TOPOLOGY OF ELLIPTIC OPERATORS

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In these notes I will deal with topological questions concerning elliptic operators, and more particularly with the index theorem. The analytical part of the theory will be reviewed only briefly and we refer to L. Nirenberg's lectures [2] for more details.

I. **The symbol and the index of an elliptic operator.** Throughout this section let X be a closed smooth (i.e. C^∞) manifold of dimension n, and let E, F be smooth complex vector bundles over X of fiber dimension N. The linear space of smooth cross-sections of E (resp. F) is denoted by $C^\infty(E)$ (resp. $C^\infty(F)$). Furthermore let k be a nonnegative integer.

DEFINITION. A *differential operator P (of order k) from E to F* is a linear map

$$P: C^\infty(E) \to C^\infty(F)$$

which locally, in terms of coordinates, can be expressed as a matrix of polynomials in partial derivatives of order $\leq k$. More precisely, if $x_1, \ldots, x_n$ are local coordinates of X with domain U, and if $\tau_1 : E|U \cong U \times C^N$, $\tau_2 : F|U \cong U \times C^N$ are smooth local trivializations of E, resp. F, then the linear operator $C^\infty(E) \to (C^\infty(U))^N = C^\infty(F|U)$ induced by P has the form $\sum_{|\alpha| \leq k} a_\alpha(x) D_x^\alpha$, where for each multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $|\alpha| = \sum \alpha_i$, a_α is an $N \times N$-matrix of elements of $C^\infty(U)$ (i.e. smooth complex valued functions), and D_x^α denotes the partial derivative $\partial^{\alpha_1}/\partial x_1^{\alpha_1} \ldots \partial^{\alpha_n}/\partial x_n^{\alpha_n}$.

P is *elliptic* (*of order* k) if for each local representation as above, the highest order term

$$p_k(x, \xi) = \sum_{|\alpha| = k} a_\alpha(x) \xi^\alpha$$

is a nonsingular matrix for $x \in U$ and $\xi \in \mathbf{R}^n - \{0\}$.

The system of functions $p_k(x, \xi)$, given for every local representation of P as above, is called the *leading symbol of P*.

There is an intrinsic geometrical interpretation of the leading symbol of an elliptic differential operator P. We fix a Riemannian metric on X. Thus the tangent bundle $T(X)$ and the cotangent bundle $T^*(X)$ are canonically isomorphic, and we denote by $B(X)$ (resp. $S(X)$) the corresponding unit ball bundle (resp. sphere bundle). If $\pi^*(E)$, $\pi^*(F)$ are the pullbacks of E, F under the projection $\pi : S(X) \to X$, then the leading symbol of P determines a well-defined vector bundle isomorphism over $S(X)$:

$$\sigma(P) : \pi^*(E) \xrightarrow{\cong} \pi^*(F),$$

[1] Notes by U. Koschorke.

given by $p_k(x, \xi): E_x \cong F_x$. This isomorphism $\sigma(P)$ gives a geometrical way of looking at the leading symbol of P.

It is a classical result that the dimension of both the kernel and the cokernel of an elliptic differential operator P are finite. The difference of the dimensions,

$$\text{index } P = \dim \ker P - \dim \operatorname{coker} P,$$

is known to depend only on the highest order term, i.e. $\sigma(P)$. Thus it is natural to try to connect the global data given by the symbol to other global data, such as the index. In fact our main goal is to give an explicit formula expressing the index of an elliptic operator P in terms of $\sigma(P)$.

For this purpose it is convenient to enlarge our class of operators by introducing pseudo-differential operators (cf. [2]). Essentially all smooth isomorphisms from $\pi^*(E)$ to $\pi^*(F)$ can occur as symbols of such operators. On the other hand, an elliptic pseudo-differential operator still has a well-defined index which depends only on the homotopy class (through smooth isomorphisms) of its symbol. Therefore, since smooth maps on one side and continuous maps on the other side have a similar homotopy behavior, the index gives rise to an integer valued function on the set of homotopy classes of continuous isomorphisms between $\pi^*(E)$ and $\pi^*(F)$. Thus the use of pseudo-differential operators enables us to replace the set of polynomial symbols by simpler geometrical objects, and to perform more drastic deformations.

We now indicate briefly an alternative topological approach which would allow us to avoid the analytical concept of pseudo-differential operators. Let $A(N, n, k)$ be the set of $N \times N$-matrices $p(\xi)$ whose entries are homogeneous polynomials of order k in $\xi_1, \ldots, \xi_n$, such that $\det p(\xi) \neq 0$ for $\xi \in \mathbf{R}^n - \{0\}$. In an obvious way this set can be imbedded in some euclidean space and hence has a natural topology. $A(N, n, k)$ is the typical fiber of a fiber bundle over X whose sections correspond to the symbols of elliptic differential operators of order k from E to F. Thus, if we deal with deformations of symbols in the framework of elliptic differential operators we need some information on the homotopy properties of $A(N, n, k)$. However, not much is known on $A(N, n, k)$, and even the question whether it is empty or not is rather hard to decide. For example $A(N, n, 1)$ is not empty if and only if the $(N - 1)$-sphere S^{N-1} admits $(n - 1)$ linearly independent tangent vector fields. This last problem, which has been solved by J. F. Adams, is highly nontrivial.

Nevertheless the situation is not quite that hopeless. If we imbed $A(N, n, 2j)$ into $A(N, n, 2(j + 1))$ by means of the map $p(\xi) \to (\sum \xi_i^2) p(\xi)$ (which corresponds by the Fourier transform to composition with the Laplace operator), then the direct limit $\lim_{j \to \infty} A(N, n, 2j)$ can be interpreted as the set of even polynomial maps from the sphere S^{n-1} into $\mathrm{GL}(N, \mathbf{C})$ (or, equivalently, as the set of polynomial maps from the projective space P^{n-1} into $\mathrm{GL}(N, \mathbf{C})$). This set is dense in the space of all even continuous maps from S^{n-1} into $\mathrm{GL}(N, \mathbf{C})$. We could use this observation in order to deduce an index function defined on the set of homotopy classes of even continuous isomorphisms between $\pi^*(E)$ and $\pi^*(F)$. Without resorting to pseudo-differential operators we thus would simplify the computation of the index of an elliptic differential operator. However, we shall not pursue this line.

We now resume our previous study of the index of elliptic (pseudo-) differential operators and we shall try to understand the topological significance of the symbol.

