$ is a p-adic γ-ring (by (5) above).

PROPOSITION 5.1. *Let $i : R \to Z_p \otimes R$ be the canonical map and $(k, p) = 1$, then for finite dimensional $x \in R$, $i(\theta_R(x))$ has a multiplicative inverse in $Z_p \otimes R$.*

Proof. If $\dim x = n$, since θ_k is of degree k, $\varepsilon(\theta_k(x)) = k^n$ (1.7.2). Since k is prime to p, we need only show that $\varepsilon(z) = r$, $(r, p) = 1$ implies $i(z)$ has a multiplicative inverse in $Z_p \otimes R$. Certainly $r^{-1} \in Z_p \otimes R$ and $r^{-1}\, i(z) = 1 + a$ where $a \in A = Z_p \otimes B$. Since $1 + A \subset Z_p \otimes R$ is a multiplicative subgroup, $1 + a$ has a multiplicative inverse $1 + a' \in 1 + A$ and $i(z)$ has inverse $r^{-1}(1 + a') \in Z_p \otimes R$.

We may now extend the domain of θ_k to give a map $\theta_k : R \to Z_p \otimes R$ homomorphic from addition to multiplication by

(6) $\theta_k(x - y) = i\,\theta_k(x)[i\,\theta_k(y)]^{-1}$ for x, y finite dimensional.

If $\varepsilon' : Z_p \otimes R \to Z_p$ is induced by the augmentation $\varepsilon : R \to Z$, we see

$$\varepsilon'(\theta_k(x - y)) = k^{\varepsilon(x) - \varepsilon(y)}$$

So if $\varepsilon(x) = \varepsilon(y)$, then $\theta_k(x - y) \in 1 + Z_p \otimes B = 1 + A$. This implies:

PROPOSITION 5.2. *θ_k induces a homomorphism $\theta_k : B \to 1 + A$ where $A = Z_p \otimes B$.*

PROPOSITION 5.3. *The following diagram commutes*:

$$\begin{array}{ccc} B & & \\ \downarrow{\scriptstyle i} & \searrow{\scriptstyle \theta_k} & \\ A & \xrightarrow{\rho_k} & 1 + A \end{array}$$

Proof. It is sufficient to show $\theta_k(x - n) = \rho_k\, i(x - n)$ where $\dim x = n$, i.e.

$$\theta_k(x) = \rho_k\, i(x - n)\theta_k(n) \text{ in } Z_p \otimes R.$$

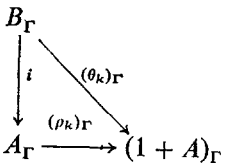
From (1), (2) we need only show

$$\lambda_{-u}(x) = \gamma_{u/(1-u)}(x - n)(1 - u)^n \text{ in } Z_p[\zeta] \otimes_{Z_p} (Z_p \otimes R)$$

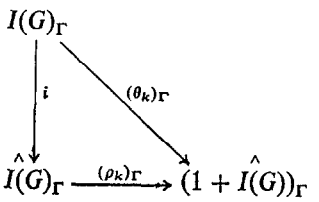
where ζ is a primitive kth root of unity. But $\gamma_t(x - n)$ is a polynomial in t of degree $\leq n$, and $\lambda_s = \gamma_{s/(1+s)}$. The result follows.

Since ρ_k, θ_k commute with the Adams operations, if Γ acts through a finite quotient as in proposition 2.3 then ρ_k, θ_k induce a commutative diagram of coinvariants:

COROLLARY 5.4. *If the Adams operations are periodic on B, i.e.* $\psi^{N+r} = \psi^r$ *for some* $N = p^e$, *then the following diagram commutes*:



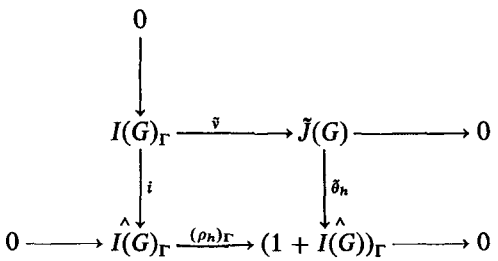
COROLLARY 5.5. *If G is a p-group and* $(k, p) = 1$



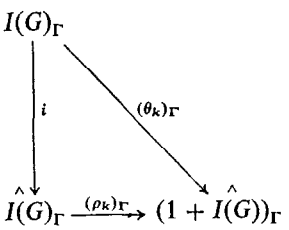
commutes.

