THOM COMPLEXES

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Introduction

THE spaces which form the title of this paper were introduced by Thom in (**16**) as a tool in his study of differentiable manifolds. In addition certain special Thom complexes have been studied by James (**10**) in connexion with Stiefel manifolds (cf. (**8**), (**9**)). The purpose of this paper is to prove a number of general results on Thom complexes, and to deduce the main theorems of James (**9**), (**10**) as immediate consequences. Our main result (3.3) is a duality theorem (in the Whitehead–Spanier S-theory) for Thom complexes over differentiable manifolds.† Besides its application this is a result of some independent interest, since it provides a satisfactory place for manifolds in S-theory.

In § 1 we introduce, for a finite CW-complex X, a finite group $J(X)$: the group of orthogonal sphere bundles over X under 'stable fibre homotopy equivalence'. If X is a sphere then $J(X)$ is just the image of the stable 'J-homomorphism' in the appropriate dimension. A general study of $J(X)$ by the methods of (**2**) will be given in a future publication.

In § 2 we consider the S-type of Thom complexes over X and examine the relation with the group $J(X)$ of § 1.

The main duality theorem is established in § 3. As an application we prove a result which was conjectured in (**11**): *the stable fibre homotopy type of the tangent sphere bundle of a differentiable manifold X depends only on the homotopy type of X.*

The stunted projective spaces introduced by James in (**10**) are studied in § 4, and their identification as Thom complexes is established. In § 5 corresponding results are proved for the quasi-projective spaces of (**10**).

Applying the general results of §§ 1–3 to the spaces of §§ 4, 5 we deduce in § 6 the main theorems of James‡ (**9**), (**10**).

1. Stable fibre homotopy type of sphere bundles

Let X be a finite CW-complex. If α is a real vector bundle over X, we can give α an essentially unique orthogonal structure. Let (α) denote the associated sphere bundle. If n is a positive integer we shall also interpret

† This has also been proved by R. Bott and A. Shapiro (unpublished).

‡ In one respect our results are slightly weaker than those of James (cf. § 6).

it as the trivial bundle $X \times \mathbf{R}^n$. The Whitney sum (or direct sum) of vector bundles α, β will be denoted by $\alpha \oplus \beta$.

We recall now the definition of *fibre homotopy equivalence.* Two fibre bundles E, E' over X with fibre F are said to be of the same *fibre homotopy type* ($E \sim E'$) if there exist fibre-preserving maps:

$$f\colon E \to E', \qquad f'\colon E' \to E,$$

and fibre-preserving homotopies:

$$h\colon E \times I \to E, \qquad h'\colon E' \times I \to E',$$

with

$$h \mid E \times 0 = f'f, \qquad h \mid E \times 1 = \text{identity},$$

$$h' \mid E' \times 0 = ff', \qquad h' \mid E' \times 1 = \text{identity}.$$

We now define two sphere bundles (α) and (α') to have the same *stable* fibre homotopy type ($(\alpha) \overset{S}{\sim} (\alpha')$) if there exist integers n, n' so that

$$(\alpha \oplus n) \sim (\alpha' \oplus n').$$

The set of equivalence classes of orthogonal sphere bundles with respect to stable fibre homotopy type will be denoted by $J(X)$. The class of (α) will be denoted by $J(\alpha)$.

LEMMA (1.1). *Let α be a real vector bundle over X. Then there exists a real vector bundle β with $\alpha \oplus \beta$ trivial.*

Proof. Let $G_{n,m}$ denote the Grassmannian of n-dimensional subspaces of $\mathbf{R}^{n+m}$. Over $G_{n,m}$ there are two real vector bundles a, b of dimensions n, m respectively and $a \oplus b$ is the trivial bundle $G_{n,m} \times \mathbf{R}^{n+m}$. If $m > \dim X$ there exists a map $f\colon X \to G_{n,m}$ with $\alpha \cong f^*a$ (14). Define $\beta = f^*b$, and we have $\alpha \oplus \beta \cong f^*(a \oplus b)$ is trivial as required.

LEMMA (1.2). *The direct sum of vector bundles induces on $J(X)$ the structure of an abelian group.*

Proof. Since the direct sum of vector bundles is commutative and associative (up to isomorphism) it is sufficient to show that $J(\alpha \oplus \beta)$ depends only on $J(\alpha)$ and $J(\beta)$. The existence of inverses will follow from (1.1). Thus we have to show:

$$\text{(i)} \qquad (\alpha) \overset{S}{\sim} (\alpha') \Rightarrow (\alpha \oplus \beta) \overset{S}{\sim} (\alpha' \oplus \beta).$$

Replacing α by $\alpha \oplus n$, α' by $\alpha' \oplus n'$ it will be sufficient finally to check:

$$\text{(ii)} \qquad (\alpha) \sim (\alpha') \Rightarrow (\alpha \oplus \beta) \sim (\alpha' \oplus \beta).$$

Let f, f', h, h' be the maps and homotopies giving the equivalence $(\alpha) \sim (\alpha')$. We define maps

$$F\colon (\alpha \oplus \beta) \to (\alpha' \oplus \beta), \qquad F'\colon (\alpha' \oplus \beta) \to (\alpha \oplus \beta)$$

by

$$F(u\cos\theta \oplus v\sin\theta) = f(u)\cos\theta \oplus v\sin\theta$$

$$(u \in (\alpha), \quad v \in (\beta), \quad 0 \leqslant \theta \leqslant 2\pi),$$

$$F'(u'\cos\theta \oplus v\sin\theta) = f'(u')\cos\theta \oplus v\sin\theta$$

$$(u' \in (\alpha'), \quad v \in (\beta), \quad 0 \leqslant \theta \leqslant 2\pi),$$

and homotopies:

$$H\colon (\alpha \oplus \beta)\times I \to (\alpha \oplus \beta), \qquad H'\colon (\alpha' \oplus \beta)\times I \to (\alpha' \oplus \beta)$$

by

$$H(u\cos\theta \oplus v\sin\theta, t) = h(u,t)\cos\theta \oplus v\sin\theta \quad (t \in I),$$

$$H'(u'\cos\theta \oplus v\sin\theta, t) = h'(u',t)\cos\theta \oplus v\sin\theta.$$

Then F, F', H, H' are all fibre-preserving, and we have

$$H \mid (\alpha \oplus \beta)\times 0 = F'F, \qquad H \mid (\alpha \oplus \beta)\times 1 = \text{identity},$$

$$H' \mid (\alpha' \oplus \beta)\times 0 = FF', \qquad H' \mid (\alpha' \oplus \beta)\times 1 = \text{identity}.$$

This completes the proof.

Let $KO(X)$ be the 'Grothendieck group' defined as in (**2**) from real vector bundles over X. From the universal property of $KO(X)$ it follows that there is a natural epimorphism $KO(X) \to J(X)$, and this will also be denoted by J.