First we consider the case when E, F are trivial line bundles. The isomorphisms between $\pi^*(E)$ and $\pi^*(F)$ are simply maps from $S(X)$ into $\mathrm{GL}(1, \mathbf{C}) = \mathbf{C}^*$ (which retracts onto the unit circle S^1), and the index is defined on the set of homotopy classes $[S(X), S^1] \cong H^1(S(X))$ (cohomology with integer coefficients). It follows from [2] that the index of an elliptic operator P is zero if for each $x \in X$ $\sigma(P)$ is constant on the sphere $S(X)_x$ over x. Hence the index vanishes on the image of τ^* in the following exact cohomology sequence

$$\begin{array}{ccccccc} \rightarrow H^1(B(X)) & \rightarrow & H^1(S(X)) & \xrightarrow{\delta} & H^2(B(X), S(X)) & \xrightarrow{j^*} & H^2(B(X)) \rightarrow \ldots \\ \wr\Vert & \nearrow_{\pi^*} & \big\downarrow \text{index} & & & & \\ H^1(X) & & \mathbf{Z} & & & & \end{array}$$

If $\dim X = n > 2$, then, by a classical result of Thom, $H^2(B(X), S(X)) = 0$, and π^* is surjective. Hence the index of every elliptic operator from the trivial line bundle into itself is zero (as proved also in [2]). If X is a Riemann surface of genus g, we have a commutative diagram of canonical homomorphisms

$$\begin{array}{ccc} H^2(B(X), S(X)) & \xrightarrow{j^*} & H^2(B(X)), \\ \wr\Vert & & \wr\Vert \\ H^0(X) & \xrightarrow{\cdot\chi(X)} & H^2(X), \\ \wr\Vert & & \wr\Vert \\ \mathbf{Z} & \longrightarrow & \mathbf{Z}, \end{array}$$

where the bottom map is given by multiplication with the Euler characteristic $2 - 2g$ of X. Thus if X is not the torus, j^* is injective and π^* is surjective, and again index $\equiv 0$. In the case of the torus however there might exist elliptic operators from the trivial line bundle into itself with nonvanishing index.

Next we allow E, F to be arbitrary line bundles. We denote by $S'(X) = B^+(X) \cup_{S(X)} B^-(X)$ the n-sphere bundle over X obtained by gluing together two copies $B^+(X)$ and $B^-(X)$ of the unit ball bundle $B(X)$. Then any isomorphism σ between $\pi^*(E)$ and $\pi^*(F)$ gives rise to a line bundle over $S'(X)$: we lift E to $B^+(X)$ and F to $B^-(X)$ and identify them along $S(X)$ by means of σ. This gives a perfectly good alternative description of σ. On the other hand each line bundle over $S(X)$ can be constructed starting from such a σ. Hence we can define the index on $H^2(S'(X), \mathbf{Z}) \cong \mathrm{Vect}^1(S'(X))$. (Here $\mathrm{Vect}^1(S'(X))$ denotes the set of isomorphism classes of complex line bundles over $S'(X)$, made into a group by means of the tensor product. The group isomorphism $\mathrm{Vect}^1(S'(X)) \cong H^2(S'(X))$ is given by the first Chern class.) Furthermore it follows from the construction that our index function vanishes on $\pi'^*(H^2(X))$ where $\pi' : S'(X) \to X$ is the bundle projection. As π' admits a section, $H^2(S'(X))$ is the direct sum of $\pi'^*(H^2(X))$ and $H^2(S'(X), B^-(X))$, and it is clear that we lose no information when we restrict the index function to $H^2(S'(X), B^-(X)) = H^2(B(X), S(X))$.

If $\dim X > 2$, again index $\equiv 0$ in this setting. However, if X is a Riemann surface other than the torus, $H^2(B(X), S(X))$ is isomorphic to Z, and for arbitrary line bundles E, F there is no longer any reason to think that the index of all elliptic operators from E to F is zero. In fact we can give a simple example of an elliptic differential operator with nonvanishing index: the operator

$$\bar{\partial} : f \to \frac{\partial f}{\partial \bar{z}} d\bar{z}$$

is elliptic, $\ker \bar{\partial}$ consists of the constants, $\operatorname{coker} \bar{\partial} \cong \ker \bar{\partial}^*$ consists of the anti-holomorphic differentials and hence (from the classical theory of Riemann surfaces) index $\bar{\partial} = 1 - g \neq 0$ (since the genus is $g \neq 1$).

Now let X again be a compact manifold of arbitrary dimension, and let E, F be N-dimensional trivial bundles over X. Then the index function takes the form

$$\text{index}: [S(X), \mathrm{GL}(N, C)] \to Z.$$

As we saw earlier, $[S(X), \mathrm{GL}(N, C)]$ has a nice cohomological interpretation when $N = 1$. For $N > 1$ however, this group is not even always commutative. It is therefore convenient to stabilize it.

Thus let

$$C^* = \mathrm{GL}(1) \subset \mathrm{GL}(2) \subset \ldots \subset \mathrm{GL}(N) \subset \ldots \subset \mathrm{GL}(2N) \subset \ldots$$

be the obvious sequence of imbeddings, and denote by $\mathrm{GL}(\infty)$ the topological group $\lim_{N\to\infty} \mathrm{GL}(N)$. If Y is a compact space and $A, B: Y \to \mathrm{GL}(N)$ are continuous maps, then by a rotation argument one can show that the product map $A \cdot B$ is homotopic to the direct sum map $A \oplus B$ as maps from Y into $\mathrm{GL}(2N)$. Hence the group

$$K^1(Y) := [Y, \mathrm{GL}(\infty)] = \lim_{N\to\infty} [Y, \mathrm{GL}(N)]$$

is abelian.

It is now easy to see the above form of the index function gives rise to a group homomorphism index: $K^1(S(X)) \to Z$. We merely have to note that if P and Q are elliptic operators of order 0, then index $(P \oplus Q)$ = index P + index Q, and $\sigma(P \oplus Q) = \sigma(P) \oplus \sigma(Q)$; for the identical operator on a trivial bundle of dimension N the index vanishes and the symbol is a constant function with value $\mathrm{Id} \in \mathrm{GL}(N)$. Furthermore, in analogy with the case $N = 1$, this homomorphism vanishes on the image of $\pi^* : K^1(X) \to K^1(S(X))$.

Finally let P be an elliptic operator from an arbitrary smooth vector bundle over X into another. As in the case of line bundles, the symbol of P determines a vector bundle $V_{\sigma(P)}$ over $S'(X) = B^+(X) \cup_{S(X)} B^-(X)$, and we obtain a homomorphism of semigroups index: $\mathrm{Vect}\,(S'(X)) \to \mathbf{Z}$, defined by $V_{\sigma(P)} \mapsto$ index P. Here $\mathrm{Vect}\,(S'(X))$ denotes the semigroup of isomorphism classes of vector bundles over $S'(X)$, with addition given by the direct sum. By the following method, due to Grothendieck, we can make $\mathrm{Vect}\,(S'(X))$ into a group.

Quite generally let A be an abelian semigroup. Define B to be the quotient of $A \times A$ under the following equivalence relation: $(a_1, a_2) \sim (a'_1, a'_2)$ if there exist $a, a' \in A$ such that $(a_1 + a, a_2 + a) = (a'_1 + a', a'_2 + a')$. Then B is an abelian group; the map $j: A \to B$ which maps $a \in A$ into the equivalence class of $(a, 0)$ is additive; and each additive map from A into a group factors uniquely through j.

In particular if $A = \mathrm{Vect}\,(Y)$ is given by the (complex) vector bundles over a compact space Y, one denotes by $K(Y)$ the corresponding group. Furthermore, if Z is a closed subspace of Y, we define $K(Y, Z)$ to be the kernel of the obvious restriction $K(Y/Z) \to K(Z/Z)$. Here Y/Z is the space obtained from Y by collapsing Z to a point, and Z/Z is this resulting point.