IV. THE MAIN THEOREM

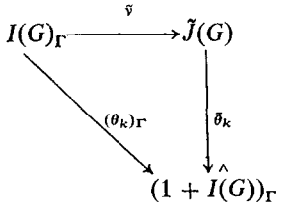
The theorem is a direct consequence of the following commutative diagram for a p-group ($p \neq 2$):



The crucial factor is the definition of $\tilde{\theta}_h$. In the light of the commutative diagram of III.5.5



it is sufficient to show $(\theta_k)_\Gamma$ factors through J(G) i.e. that there is a commutative diagram:



This needs some topological facts from equivariant K-theory which are recalled in §1. The main commutative diagram is set up in §2 and the required conclusions are drawn.

§1. EQUIVARIANT K-THEORY

In this section we recall some basic facts of equivariant K-theory from [5], [19] which will be instrumental in showing $(\theta_k)_\Gamma$ factors through $\tilde{J}(G)$ for $(k, p) = 1$ where G is any p-group. The main result of this section is that if $\phi\colon S(E) \to S(F)$ is a G-map of degree r, then there is an element $z \in R(G)$ of augmentation r such that $\theta_k(F) \cdot z = \theta_k(E) \cdot \psi^k(z)$.

Note that the results of this section are true for any finite group. Throughout §1, the symbol G will refer to an arbitrary finite group.

Let E be a unitary representation space of G, E^+ its one-point compactification. Then using K_G-theory with compact supports we can introduce $K_G(E) = K_G(E^+, +)$. It is a module over $K_G(\text{point}) = R(G)$. The main theorem of the subject, as proved in [5; (4.3)] asserts that $K_G(E)$ in a free module over $R(G)$ with a canonical generator† μ_E. Moreover the method of proof in [5] shows also that the map

$$j^* = K_G(E) \to K_G(P(E \oplus 1))$$

is injective, where $P(E \oplus 1)$ is the projective space associated to $E \oplus 1$ (1 denoting the trivial representation $\mathbf{C}$) and j^* is induced by the open inclusion $j\colon E \to P(E \oplus 1)$ given by $j(u) = (u, 1)$. If h is the class of the standard line bundle H over $P(E \oplus 1)$, the image of μ_E is [5; (4.1)]

$$j^*(\mu_E) = \sum(-1)^r h^r \lambda^r(E).$$

If $i\colon P(E) \to P(E \oplus 1)$ is the natural inclusion then $i^*j^* = 0$ and so

$$\sum(-1)^r (i^*(h))^r \lambda^r(E) = 0 \text{ in } K_G(P(E)).$$

Replacing E by $E \oplus 1$ we deduce the equation

$$\sum(-1)^r h^r \lambda^r(E \oplus 1) = 0$$

or equivalently

$$(1 - h) \sum(-1)^r h^r \lambda^r(E) = 0. \tag{1}$$

From these facts it follows that we can identify $K_G(E)$ (as λ-ring) with the $R(G)$-module denoted in I.7.3 by $R(G)_E$. To see this we map an indeterminate ξ to $h^{-1} \in K_G(P(E \oplus 1))$. This induces a homomorphism

$$\alpha\colon R(G)_E \to j^* K_G(E)$$

in which

$$\begin{aligned} \alpha\left(\sum(-1)^r \eta^{n-r} \lambda^r(E)\right) &= h^{-n}\left(\sum(-1)^r h^r \lambda^r(E)\right) \\ &= \sum(-1)^r h^r \lambda^r(E) \qquad \text{by (1)} \\ &= j^*(\mu_E). \end{aligned}$$

† In [5] this is denoted by λ_E

Here (as in I.7.3) η is the image of ξ in $R(G)_E$. Now in I.7.3 we saw that $R(G)_E$ was a free $R(G)$-module generated by the element $\sum(-1)^r\eta^{n-r}\lambda^r(E)$. Since $j^*K_G(E)$ is freely generated by $j^*(\mu_E)$ it follows that α, and hence also $(j^*)^{-1}\alpha$, is an isomorphism. We shall therefore identify $K_G(E)$ with $R(G)_E$ by this isomorphism which takes the generator μ_E into the element $\sum(-1)^r\eta^{n-r}\lambda^r(E)$, already denoted by μ_E in I.7.3.