Let S^n denote the standard n-sphere, and let H_n denote the space of homotopy equivalences of S^{n-1} (with the C–O topology). If O_n denotes as usual the group of orthogonal transformations of S^{n-1}, we have a natural inclusion map $O_n \to H_n$. According to Dold and Lashof (**6**) the H-space H_n has a 'classifying space' BH_n and the map $O_n \to H_n$ induces a map $i_n\colon BO_n \to BH_n$, where BO_n is the usual classifying space of the topological group O_n. The map i_n induces a map

$$\psi_n\colon [X, BO_n] \to [X, BH_n],$$

where $[X, Y]$ denotes the set of homotopy classes of maps $X \to Y$. The main result of (**6**) then asserts that the set of fibre homotopy types of orthogonal S^{n-1}-bundles over X is in one-to-one correspondence with the image of ψ_n.

Now, under suspension, we have natural inclusion maps $O_n \to O_{n+1}$, $H_n \to H_{n+1}$, and these give rise to natural maps:

$$\operatorname{im}\psi_n \to \operatorname{im}\psi_{n+1}.$$

From our definition of $J(X)$, and the result of (**6**) mentioned above it follows that

$$J(X) = \lim_{n\to\infty} \operatorname{im}\psi_n.$$

For completeness we shall now prove the following well-known result.

LEMMA (1.3). $\pi_r(BH_n) \cong \pi_{r-1}(H_n) \cong \pi_{n+r-2}(S^{n-1})$ *for* $2 \leqslant r \leqslant n-2$, *and* $\pi_1(BH_n) \cong \pi_0(H_n) \cong \mathbf{Z}_2$ $(n \geqslant 1)$, *where* $\pi_0(H_n)$ *is the group of components of* H_n.

Proof. The isomorphism $\pi_r(BH_n) \cong \pi_{r-1}(H_n)$, $r \geqslant 1$, follows from the existence of the universal H_n-bundle over BH_n with contractible total space (**6**). Now it is clear that H_n has two components H_n^+, H_n^- representing maps $S^{n-1} \to S^{n-1}$ of degree $+1$, -1 respectively. Hence $\pi_0(H_n) \cong \mathbf{Z}_2$ as asserted. Consider now the space F_n of maps $(S^{n-1}, a) \to (S^{n-1}, a)$, where a is the standard base point. Let F_n^d denote the component representing maps of degree d. Then we have a natural inclusion map $F_n^1 \to H_n^+$. Moreover if $f \in H_n^+$ and if we define $\pi\colon H_n^+ \to S^{n-1}$ by $\pi(f) = f(a)$, then π is a fibre-map with fibre F_n^1. Hence we have an exact homotopy sequence:

$$\ldots \to \pi_{r+1}(S^{n-1}) \to \pi_r(F_n^1) \to \pi_r(H_n^+) \to \pi_r(S^{n-1}) \to \ldots .$$

If $r \leqslant n-3$ we have $\pi_{r+1}(S^{n-1}) = \pi_r(S^{n-1}) = 0$, and so:

$$\pi_r(H_n) \cong \pi_r(H_n^+) \cong \pi_r(F_n^1) \quad (1 \leqslant r \leqslant n-3). \tag{1}$$

Now $F_n = \Omega^{n-1}(S^{n-1})$, where Ω^{n-1} denotes the iterated loop space. For $n \geqslant 2$ therefore F_n is an H-space and so

$$\pi_r(F_n^d) \cong \pi_r(F_n^0) \quad (d \in \mathbf{Z},\ r \geqslant 1). \tag{2}$$

Moreover we have the standard isomorphism

$$\pi_r(F_n^0) \cong \pi_{n+r-1}(S^{n-1}) \quad (r,\ n \geqslant 1). \tag{3}$$

From (1), (2), and (3) we obtain finally

$$\pi_r(H_n) \cong \pi_{n+r-1}(S^{n-1}) \quad (1 \leqslant r \leqslant n-3).$$

This completes the proof of the lemma.

Since we have the suspension isomorphism

$$\pi_{n+k}(S^n) \cong \pi_{n+k+1}(S^{n+1}) \quad (n \geqslant k+2)$$

it follows from (1.3) that $J(X)$ may be identified with the image of

$$\psi_n\colon [X, BO_n] \to [X, BH_n]$$

in the stable range $(n-2 \geqslant \dim X)$.

If we put $BO = \bigcup_n BO_n$, $BH = \bigcup_n BH_n$ (with the weak topology) we see that $J(X)$ may be identified with the image of

$$\psi\colon [X, BO] \to [X, BH].$$

Now the Whitney sum makes BO a (weak) H-space, so that $[X, BO]$ has a group structure. In fact $[X, BO]$ is isomorphic with the kernel of the rank homomorphism $KO(X) \to H^0(X, \mathbf{Z})$ and we have the canonical decomposition $KO(X) \cong [X, BO] \oplus H^0(X, \mathbf{Z})$ (cf. (**7**)). The composite map $KO(X) \to [X, BO] \to \operatorname{im}\psi$ may then be identified with the homomorphism $J\colon KO(X) \to J(X)$ mentioned earlier.

Suppose now $X = S^r$. Then we have one-to-one correspondences

$$[S^r, BO] \leftrightarrow \overline{\pi_r(BO)},$$
$$[S^r, BH] \leftrightarrow \overline{\pi_r(BH)},$$

where $\bar{\pi}_r$ denotes the set of equivalence classes in π_r under the operation of π_1. Now if Y is an H-space we have the following well-known facts:

(i)† $\pi_1(Y)$ operates trivially on $\pi_r(Y)$,

(ii) the group structure in $\pi_r(Y)$ coincides with that induced by the H-space structure of Y.

Since BO is an H-space (i) and (ii) show that we have a group isomorphism:

$$[S^r, BO] \cong \pi_r(BO).$$

Since $\pi_1(BO) \cong \pi_1(BH) \cong \mathbf{Z}_2$ it follows also that $\pi_1(BH)$ operates trivially on $\operatorname{im} \psi$. Hence $J(S^r)$ is isomorphic to the image of

$$\pi_r(BO) \to \pi_r(BH)$$

and hence (**6**) to the image of

$$\pi_{r-1}(O) \to \pi_{r-1}(H).$$

Using (1.3) this last homomorphism becomes

$$\pi_{r-1}(O) \to G_{r-1}, \quad \text{for } r \geqslant 2, \tag{4}$$

where G_{r-1} denotes the stable homotopy group $\pi_{n+r-1}(S^n)$. Now (4) is just the stable J-homomorphism of G. W. Whitehead (**18**). Hence we have established the following proposition which justifies our notation for $J(X)$.

PROPOSITION (1.4). *$J(S^r)$ is isomorphic to the image of the stable J-homomorphism*

$$J: \pi_{r-1}(O) \to G_{r-1} \quad (r \geqslant 2).$$

(1.4) shows that $J(S^r)$ is finite. More generally we have:

PROPOSITION (1.5). *$J(X)$ is a finite group.*

Proof. We have just shown that, for $n-2 \geqslant \dim X$, $J(X) \subset [X, BH_n]$. Since X is a finite complex it will be sufficient, by obstruction theory, to know that $\pi_r(BH_n)$ is finite for $1 \leqslant r \leqslant n-2$. Now by (1.3) we have

$$\pi_1(BH_n) \cong \mathbf{Z}_2,$$
$$\pi_r(BH_n) \cong \pi_{n+r-2}(S^{n-1}) \quad (r \geqslant 2),$$

† For $Y = BO$ this can also be seen directly by observing that, for odd n, the component of O_n not containing the identity contains the element -1 of the centre. Conjugation by this element is therefore trivial. Since $BO = \lim BO_n$ through odd values, (i) follows.

and the stable homotopy group $\pi_{n+r-2}(S^{n-1})$ $(r \leqslant n-2)$ is finite (**12**). This completes the proof.