We can now interpret our index function as a group homomorphism from $K(S'(X))$ into $\mathbf{Z}$. Again, as in the case of line bundles, we lose no information when we restrict it to $K(S'(X), B^-(X)) = K(B(X), S(X))$. Thus we will consider it from now on as a group homomorphism index: $K(B(X), S(X)) \to \mathbf{Z}$.

In examining the topological role of the symbol of elliptic operators we have been led to introduce the functors K and K^1. In fact they fit into an extraordinary cohomology theory—K-theory—which is the appropriate setting for the study of the index. In particular we have coboundary operators, and our two last interpretations of the index are related to one another by the following commutative diagram

$$\begin{array}{ccc} K^1(S(X)) & \xrightarrow{\delta} & K(B(X), S(X)) \\ & \text{index} \searrow \quad \swarrow \text{index} & \\ & \mathbf{Z} & \end{array}$$

II. **Classical Examples.** We will now give some concrete examples of elliptic differential operators. This will not only motivate our investigation of the index problem, but also give us some idea what the index formula should look like.

First we study purely locally homogeneous elliptic differential operators with constant coefficients. Such an operator can be given by its symbol over one base point, i.e. by an $N \times N$ matrix $p(\xi)$ (whose entries are homogeneous polynomials of a fixed degree in $\xi = (\xi_1, \ldots, \xi_n)$) such that $\det(p(\xi)) \neq 0$ for $\xi \neq 0$. In this terminology e.g. the Laplace operator Δ corresponds to $\sum \xi_i^2 \cdot \mathrm{Id}$.

Since many important elliptic operators are of order 1 or can be decomposed into first order operators, it is reasonable to ask whether $-\Delta$ admits a square root. Thus we have to look for $N \times N$-matrices $A_1, \ldots, A_n$ such that

$$\left(\sum_{i=1}^{n} A_i \xi_i\right)^2 = -(\sum \xi_i^2)\,\mathrm{Id},$$

or, equivalently, such that

$$(*) \qquad A_i^2 = -\mathrm{Id} \quad \text{for } i = 1, \ldots, n; \qquad A_i A_j = -A_j A_i \quad \text{for } i \neq j.$$

These relations generate abstractly the complexified Clifford algebra $C_n = \mathrm{Cliff}_{\boldsymbol{C}}(\boldsymbol{R}^n)$ of $\boldsymbol{R}^n$, and the solutions of (*) correspond to the representations of C_n on $\boldsymbol{C}^N$. If n is even, C_n is simple and hence a matrix algebra. Therefore the equations (*) have a solution with $N^2 = 2^n = \dim C_n$, i.e. $N = 2^{n/2}$. The corresponding first order differential operator is called the *Dirac operator.*

Since C_n is simple for even n, there is no square root of $-\Delta$ when $0 < N^2 < 2^n$. However, it follows from the general theory of Clifford algebras that the Dirac operator can be written in the form

$$\begin{pmatrix} 0 & D^+ \\ D^- & 0 \end{pmatrix}$$

where D^+ and D^- are adjoint to one another. Hence D^+ is a solution to the modified condition $D^+ \cdot D^{+*} = -\Delta$.

We will later give an explicit description of the Dirac operator and of D^+, and we will see that e.g. for $n = 2$, D^+ is just the Cauchy-Riemann operator $\bar{\partial}$. Now we only state two important properties of D^+. First, D^+ has minimal N, i.e. D^+ is defined on systems of $N = 2^{n/2-1}$ complex valued functions, and there is no first-order elliptic differential operator over an open subset of $\boldsymbol{R}^n$ defined on a small number of functions. This follows from the results of J. F. Adams, and from the remarks in §I concerning the space $A(N, n, 1)$. Second, we have the "*local Bott theorem*": the symbol of D^+

$$\sigma(D^+) : S^{2n-1} \to \mathrm{GL}(2^{n-1}, \boldsymbol{C})$$

generates the infinite cyclic group $\pi_{2n-1}(\mathrm{GL}(\infty))$.

Next we indicate how, in the setting of exterior differential forms, one can find another square root of the Laplace operator. For $0 \leq p \leq n$ let Ω^p denote the space of differentiable p-forms over an open set $U \subset \boldsymbol{R}^n$, and let $d : \Omega^p \to \Omega^{p+1}$ be the exterior derivation. We are going to make d into a square system. Thus, for p-forms α, β with compact support, define an inner product by

$$\langle \alpha, \beta \rangle = \quad \alpha \wedge * \beta,$$

where $* : \Omega^p \to \Omega^{n-p}$ is the $C^\infty(U)$-linear operator with $*(dx^1 \wedge \ldots \wedge dx^p) = dx^{p+1} \wedge \ldots \wedge dx^n$ etc. We can then define the formal adjoint $d^* : \Omega^{p+1} \to \Omega^p$ of d by the following condition: $\langle d\alpha, \beta \rangle = \langle \alpha, d^*\beta \rangle$ for $\alpha \in \Omega^p$, $\beta \in \Omega^{p+1}$ with compact support. Clearly

$$D = d + d^* : \Omega = \sum_{p=0}^{n} \Omega^p = C^\infty(U \times \Lambda^* C^n) \to \Omega,$$

is a first order differential operator, and we have

$$D^2 = DD^* = dd^* + d^*d = -\Delta.$$

Thus, being a square root of the Laplacian, D is elliptic. Since $\dim(\Lambda^* C^n) = 2^n$, the symbol of D corresponds to the regular representation of $\mathrm{Cliff}_C(R^n)$ on itself.

Similarly if U is an open subset of $R^{2n} = C^n$ let $\Omega^{0,p}$ denote the space of exterior differentials of the form $a_{i_1,\dots,i_p}\, d\bar z_{i_1} \wedge \dots \wedge d\bar z_{i_p}$. Then the exterior derivation $\bar\partial$ gives rise to a first order elliptic operator $\bar\partial + \bar\partial^*$ whose square is the Laplace operator. Hence the symbol of $\bar\partial + \bar\partial^*$ defines a (nonzero) homomorphism from $C_{2n} = \mathrm{Cliff}_C(R^{2n})$ into the matrix algebra $\mathrm{End}_C(\Lambda^*(C^n))$. As both algebras have dimension 2^{2n} and as a full matrix algebra is simple, it follows that this homomorphism is in fact an isomorphism

$$\mathrm{Cliff}_C(R^{2n}) \cong \mathrm{End}_C(\Lambda^*(C^n)).$$

This gives an explicit description of the minimal representation of C_{2n} which we used when defining the Dirac operator. Thus $\bar\partial + \bar\partial^*$ is just the Dirac operator, and its splitting into

$$\begin{pmatrix} 0 & \bar D^+ \\ \bar D^- & 0 \end{pmatrix}$$

can be given by

$$\bar D^+ = (\bar\partial + \bar\partial^*)\Big| \sum_{p=0}^{n} \Omega^{0,2p}, \qquad \bar D^- = (\bar\partial + \bar\partial^*)\Big| \sum_{p=0}^{n} \Omega^{0,2p+1}.$$

Next we study global versions of some of the local elliptic operators investigated so far. Thus let X be a closed Riemannian manifold. Then we have the exterior differentiation d operating on the space Ω of (complexified) global exterior differential forms, and the Riemannian structure defines an adjoint d^*. Also $D = d + d^*$ is an elliptic differential operator over X and hence has an index. Since D is selfadjoint, this index vanishes.