From (I.7.3) we can now read off the action of ψ^k on $K_G(E)$. Namely we have

(2) $\psi^k(z\mu_E) = \psi^k(z) \cdot \psi^k(\mu_E)$

(3) $\psi^k(\mu_E) = \theta_k(E) \cdot \mu_E$

Remark. The structure of $K_G(P(E))$ is also known—it can be easily deduced from the main theorem of [5] by various methods—and one has

$$K_G(E) \cong R(G)[\xi]/I(E)$$

where $I(E)$ is the ideal introduced in I.7.3. For our purpose however this is not really needed.

Suppose now that E, F are two unitary representations of G, and suppose we have a G-map $\phi\colon S(E) \to S(F)$. Extending radially to the balls ϕ induces the suspension $\Omega\phi$: $B(E)/S(E) \to B(F)/S(F)$. Since we have the obvious identification of $B(E)/S(E)$ with E^+ we obtain an $R(G)$-homomorphism of λ-rings

$$\phi^! : K_G(F) \to K_G(E).$$

Since these are free modules there is a unique $z \in R(G)$ such that

$$\phi^!(\mu_E) = z\mu_E$$

Applying ψ^k to this and using the formula (1) for E and also for F we obtain

$$\theta_k(F)z\mu_E = \theta_k(E)\psi^k(z)\mu_E.$$

Since μ_E is a free generator we deduce

$$\theta_k(F)z = \theta_k(E)\psi^k(z)$$

It remains to show that $\varepsilon(z) = \deg \phi$. This is straightforward.

The inclusion of the trivial group 1 in G induces the maps

$$f_E\colon K_G(E) \to K(E), \quad \varepsilon\colon R(G) \to R(1) = \mathbf{Z}$$

which forget the G-structure. Since $K_G(E)$ is an $R(G)$-module in a natural way, f_E is equivariant, that is to say

(4) $$f_E(z\mu_E) = \varepsilon(z)\mu_E \qquad z \in R(G).$$

$\phi\colon S(E) \to S(F)$ induces $\phi^!\colon K(F) \to K(E)$ where by II.1.1.,

(5) $$\phi^!(\mu_F) = \deg \phi \cdot \mu_E.$$

By naturality, the diagram

$$\begin{array}{ccc} K_G(F) & \xrightarrow{\phi^!} & K_G(E) \\ \downarrow{\scriptstyle f_F} & & \downarrow{\scriptstyle f_E} \\ K(F) & \xrightarrow{\phi^!} & K(E) \end{array} \quad \text{commutes.}$$

From (4), (5), $\deg \phi = \varepsilon(z)$.

We summarize the results above in:

PROPOSITION 1.1. *Let E, F be unitary representations of a finite group G. If $\phi\colon S(E) \to S(F)$ is a G-map of degree r, there is an element $z \in R(G)$ of augmentation r such that*

$$\theta_k(F)z = \theta_k(E)\psi^k(z).$$

§2. THE COMMUTATIVE DIAGRAM

From proposition 1.1, given a G-map $\phi\colon S(E) \to S(F)$ of degree r, there is an element $z \in R(G)$ such that $\varepsilon(z) = r$ and $\theta_k(F)z = \theta_k(E)\psi^k(z)$. Suppose G is a p-group. If r is prime to p, since $\varepsilon(\psi^k(z)) = r$, $r^{-1}z$ and $\psi^k(r^{-1}z) \in 1 + \widehat{I(G)} \subset Z_p \otimes R(G)$. For $(k, p) = 1$, we then have, in $1 + \widehat{I(G)}$, the equation:

$$\theta_k([E] - [F]) = r^{-1}z.\,[\psi^k(r^{-1}z)]^{-1}. \tag{1}$$

If $c\colon 1 + \widehat{I(G)} \to (1 + \widehat{I(G)})_\Gamma$ is the canonical map, we therefore have:

LEMMA 2.1. Let G be a p-group and k be prime to p. If there is a G-map $\phi\colon S(E) \to S(F)$ of degree prime to p, then $[E] - [F]$ is in the kernel of the map $c\theta_k\colon I(G) \to (1 + \widehat{I(G)})_\Gamma$.

