Math. Ann. 326, 589–623 (2003) **Mathematische Annalen**

DOI: 10.1007/s00208-003-0439-4

Computing twisted signatures and L-classes of stratified spaces

Markus Banagl · Sylvain E. Cappell · Julius L. Shaneson

Received: 15 May 2002 / Revised version: 23 September 2002 /
Published online: 8 April 2003 –

Abstract. This paper establishes an Atiyah-type characteristic class formula for both the twisted signature and the twisted L-classes of a singular space. The result, in its present form, applies to coefficient systems that satisfy a transversality condition near the singularities. This condition is frequently automatically satisfied, e.g. on supernormal spaces or for local systems arising in certain geometric mapping situations. We obtain applications to stratified maps by combining our formula with the Cappell-Shaneson signature formula.

Mathematics Subject Classification (2000): 32S50, 32S60, 55N25, 55N33, 55R55, 57N80, 57R20, 57R45, 58A35, 58K10

1. Introduction

Let X be a closed oriented Whitney stratified normal Witt space of even dimension, for example a compact normal complex algebraic variety. Let Σ denote the singular set of X and suppose $\mathcal{S}$ is a local coefficient system (locally constant sheaf) on the top stratum $X - \Sigma$ equipped with a nondegenerate bilinear (symmetric or anti-symmetric) pairing $\phi : \mathcal{S} \times \mathcal{S} \to \mathbb{R}_{X-\Sigma}$ (with $\mathbb{R}_{X-\Sigma}$ the constant sheaf with stalk $\mathbb{R}$ on $X - \Sigma$). If $\mathcal{S}$ is strongly transverse to Σ (definition 3) then it possesses a K-theory signature $[\mathcal{S}]_K$ in the K-theory of X (corollary 2) and our central result (theorem 1) asserts that the twisted signature $\sigma(X; \mathcal{S})$ of X with coefficients in $\mathcal{S}$ can be computed as

$$\sigma(X; \mathcal{S}) = \epsilon_*(\widetilde{\mathrm{ch}}([\mathcal{S}]_K) \cap L(X)), \tag{1}$$

M. Banagl
Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA.
(e-mail: `banagl@math.wisc.edu`)

S.E. Cappell
Courant Institute of Mathematical Sciences, New York University, New York, NY 10012, USA.
(e-mail: `cappell@cims.nyu.edu`)

J.L. Shaneson
Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104, USA.
(e-mail: `shaneson@math.upenn.edu`)

All authors supported in part by NSF grants.

where $\widetilde{\mathrm{ch}}$ is a modified Chern-character and $L(X)$ the Goresky-MacPherson L-class of X. More generally, we prove in theorem 3 that the twisted L-class $L(X;\mathcal{S})$ can be calculated as

$$L(X;\mathcal{S}) = \widetilde{\mathrm{ch}}([\mathcal{S}]_K) \cap L(X) \tag{2}$$

We discuss a corollary for supernormal spaces, and apply the characteristic class formula to stratified maps. This is done by combining (1) with the Cappell-Shaneson signature formula, reviewed in section 1.3, yielding an expression for the signature of the source of a stratified map in terms of the L-classes of the closed strata of the target and the Chern classes of the various local systems associated to the map (theorem 4). We provide illustrative examples by considering situations such as fiber bundle projections to a singular base — the terms arising from the singular strata will be seen to vanish in this situation (theorem 5) — and stratified maps (with changing fiber) to manifolds (proposition 5). In a future paper, we will verify that large classes of orbit spaces of smooth group actions are supernormal, so that strong transversality is satisfied by any local system and the formulae presented here can be applied to compute invariants.

The paper is organized as follows: Section 1.1 reviews Atiyah's classical formula for the signature of the total space of a nonsingular fiber bundle. This is supplemented in section 1.2 by a brief discussion of Werner Meyer's abstract version of that formula which applies to any given local system, not necessarily of geometric origin. Section 1.3 serves to describe some key results of [CS91], in particular the version of the singular signature formula that will be used in this paper. Section 2 discusses the homology L-classes of a Witt space with boundary and clarifies their relation to the classes of the boundary. Section 3 reviews that at odd primes, Witt bordism classes are representable by smooth manifolds. Section 4 defines the notion of "local system with duality" that we require (Poincaré local systems, 4.1), recalls how such systems give rise to a signature invariant via intersection homology, and introduces a transversality condition stated in terms of the representation of a local system by the monodromy functor it induces on the fundamental groupoid (4.2). Furthermore we show that a transverse Poincaré local system has a global K-theory signature. Section 5 contains a statement and proof of the twisted signature formula (1) and section 6 discusses the L-class formula (2). Finally, section 7 gives various applications (notably to stratified maps and singularities of maps), and in section 8 calculations for a concrete example are carried out.

1.1. Atiyah's formula

Let E be an oriented closed smooth manifold, the total space of a fiber bundle $F \to E \to B$. It is a classical result of Chern, Hirzebruch and Serre [CHS57] that if B is simply connected (or the fundamental group of B acts trivially on the cohomology of F), then the signatures satisfy the multiplicative formula

$$\sigma(E) = \sigma(B)\sigma(F).$$

Kodaira [Kod67], Atiyah [Ati69] and Hirzebruch [Hir69] constructed various examples of fiber bundles in which $\pi_1(B)$ acts nontrivially on $H^*(F;\mathbb{R})$ and the signature is not multiplicative. In the case where both B and F are even-dimensional, the Hirzebruch signature theorem and the Atiyah-Singer index theorem were used by Atiyah [Ati69] to obtain a characteristic class formula for $\sigma(E)$ involving a contribution from the $\pi_1(B)$-action on $H^*(F)$: Let $k = \dim(F)/2$. The flat (symmetric if k even, anti-symmetric if k odd) bundle $\mathcal{S}$ over B with fibers $H^k(F_x;\mathbb{R})$ $(x \in B)$, has a real (resp. complex) K-theory signature $[\mathcal{S}]_K \in KO(B)$ for k even (resp. $KU(B)$ for k odd) and the twisted signature theorem is