D does not preserve the degree of a differential form; however, it preserves the parity. Thus, if we define $\Omega^+ = \sum_{p\,\mathrm{even}} \Omega^p$, $\Omega^- = \sum_{p\,\mathrm{odd}} \Omega^p$, we can split D into two operators

$$D^+ : \Omega^+ \to \Omega^-, \qquad D^- : \Omega^- \to \Omega^+$$

which are adjoint to one another. D^+ is elliptic (since D is), and its index is given by

$$\mathrm{index}\, D^+ = \dim\ker D^+ - \dim\ker D^-.$$

Next we show that $\ker D = \ker D^2$. We have only to see that for $u \in \ker D^2$ Du vanishes. But this is clear because $0 = \langle D^2 u, u\rangle = \langle D^* D u, u\rangle = \langle Du, Du\rangle$.

Now $D^2 = -\Delta$ is the Laplace operator. By definition its kernel is formed by the harmonic forms on X. Thus, if H^p denotes the space of harmonic p-forms,

$$\ker D^+ = \sum_{p \text{ even}} H^p, \qquad \ker D^- = \sum_{p \text{ odd}} H^p,$$

and we have

$$\operatorname{index} D^+ = \sum (-1)^p \dim H^p.$$

According to the main theorem of Hodge theory, H^p is isomorphic to the cohomology group $H^p(X, \mathbf{C})$. Thus $B^p = \dim H^p$ is the pth Betti number of X, and the index of D^+ is just the Euler characteristic $\chi(X) = \sum(-1)^p B^p$ of X.

It is a well-established result in algebraic topology that $\chi(X)$ vanishes if X admits a tangent vector field without zeros. Now we can use the above index interpretation of $\chi(X)$ to give a proof of this fact which stresses more the operator point of view.[2] We first observe that the symbol $\sigma(D)$ is given by exterior and interior multiplication on the left in $\Lambda^*(T^*(X))$. A nonvanishing vector field on X gives rise, by exterior and interior multiplication on the right, to a bundle isomorphism which commutes with $\sigma(D)$, and interchanges $\sigma(D^+)$ with $\sigma(D^-)$. Therefore $\operatorname{index} D^- = \operatorname{index} D^+$. On the other hand, since $D^- = D^{+*}$, $\operatorname{index} D^- = -\operatorname{index} D^+$. We thus obtain finally that $\chi(X) = \operatorname{index} D^+ = 0$.

Now assume that X is an orientable compact Riemannian manifold with $\dim X = 4k$. Then there is a canonical isomorphism $*: \Omega^p \to \Omega^{4k-p}$ such that on $\Omega^p \quad (*)^2 = (-1)^p$. If we define

$$\tau : \Omega \to \Omega$$

by $\tau|\Omega^p = (-1)^{(p(p-1)+2k)/2} \cdot *$, τ is an involution. Hence Ω splits into the eigenspace Ω_+ of $+1$ and the eigenspace Ω_- of -1. Since $D\tau = -\tau D$, D also can be decomposed into the direct sum of the elliptic operators $D_+ : \Omega_+ \to \Omega_-$, $D_- : \Omega_- \to \Omega_+$. D_+ and D_- are adjoint to one another, and therefore

$$\operatorname{index} D_+ = \dim\ker D_+ - \dim\ker D_- .$$

Clearly $\ker D_\pm = \{\alpha \text{ harmonic form} | \tau = \pm\tau\alpha\}$. Moreover we may restrict our attention to real harmonic $2k$-forms, because the contributions to the index from the other dimensions cancel out, and on H^{2k} $\tau = *$ is real. Now if $\alpha \neq 0$ is a real harmonic $2k$-form with $\alpha = \pm *\alpha$, then $\pm \int_X \alpha \wedge \alpha = \int_X \alpha \wedge *\alpha > 0$, and hence the corresponding expression in real cohomology $\pm(\alpha \cup \alpha)[X]$ is positive. Thus $\operatorname{index} D_+$ is the difference of the dimensions of maximal subspaces of $H^{2p}(X, \mathbf{R})$ on which the nondegenerate quadratic form $\alpha \mapsto (\alpha \cup \alpha)[X]$ is positive, resp. negative, definite. In other words the index of D_+ equals the signature (or index) of the manifold X.

Let us remark here that one can also interpret $\operatorname{sign}(X)$ as the index of the elliptic operator $d^*d - dd^* = (d + d^*)(d - d^*) : \Omega_+^{2k} \to \Omega_-^{2k}$.

[2] For fuller details and generalizations of this argument see: M. F. Atiyah, *Vector fields on manifolds*, Arbeitsgemeinschaft für Forschung des Landes Nordrhein-Westfalen (1969).

The signature sign (X) of X is related to the Euler characteristic by sign $(X) \equiv \chi(X) \pmod 2$. However, sign (X) is a much more sophisticated invariant. For example if X has a nonvanishing vector field, we can still have sign $(X) \neq 0$. The argument which shows that in this case $\chi(X)$ vanishes, does not necessarily extend to the signature, because the definition of D_+ involves the Riemannian metric on X in a much more essential way than the definition of D^+ does. However, if the vector field preserves the Riemannian metric, i.e. if it is an infinitesimal isometry, then sign $(X) = 0$.

As an example we determine $\chi(X)$ and sign (X) when X is the projective space $P^2(\mathbf{C})$. One knows that there is a class $c_1 \in H^2(P^2(\mathbf{C}), \mathbf{R})$ such that $H^*(P^2(\mathbf{C}), \mathbf{R})$ is the truncated polynomial algebra $\mathbf{R}[c_1]/(c_1^3)$. Hence $\chi(P^2(\mathbf{C})) = 3$ and sign $(P^2(\mathbf{C})) = 1$.

In the last examples we assumed the main theorem of Hodge theory without proof. However, we can deduce it, and even a more general theorem on elliptic complexes, from [2].

First we digress to show how the regularity results of [2], expressed in terms of the Sobolev spaces H^s, lead quickly to statements in C^∞. Thus let

$$P: C^\infty(E) \to C^\infty(F)$$

be an elliptic pseudo-differential operator of order m and let

$$P_s: H^s(E) \to H^{s-m}(F)$$

be the induced operator on the Sobolev spaces. Let P^t, P_s^t denote the corresponding transposed operators. Then the regularity results in [2] assert that

(i) Ker $P_s^t =$ Ker P^t for all s.

(ii) P_s has closed range.

Thus if $f \in C^\infty(F) \subset H^{s-m}(F)$ we can write $f = P_s(e) + h$, where $e \in H^s(E)$ and $h \in \text{Ker } P^t \subset C^\infty(F)$. By the regularity (hypoellipticity) of P this equation implies that $e \in C^\infty(E)$. Thus we deduce that $C^\infty(F)$ is the direct sum of $P(C^\infty(F))$ and Ker P^t.