Remark. It is possible, though a little more complicated, to give an obstruction theory proof of (1.5) which avoids the use of (**6**).

2. S-type of Thom complexes

All spaces X, Y considered here will be finite CW-complexes, subspaces will be subcomplexes. According to (**19**) such a space has the homotopy type of a finite polyhedron. In view of the triangulation theorem (**5**) every compact differentiable† manifold, with or without boundary, is a finite polyhedron, and so belongs to our class of spaces.

If Y is a subspace of X we shall denote by X/Y the space obtained from X by collapsing Y to a point. We note that X/Y has a canonical base point, namely, the point corresponding to Y. If Y is the empty set we adopt the convention that $X/\emptyset$ is the disjoint union of X and the canonical base point.

We recall that, if X, Y are spaces with base points x_0, y_0, their 'smash product' $X \divideontimes Y$ is defined as

$$X \divideontimes Y = X \times Y / X \times y_0 \cup x_0 \times Y.$$

It is again a space with canonical base point.

We choose a standard base point for the 1-sphere S^1 and then define the suspension of a space X with base point to be $S^1 \divideontimes X$. The n-sphere S^n is then homeomorphic to $S^1 \divideontimes S^1 \divideontimes \ldots \divideontimes S^1$ (n times) and the n-fold suspension of X may therefore be identified with $S^n \divideontimes X$.

Let α be a real vector bundle over X. The *Thom complex* X^α of α is defined to be the one-point compactification of α. It is thus a space with canonical base point. If α is given an orthogonal structure, and if A, $\dot{A}$ are the associated bundles with fibres the unit ball and unit sphere respectively, then it is easy to see that $A/\dot{A}$ is homeomorphic to X^α in a natural way (base points corresponding).

If $\alpha = 0$ is the zero-dimensional bundle, then $X^0 = X/\emptyset$ is the disjoint union of X and a base point.‡

We shall suppose now that the cell-structure of X is such that the restriction of α to the closure of each cell is trivial. Since X has the same homotopy type as a polyhedron, and since we are only interested in the homotopy type of X^α, this is not an essential restriction. Under this hypothesis it follows that X^α may be given a cell structure where there is one cell of dimension $k+n$ for each cell of X of dimension k, n being the dimension of α $(n > 0)$. This leads to the Thom–Gysin isomorphism (cf. (**15**)).

† Differentiable will mean of class C^∞.

‡ The one-point compactification of a *compact* space X must be interpreted as X^0.

LEMMA (2.1). *There are natural isomorphisms*

$$H^q(X, A) \cong \tilde{H}^{q+n}(X^\alpha, \mathbf{Z}),$$
$$H_q(X, A) \cong \tilde{H}_{q+n}(X^\alpha, \mathbf{Z}),$$

where A is the system of local integer coefficients defined by α, and $\tilde{H}$ denotes cohomology or homology modulo the base point.

The local coefficient system A may be defined as follows. We recall first that an *orientation* of an n-dimensional real vector space V is a connected component of the space of isomorphisms $V \to \mathbf{R}^n$. The *orientation bundle* of α is the bundle whose fibre at $x \in X$ is the set of (two) orientations of α_x. The orientation bundle of α is thus a double covering of X, i.e. a principal $\mathbf{Z}_2$-bundle. Identifying $\mathbf{Z}_2$ with the group of automorphisms of $\mathbf{Z}$ we may consider the associated bundle with fibre $\mathbf{Z}$. This bundle of rings is, by definition, the local coefficient system A. The orientation bundle of α may then be identified with the bundle of units of A.

LEMMA (2.2). *Let α, β be real vector bundles over X, and let A, B be the corresponding local coefficient systems. Then*

(i) $$A \cong B \Leftrightarrow \omega_1(\alpha) = \omega_1(\beta);$$

(ii) $$H^0(X, A) \cong H^0(X, \mathbf{Z}) \Leftrightarrow \omega_1(\alpha) = 0,$$

where ω_1 denotes the first Stiefel–Whitney class.

Proof. Since the orientation bundle of α may be identified with the bundle of units of A it follows that A is trivial if and only if α is orientable. This proves (i) when B is trivial, and the general case follows at once since $A \cong B \Leftrightarrow A \otimes B$ is trivial. Now $H^0(X, A) \cong H^0(X, \mathbf{Z})$ is equivalent to saying that A has a non-zero section over each component of X. This in turn is equivalent to the triviality of A and now (ii) follows from (i).

If α, β are real vector bundles over X, Y respectively then $\alpha \times \beta$ is a vector bundle over $X \times Y$, the fibre at (x, y) being $\alpha_x \oplus \beta_y$.

LEMMA (2.3). *Let α, β be real vector bundles over X, Y respectively, then we have a natural homeomorphism $X^\alpha \divideontimes Y^\beta \to (X \times Y)^{\alpha \times \beta}$.*

Proof. We have a natural map f: $X^\alpha \times Y^\beta \to (X \times Y)^{\alpha \times \beta}$ in which $f^{-1}(c) = a \times Y^\beta \cup X^\alpha \times b$, and f is a homeomorphism elsewhere (a, b, c being the base points of X^α, Y^β, $(X \times Y)^{\alpha \times \beta}$ respectively). Hence f induces a homeomorphism

$$X^\alpha \divideontimes Y^\beta \to (X \times Y)^{\alpha \times \beta}$$

as required.

A special case of (2.3) arises when Y is a point, so that β is trivial of dimension n say. Then $Y^\beta = S^n$, and so (2.3) yields:

LEMMA (2.4). *$S^n(X^\alpha)$ is naturally homeomorphic to $X^{\alpha \oplus n}$.*

In view of (2.4) the S-type of a Thom complex X^α depends only on the element of $KO(X)$ determined by α. Since by (1.1) every element of $KO(X)$ is expressible as† $\alpha - n$ for some integer n and some bundle α, it follows that we may extend our notation and speak of the S-type of X^β, for any $\beta \in KO(X)$. By definition this means the S-type of X^α for any bundle α with $\beta = \alpha - n$ in $KO(X)$.