COROLLARY 2.2. *If E and F are J-equivalent, then $[E] - [F]$ is in the kernel of $c\theta_k$.*

Recalling that $T(G)$ is the subgroup of $I(G)$ consisting of elements $[E] - [F]$ where E, F are J-equivalent, we have:

COROLLARY 2.3. $T(G) \subset \ker c\theta_k$.

But the following diagram commutes:

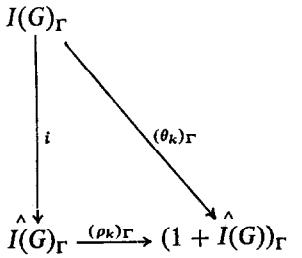
$$\begin{array}{ccc} I(G) & \xrightarrow{\theta_k} & 1 + \hat{I}(G) \\ \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle c} \\ I(G)_\Gamma & \xrightarrow{(\theta_k)_\Gamma} & (1 + \hat{I}(G))_\Gamma \end{array}$$

where d is the canonical map. So $T(G) \subset \ker (\theta_k)_\Gamma d$. Immediately we see that $(\theta_k)_\Gamma$ factors through the group $\tilde{J}(G) = I(G)/T(G)$. Define $\tilde{\theta}_k\colon \tilde{J}(G) \to (1 + \widehat{I(G)})_\Gamma$ by $\tilde{\theta}_k(x + T(G)) = \theta_k(x)$ for $x \in I(G)$ then:

PROPOSITION 2.4. *The following diagram commutes for a p-group G, where $\tilde{v}$ is the canonical epimorphism of* II.4.3.

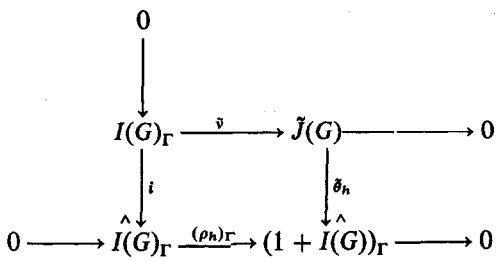
$$\begin{array}{ccc} I(G)_\Gamma & \xrightarrow{\tilde{v}} & \tilde{J}(G) \\ & {\scriptstyle (\theta_k)_\Gamma} \searrow & \downarrow{\scriptstyle \tilde{\theta}_k} \\ & & (1 + \hat{I}(G))_\Gamma \end{array}$$

From III.5.5 we have the commuting diagram:



Fitting together diagrams (3), (4) with $k = h$ (the generator of Γ), we find the required diagram for a p-group of odd order:

(5)



We remember $(\rho_h)_\Gamma$ is an isomorphism by III.4.3, $\tilde{v}$ is an epimorphism by II.4.3, and i is the canonical map from a finitely generated free abelian group (by II.3.3) to its p-adic completion and is therefore a monomorphism.

LEMMA 2.5. *For a p-group of odd-order, $\tilde{v}$ is an isomorphism.*

Proof. Chase an element round diagram (5).

Recalling $R(G)_\Gamma = \mathbf{Z} + I(G)_\Gamma$, $J(G) = \mathbf{Z} + \tilde{J}(G)$ and v: $R(G)_\Gamma \to J(G)$ is given by $v(n + a) = n + \tilde{v}(a)$ for $a \in I(G)_\Gamma$, lemma 2.5 implies the main theorem, (since $I(G)_\Gamma = I(G)_{\Gamma_N}$ by III.2.4, where $N = |G|$).

THEOREM 2.6. *For a p-group of odd order $N = p^e$, v: $R(G)_{\Gamma_N} \to J(G)$ is an isomorphism.*

We recall the remark of II, §4. This expresses the possibility that $E \oplus C$, $F \oplus C$ may be J-equivalent whereas E, F are not. By 2.6, if G is a p-group of odd order, if $[E] - [F] = [E \oplus C] - [F \oplus C] \in T(G)$, then $[E] - [F] \in W(G)$. For conjugate representations, we may construct J-equivalences using II.4.2 and so E, F are truly J-equivalent.