Now let

$$0 \to C^\infty(E^0) \xrightarrow{d_0} C^\infty(E^1) \xrightarrow{d_1} C^\infty(E^2) \xrightarrow{d_2} \dots \to C^\infty(E^m) \to 0$$

be a complex of differential operators over the closed manifold X, i.e. $E^0, E^1, \dots, E^m$ are smooth complex vector bundles, and $d_0, d_1, \dots$ are differential operators of a fixed order such that $d \cdot d = 0$. If for each $x \in X$ and $\xi \in T_x^*(X) - \{0\}$ the leading symbol sequence

$$0 \to E_x^0 \xrightarrow{\sigma(d_0)_\xi} E_x^1 \xrightarrow{\sigma(d_1)_\xi} E_x^2 \to \dots \to E_x^m \to 0$$

is exact, then the complex is called *elliptic*. This definition extends in a natural way the notion of elliptic operators ($m = 1$). On the other hand an elliptic complex can easily be transformed into an elliptic operator. If an elliptic complex as above is given, we can choose hermitian inner products on $E^0, \dots, E^m$ and a measure on

X and thus introduce adjoint operators d_i^*. Clearly the exactness of the symbol sequence implies the ellipticity of the operator

$$D = d + d^* : C^\infty(\bigoplus_0^m E^i) \to C^\infty(\bigoplus_0^m E^i).$$

Now define the ith cohomology H^i of our complex by

$$H^i\,(\text{complex}) = \operatorname{Ker} d_i / \operatorname{Image} d_{i-1}.$$

We claim that this vector space is of finite dimension, or more precisely that the obvious map

$$\operatorname{Ker}(D^2 | C^\infty(E^i)) \subset \operatorname{Ker} d_i \to H^i$$

is an isomorphism. In fact as we have already explained the regularity theorems in [2] imply that each $u \in C^\infty(E^i)$ can be written in the form $u = d_{i-1}v + d_i^* w + h$, where h is harmonic, i.e. $D^2(h) = 0$. In particular if $u \in \operatorname{Ker} d_i$, then $d_i d_i^* w = 0$ and hence is exact off the zero section of $T^*(X)$. A Riemannian metric on X boundary and a harmonic element. In addition this decomposition is unique because the image of d is orthogonal to the kernel of d^* and thus to the kernel of D^2. This proves the assertion above. Thus we can define the Euler characteristic of the elliptic complex by

$$\chi = \sum(-1)^i \dim H^i\,(\text{complex}).$$

This number generalizes naturally the notion of the index of an elliptic operator. On the other hand it is the index of the elliptic operator

$$D^+ : C^\infty(\bigoplus_{i\ \text{even}} E^i) \to C^\infty(\bigoplus_{i\ \text{odd}} E^i)$$

associated to the complex.

A classical example of an elliptic complex is the complex Ω of exterior forms over a closed manifold X

$$0 \to \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \to \dots \to \Omega^n(X) \to 0.$$

Its symbol sequence is given by exterior multiplication with cotangent vectors, and hence is exact off the zero section of $T^*(X)$. A riemannian metric on X determines a choice of the structures needed in the construction above, and we have a canonical isomorphism between the space of harmonic p-forms on X and $H^p(\Omega)$. Together with de Rham's theorem, which gives an isomorphism between $H^p(\Omega)$ and $H^p(X, C)$, this yields the main result of Hodge's theory.

If X is a compact, complex analytic manifold of complex dimension n, the Dolbeault complex

$$0 \to \Omega^0 \xrightarrow{\bar\partial} \Omega^{0,1} \xrightarrow{\bar\partial} \Omega^{0,2} \to \dots \to \Omega^{0,n} \to 0$$

provides an even more interesting and delicate example of an elliptic complex. Here the symbol sequence is given by exterior multiplication in the complex exterior algebra of $\overline{T}^*(X)$. If we introduce a metric, we again get an isomorphism between the space of harmonic p-forms and $H^p(\text{complex}) = \operatorname{Ker} \bar{\partial}^{0,p}/\operatorname{Im} \bar{\partial}^{0,p-1} \cong H^p(X, \mathcal{O})$, where $\mathcal{O}$ is the sheaf of holomorphic functions. Thus the Euler characteristic of X with coefficients in $\mathcal{O}$

$$\chi(X, \mathcal{O}) = \sum (-1)^p \dim H^p(X, \mathcal{O})$$

is the index of the elliptic operator associated to the Dolbeault complex.

In the more special case when X is algebraic, i.e. a complex analytic submanifold of a projective space, the number, $g_p = \dim H^p(X, \mathcal{O}) =$ dimension of the space of holomorphic p-forms, can be interpreted as a higher dimensional genus (for Riemann surfaces g_1 is just the usual genus). $\chi(X, \mathcal{O}) = 1 - g_1 + g_2 - g_3 + \cdots$ is also called the arithmetic genus of X.

A holomorphic vector bundle over a complex manifold X gives rise to a generalized Dolbeault complex,

$$0 \to \Omega^0(V) \overset{\bar{\partial}_V}{\to} \Omega^{0,1}(V) \overset{\bar{\partial}_V}{\to} \Omega^{0,2}(V) \to \cdots.$$

Here $\Omega^{0,p}(V)$ is defined by the use of the tensor product; for instance $\Omega^0(V) = C^\infty(V)$. Locally the definition of $\bar{\partial}_V$ is clear. Since it annihilates the holomorphic transition matrices of V, the extension to a global operator $\bar{\partial}_V$ poses no problem.

The index of the elliptic operator $\bar{D}_V^+$ associated to the complex above has the form

$$\operatorname{index} \bar{D}_V^+ = \chi(X, V) = \sum (-1)^p \dim H^p(X, \mathcal{O}(V)).$$

In favorable situations $H^p(X, \mathcal{O}(V)) = 0$ for $p > 0$, and $\chi(X, V) = \dim H^0(X, \mathcal{O}(V))$. Now $H^0(X, \mathcal{O}(V))$ is the space of holomorphic sections of V, and it is important to determine its dimension. If V is a line bundle with a holomorphic section $\phi \not\equiv 0$, this question can be reduced to another classical problem: the holomorphic sections ψ of V are in one-to-one correspondence with the meromorphic functions ψ/ϕ on X with a prescribed set of poles.

These few remarks indicate that a formula expressing index $\bar{D}_V^+ = \chi(X, V)$ in terms of topological invariants of X and V is of fundamental interest. The solution, which includes the Riemann-Roch Theorem, has been given by F. Hirzebruch [3]. For example if V is a trivial line bundle we get

$$\chi(X, V) = \tfrac{1}{2}(c_1(X))[X] \quad \text{for } \dim X = 1,$$
$$\chi(X, V) = \tfrac{1}{12}(c_1^2(X) + c_2(X))[X] \quad \text{for } \dim X = 2 \text{ etc.},$$

where $c_i(X) \in H^{2i}(X)$ is the ith Chern class of TX. Hirzebruch proved his formulas only for algebraic manifolds; however the general index theorem implies that they are still valid in the general case of a complex manifold X.

Hirzebruch gave also a formula for the index of the operator D_+ canonically associated to a real oriented Riemannian manifold X of dimension $4k$. He showed that the signature of X equals the L-genus. Thus the index of D_+ has the form $\frac{1}{3}p_1[X]$ for $k = 1$, $(1/45)(7p_2 - p_1^2)[X]$ for $k = 2$ etc., where p_i is the ith Pontrjagin class of X. The polynomials used here are related to, but different from, the polynomials involved in the formulas for index $D^+ = \chi(X, V)$ above.