The relation of § 1 to the S-type of Thom complexes is explained by the following results:

LEMMA (2.5). *Let α, β be real vector bundles over X, (α), (β) the associated sphere bundles. Suppose (α) and (β) are of the same fibre homotopy type. Then X^α and X^β are of the same homotopy type.*

Proof. Let $\dot{A} = (\alpha)$ be the boundary of the unit ball bundle A associated to α. Similarly for B, $\dot{B}$ with respect to β. Then $X^\alpha = A/\dot{A}$, $X^\beta = B/\dot{B}$. Let $f\colon \dot{A} \to \dot{B}$, $g\colon \dot{B} \to \dot{A}$ be the given fibre homotopy equivalences. Then f, g can be extended radially to maps:

$$F\colon (A,\dot{A}) \to (B,\dot{B}), \qquad G\colon (B,\dot{B}) \to (A,\dot{A}),$$

and these induce maps

$$\bar{F}\colon A/\dot{A} \to B/\dot{B}, \qquad \bar{G}\colon B/\dot{B} \to A/\dot{A}.$$

Let $h\colon \dot{A}\times I \to \dot{A}$, $k\colon \dot{B}\times I \to \dot{B}$, be the given homotopies of $gf \simeq 1, fg \simeq 1$. Extend these radially and we get homotopies:

$$H\colon (A\times I, \dot{A}\times I) \to (A,\dot{A}), \qquad K\colon (B\times I, \dot{B}\times I) \to (B,\dot{B}),$$

of $GF \simeq 1$, $FG \simeq 1$. These induce homotopies

$$\bar{H}\colon A/\dot{A}\times I \to A/\dot{A}, \qquad \bar{K}\colon B/\dot{B}\times I \to B/\dot{B}$$

of $\bar{G}\bar{F} \simeq 1$, $\bar{F}\bar{G} \simeq 1$ as required.

PROPOSITION (2.6). *Let α, β be real vector bundles over X, and suppose $J(\alpha) = J(\beta)$. Then X^α and X^β are of the same S-type.*

Proof. By definition $J(\alpha) = J(\beta)$ means there exist integers m, n so that $(\alpha \oplus m)$ and $(\beta \oplus n)$ are of the same fibre homotopy type. Hence by (2.5) $X^{\alpha\oplus m}$ and $X^{\beta\oplus n}$ are of the same homotopy type. But by (2.4) $X^{\alpha\oplus m} = S^m(X^\alpha)$, $X^{\beta\oplus n} = S^n(X^\beta)$. Hence X^α and X^β are of the same S-type.

LEMMA (2.7).‡ *Let α, β be real vector bundles over X and suppose $J(\alpha) = J(\beta)$. Then $\omega_1(\alpha) = \omega_1(\beta)$.*

† Here and elsewhere we shall use the same symbol for a vector bundle and for the corresponding element of $KO(X)$. The two meanings of the symbol $J(\alpha)$ then coincide.

‡ This lemma is also a consequence of the general results of (**15**), on Stiefel–Whitney classes ω_i for all i.

Proof. Since J and ω_1 define homomorphisms $KO(X) \to J(X)$, $KO(X) \to H_1(X, \mathbf{Z}_2)$ respectively it is sufficient to consider the case $\beta = 0$. Let $\dim \alpha = n$, and let A be the local coefficient system defined by α. If $J(\alpha) = 0$ then by (2.6) X^α is of the same S-type as X^0, and so

$$H^0(X, A) \cong \tilde{H}^n(X^\alpha, \mathbf{Z}), \quad \text{by (2.1)},$$
$$\cong H^0(X, \mathbf{Z}).$$

Hence, by (2.2 (ii)), $\omega_1(\alpha) = 0$.

Let Y be a space with base point y_0. Following James (**10**) we shall say that Y is *reducible* (*S-reducible*) if there is a map (S-map) $f\colon (S^n, a) \to (Y, y_0)$ inducing isomorphism of $\tilde{H}_q$ for $q \geqslant n$. Dually we shall say that Y is *coreducible* (*S-coreducible*) if there is a map (S-map) $f\colon (Y, y_0) \to (S^n, a)$ inducing isomorphism of $\tilde{H}^q$ for $q \leqslant n$.

It is clear that reducibility and coreducibility are properties of homotopy type, that S-reducibility and S-coreducibility are properties of S-type, and that Y is S-reducible if and only if its S-dual, in the sense of Spanier–Whitehead (**13**), is S-coreducible.

From cohomology considerations, using (2.1) and (2.2), we see that a Thom complex X^α over a connected space X cannot be S-coreducible unless α is orientable.

If X is a compact connected differentiable manifold of dimension q, α a real vector bundle of dimension n, we have

$$\tilde{H}_{q+n}(X^\alpha, \mathbf{Z}) \cong H_q(X, A), \quad \text{by (2.1)},$$
$$\cong H^0(X, A \otimes T) \quad \text{(Poincaré duality)},$$

where T is the local coefficient system defined by the tangent bundle of X. From (2.2 (ii)) it follows that X^α cannot be S-reducible unless

$$\omega_1(\alpha) = \omega_1(X).$$

PROPOSITION (2.8). *Let α be a real vector bundle over a connected space X. Then X^α is S-coreducible if and only if $J(\alpha) = 0$.*

Proof.† If α is not orientable then X^α cannot be S-coreducible, as observed above, and $J(\alpha) \neq 0$ by (2.7). Hence we may suppose α orientable, and it will be sufficient to prove that, if $\dim \alpha = n$ is large, then X^α is coreducible if and only if $(\alpha) \sim (n)$. Let A, $\dot{A}$ be as in the proof of (2.5) and let $x \in X$. Then from the exact cohomotopy sequences of A, $\dot{A}$ and A_x, $\dot{A}_x$ (which are valid for large n), we have the commutative diagram:

$$\begin{array}{ccc} \pi^{n-1}(\dot{A}) & \cong & \pi^n(A, \dot{A}) \\ \theta \big\downarrow & & \phi \big\downarrow \\ \pi^{n-1}(\dot{A}_x) & \cong & \pi^n(A_x, \dot{A}_x). \end{array}$$

† This proof appeared in an earlier (unpublished) version of (**11**).

Now, according to a criterion of Dold (**5a**, 121, Corollary 2), $(\alpha) \sim (n)$ if and only if there is a map $f\colon \dot{A} \to S^{n-1}$ whose restriction to $\dot{A}_x$ is a homotopy equivalence. Thus, in terms of our diagram,

$$(\alpha) \sim (n) \Leftrightarrow \theta \text{ is an epimorphism,}$$
$$\Leftrightarrow \phi \text{ is an epimorphism.}$$

On the other hand, by (2.1), $H^r(A, \dot{A}; \mathbf{Z}) \to H^r(A_x, \dot{A}_x; \mathbf{Z})$ is an isomorphism for $0 \leqslant r \leqslant n$, from which it follows that

$$\phi \text{ is an epimorphism} \Leftrightarrow A/\dot{A} \text{ is coreducible.}$$

This completes the proof.

PROPOSITION (2.9). *Let α be a real vector bundle over a connected space X. Then $J(\alpha) = 0$ if and only if X^α and X^0 have the same S-type.*

Proof. In one direction this is a special case of (2.6). Conversely, suppose that X^α and X^0 have the same S-type. Since X is connected X^0 is coreducible, hence X^α is S-coreducible, and so $J(\alpha) = 0$ by (2.8).

PROPOSITION (2.10). *Let α be a real vector bundle over a connected space X. Then there exists a positive integer q such that $X^{n\alpha}$ has the S-type of X^0 if and only if n is a multiple of q.*

Proof. This follows at once from (2.9) and (1.5). The integer q is the order of $J(\alpha)$ in the finite group $J(X)$.