$$\sigma(E) = \langle \mathrm{ch}([\mathcal{S}]_K) \cup \widetilde{\ell}(B), [B]\rangle$$

with $\widetilde{\ell}$ the modification of the Hirzebruch ℓ-genus defined by

$$\widetilde{\ell}(B) = \prod_{i=1}^{r} \frac{y_i}{\tanh y_i/2} \in H^{4*}(B;\mathbb{Q})$$

and $y_1, \dots, y_r$ notional elements of degree 2 such that the i-th Pontrjagin class of the tangent bundle of B is the i-th elementary symmetric function in $y_1^2, \dots, y_r^2$.

1.2. Meyer's generalization

For a complex vector bundle ξ over a base space B, let

$$\widetilde{\mathrm{ch}}(\xi) \in H^{2*}(B;\mathbb{Q})$$

denote the modified Chern character $\widetilde{\mathrm{ch}} = \mathrm{ch} \circ \psi^2$ obtained by composition with the second Adams operation ψ^2, that is, if ξ has total Chern class

$$c(\xi) = (1+y_1)\cdots(1+y_r)$$

$(r = \mathrm{rank}\xi)$, then

$$\mathrm{ch}(\xi) = \sum_{i=1}^{r} e^{y_i} \in H^{2*}(B;\mathbb{Q})$$

and $\psi^2(\xi)$ is a bundle with

$$\mathrm{ch}(\psi^2(\xi)) = \sum_{i=1}^{r} e^{2y_i}.$$

Hence,

$$\widetilde{\mathrm{ch}}(\xi)_{2k} = 2^k \,\mathrm{ch}(\xi)_{2k} \in H^{2k}(B; \mathbb{Q}).$$

(see [HBJ92]).

Now W. Meyer [Mey72] considers a locally constant sheaf $\mathcal{S}$, not necessarily arising from a fiber bundle projection, over a closed oriented smooth manifold B of even dimension such that the stalks $\mathcal{S}_x$ ($x \in B$) are nondegenerate (anti-) symmetric bilinear forms over $\mathbb{R}$. The twisted signature $\sigma(B; \mathcal{S}) \in \mathbb{Z}$ is defined to be the signature of the nondegenerate form on the sheaf cohomology group $H^{\dim(B)/2}(B; \mathcal{S})$. The twisted signature formula of [Mey72] is

$$\sigma(B; \mathcal{S}) = \langle \widetilde{\mathrm{ch}}([\mathcal{S}]_K) \cup \ell(B), [B] \rangle,$$

where ℓ is the original Hirzebruch ℓ-genus.

1.3. The Cappell-Shaneson signature formula

The previously discussed formulae apply to fiber bundle projections between manifolds. The invariants associated to general stratified maps between stratified singular spaces were investigated by Cappell and Shaneson in [CS91]. The fiber is, in particular, allowed to vary and one is interested in describing the error introduced by the singular contributions in the known formulae for nonsingular maps. We shall review some of the results obtained in [CS91].

Let $f : Y^m \longrightarrow X^n$ be a stratified map of compact oriented Whitney stratified spaces with only even-codimensional strata, $m - n$ even. Let Σ denote the singular set of X and let $\mathcal{X}$ denote the set of components of pure strata of X of codimension at least 2. For $z \in Z \in \mathcal{X}$, let $N(z)$ denote the normal slice to Z at z and $Lk(z) = \partial N(z)$ be the link. Define E_z to be the stratified pseudomanifold obtained by taking the pre-image under f of the normal slice at z and collapsing the boundary to a point, that is,

$$E_z = f^{-1}N(z) \cup_{f^{-1}Lk(z)} c(f^{-1}Lk(z))$$

(cA denotes the cone on a space A, c the cone-point). The space E_z is of dimension $(m - n) + \operatorname{codim} Z = 2d(Z)$ and has only strata of even codimension.

Now assume we are given a self-dual complex of sheaves $\mathbf{S}^\bullet \in D^b(Y)$, where $D^b(Y)$ denotes the constructible bounded derived category. On E_z, we consider the sheaf

$$\mathbf{S}^\bullet(z) = \tau^{\{c\}}_{\leq -d(Z)-1} R i_{z*} j_z^! \mathbf{S}^\bullet,$$

where

$$E_z \overset{i_z}{\hookleftarrow} E_z - \{c\} \overset{j_z}{\hookrightarrow} Y$$

are inclusions and $\tau^{\{c\}}_{\leq -d(Z)-1}$ is truncation over the cone-point. The self-duality of $\mathbf{S}^\bullet$ induces self-duality for $\mathbf{S}^\bullet(z)$ and we have a nondegenerate bilinear pairing on middle-dimensional hypercohomology

$$\phi_z : \mathcal{H}^{-d}(E_z; \mathbf{S}^\bullet(z)) \times \mathcal{H}^{-d}(E_z; \mathbf{S}^\bullet(z)) \to \mathbb{R}.$$

Letting z vary over Z, we thus have a local system $\mathcal{S}^Z_f$ over Z with stalk

$$(\mathcal{S}^Z_f)_z = \mathcal{H}^{-d}(E_z; \mathbf{S}^\bullet(z))$$

and pairing ϕ_z on each stalk $(\mathcal{S}^Z_f)_z$. Let $\mathbf{IC}^\bullet_{\bar{m}}(\overline{Z}; \mathcal{S}^Z_f)$ denote the lower-middle perversity intersection chain complex on the closure $\overline{Z}$ of Z with coefficients in the local system $\mathcal{S}^Z_f$. Over the top stratum $X - \Sigma$, a local system $\mathcal{S}^{X-\Sigma}_f$ is given by considering the stalks ($z \in X - \Sigma$)

$$(\mathcal{S}^{X-\Sigma}_f)_z = \mathcal{H}^{-\frac{1}{2}(m-n)}(f^{-1}(z); j_z^! \mathbf{S}^\bullet),$$

$j_z : f^{-1}(z) \hookrightarrow Y$, together with the appropriate nondegenerate pairings.