III. The index theorem. In this section we will state the index theorem for elliptic operators, and outline its proof. In other words, we will give an explicit description of the homomorphism index: $K(B(X), S(X)) \to Z$ (defined in §I) in terms of the topological invariants of the compact manifold X. For this purpose we will use the language of K-theory which seems to be most appropriate to the problem. However, it is not difficult to transform the result into a formula involving Chern character, Todd class, and other objects of algebraic topology which might be more familiar.

First we recall some additional notions and results of K-theory. In §I we introduced, for compact Y, the group $K(Y)$ of virtual complex vector bundles over Y. For a locally compact space Y we now define $K(Y)$ to be the kernel of the restriction map $K(Y^+) \to K(+) = Z$, where Y^+ denotes the one-point compactification of Y (with the obvious base point $+$). With this notation if Z is a closed subset of a compact space Y, then $K(Y, Z) = K(Y - Z)$. We can interpret elements of this group as equivalence classes of isomorphisms σ of the form $\sigma: E|Z \cong F|Z$, where E, F are complex vector bundles over Y.

In the special case when X is a compact manifold (with a Riemannian metric), the tangent bundle TX is diffeomorphic to its open ball bundle, and hence $K(TX) = K(B(X), S(X))$. The description of this group by means of isomorphisms σ is particularly meaningful, because here such isomorphisms correspond to symbols of elliptic operators. The index homomorphism thus takes the following form:

$$\text{index}: K(T(X)) \to Z$$
$$[\sigma(P)] \mapsto \text{index } P.$$

We shall refer to this as the *analytical index*.

The tensor product of vector bundles defines a ring structure on $K(X)$, and via the projection, $K(T(X))$ is a module over $K(X)$. A typical example of this module multiplication is the following: if X is a complex manifold, let $[V] \in K(X)$ be represented by a holomorphic vector bundle V, and let $[\sigma(\bar{\partial})]$ (resp. $[\sigma(\bar{\partial}_V)]$) $\in K(T(X))$ be given by the symbol of the operator $\bar{D}^+$ (resp. $\bar{D}_V^+$) corresponding to the Dolbeault complex (of V). Then

$$[\sigma(\bar{\partial}_V)] = [\sigma(\bar{\partial})] \cdot [V].$$

Moreover we can assert the following "*global Bott theorem*": For a compact complex manifold X, $K(T(X))$ is a free module over $K(X)$ with generator $[\sigma(\bar{\partial})]$, i.e. we have an isomorphism of the form

$$K(X) \to K(T(X)) \qquad [V] \mapsto [\sigma(\bar{\partial})] \cdot [V].$$

These results which can be extended to quasi-complex manifolds, show that the situation in the Riemann-Roch theorem is quite typical for the index question. In fact the partial answers contained in the classical Riemann-Roch-Hirzebruch theorem provide the main guide to the general index formula.

The global Bott theorem above holds in greater generality: if E is a complex vector bundle over any (compact) space X, then there is an isomorphism $K(X) \cong K(E)$ given by multiplication with the fundamental element $\lambda_E \in K(E)$ which is derived from the exterior algebra of E. For X = point this assertion is equivalent to the "local Bott theorem" stated in connection with the Dirac operator. The isomorphism $K(X) \cong K(E)$ also holds for locally compact X.

Now we are well prepared to define, for a closed smooth manifold X, a homomorphism i_t from $K(T(X))$ into Z, called the *topological index*, which will turn out to coincide with the analytical index. Embed X in a high dimensional euclidean space E and choose a tubular neighborhood N. Thus N is open in E, but can be thought of as a (real) vector bundle over X, with projection p. Then i_t is defined to be the composition of three homomorphisms:

$$i_t: K(T(X)) \xrightarrow{i_3} K(T(N)) \xrightarrow{i_1} K(TE) \xrightarrow{i_2} Z.$$

Here i_1 is induced by the map $(TE)^+ \to (TN)^+$ which collapses the complement of the open set TN in TE into the compactifying point $+ \in (TN)^+$. The tangent bundle of the euclidean space E can be identified with $E \otimes_R C$, and i_2 is defined to be the canonical isomorphism from $K(E \otimes_R C)$ to $K(\text{point}) = Z$, given by the (local) Bott theorem. A similar (but global) argument leads to the definition of i_3: since $p: N \to X$ is a real vector bundle over X, $Tp: TN \to TX$ can be considered as a complex vector bundle (for $x \in X$ and $\xi \in T_x(X)$ the fiber $(Tp)^{-1}(\xi)$ is $N_x \oplus T(N_x) \cong N_x \otimes_R C$, where $N_x = p^{-1}(x)$). Then the (generalized) global Bott theorem above yields the isomorphism $i_3: K(TX) \to K(TN)$.

The homomorphism i_t is well defined and in particular independent of the choice of the embedding. In fact let $f: X \to E$, $g: X \to E'$ be two embeddings of X into euclidean spaces. Clearly the product embedding $f \times g: X \to E \times E'$ is homotopic through embeddings to both $f \times 0$ and $0 \times g$. Thus, since the construction above is stable and homotopy invariant, we obtain the same topological index, whether we use f or g.

Now we can state the central theorem of these notes.

INDEX THEOREM. *The analytical index equals the topological index.*

We now proceed to outline the proof of the index theorem. The idea is to interpret the transformations i_1, i_2 and i_3 (introduced in the construction of the topological index merely at the symbol level) as operations on operators themselves. At the same time we have to show that the analytical index is not changed by these operations. Thus conceptually the proof of the index theorem is rather simple: given an elliptic operator on X one has to construct an operator of the same index first on N, and finally on the sphere E^+, where the computation of the index is much easier. This method has much in common with A. Grothendieck's approach to the Riemann-Roch problem.

As a preliminary step in the proof, we have to introduce, on noncompact manifolds, an appropriate class of elliptic operators which still have finite dimensional solution spaces. Thus, if U is a possibly noncompact manifold, let P be an elliptic pseudo-differential operator of order zero on U which equals the identity at ∞, i.e. outside a compact subset K of U. This means $P\phi = \phi$ for all smooth sections ϕ with $\operatorname{supp}\phi \cap K = \emptyset$. We furthermore put the same condition on

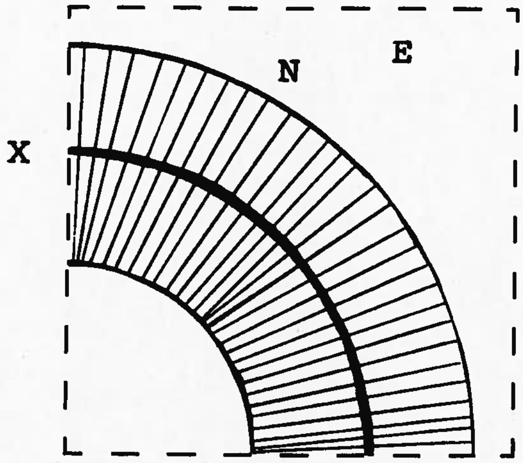


FIGURE 1

the transpose P^t. An equivalent way of describing these conditions is to say that P is represented by a distribution $K(x, y)$ on $U \times U$ which outside $K \times K$ is a delta function on the diagonal.