Returning to the diagram (5), trivial diagram chasing shows:

PROPOSITION 2.7. $\tilde{\theta}_h : \tilde{J}(G) \to \widehat{(1 + I(G))_\Gamma}$ *is a monomorphism.*

From lemma 2.1, if there is a G-map ϕ: $S(E) \to S(F)$ of degree prime to p, then the image of $[E] - [F]$ in $\tilde{J}(G)$ is in the kernel of $\tilde{\theta}_h$. Proposition 2.7 gives:

THEOREM 2.8. *If there is a G-map ϕ: $S(E) \to S(F)$ of degree prime to p, then E and F are J-equivalent.*

We now consider $W(G)$, the subgroup of $I(G)$ generated by $\{x - \alpha x\}$, $x \in R(G)$, $\alpha \in \Gamma$. If E is irreducible, it is trivial to see that αE is irreducible also. Thus if $\xi_1, \ldots, \xi_m$ are the

classes of irreducible representations, they split up into equivalence classes where each class consists of an irreducible representation ξ_r and the elements of the form $\alpha\xi_r$ for $\alpha \in \Gamma$.

As in chapter II §3, we order the classes or irreducible representations $\xi_1, \ldots, \xi_s, \ldots, \xi_m$ so that no two of $\xi_1, \ldots, \xi_s$ are conjugate under the action of Γ and their conjugates exhaust $\xi_1, \ldots, \xi_m$. Then $W(G)$ is generated by elements of the form $\{\xi_i - \alpha\xi_i\}$ where $1 \leq i \leq s$, $\alpha \in \Gamma$.

Suppose now E, F are J-equivalent and E is irreducible. We may suppose $[E] = \xi_s$. If $[F] = \sum_{i=1}^{m} n_i \xi_i$, then $[E] - [F] = \xi - \sum n_i \xi_i \in W(G)$ by theorem 2.6. Since $n_i \geq 0$ for $i = 1, \ldots, m$, immediately we see that $[F] = \alpha\xi_s$ for some $\alpha \in \Gamma$. Putting this fact together with theorem 2.8 and II.4.2 (which says conjugate representations of a p-group are J-equivalent) we have:

THEOREM 2.9. *If G is a p-group of odd order and E is an irreducible unitary representation of G, then for any unitary representation F, there is a G-map ϕ: $S(E) \to S(F)$ of degree prime to p if, and only if, $F = \alpha E$ for some $\alpha \in \Gamma$.*

V. THE REAL CASE

If U, V are orthogonal representations of G, as in the complex case, we may formulate the problem of when there may exist a G-map of unit spheres $S(U) \to S(V)$.

For a finite group G of odd order, it is easy to show that a non-trivial irreducible orthogonal representation space is the underlying real space (having real dimension equal to twice the complex dimension) of an irreducible unitary representation. Using the isomorphism $R(G)_{\Gamma_N} \cong J(G)$, of IV. 2.7, it is a straightforward matter to deduce the theorem:

'*Given two orthogonal representations U, V of a p-group G ($p \neq 2$), where U is irreducible, then there exists a G-map θ: $S(U) \to S(V)$ of degree prime to p if, and only if, U is conjugate to V, that is to say if, and only if $U = \psi^k(V)$ for k prime to p.*'

It will be the purpose of Part V to prove this theorem.

§1. REAL REPRESENTATIONS

Let G be a finite group of order N; let Γ_N be the Galois group of $\mathbf{Q}(\omega)$ over $\mathbf{Q}$ where ω is a primitive Nth root of unity. Denote by $RO(G)$ the real (orthogonal) representation ring of G. This is the free additive abelian group generated by the isomorphism classes of irreducible real (or orthogonal) representations. As in the case of the complex representation ring $R(G)$, the tensor product makes $RO(G)$ into a ring and exterior powers give $RO(G)$ a λ-ring structure.

If U is a real representation space, then $U \otimes \mathbf{C}$ is a complex representation space. This construction gives rise to a λ-homomorphism c: $RO(G) \to R(G)$ which is easily seen to be a monomorphism, and so $RO(G)$ is a special λ-ring. The Adams operations are defined on $RO(G)$ and commute with the map c since it is a λ-homomorphism. If χ is the character of the real representation U, then the complexification $cU = U \otimes \mathbf{C}$ has the same character.

This shows that the character $\psi^k(\chi)$ of the representation $\psi^k(U)$ is given by $[\psi^k(\chi)](g) = \chi(g^k)$ as in II.3.1., and in particular the Adams operators are periodic in k on $RO(G)$, in that $\psi^{N+k} = \psi^k$. For this reason, $RO(G)$ is a Γ_N-module where the action of k, for $(k, N) = 1$, is given by $x \mapsto \psi^k(x)$.