3. The duality theorem

We shall require a lemma on the embeddings of manifolds with boundary which is a simple consequence of the theorems of Whitney (**20**).

LEMMA (3.1). *Let X be a compact differentiable manifold with boundary Y. Then, if $n > 2 \dim X$, we can embed X differentiably in $\mathbf{R}^n$ so that*

(i) *$X - Y$ lies in the open unit cube $0 < x_i < 1$ $(1 \leqslant i \leqslant n)$;*

(ii) *Y lies in the open face $0 < x_i < 1$ $(1 \leqslant i \leqslant n-1)$, $x_n = 0$;*

(iii) *X intersects $x_n = 0$ normally in Y.*

Proof. By Theorem 1 of (**20**) we can find a differentiable embedding $g\colon Y \to \mathbf{R}^{n-1}$. Define a differentiable embedding $f\colon Y \times I \to \mathbf{R}^n$ by $f(y,t) = (g(y), t)$, where I denotes the unit interval $0 \leqslant t \leqslant 1$. Now we can find a neighbourhood Z of Y in X which is diffeomorphic to $Y \times I$, Y corresponding to $Y \times 0$. Then applying Theorem 5 of (**20**) to the closed submanifold $Z - Y$ of $X - Y$ we deduce the existence of a differentiable embedding $F\colon X - Y \to \mathbf{R}^n_+$ with $F = f$ on $Z - Y$, where $\mathbf{R}^n_+$ denotes the subspace of $\mathbf{R}^n$ with $x_n > 0$. The pair F, f then defines a differentiable embedding of X in $\mathbf{R}^n$, which, after a change of scale, has all the required properties.

Next we prove a duality theorem for manifolds with boundary.

PROPOSITION (3.2). *Let X be a compact differentiable manifold with boundary Y, and let τ denote the tangent bundle of X. Then $X^{-\tau}$ is the S-dual of X/Y.*

Proof. Let X be embedded in the closed unit cube I^n of $\mathbf{R}^n$ as in (3.1). Let P be the point of $\mathbf{R}^{n+1}$ with coordinates $(0,\ldots,0,1)$ and form the join of P to I^n. This is a pyramid whose boundary S^n is homeomorphic to the n-sphere. The space $X \cup PY$ (where PY denotes the join of P to Y) has the homotopy type of X/Y and is embedded in S^n. Hence an S-dual of X/Y is given by a deformation retract of $Z = S^n-(X \cup PY)$ (cf. (**13**)). Now by projection from P the space Z may be deformed into I^n-X, and this in turn may be deformed into $\overline{I^n-A}$, where A is a normal tubular neighbourhood of X in I^n. There is no trouble on the boundary of I^n because of the property (iii) of (3.1). Since I^n is contractible and the pair $(I^n, \overline{I^n-A})$ has the homotopy extension property† $I^n/\overline{I^n-A}$ has the homotopy type of the suspension of $\overline{I^n-A}$, and hence is also an S-dual of X/Y. But $I^n/\overline{I^n-A} = A/\dot{A}$, where $\dot{A} = \overline{I^n-A} \cap A$ is the closure of that part of the boundary of A which is interior to I^n. If ν is the normal bundle of X in I^n, A and $\dot{A}$ may be identified with the unit ball and unit sphere bundles associated to ν. Hence $A/\dot{A} = X^{\nu}$ is an S-dual of X/Y. Since $\nu \oplus \tau$ is trivial this concludes the proof.

Remark. If one is prepared to allow the triangulability of manifolds with 'corners' (e.g. a square), or at least that these are finite CW-complexes, then this proof can be simplified using the fact (**13**) that if $DY \supset DX$ is S-dual to $Y \subset X$ then DY/DX is S-dual to X/Y (all spaces being finite CW-complexes).

We are now in a position to deduce the duality theorem for Thom complexes.

THEOREM (3.3). *Let X be a compact differentiable manifold (without boundary) with tangent bundle τ. Let α be a real vector bundle over X. Then the S-dual of X^{α} is $X^{-\alpha-\tau}$.*

Proof. It is well known that we may give α a differentiable structure. For example we may take a continuous map $f\colon X \to G_{n,m}$ which induces α ($G_{n,m}$ denoting as before a real Grassmannian), and approximate it by a differentiable map g. The map g will induce a differentiable bundle over X which is equivalent to α. Now let $A, \dot{A}$ be the bundles associated to α with fibre the unit ball and unit sphere respectively. Then A is a compact differentiable manifold with boundary $\dot{A}$. By (3.2) the S-dual of $X^{\alpha} = A/\dot{A}$

† This is equivalent to $(A, \dot{A})$ having the homotopy extension property, and this is true for any sphere bundle.

is A^{-t}, where t is the tangent bundle of A. Let $\pi: A \to X$ be the projection map, then it is clear that $t \simeq \pi^*(\tau \oplus \alpha)$. Since π is a homotopy equivalence it follows that A^{-t} and $X^{-\alpha-\tau}$ are of the same S-type, and hence $X^{-\alpha-\tau}$ is the S-dual of X^α as required.

Remark. The special case of (3.3) with $\alpha = 0$ (or equivalently the special case of (3.2) with $Y = \emptyset$) is proved in (**11**).

PROPOSITION (3.4). *Let X be a compact connected differentiable manifold with tangent bundle τ. Let α be a real vector bundle over X. Then X^α is S-reducible if and only if $J(\alpha) = -J(\tau)$.*

Proof. X^α is S-reducible if and only if its S-dual $X^{-\alpha-\tau}$ (3.3) is S-coreducible. The required result now follows from (2.8).

We propose now to give an application of (3.4). First we make a few remarks on orientability. Let X, Y be compact differentiable manifolds with tangent bundles ξ, η and let A, B be the local coefficient systems defined by ξ, η. A map $f: Y \to X$ is said to be *orientable* if $B \simeq f^*A$. In view of (2.2) (i), f is orientable if and only if $w_1(Y) = f^*w_1(X)$, where $w_1(Y)$ denotes as usual $\omega_1(\eta)$ and similarly for X. The map f is said to be *oriented* if we are given an isomorphism $B \simeq f^*A$. An oriented map f induces a homomorphism:

$$f_*: H_*(Y, B) \to H_*(X, A).$$

THEOREM (3.5). *Let X, Y be compact differentiable manifolds with tangent bundles ξ, η respectively. Let A, B be the local coefficient systems defined by ξ, η. Let $f: Y \to X$ be an oriented map which induces an isomorphism*

$$f_*: H_*(Y, B) \to H_*(X, A).$$

Then

$$J(\eta) = J(f^*\xi).$$

Proof. Let α be a real vector bundle over X with $\alpha \oplus \xi$ trivial (1.1) and suppose $\dim \alpha > 1$. Then $w_1(\alpha) = w_1(\xi)$ and so, by (2.2) (i), we may choose an isomorphism of A with the local coefficient system of α. Put $\beta = f^*\alpha$, then the local coefficient system of β may be identified with f^*A and hence with B, since f is oriented. Also f induces a map $F: Y^\beta \to X^\alpha$. Then, by the naturality of the Thom–Gysin isomorphism (2.1) we obtain a commutative diagram:

$$\begin{array}{ccc} \tilde{H}_*(Y^\beta, \mathbf{Z}) & \xrightarrow{F_*} & \tilde{H}_*(X^\alpha, \mathbf{Z}) \\ \wr\| & & \wr\| \\ H_*(Y, B) & \xrightarrow{f_*} & H_*(X, A). \end{array}$$