These restrictions for P guarantee that the main results of the theory of elliptic operators are still valid. In particular the kernel and the cokernel of P are finite dimensional. Furthermore each element of $K(TU)$ can be represented as the symbol class $[\sigma(P)]$ of such an operator P, and the index of P gives rise to a well-defined homomorphism index: $TU \to Z$.

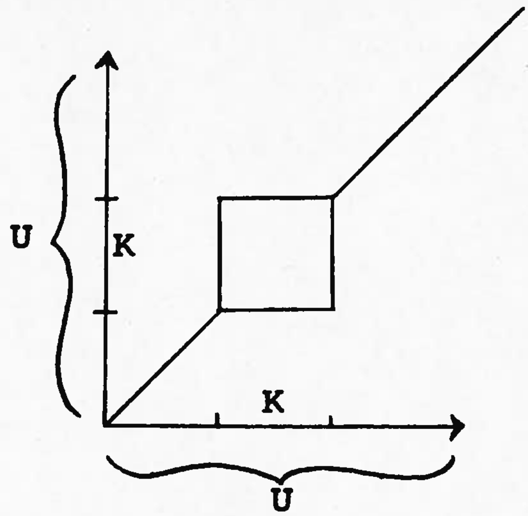
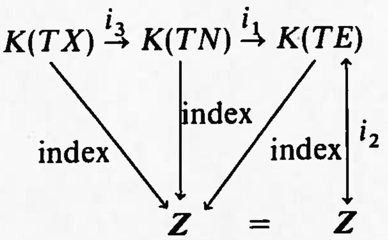


FIGURE 2

If $U \subset V$ is open in a manifold V, we can extend an operator P as above to an operator P_V on V by taking the identity outside U. If $P_V \phi = 0$ on V, then $\operatorname{supp} \phi \subset K \subset U$, since $P\phi = \phi$ outside K. Hence Ker P_V is isomorphic to Ker P by the obvious restriction map. The same applies to the transposed P', and hence index P = index P_V.

Now we come back to the situation in the construction of the topological index. Our extension of the analytical index function to noncompact manifolds now yields the following diagram:



The index theorem is proved once we have shown that all three triangles in this diagram are commutative. Hence we proceed in three steps.

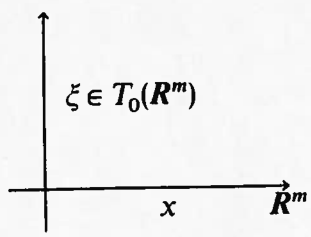
(1) Since N is an open submanifold of E, we can apply the above extension argument, and it is clear that i_1 commutes with the analytical index.

(2) In order to prove the commutativity of the triangle involving i_2 we have to represent the fundamental generator $[\lambda]$ of $K(TE)$ by an elliptic operator and to show that its index is equal to 1. Here λ is derived from the exterior algebra of $TE = \boldsymbol{C}^m$ ($m = \dim_{\boldsymbol{R}} E$) in the usual way: by means of the hermitian metric in $\boldsymbol{C}^m$ we make the sequence (Λ):

$$0 \to \Lambda^0(\boldsymbol{C}^m) \xrightarrow{\wedge u} \Lambda^1(\boldsymbol{C}^m) \xrightarrow{\wedge u} \Lambda^2(\boldsymbol{C}^m) \to \dots \xrightarrow{\wedge u} \Lambda^m(\boldsymbol{C}^m) \to 0$$

of vector bundle homomorphisms over $\boldsymbol{C}^m$ (exact over $u \neq 0$) into a single vector bundle isomorphism over $\boldsymbol{C}^m - \{0\}$:

$$\lambda : \Lambda^{\text{even}}(\boldsymbol{C}^m) \cong \Lambda^{\text{odd}}(\boldsymbol{C}^m).$$



We already know the sequence (Λ) as the local symbol sequence of the Dolbeault complex. However, as we need to get an operator on $E = \boldsymbol{R}^m$ and not on $\boldsymbol{C}^m$ it is necessary to stress rather the real point of view. Thus we write $u \in \boldsymbol{C}^m$ as $x + i\xi \in \boldsymbol{R}^m \oplus T_0(\boldsymbol{R}^m)$, and ($\Lambda$) takes the form

$$0 \to \Lambda^0(\boldsymbol{R}^m) \otimes \boldsymbol{C} \xrightarrow{\wedge(x+i\xi)} \Lambda^1(\boldsymbol{R}^m) \otimes \boldsymbol{C} \to \dots \to \Lambda^m(\boldsymbol{R}^m) \otimes \boldsymbol{C} \to 0.$$

For $x = 0$ we get the local symbol sequence of the de Rham complex.

The main problem is now to find an elliptic operator P with $[\sigma(P)] = [\lambda]$ which is sufficiently simple for us to compute its index. Since P is supposed to be the identity at ∞, P cannot have constant coefficients, nor can it be a *differential* operator (since it is of order zero).

We first consider the case $m = 1$. Thus λ has the form

$$C \cong \Lambda^0(R^1) \otimes C \xrightarrow{\cdot(x+i\xi)} \Lambda^1(R^1) \otimes C = C.$$

As it stands this does not represent a pseudo-differential operator. However, we can deform λ into a map p with

$$\begin{cases} p(x,\xi) = 1 & \text{for } |x| \geq 1, \\ p(x,\lambda\xi) = p(x,\xi) & \text{for } \lambda > 0, \\ p(x,\xi) \neq 0 & \text{for } \xi \neq 0. \end{cases}$$