Now a principal result of [CS91] asserts that the self-dual sheaf

$$f_* \mathbf{S}^\bullet[-\tfrac{1}{2}(m-n)]$$

is cobordant to

$$\mathbf{IC}^\bullet_{\bar{m}}(X; \mathcal{S}^{X-\Sigma}_f) \oplus \sum_{Z \in \mathcal{X}} \mathbf{IC}^\bullet_{\bar{m}}(\overline{Z}; \mathcal{S}^Z_f)^X[\tfrac{1}{2} \operatorname{codim} Z]$$

(for a sheaf $\mathbf{A}^\bullet$ on $\overline{Z} \subset X$, $(\mathbf{A}^\bullet)^X$ denotes extension by zero to X).

Let us specialize to $\mathbf{S}^\bullet = \mathbf{IC}^\bullet_{\bar{m}}(Y)$. Then

$$\mathbf{S}^\bullet(z) = \mathbf{IC}^\bullet_{\bar{m}}(E_z)$$

and

$$(\mathcal{S}^Z_f)_z = \mathcal{H}^{-d}(E_z; \mathbf{S}^\bullet(z)) = IH^{\bar{m}}_d(E_z).$$

Taking signatures, we get the signature formula

$$\sigma(Y) = \sigma(X; \mathcal{S}^{X-\Sigma}_f) + \sum_{\mathcal{X}} \sigma(\overline{Z}; \mathcal{S}^Z_f) \tag{3}$$

(see definition 2 for the twisted signature $\sigma(-;-)$). Thus the sum over $\mathcal{X}$ represents correction terms contributed by the singularities of the map. In section 7.2 of the present paper, we apply our characteristic class formula for the twisted signature of singular spaces (theorem 1) to formula (3) to calculate the terms on the right hand side for maps f whose coefficient systems $\mathcal{S}_f$ are strongly transverse to the singularities of X.

2. L-Classes for singular spaces with boundary

The present section reviews the construction of homology L-classes of stratified spaces with boundary as well as their relation to the L-classes of the boundary. We adopt the Thom-Pontrjagin construction approach using global transversality as employed by Goresky and MacPherson [GM80], although more refined methods exist: According to [CSW91], see also [CS91], every self-dual complex of sheaves $\mathbf{S}^\bullet \in D^b(X)$ on a closed oriented pseudomanifold X of dimension n has a set of L-classes

$$L_i(\mathbf{S}^\bullet) \in H_i(X; \mathbb{Q}),$$

uniquely determined by the requirements $L_0(\mathbf{S}^\bullet) = \sigma(\mathbf{S}^\bullet) \in H_0(X) \cong \mathbb{Z}$ (X connected, σ the signature of $\mathbf{S}^\bullet$) and $L_{i-n+m}(j^!\mathbf{S}^\bullet) = j^! L_i(\mathbf{S}^\bullet)$ for $j : Y^m \hookrightarrow X^n$ a normally nonsingular inclusion with trivial normal bundle, $j^! : H_i(X; \mathbb{Q}) \to H_{i-n+m}(Y; \mathbb{Q})$ the map given by intersection of cycles with Y (see section 6, note that $j^!\mathbf{S}^\bullet$ is a self-dual sheaf on Y). The intersection chain sheaf $\mathbf{S}^\bullet = \mathbf{IC}^\bullet_{\bar{m}}(X)$ is self-dual for X a Witt space and we obtain L-classes

$$L_i(X) = L_i(\mathbf{IC}^\bullet_{\bar{m}}(X)) \in H_i(X; \mathbb{Q})$$

for Witt spaces X. If $n - i$ is odd, then $L_i = 0$.

Let $(M^n, \partial M)$ be a compact oriented smooth manifold with boundary. The Hirzebruch L-classes ℓ_i of M are the L-classes of the tangent bundle TM of M in cohomology

$$\ell_i(M) = \ell_i(TM) \in H^{4i}(M; \mathbb{Q}).$$

By Poincaré duality $H^{4i}(M) \cong H_{n-4i}(M, \partial M)$, and we have dual homology L-classes

$$L_i(M) \in H_i(M, \partial M; \mathbb{Q}),$$

so that $L_{n-4i} = \ell_i \cap [M]$, where $[M] \in H_n(M, \partial M)$ is the fundamental class. This illustrates that the correct address for L-classes of singular spaces X with boundary is relative homology $H_*(X, \partial X)$.