The periodicity of the Adams operations makes it possible to define ψ^k acting on $R(G)$ and $RO(G)$ for any integer k, possibly negative. In particular, if U is real, $\psi^{-1}(U) = U$ and if E is complex, $\psi^{-1}(E) = \bar{E}$, the complex conjugate of E.

Note that this implies the action of the subgroup $\{\pm 1\}$ of Γ_N is trivial on $RO(G)$ and so the Γ_N-action is via the quotient group $\Gamma_N' = \Gamma_N/\{\pm 1\}$.

Define $r: R(G) \to RO(G)$ to be induced by taking the underlying real structure of a complex representation. If E is an n-dimensional complex representation, then rE has (real) dimension $2n$. If we choose a basis in E so that the action of $g \in G$ is given by an $n \times n$ complex (unitary) matrix $\rho(g)$, writing $\rho(g) = X(g) + iY(g)$ where $X(g)$, $Y(g)$ are real $n \times n$ matrices, then the action of g on rE has a $2n \times 2n$ real (orthogonal) matrix

$$\begin{pmatrix} X(g) & Y(g) \\ -Y(g) & X(g) \end{pmatrix}$$

Evidently r is an additive homomorphism, but it cannot preserve multiplication since it doubles the dimension.

By considering characters, we see

$$rc\colon\ RO(G) \to RO(G) \text{ is multiplication by } 2$$

and

$$cr\colon\ R(G) \to R(G) \text{ takes } x \mapsto x + \bar{x},$$

which may be written formally as $cr = \psi^1 + \psi^{-1}$.

Since c is a λ-homomorphism, we have $\psi^k c = c\psi^k$. The map r is certainly not a λ-homomorphism, nevertheless we still have $r\psi^k = \psi^k r$, for any integer k. To prove this, since c is a monomorphism, we need only show $cr\,\psi^k = c\,\psi^k r$. But $(cr)\psi^k = \psi^1\psi^k + \psi^{-1}\psi^k = \psi^k + \psi^{-k}$ and $c\psi^k r = \psi^k cr = \psi^k(\psi^1 + \psi^{-1}) = \psi^k + \psi^{-k}$.

Hence we have:

PROPOSITION 1.1. *$r\colon R(G) \to RO(G)$ and $c\colon RO(G) \to R(G)$ are (additive group) homomorphisms commuting with Γ_N-action.*

PROPOSITION 1.2 ([18] §222). *The only irreducible complex representation of a group of odd order whose character takes real values is the trivial representation.*

Proof. See [10] page 223, or [18] page 294.

From proposition 1.2 we may immediately deduce:

PROPOSITION 1.3. *A real irreducible representation of a group of odd order is either the identity or the underlying real structure of an irreducible complex representation.*

§2. J-EQUIVALENCE OF ORTHOGONAL REPRESENTATIONS

Let U be an orthogonal representation of G, then $S(U)$ is defined to be the unit sphere of U (vectors of length 1). $S(U)$ is a G-space.

Definition 2.1. If U, V are orthogonal representations of G, then U, V are said to be J-equivalent if there exist G-maps $\theta: S(U) \to S(V)$, $\phi: S(V) \to S(U)$ of degree prime to the order of G.

Write $U \sim V$ if U, V are J-equivalent.

PROPOSITION 2.2. *J-equivalence is an equivalence relation on the set of isomorphism classes of orthogonal representations such that $U_1 \sim V_1$, $U_2 \sim V_2$ implies $U_1 \oplus U_2 \sim V_1 \oplus V_2$.*

Proof. Analogous to II.2.5.

PROPOSITION 2.3. *If U, V are orthogonal representations of G, then $U \sim V \Rightarrow cU \sim cV$ and if E, F are complex representations, then $E \sim F \Rightarrow rE \sim rF$.*

Proof. $S(cU)$ is homeomorphic to $S(U) \circ S(U)$ as a G-space, and given a G-map $\theta : S(U) \to S(V)$ of degree k, then $\theta \circ \theta: S(U) \circ S(U) \to S(V) \circ S(V)$is a G map of degree k^2. This proves the first part.

$S(rE)$ is the same G-space as $S(E)$ and so the second part is trivial.