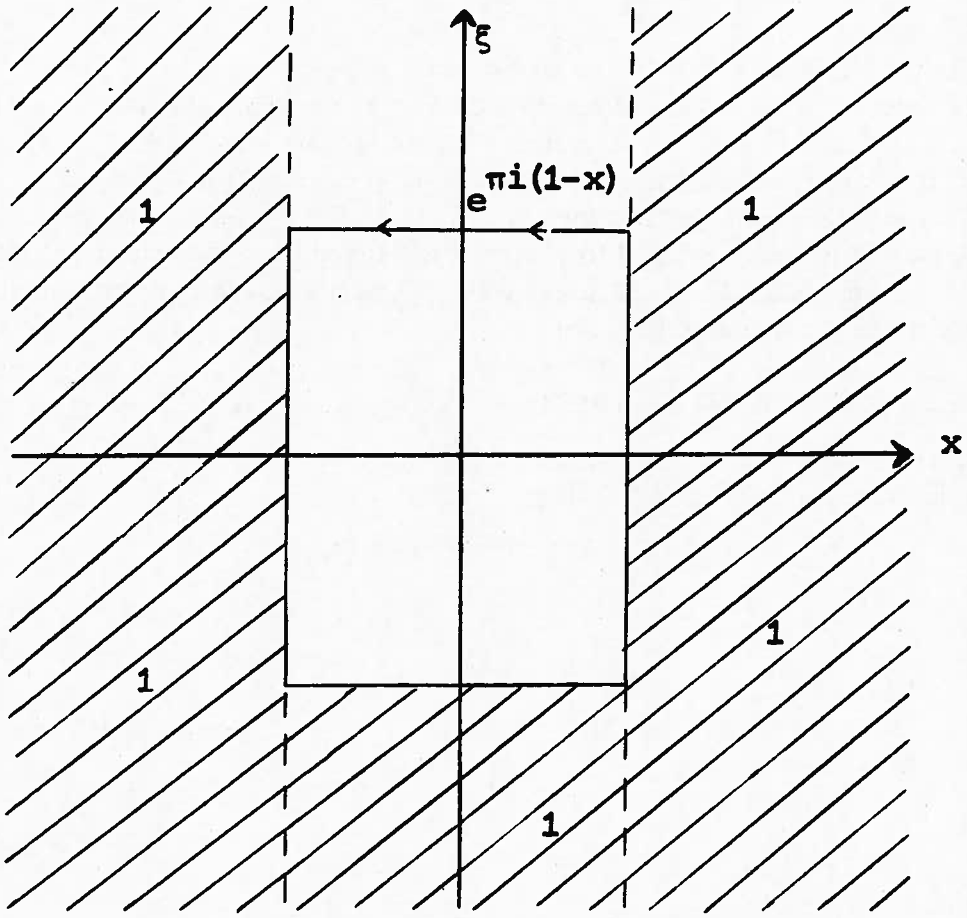


FIGURE 3

This map defines the operator we need, and it is possible to show by straightforward computation that its index is 1.

For $m > 1$ this method however is inconvenient, and we use a different technique. If m is even, we interpret $\boldsymbol{R}^m$ as the top hemisphere of S^m and extend (Λ) trivially on the bottom. Furthermore, if we deform the symbol of the de Rham complex on S^m in a way to make it constant on the equator S^{m-1}, we obtain two copies of (Λ). Thus index $(2[\lambda]) = \text{index } D^+ = \chi(S^m) = 2$, and therefore index $([\lambda]) = 1$. If m is odd, we have to combine the results for $m = 1$ and for even m in order to see also that index $([\lambda])$ is 1.

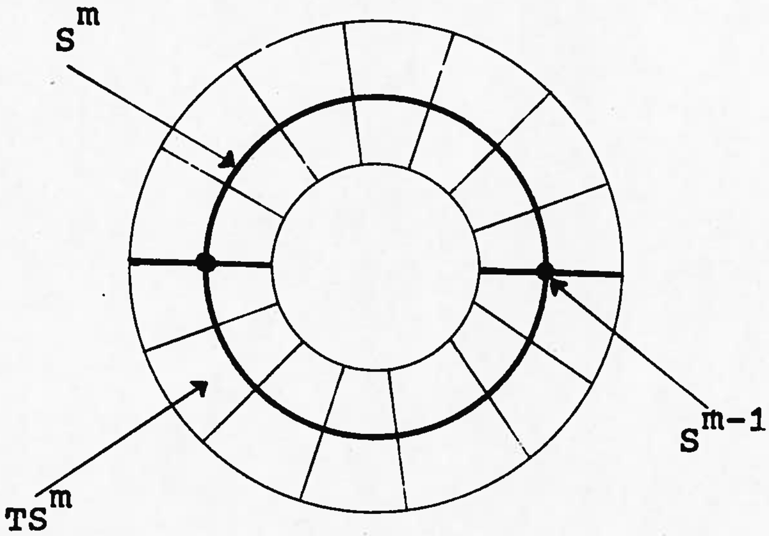


FIGURE 4

A third method of calculating index $([\lambda])$ allows us to construct an operator directly from λ without first deforming it. However, we have to introduce the weight function $e^{(|x|^2)/2}$ and the corresponding L^2-space, and we need a more general class of elliptic operators (with certain growth conditions at ∞ instead of our drastic restrictions). For example, on $\boldsymbol{R}^1$ the operator we look for is $x + d/dx$. Its symbol (in some more general sense) is $x + i\xi$, and its index in this setting turns out to be 1. While the previous methods of computing index $([\lambda])$ were rather topological, the delicacy of this last approach lies in the analysis, and we refer to [**4**] for more details.

For further applications we observe that, because of the spherical symmetry of $e^{(|x|^2)/2}$, $[\lambda]$ can be represented by an operator whose null space etc. is invariant under the action of the orthogonal group on $\boldsymbol{R}^m$. If we are careful enough with the choice of deformations, we can deduce this fact also by the more topological methods above.

(3) We now proceed to the last and most delicate step of the proof: we have to show that the homomorphism $i_3 : K(TX) \to K(TN)$ preserves the analytical index. First we assume that N is trivial, i.e. $N = X \times \boldsymbol{R}^k$. Then i_3 is realized by multiplication of operators on X with the fundamental operator on $\boldsymbol{R}^k$ (or on S^k, if we prefer to work with the compact manifold $X \times S^k$). According to the

last section the index of the fundamental operator on S^k is 1. Hence the case of trivial N is sufficiently dealt with once we know that in fact quite generally the analytical index is multiplicative.

So let Y, Z be compact manifolds, and let (E) and (F) be elliptic complexes on Y and Z respectively:

$$(E): 0 \to E^0 \xrightarrow{d_E} E^1 \xrightarrow{d_E} E^2 \to \dots$$

$$(F): 0 \to F^0 \xrightarrow{d_F} F^1 \to \dots \quad .$$

Then their product (G) is defined by $G^i = \sum_j E^j \otimes F^{i-j}$ and $d_G = d_E \otimes 1 + (-1)^j 1 \otimes d_F$. ($d_E \otimes 1$ denotes partial differentiation by d_E on the y-variables.) If d_E, d_F are differential operators so is d_G and (G) is elliptic. If d_E, d_F are only pseudo-differential then d_G is not pseudo-differential. However, as explained by Nirenberg [2] d_G is a limit of pseudo-differential operators (if order $d_E, d_F > 0$) and everything still works out nicely.

Now we want to see that the index of the elliptic operator associated to G is the product of the indices of the corresponding operators for E and F, i.e. that the Euler characteristics of the above complexes satisfy the equality $\chi(G) = \chi(E) \cdot \chi(F)$. In the special case when E and F are the de Rham complexes of Y and Z, this follows from the isomorphism $H^*(Y \times Z; C) \cong H^*(Y, C) \otimes H^*(Z, C)$ and can be shown by the use of harmonic forms. The same can be carried through for elliptic complexes in general.

If N is not trivial we must generalize the argument above. In the general setting of fiber bundles the index of elliptic operators does not always multiply. However, in our special case we can use the strong symmetry properties of the fundamental operator on the sphere. Thus coordinate changes preserve the fact that i_3 commutes with the analytical index, and we obtain the full global commutativity of the left hand triangle in the diagram in § III(2). This completes the proof.

Let us now add some final remarks on possible extensions and further development of the index theory. First one can study the situation when a compact Lie-group acts on X and when the elliptic operator under consideration is compatible with this action. The above proof can be generalized to this situation, and one can deduce fixed point formulae connecting fixed points of the action of G with global invariants. Another generalization is to extend the index theorem to families of elliptic operators parametrised by a compact space M. In this case the index is an element of $K(M)$. Finally, for a real, skew adjoint operator one has an analog of the index with more refined properties. The dimension of the kernel equals the dimension of the cokernel and is a mod 2 homotopy invariant. An appropriate extension of the index theorem gives a topological description of this invariant.

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