Now let $(X^n, \partial X)$ be a compact oriented Whitney stratified Witt space with boundary and S^k be a k-sphere with base-point $p \in S^k$. The cohomotopy set $\pi^k(X, \partial X) = [(X, \partial X), (S^k, p)]$ is a group for $2k > n + 1$ and in that range the Hurewicz map is rationally an isomorphism

$$\pi^k(X, \partial X) \otimes \mathbb{Q} \cong H^k(X, \partial X; \mathbb{Q}) \tag{4}$$

Fix a point $q \in S^k$, $q \neq p$. A given continuous map $f : (X, \partial X) \to (S^k, p)$ is homotopic rel ∂X to a map $\tilde{f}$, the restriction of a smooth map on an open neighborhood of X in the ambient manifold implicit in the Whitney stratification, such that

$\tilde{f}$ is transverse regular to q and $\tilde{f}^{-1}(q) \subset \operatorname{int} X$ is transverse to each stratum of X (in fact the modification of f takes place only in $f^{-1}(U_q) \subset \operatorname{int} X$, where $U_q \subset S^k$ is a small open neighborhood of q). Transversality implies that $\tilde{f}^{-1}(q)$ is a Witt space. Thus the intersection chain sheaf $\mathbf{IC}^\bullet_{\bar{m}}(\tilde{f}^{-1}(q))$ is self-dual and $\tilde{f}^{-1}(q)$ has a well defined signature $\sigma(\tilde{f}^{-1}(q))$. Alternatively, observe that as transversality implies the normal nonsingularity of the inclusion $i : \tilde{f}^{-1}(q) \hookrightarrow X$, the restriction $i^!\mathbf{IC}^\bullet_{\bar{m}}(X)$ is a self-dual complex of sheaves on $\tilde{f}^{-1}(q)$ and hence has a signature. If $f, g : (X, \partial X) \to (S^k, p)$ are homotopic transverse maps, then the pre-image $H^{-1}(q) \subset \operatorname{int} X$ under a transverse homotopy rel ∂X, $H : X \times [0, 1] \to S^k$ is a Witt cobordism between $f^{-1}(q)$ and $g^{-1}(q)$, so that the map

$$\begin{aligned} \lambda_k(X) : \pi^k(X, \partial X) &\to \quad \mathbb{Z} \\ [f] \quad &\mapsto \sigma(\tilde{f}^{-1}(q)) \end{aligned}$$

is a well defined homomorphism. Under the identification (4), $\lambda_k(X)$ induces a map

$$\lambda_k(X) \otimes \mathbb{Q} : H^k(X, \partial X; \mathbb{Q}) \longrightarrow \mathbb{Q}$$

defining an element $L_k(X) \in H_k(X, \partial X; \mathbb{Q}) \cong \operatorname{Hom}(H^k(X, \partial X; \mathbb{Q}), \mathbb{Q})$, the *L-class of the Witt space* $(X, \partial X)$. The restriction $2k > n + 1$ is removed by considering products of $(X, \partial X)$ with spheres.

The main purpose of this section is to establish a relation between the L-class of X and the L-class of the boundary ∂X. What we will prove is that $L_{k+1}(X)$ hits $L_k(\partial X)$ under the boundary homomorphism on homology. Consequently, the pushforward of the L-class of the boundary into $H_*(X)$ vanishes.

Proposition 1. *Let* $(X, \partial X)$ *be a compact oriented Whitney stratified Witt space with boundary,* $L_{k+1}(X)$ *its* $(k + 1)$*-th L-class and* $L_k(\partial X)$ *the k-th L-class of the boundary. With* $\partial_* : H_{k+1}(X, \partial X; \mathbb{Q}) \to H_k(\partial X; \mathbb{Q})$ *the homology boundary operator, we have*

$$\partial_* L_{k+1}(X) = L_k(\partial X).$$

Proof. Given $f : \partial X \to S^k$ transverse to $p \in S^k$. We shall describe how the cohomotopy coboundary operator

$$\delta^* : \pi^k(\partial X) \longrightarrow \pi^{k+1}(X, \partial X)$$

acts on $[f]$.
Write $c\partial X$ for the cone on ∂X and view $D^{k+1} \cong cS^k$. Then f extends over the cones as

$$cf : c\partial X \longrightarrow cS^k \cong D^{k+1}.$$

Let $q = (p, \frac{1}{2}) \in cS^k$ and N be an open collar neighborhood of ∂X in X. Consider the collapse maps

$$X \longrightarrow X/(X - N) \cong c\partial X$$

and

$$D^{k+1} \longrightarrow D^{k+1}/S^k \cong S^{k+1}.$$

Denote the images of p and q under the latter collapse again by $p, q \in S^{k+1}$. Then $\delta^*[f]$ is represented by the composition

$$g : (X, \partial X) \to (c\partial X, \partial X) \xrightarrow{cf} (D^{k+1}, S^k) \to (S^{k+1}, p).$$

Observe that g is transverse to q, since in fact $g^{-1}(q) = f^{-1}(p) \times \{\frac{1}{2}\}$ when regarded as a subvariety of the collar $N \cong \partial X \times [0, 1)$. If

$$\lambda_k(\partial X) : \pi^k(\partial X) \longrightarrow \mathbb{Z}$$

is the L-class of ∂X and

$$\lambda_{k+1}(X) : \pi^{k+1}(X, \partial X) \longrightarrow \mathbb{Z}$$

is the L-class of X, then

$$\lambda_{k+1}(X)(\delta^*[f]) = \lambda_{k+1}(X)[g] = \sigma(g^{-1}(q)) = \sigma(f^{-1}(p)) = \lambda_k(\partial X)[f]$$

and so

$$\lambda_{k+1}(X) \circ \delta^* = \lambda_k(\partial X).$$

The commutative diagram

$$\begin{tikzcd} \pi^k(\partial X) \otimes \mathbb{Q} \arrow[r, "\delta^* \otimes \mathbb{Q}"] \arrow[d, "\cong"'] & \pi^{k+1}(X, \partial X) \otimes \mathbb{Q} \arrow[d, "\cong"] \\ H^k(\partial X; \mathbb{Q}) \arrow[r, "\delta^*_H"'] & H^{k+1}(X, \partial X; \mathbb{Q}) \end{tikzcd}$$

shows

$$(\lambda_{k+1}(X) \otimes \mathbb{Q}) \circ \delta^*_H = \lambda_k(\partial X) \otimes \mathbb{Q}.$$