Since f_* is an isomorphism so is F_*. Since $\dim \beta = \dim \alpha > 1$, X^α and Y^β are simply-connected. Hence, by the theorem of J. H. C. Whitehead (**19**) F is a homotopy equivalence. Since $J(\alpha) = -J(\xi)$, X^α is S-reducible (3.4). Hence Y^β, being of the same homotopy type as X^α, is S-reducible, and so by (3.4) $J(\beta) = -J(\eta)$. But $J(\beta) = J(f^*\alpha) = -J(f^*\xi)$. Hence $J(\eta) = J(f^*\xi)$ as required.

Since w_1 is a homotopy invariant (**15**) a homotopy equivalence is necessarily orientable. Hence as a corollary of (3.5) we deduce:

THEOREM (3.6). *Let X, Y be compact differentiable manifolds with tangent bundles ξ, η. Let $f: Y \to X$ be a homotopy equivalence. Then $J(\eta) = J(f^*\xi)$.*

Remark. One can give a direct proof of (3.5) and (3.6) without using (3.3). However the proof given here is somewhat simpler.

4. Stunted projective spaces

We recall here some definitions and notation of (**10**), adapting them to our purpose.

Let F denote one of the three basic fields: $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{K}$ (quaternions). Let $V = F^k$ be regarded as a right F-module and let P_k be the associated right projective space. Thus we have a principal F^*-bundle† $\pi: V - O \to P_k$, which we shall denote by M, and an associated right line-bundle (fibre F, group F^* operating on F on the left) which we denote by L. We may also consider the left line-bundle $L^* = \mathrm{Hom}_F(L, F)$. This defines a real vector bundle ξ of dimension d, where $d = \dim_{\mathbf{R}} F$.

The 'stunted projective space' $P_{n,k}$ is defined by

$$P_{n,k} = P_n/P_{n-k}.$$

James excludes the case $n = k$, but with our conventions this case presents no anomalies.

LEMMA (4.1). *Let T_x denote the tangent space of P_k at a point x. Then we have a canonical isomorphism (of real vector spaces)*

$$\psi_x: (V/L_x) \otimes_F L_x^* \to T_x.$$

Proof. Applying the construction of § 2 of (**1**) to the principal bundle M we get an exact sequence of vector spaces:

$$0 \to S_x \to Q_x \to T_x \to 0,$$

where an element of Q_x is a field of tangent vectors of $M = V - O$ along $\pi^{-1}(x)$, invariant under F^*, and S_x is the subspace of fields tangent to the fibre $\pi^{-1}(x)$. If $y \in V$, then the tangent space to V at y may be identified with V itself by translation. A vector field $v(y)$ defined for $y \in \pi^{-1}(x)$ is then invariant under F^* if

$$v(y\lambda) = v(y)\,\lambda \quad \text{for} \quad \lambda \in F^*.$$

† F^* denotes the multiplicative group of non-zero elements of F.

We have a canonical isomorphism $\theta\colon V \otimes_F L_x^* \to Q_x$ defined by

$$\theta(v \otimes \phi)(y) = v\,\phi(y), \quad v \in V, \quad \phi \in \mathrm{Hom}_F(L_x, F), \quad y \in \pi^{-1}(x).$$

Clearly θ induces an isomorphism

$$L_x \otimes_F L_x^* \to S_x,$$

and hence an isomorphism of quotients

$$\psi_x\colon (V/L_x) \otimes_F L_x^* \to T_x$$

as required.

LEMMA (4.2). *Let T denote the (real) tangent bundle of P_k. Then we have an isomorphism of real vector bundles: $T \cong (V/L) \otimes_F L^*$, where V denotes the trivial bundle $P_k \times V$.*

Proof. This follows at once from (4.1). We have only to observe that ψ_x varies continuously with x.

PROPOSITION (4.3). *$P_{n,k}$ is homeomorphic to the Thom complex $P_k^{(n-k)\xi}$.*

Proof. By definition $P_{n,k} = P_n/P_{n-k}$. Let P_k be complementary to P_{n-k} in P_n. In terms of vector spaces this corresponds to a decomposition $F^n = U \oplus W$ with $\dim_F U = k$, $\dim_F W = n-k$. For each point $x \in P_k$ we consider the vector space $W \oplus L_x$ and the associated projective space A_x of dimension $(n-k)$. The tangent space of A_x at x may be identified, on the one hand with the affine space $A_x - A_x \cap P_{n-k}$, and on the other hand (by (4.1)) with

$$((W \oplus L_x)/L_x) \otimes_F L_x^* \cong W \otimes_F L_x^*.$$

Hence, allowing x to vary in P_k, we see that $P_n - P_{n-k}$ may be identified with the bundle space of the vector bundle $W \otimes_F L^*$ over P_k (W denoting the trivial bundle $P_k \times W$). Since W is a trivial bundle we have

$$W \otimes_F L^* \cong L^* \oplus L^* \oplus \ldots \oplus L^* \quad (n-k) \text{ times},$$

and so the underlying real vector bundle is $(n-k)\xi$. Clearly P_n/P_{n-k} is the one-point compactification of $P_n - P_{n-k}$, i.e. of the bundle space of $(n-k)\xi$. Thus P_n/P_{n-k} is homeomorphic to the Thom complex $P_k^{(n-k)\xi}$.

LEMMA (4.4). *The real vector bundle $L \otimes_F L^*$ is isomorphic to the bundle η with fibre the Lie algebra F of F^* associated to M by the adjoint representation (in brief: $\eta = \mathrm{ad}(M)$).*

Proof. In the proof of (4.1) we obtained a canonical isomorphism $L_x \otimes_F L_x^* \to S_x$. In (1) it is proved† as a special case of a general result on principal bundles, that S_x is canonically isomorphic to η_x. This gives an

† Actually in (1) all structures are complex but the argument is identical for real bundles, real Lie groups, etc.

isomorphism $L_x \otimes_F L_x^* \to \eta_x$ which varies continuously and so defines an isomorphism of vector bundles $L \otimes_F L^* \cong \eta$.

Since the centre of F always contains $\mathbf{R}$ it follows that we can decompose η in the form $\eta = 1 \oplus \zeta$. In the commutative cases ($F = \mathbf{R}$, $\mathbf{C}$) we even have ζ trivial, but this is not so for $\mathbf{K}$.

LEMMA (4.5). *Let T denote the tangent bundle of P_k. Then $T \oplus \eta \cong k\xi$.*

Proof. This follows from (4.2) and (4.4) and the fact that all exact sequences of real vector bundles split topologically (cf. (**1**)).

Remark. The well-known formula for the characteristic classes of P_k follows easily from (4.5). One has just to find the characteristic classes of η.