Define $TO(G)$ to be the additive subgroups of $RO(G)$ consisting of elements of the form $[U] - [V]$ where U, V are J-equivalent and $[U]$ denotes the isomorphism class of U. Let $JO(G) = RO(G)/TO(G)$ and then, using the notation of II §2, we have:

COROLLARY 2.4. $r(T(G)) \subset TO(G)$
$c(TO(G)) \subset T(G)$

COROLLARY 2.5. *The following diagrams commute, where the vertical maps are the canonical epimorphisms:*

$$\begin{array}{ccc} R(G) & \xrightarrow{r} & RO(G) \\ \downarrow & & \downarrow \\ J(G) & \xrightarrow{r} & JO(G) \end{array} \qquad \begin{array}{ccc} RO(G) & \xrightarrow{c} & R(G) \\ \downarrow & & \downarrow \\ JO(G) & \xrightarrow{c} & J(G) \end{array}$$

We already know for an M-group G of order N, that if E is a unitary representation of G and $(k, N) = 1$, then $E \sim \psi^k(E)$. (II.4.2). For an M-group of odd order, propositions 1.1, 1.3 and 2.3 imply that if U is a non-trivial irreducible real representation, then $U \sim \psi^k(U)$. Also the trivial real representation $\underline{1}$ satisfies $\psi^k(\underline{1}) = \underline{1}$ and the identity map from $S(\underline{1})$ to $S(\underline{1})$ demonstrates $\underline{1} \sim \psi^k(\underline{1})$. Since J-equivalence is additive, we see the canonical map $RO(G) \to JO(G)$ factors through $RO(G)_{\Gamma_N}$. Let μ: $RO(G)_{\Gamma_N} \to JO(G)$ be the induced map, then we immediately deduce:

PROPOSITION 2.6. *For an M-group G of odd order, the following diagrams commute:*

$$\begin{array}{ccc} R(G)_{\Gamma_N} & \xrightarrow{r^{\Gamma_N}} & RO(G)_{\Gamma_N} \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle \mu} \\ J(G) & \xrightarrow{r} & JO(G) \end{array} \qquad \begin{array}{ccc} RO(G)_{\Gamma_N} & \xrightarrow{c^{\Gamma_N}} & R(G)_{\Gamma_N} \\ \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle v} \\ JO(G) & \xrightarrow{c} & J(G) \end{array}$$

THEOREM 2.7. *For a p-group G ($p \neq 2$)*

$$\mu\colon RO(G)_{\Gamma_N} \to JO(G) \text{ is an isomorphism.}$$

Proof. Since $RO(G) \to JO(G)$ is epimorphic, so is

$$\mu\colon RO(G)_{\Gamma_N} \to JO(G).$$

For a p-group G ($p \neq 2$), we have $v\colon R(G)_{\Gamma_N} \to J(G)$ is an isomorphism (IV.2.6).

Suppose $\mu(x) = 0$ for $x \in RO(G)_{\Gamma_N}$, then

$$c\mu(x) = 0, \text{ i.e. } vc_{\Gamma_N}(x) = 0.$$

This gives $c_{\Gamma_N}(x) = 0$ since v is an isomorphism. Hence $r_{\Gamma_N} c_{\Gamma_N}(x) = 0$, i.e. $(rc)_{\Gamma_N}(x) = 0$. But $(rc)_{\Gamma_N}(x) = 2x = 0$, and since there is no torsion in $RO(G)_{\Gamma_N}$, we have $x = 0$ and μ is monomorphic. This shows $\mu\colon RO(G)_{\Gamma_N} \to JO(G)$ is an isomorphism.

The remark during the proof of theorem 2.7 '$RO(G)_{\Gamma_N}$ has no torsion' is a direct result of the fact that Γ_N permutes the (free) generators of $RO(G)$, which are, of course, the classes of the irreducible real representations. As in the complex case, we may use this fact to deduce from theorem 2.7 the following theorem (proof analogous to IV. 2.9):

THEOREM 2.8. *If G is a p-group of odd order and U is an irreducible orthogonal representation of G, then for any orthogonal representation V, there is a G-map $\phi\colon S(U) \to S(V)$ of degree prime to p if, and only if, U and V are conjugate, that is to say if, and only if, $U = \psi^k(V)$ for $(k, p) = 1$.*

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