In other words, if $L_{k+1}(X) \in \operatorname{Hom}(H^{k+1}(X, \partial X), \mathbb{Q})$ is the element defined by $\lambda_{k+1}(X) \otimes \mathbb{Q}$ and $L_k(\partial X) \in \operatorname{Hom}(H^k(\partial X), \mathbb{Q})$ is the element defined by $\lambda_k(\partial X) \otimes \mathbb{Q}$, then

$$\operatorname{Hom}(\delta^*_H, \mathbb{Q})(L_{k+1}(X)) = L_k(\partial X)$$

and the commutative square

$$\begin{array}{ccc} \mathrm{Hom}(H^{k+1}(X,\partial X),\mathbb{Q}) & \xrightarrow{\mathrm{Hom}(\delta_H^*,\mathbb{Q})} & \mathrm{Hom}(H^k(\partial X),\mathbb{Q}) \\ \cong\uparrow & & \uparrow\cong \\ H_{k+1}(X,\partial X;\mathbb{Q}) & \xrightarrow[\partial_*]{} & H_k(\partial X;\mathbb{Q}) \end{array}$$

implies

$$\partial_* L_{k+1}(X) = L_k(\partial X).$$

□

3. Representability of Witt spaces

We show that results of Sullivan and Siegel imply that at odd primes, Witt bordism is representable by smooth oriented bordism. This fact will be used subsequently to pull back calculations on singular spaces to calculations on smooth spaces.

Let $\Omega_*^{SO}(X, A)$ denote bordism of smooth oriented manifolds and let $\Omega_*^{Witt}(X, A)$ denote Witt space bordism.

Proposition 2. *For compact PL-pairs* (X, A), *the natural map*

$$\Omega_*^{SO}(X, A) \otimes \mathbb{Z}[\tfrac{1}{2}] \longrightarrow \Omega_*^{Witt}(X, A) \otimes \mathbb{Z}[\tfrac{1}{2}]$$

is surjective.

Proof. Considering the signature as a map $\sigma : \Omega_*^{SO}(pt) \to \mathbb{Z}[\frac{1}{2}]$ makes $\mathbb{Z}[\frac{1}{2}]$ into an $\Omega_*^{SO}(pt)$-module and we can form the homology theory

$$\Omega_*^{SO}(X, A) \otimes_{\Omega_*^{SO}(pt)} \mathbb{Z}[\tfrac{1}{2}].$$

Let $ko_*(X, A)$ denote connected KO homology. Sullivan [Sul70] constructs a natural isomorphism of homology theories

$$\Omega_*^{SO}(X, A) \otimes_{\Omega_*^{SO}(pt)} \mathbb{Z}[\tfrac{1}{2}] \xrightarrow{\simeq} ko_*(X, A) \otimes \mathbb{Z}[\tfrac{1}{2}]$$

(for compact PL-pairs). Siegel [Sie83] shows that Witt spaces provide a geometric description of connected KO homology at odd primes: He constructs a natural isomorphism of homology theories

$$\Omega_*^{Witt}(X, A) \otimes \mathbb{Z}[\tfrac{1}{2}] \xrightarrow{\simeq} ko_*(X, A) \otimes \mathbb{Z}[\tfrac{1}{2}].$$

Now $\Omega_*^{SO}(X, A) \otimes_{\Omega_*^{SO}(pt)} \mathbb{Z}[\frac{1}{2}]$ being a quotient of $\Omega_*^{SO}(X, A) \otimes \mathbb{Z}[\frac{1}{2}]$ yields a natural surjection

$$\Omega_*^{SO}(X, A) \otimes \mathbb{Z}[\tfrac{1}{2}] \longrightarrow \Omega_*^{SO}(X, A) \otimes_{\Omega_*^{SO}(pt)} \mathbb{Z}[\tfrac{1}{2}].$$

The statement follows from the commutative diagram

$$\begin{array}{ccc} \Omega_*^{SO}(X,A)\otimes_{\Omega_*^{SO}(pt)}\mathbb{Z}[\tfrac{1}{2}] & \xrightarrow{\simeq} & ko_*(X,A)\otimes\mathbb{Z}[\tfrac{1}{2}] \\ \uparrow & & \uparrow{\cong} \\ \Omega_*^{SO}(X,A)\otimes\mathbb{Z}[\tfrac{1}{2}] & \longrightarrow & \Omega_*^{Witt}(X,A)\otimes\mathbb{Z}[\tfrac{1}{2}] \end{array}$$

□

4. The twisted signature

4.1. Poincaré Local Systems

Let $(X^n, \partial X)$ be a pseudomanifold with (possibly empty) boundary and filtration

$$X^n = X_n \supset X_{n-2} \supset X_{n-3} \supset \ldots \supset X_0 \supset \varnothing,$$

where the strata are indexed by dimension, the $X_i \cap \partial X$ stratify ∂X, and the $X_i - \partial X$ stratify $X - \partial X$; $\Sigma = X_{n-2}$ is the singular set. If $\partial X = \varnothing$, we assume n even, if $\partial X \neq \varnothing$, we assume n odd. Let $\mathbb{R}_X$ denote the constant sheaf with stalk $\mathbb{R}$ on X.

Definition 1. *A* Poincaré local system on X *is a locally constant sheaf* $\mathcal{S}$ *on* X *together with a nondegenerate bilinear pairing* $\phi : \mathcal{S} \times \mathcal{S} \to \mathbb{R}_X$. *In the case* $\partial X = \varnothing$, *the pairing is required to be symmetric if* $n \equiv 0 (mod\, 4)$ *and anti-symmetric if* $n \equiv 2 (mod\, 4)$. *When* $\partial X \neq \varnothing$, *the pairing is required to be symmetric if* $n \equiv 1 (mod\, 4)$ *and anti-symmetric if* $n \equiv 3 (mod\, 4)$.