5. Quasi-projective spaces

We start with some generalities. By a group we shall mean a compact Lie group, and a subgroup will mean a closed subgroup.

Let G be a group, H a subgroup, N the normalizer of H in G. Then $G \to G/N$ is a principal N-bundle (N operating on G on the right). Since N operates on H (by conjugation) as a group of automorphisms we can form the associated bundle $E = G \times_N H$ over G/N with fibre H. Moreover this is a bundle of groups and so in particular it has an identity cross-section s.

We now define a map
$$f\colon G \times_N H \to G$$
by $f(g, h) = ghg^{-1}$. This is well-defined, for if $n \in N$,
$$f(gn, n^{-1}hn) = gn\, n^{-1}hn\, n^{-1}g^{-1} = ghg^{-1} = f(g, h).$$
Clearly $f(s) = e$ (the identity of G), and f restricted to each fibre E_{gN} gives an isomorphism onto the subgroup gHg^{-1} of G. Thus $f(E) = \bigcup_\sigma H^\sigma$, where σ runs through the inner automorphisms of G. The following lemma is therefore obvious:

LEMMA (5.1). *Suppose H is a subgroup of G with the property that, for any inner automorphism σ of G, we have $H = H^\sigma$ or $H \cap H^\sigma = e$. Then $f(E) = E/s$ (the space obtained from $E = G \times_N H$ by collapsing the identity cross-section s).*

Examples of pairs (G, H) satisfying the hypothesis of (5.1) are provided by the classical groups. Following (**8**) we let O_n denote the group of automorphisms of F^n ($F = \mathbf{R}$, $\mathbf{C}$, or $\mathbf{K}$) which preserve the standard scalar product. We take $G = O_n$ and $H = O_1$, this being embedded in O_n via the standard embedding of F^1 in F^n. It is clear that the conditions of (5.1) are satisfied. Moreover, $H = O_1 \times I_{n-1}$ (I_{n-1} being the unit matrix), $N = O_1 \times O_{n-1}$, and $G/N = P_n$. Thus E is a sphere bundle over P_n, the dimension of the sphere being $(d-1)$, where $d = \dim_{\mathbf{R}} F$. By definition E is associated to $G \to G/N$ by the operation of N on H (conjugation). Since $N = H \times O_{n-1}$ it follows that E is also the bundle associated to the Hopf bundle $O_n/O_{n-1} \to P_n$ by the operation of H on itself (conjugation). Now the Hopf bundle $O_n/O_{n-1} \to P_n$ is just the orthogonal reduction of the principal F^*-bundle M of § 4. Hence E is the sphere bundle associated to the vector bundle $\eta = \mathrm{ad}(M)$. As remarked in § 4 we have a decomposition $\eta = 1 \oplus \zeta$ in which the factor 1 comes from $\mathbf{R} \subset F$ and so corresponds to the identity cross-section s of E. Hence E/s is just the Thom complex of ζ. We have then established, as a special case of (5.1):

LEMMA (5.2). *Let X_n be the subspace of O_n defined by $X_n = \bigcup_\sigma O_1^\sigma$ where σ runs through the inner automorphisms of O_n. Then X_n is homeomorphic to P_n^ζ.*

In (**10**) James defines an explicit subspace Q_n of O_n, which he calls a *quasi-projective space*. We shall now establish:

PROPOSITION (5.3). *The spaces Q_n of* (**10**) *and X_n of* (5.2) *coincide. Hence Q_n is homeomorphic to P_n^ζ.*

Proof. In terms of matrices a point of O_1 is given by a diagonal matrix:

$$\Lambda = \begin{bmatrix} \lambda & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & \cdot \quad 1 \end{bmatrix} \quad (\lambda \in F,\ |\lambda| = 1).$$

Hence a typical conjugate point is given by a matrix $B\Lambda\bar{B}'$ $(B \in O_n)$. If $(y_1,\ldots,y_n)$ is the first column of B we find:

$$(\bar{B}'\Lambda B)_{ij} = \delta_{ij} - y_i(1-\lambda)\bar{y}_j \quad (1 \leqslant i \leqslant j \leqslant n). \tag{1}$$

In James's definition a point of Q_n' is given by the matrix A with

$$A_{ij} = \delta_{ij} - 2x_i(1+x_0)^{-2}\bar{x}_j \quad (1 \leqslant i \leqslant j \leqslant n), \tag{2}$$

where
$$\sum_0^n |x_i|^2 = 1, \qquad x_0 + \bar{x}_0 = 0. \tag{3}$$

If we put
$$\left.\begin{aligned} \lambda &= -(1-x_0)^2(1+x_0)^{-2} \\ y_i &= (1+x_0^2)^{-\frac{1}{2}}x_i \quad (1 \leqslant i \leqslant n) \\ &= \Big(\sum_1^n |x_i|^2\Big)^{-\frac{1}{2}} x_i \end{aligned}\right\}, \tag{4}$$

it is easy to check that $A = B\Lambda\bar{B}'$, showing that $Q_n' \subset X_n$. Moreover, given λ, y_i with $|\lambda| = 1$, $\sum_1^n |y_i|^2 = 1$, we can find $x_0, x_1,..., x_n$ to satisfy (3) and (4), except when $F = \mathbf{R}$ and $\lambda = 1$, i.e. when A is the unit matrix. But by definition (10) $Q_n = Q_n'$, $F \neq \mathbf{R}$, and $Q_n = Q_n' \cup e$ (e the identity of O_n) for $F = \mathbf{R}$. Thus $Q_n = X_n$ as asserted in all cases.

We have natural inclusions $Q_{n-k} \subset Q_n$ and James defines the *stunted quasi-projective space* $Q_{n,k} = Q_n/Q_{n-k}$. The inclusion $Q_{n-k} \subset Q_n$ corresponds in the homeomorphism† $Q_n \sim P_n^\zeta$ of (5.3) to the natural inclusion $P_{n-k} \subset P_n$. Here we use the same symbol ζ for all dimensions of projective space, since the ζ on P_{n-k} is just the bundle induced by the ζ on P_n (this being true of their principal bundles M).

Hence

$$\begin{aligned} Q_n - Q_{n-k} &\sim P_n^\zeta - P_{n-k}^\zeta \\ &\sim \text{bundle space of } \zeta \text{ over } (P_n - P_{n-k}) \\ &\sim \text{bundle space of } \zeta \oplus (n-k)\xi \text{ over } P_k \end{aligned}$$

by (4.3). Taking the one-point compactifications we obtain finally:

PROPOSITION (5.4). *$Q_{n,k}$ is homeomorphic to the Thom complex $P_k^{(n-k)\xi\oplus\zeta}$.*

6. The theorems of James

We are now in a position to apply the general results on Thom complexes proved in §§ 1–3 to the spaces $P_{n,k}$, $Q_{n,k}$.