Let $(\mathcal{S}, \phi)$ be a Poincaré local system on $X - \Sigma$ (the top stratum of X). The pairing ϕ, being nondegenerate, induces an isomorphism

$$\phi : \mathbf{Hom}(\mathcal{S}, \mathbb{R}_{X-\Sigma}) \xrightarrow{\simeq} \mathcal{S}$$

Now assume $\partial X = \varnothing$. As $X - \Sigma$ is a manifold,

$$\mathcal{D}\mathcal{S}[-n] = \mathbf{Hom}(\mathcal{S}, \mathbb{R}_{X-\Sigma}) \otimes \mathcal{O}_{X-\Sigma},$$

where $\mathcal{O}_{X-\Sigma}$ is the orientation sheaf on $X - \Sigma$ and $\mathcal{D}$ the Borel-Moore-Verdier dualizing functor. Thus ϕ induces an isomorphism

$$\phi : \mathcal{D}\mathcal{S}[-n] \cong \mathcal{S} \otimes \mathcal{O}_{X-\Sigma}.$$

An orientation for X is an isomorphism $\mathcal{O}_{X-\Sigma} \cong \mathbb{R}_{X-\Sigma}$. Assuming X to be oriented, it follows that $\phi : \mathcal{S} \times \mathcal{S} \to \mathbb{R}_{X-\Sigma}$ induces a self-duality isomorphism

$$\phi : \mathcal{D}\mathcal{S}[-n] \cong \mathcal{S} \tag{5}$$

Let

$$\mathbf{IC}^\bullet_{\bar m}(X;\mathcal{S}) = \tau_{\le \bar m(n)-n} Ri_{n*} \dots \tau_{\le \bar m(2)-n} Ri_{2*}\mathcal{S}[n]$$

be the Goresky-MacPherson-Deligne extension of $\mathcal{S}$ to all of X, i.e. the intersection chain complex of X with coefficients in $\mathcal{S}$ (here $i_k : X - X_{n-k} \hookrightarrow X - X_{n-k-1}$ is the inclusion). For certain classes of spaces, the self-duality isomorphism (5) will extend to a self-duality isomorphism for $\mathbf{IC}^\bullet_{\bar m}(X;\mathcal{S})$. Consider the case of a space X with only even-codimensional strata. Then ϕ induces

$$\bar\phi : \mathcal{D}\mathbf{IC}^\bullet_{\bar m}(X;\mathcal{S})[n] \cong \mathbf{IC}^\bullet_{\bar m}(X;\mathcal{S})$$

without any further obstruction; $\bar\phi$ will be symmetric ($\mathcal{D}\bar\phi[n] = \bar\phi$) if ϕ is, and anti-symmetric ($\mathcal{D}\bar\phi[n] = -\bar\phi$) if ϕ is. The case of X being a Witt space is slightly more subtle: ϕ will extend if and only if

$$IH^{\bar m}_k(Lk(x);\mathcal{S}) = 0,$$

for all $x \in X_{n-2k-1} - X_{n-2k-2}$, $k \ge 1$. This condition is satisfied if, for instance, $\mathcal{S}$ is constant on links of odd-codimensional strata.

Now assume X to be closed.

Definition 2. *Let $(\mathcal{S},\phi)$ be a Poincaré local system on $X-\Sigma$ such that a self-dual extension $(\boldsymbol{IC}^\bullet_{\bar m}(X;\mathcal{S}),\bar\phi)$ exists. The* twisted signature $\sigma(X;\mathcal{S})$ of X with coefficients in $\mathcal{S}$ *is the signature of the pairing induced by $\bar\phi$ on the middle dimensional hypercohomology of $\boldsymbol{IC}^\bullet_{\bar m}(X;\mathcal{S})$:*

$$\sigma(X;\mathcal{S}) = \sigma(\boldsymbol{IC}^\bullet_{\bar m}(X;\mathcal{S}),\bar\phi).$$

4.2. Strongly Transverse Poincaré Local Systems

Let $(\mathcal{S},\phi)$ be a Poincaré local system of stalk dimension m on the space X^n and let $\Pi_1(X)$ denote the fundamental groupoid of X. By $\mathfrak{Vect}_m$ denote the category whose objects are pairs (V,ψ), with V an m-dimensional real vector space and $\psi : V \times V \to \mathbb{R}$ a nondegenerate bilinear pairing (symmetric if $n \equiv 0(4)$ and anti-symmetric if $n \equiv 2(4)$), and whose morphisms are linear maps preserving the pairings:

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{Vect}_m}&((V_1,\psi_1),(V_2,\psi_2)) = \\ &\{A : V_1 \to V_2 \text{ linear } | \psi_2(Av, Aw) = \psi_1(v,w),\, v,w \in V_1\}.\end{aligned}$$

The system $(\mathcal{S},\phi)$ induces a covariant functor

$$\mu(\mathcal{S}) : \Pi_1(X) \longrightarrow \mathfrak{Vect}_m$$

as follows: For $x \in X$, let

$$\mu(\mathcal{S})(x) = (\mathcal{S}_x, \phi_x)$$

and for a path class $[\omega] \in \pi_1(X, x_1, x_2) = \mathrm{Hom}_{\Pi_1(X)}(x_2, x_1)$, $\omega : I \to X$, $\omega(0) = x_1$, $\omega(1) = x_2$, define the linear operator

$$\mu(\mathcal{S})[\omega] : \mu(\mathcal{S})(x_2) \longrightarrow \mu(\mathcal{S})(x_1)$$

to be the composition

$$\mu(\mathcal{S})(x_2) = \mathcal{S}_{\omega(1)} \cong (\omega^*\mathcal{S})_1 \underset{\text{restr}}{\overset{\cong}{\leftarrow}} \Gamma(I, \omega^*\mathcal{S}) \underset{\text{restr}}{\overset{\cong}{\rightarrow}} (\omega^*\mathcal{S})_0 \cong \mathcal{S}_{\omega(0)} = \mu(\mathcal{S})(x_1).$$

If we choose a base-point $x \in X$, then restricting $\mu(\mathcal{S})$ to the fundamental group $\pi_1(X, x) = \mathrm{Hom}_{\Pi_1(X)}(x, x)$ gives an assignment of a linear automorphism on the stalk $\mathcal{S}_x$,

$$\mu(\mathcal{S})_x(g) : \mathcal{S}_x \longrightarrow \mathcal{S}_x,$$

preserving the pairing $\phi_x : \mathcal{S}_x \times \mathcal{S}_x \to \mathbb{R}$, to each $g \in \pi_1(X, x)$. Thus one obtains the monodromy representation

$$\mu(\mathcal{S})_x : \pi_1(X, x) \longrightarrow O(p, q; \mathbb{R})$$

when $n \equiv 0(4)$ ($p + q = m$ is the rank of $\mathcal{S}$, $p - q$ the signature of ϕ_x), and

$$\mu(\mathcal{S})_x : \pi_1(X, x) \longrightarrow Sp(2r; \mathbb{R})$$

when $n \equiv 2(4)$ ($m = 2r$ is the rank of $\mathcal{S}$). Conversely, a given functor $\mu : \Pi_1(X) \to \mathfrak{Vect}_m$ determines a Poincaré local system: Let X_0 be a path component of X, and $x_0 \in X_0$. Then $\pi(X_0, x_0)$ acts on $\mu(x_0) = (V, \phi)$ by the restriction μ_{x_0} and we have the associated local system

$$\mathcal{S}|_{X_0} = \widetilde{X_0} \times_{\pi_1(X_0, x_0)} V$$

over X_0 with an induced pairing ϕ, where $\widetilde{X_0}$ denotes the universal cover of X_0.

Definition 3. *Let X be a stratified pseudomanifold with singular set Σ and let $\mathcal{X}$ denote the set of components of open strata of X of codimension at least* 2. *Each $Z \in \mathcal{X}$ has a link $Lk(Z)$. Call a Poincaré local system $\mathcal{S}$ on $X - \Sigma$* strongly transverse to Σ *if the composite functor*

$$\Pi_1(Lk(Z) - \Sigma) \xrightarrow{incl_*} \Pi_1(X - \Sigma) \xrightarrow{\mu(\mathcal{S})} \mathfrak{Vect}_m$$

is isomorphic to the trivial functor for all $Z \in \mathcal{X}$.

The following two auxiliary lemmata have the same sheaf-theoretic identity as a conclusion, but apply in different geometric contexts ($Sh(A)$ denotes the abelian category of sheaves on a space A).

Lemma 1. *Let X be a topological space and $U_1, U_2 \subset X$ open subsets. Consider the diagram of open inclusions*

$$\begin{tikzcd} U_1 \arrow[d, "i"'] & U_1 \cap U_2 \arrow[l, "j|"'] \arrow[d, "i|"] \\ X & U_2 \arrow[l, "j"] \end{tikzcd}$$

If $\boldsymbol{A} \in Sh(U_1)$, then

$$j^* i_* \boldsymbol{A} \cong i|_* j|^* \boldsymbol{A}.$$

Proof. We show that the two sheaves have isomorphic canonical presheaves. Let $V \subset U_2$ be open in U_2. As V is then open in X as well, we have

$$\Gamma(V, j^* i_* \mathbf{A}) = \Gamma(V, i_* \mathbf{A}) = \Gamma(V \cap U_1, \mathbf{A})$$

As $V \cap U_1$ is open in U_1, we obtain on the other hand

$$\Gamma(V, i|_* j|^* \mathbf{A}) = \Gamma(V \cap U_1, j|^* \mathbf{A}) = \Gamma(V \cap U_1, \mathbf{A}).$$

□

Lemma 2. *Let X, Y be pseudomanifolds, $U \subset X$ open, and $j : Y \hookrightarrow X$ a normally nonsingular inclusion with trivial normal bundle. Consider the diagram of inclusions*

$$\begin{tikzcd} U \arrow[d, "i"'] & U \cap Y \arrow[l, "j|"'] \arrow[d, "i|"] \\ X & Y \arrow[l, "j"] \end{tikzcd} \tag{6}$$

If $\boldsymbol{A} \in Sh(U)$ is locally constant, then

$$j^* i_* \boldsymbol{A} \cong i|_* j|^* \boldsymbol{A}.$$

Proof. Let $E(\nu)$ be an open tubular neighborhood of Y in X, the total space of the normal bundle ν of j. Choose a stratum preserving trivialization ϕ of ν,

$$\phi : E(\nu) \xrightarrow{\cong} Y \times \mathbb{R}^k$$

(where k is the codimension of j) such that there is a commutative diagram

$$\begin{tikzcd}
E(\nu) \arrow[rrr, "\cong", "\phi"'] & & & Y \times \mathbb{R}^k \\
& U \cap E(\nu) \arrow[ul, "i_0"'] \arrow[r, "\cong", "\phi|"'] & (U \cap Y) \times \mathbb{R}^k \arrow[ur, "i| \times 1_{\mathbb{R}^k}"'] & \\
& U \cap Y \arrow[u, "j_1|"] \arrow[dl, "i|"'] \arrow[r, "\cong", "1_{U \cap Y}"'] & (U \cap Y) \times \{0\} \arrow[u, "1_{U \cap Y} \times 0"'] \arrow[dr, "i| \times 0"] & \\
Y \arrow[uuu, "j_1"] \arrow[rrr, "\cong", "1_Y"'] & & & Y \times \{0\} \arrow[uuu, "1_Y \times 0"']
\end{tikzcd}$$