We have established the homeomorphisms:

$$P_{n,k} \sim P_k^{(n-k)\xi} \quad (4.3),$$
$$Q_{n,k} \sim P_k^{(n-k)\xi\oplus\zeta} \quad (5.4).$$

The spaces on the left are only defined for $n > k \geqslant 1$. However, the Thom complexes on the right have been defined (§ 2) for $k \geqslant 1$ and all integers n, provided that we are only interested in S-types. We use (4.3) and (5.4) to extend similarly the definition of the S-types of‡ $P_{n,k}$ and $Q_{n,k}$.

THEOREM (6.1).§ *$Q_{n,k}$ and $P_{k-n,k}$ are S-duals.*

Proof. We have by (4.3) and (5.4) $Q_{n,k} \sim P_k^{(n-k)\xi\oplus\zeta}$, $P_{k-n,k} \sim P_k^{-n\xi}$. Hence, by (3.3), $Q_{n,k}$ is S-dual to P_k^α where

$$\begin{aligned} \alpha &= -(n-k)\xi - \zeta - T \quad \text{in } KO(X) \\ &= -(n-k)\xi - \zeta - (k\xi - \eta) \quad \text{by (4.5)} \\ &= -n\xi + 1. \end{aligned}$$

Since P_k^α and $P_k^{\alpha-1}$ have the same S-type (2.4) the theorem is proved.

† In this section and the next we denote a homeomorphism by $\sim$.

‡ Thus $P_{n,k}$ stands for the S-type of $P_{n+r,k}$, where $n+r > k$ and $rJ(\xi) = 0$.

§ This result was conjectured by James (unpublished).

THEOREM (6.2). *$Q_{n,k}$ is S-reducible if and only if n is a multiple of the order of $J(\xi)$.*

Proof. Since $Q_{n,k} \sim P_k^{(n-k)\xi\oplus\zeta}$ (5.4) we can apply (3.4) and we obtain:

$$\begin{aligned} Q_{n,k}\ S\text{-reducible} &\Leftrightarrow J\{(n-k)\xi+\zeta\} = -J(T) \\ &\Leftrightarrow J(n\,\xi) = 0 \quad \text{by (4.5)} \\ &\Leftrightarrow nJ(\xi) = 0 \quad \text{(q.e.d.).} \end{aligned}$$

Remark. Since $J(P_k)$ is a finite group (1.5), $J(\xi)$ is of finite order q_k say. Then (6.2) asserts that $Q_{n,k}$ is S-reducible if and only if $n \equiv 0 \bmod q_k$. In particular, values of n for which $Q_{n,k}$ is S-reducible always exist. Theorem (6.2) includes one of the main results of (**10**). Most of the other results of (**10**) are simple consequences of (6.1), (6.2) and the results of §§ 1–3.

The interest of the spaces $Q_{n,k}$ lies in their relation to the Stiefel manifolds $O_{n,k} = O_n/O_{n-k}$ as explained in (**10**). There is a commutative diagram of maps

$$\begin{tikzcd} Q_{n,k} \arrow[r, "i"] \arrow[dr, "\rho"'] & O_{n,k} \arrow[d, "\pi"] \\ & S^{dn-1} \end{tikzcd}$$

where i is an inclusion induced by the inclusion $Q_n \subset O_n$, π is the fibre map $O_{n,k} \to O_{n,1} = S^{dn-1}$ and ρ is the 'cofibre map' $Q_{n,k} \to Q_{n,1} = S^{dn-1}$. By determining the connectivity of the pair $(O_{n,k}, Q_{n,k})$ James in (8.2) of (**10**) proves:

LEMMA (6.3). *$Q_{n,k}$ is reducible if and only if $O_{n,k} \to S^{dn-1}$ has a cross-section and, for $F = \mathbf{R}$, $n \geqslant 2k$ or $k = 1$.*

Now $Q_{n,k}$ is r-connected where $r = d(n-k+1)-2$. Hence by (2.6) of (**17**)

$$\pi_{dn-1}(Q_{n,k}) \to \pi^S_{dn-1}(Q_{n,k}) \quad \text{(the stable group)}$$

is an epimorphism for $2r \geqslant dn-2$, i.e. for $n \geqslant 2(k-1)+2/d$. Now from cohomology considerations $Q_{n,k}$ cannot be reducible unless $n \geqslant 2k$, or $k = 1$. Hence we have

LEMMA (6.4). *$Q_{n,k}$ is reducible if and only if it is S-reducible and $n \geqslant 2k$ or $k = 1$.*

By (6.2), if $Q_{n,k}$ is S-reducible, n is a multiple of q_k (the order of $J(\xi)$ in $J(P_k)$). The results of (**3**) or (**4**) show that in the complex and quaternionic cases $q_k \geqslant 2k$ for $k > 1$. Hence from (6.2), (6.3), and (6.4) we obtain finally:

THEOREM (6.5). *In the real case suppose $n \geqslant 2k$. Then (for all cases) $O_{n,k} \to S^{dn-1}$ has a cross-section if and only if n is a multiple of q_k, where q_k is the order of $J(\xi)$ in $J(P_k)$.*

In one respect this result is weaker than results of (**9**). In the real case we make no assertion for the case $n < 2k$ whereas James proves one half of (6.5) in all cases.

The problem of determining the actual numbers q_k in (6.5) is thus seen to be a special case of the more general problem of determining the structure of the group $J(X)$. An attack on this problem by the methods of (**2**) and (**3**) will be given in a future publication.

We conclude with a few remarks on the relation between the methods of this paper and those of James. The main tool of James is the 'intrinsic join'. Actually he defines several 'intrinsic joins', two of them being maps

$$Q_{n,k} * P_{m,k} \to P_{m+n,k},$$
$$Q_{n,k} * Q_{m,k} \to Q_{m+n,k},$$

where $X * Y$ denotes the join of X and Y. Since $X * Y$ and $S(X \divideontimes Y)$ are of the same homotopy type these two maps, in view of (2.4), (4.3), and (5.4), give rise to maps

$$P_k^{(n-k)\xi \oplus \eta} \divideontimes P_k^{(m-k)\xi} \to P_k^{(m+n-k)\xi},$$
$$P_k^{(n-k)\xi \oplus \eta} \divideontimes P_k^{(m-k)\xi \oplus \zeta} \to P_k^{(m+n-k)\xi \oplus \zeta}.$$

In view of (4.5) both these are maps of the type

$$f\colon X^{\alpha} \divideontimes X^{\beta} \to X^{\alpha+\beta+\tau},$$

where X is a compact differentiable manifold with tangent bundle τ. By (3.3), and the fact that the smash product commutes with duality, the dual of f is an S-map of the form:

$$Df\colon X^{\gamma+\delta} \to X^{\gamma} \divideontimes X^{\delta},$$

where $\gamma = -\alpha - \tau$, $\delta = -\beta - \tau$. Now the diagonal map $X \to X \times X$ induces a map

$$\Delta\colon X^{\gamma+\delta} \to (X \times X)^{\gamma \times \delta} \sim X^{\gamma} \divideontimes X^{\delta} \quad \text{by (2.3).}$$

It would seem reasonable to suppose that Df and Δ are S-homotopic.

In addition to the maps just mentioned James also defines an intrinsic join for the Stiefel manifolds $O_{n,k}$. This has no counterpart here, since we are not directly concerned with Stiefel manifolds.

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