(all vertical and diagonal maps are inclusions, all horizontal maps are homeomorphisms.)
The argument is based on the following factorization of diagram (6):

$$\begin{tikzcd}
U \arrow[ddd, "i"'] & U \cap E(\nu) \arrow[l, "j_2|"'] \arrow[ddd, "i_0"'] \arrow[dr, "\phi|"', "\cong"] & & U \cap Y \arrow[ll, "j_1|"'] \arrow[dl, "1_{U \cap Y} \times 0"] \arrow[ddd, "i|"] \\
& & (U \cap Y) \times \mathbb{R}^k \arrow[d, "i| \times 1"'] & \\
& & Y \times \mathbb{R}^k & \\
X & E(\nu) \arrow[l, "j_2"'] \arrow[ur, "\phi", "\cong"'] & & Y \arrow[ll, "j_1"'] \arrow[ul, "1_Y \times 0"']
\end{tikzcd}$$

(all maps are inclusions, except ϕ, $\phi|$.)
Suppose $\mathbf{B}$ is a locally constant sheaf on $(U \cap Y) \times \mathbb{R}^k$. Then $\mathbf{B} \cong ((1_{U \cap Y} \times 0)^*\mathbf{B}) \times \mathbb{R}^k$ and $(i| \times 1)_*((1_{U \cap Y} \times 0)^*\mathbf{B} \times \mathbb{R}^k) \cong (i|_*(1_{U \cap Y} \times 0)^*\mathbf{B}) \times \mathbb{R}^k$, so that $(1_Y \times 0)^*(i| \times 1)_*\mathbf{B} \cong (1_Y \times 0)^*((i|_*(1_{U \cap Y} \times 0)^*\mathbf{B}) \times \mathbb{R}^k) = i|_*(1_{U \cap Y} \times 0)^*\mathbf{B}$. By commutativity, we have $\phi_* i_{0*} = (i| \times 1)_* \phi|_*$. Since ϕ and $\phi|$ are homeomorphisms, $\phi_* \cong (\phi^{-1})^*$ and $\phi|_* \cong (\phi|^{-1})^*$, and thus

$$(\phi^{-1})^* i_{0*} \cong (i| \times 1)_*(\phi|^{-1})^*.$$

By lemma 1, $j_2^* i_* \mathbf{A} \cong i_{0*} j_2|^* \mathbf{A}$ (as all these inclusions are open). Putting $\mathbf{B} = (\phi|^{-1})^* j_2|^* \mathbf{A}$, (which is locally constant), we calculate

$$\begin{aligned}
j^* i_* \mathbf{A} &\cong j_1^* j_2^* i_* \mathbf{A} \cong j_1^* i_{0*} j_2|^* \mathbf{A} \cong (1_Y \times 0)^*(\phi^{-1})^* i_{0*} j_2|^* \mathbf{A} \\
&\cong (1_Y \times 0)^*(i| \times 1)_*(\phi|^{-1})^* j_2|^* \mathbf{A} = (1_Y \times 0)^*(i| \times 1)_* \mathbf{B} \\
&\cong i|_*(1_{U \cap Y} \times 0)^* \mathbf{B} \cong i|_*(1_{U \cap Y} \times 0)^*(\phi|^{-1})^* j_2|^* \mathbf{A} \cong i|_* j_1|^* j_2|^* \mathbf{A} \\
&\cong i|_* j|^* \mathbf{A}.
\end{aligned}$$

□

Lemma 3. *If* $(X, \partial X)$ *is a connected normal pseudomanifold with singular set* Σ, *then* $X - \Sigma$ *is connected.*

Proof. Let $Z \in \mathcal{X}$ be a component of the bottom stratum of X. Then Z is a manifold with (possibly empty) boundary and has a closed neighborhood N_Z in X such that

$$Y = cl(\partial N_Z - \partial X)$$

is the total space of a fiber bundle over Z:

$$\begin{array}{ccc} Lk(Z) & \to & Y \\ & & \downarrow \\ & & Z \end{array}$$

Now the link $Lk(Z)$ is connected by normality, thus Y is connected. Let $x \in cl(X - N_Z)$ and connect x to some point in Z by a path $\omega : I \to X$ using X connected. There exists a $t_0 \in I$ with $\omega(t_0) \in Y$, for example

$$t_0 = \inf\{t \in I | \omega(t) \in N_Z\}.$$

Thus every point of $cl(X - N_Z)$ is in the same component as Y and so $cl(X - N_Z)$ is connected, since Y is connected.
Now apply the same argumentation to the connected normal pseudomanifold $X' = cl(X - N_Z)$, cutting out a neighborhood of a component Z' of the bottom stratum of X'. After having removed neighborhoods of all singular strata, we end up with the connected nonsingular space

$$cl(X - N_\Sigma),$$

where N_Σ is a closed neighborhood of Σ in X. The statement follows as $X - \Sigma$ is homotopy equivalent to $cl(X - N_\Sigma)$. $\square$

Lemma 4. *If X is a normal pseudomanifold, then $i_*\mathbb{R}^m_{X-\Sigma} \cong \mathbb{R}^m_X$, where $i : X - \Sigma \hookrightarrow X$ is the inclusion.*

Proof. We show that the canonical presheaf of $i_*\mathbb{R}^m_{X-\Sigma}$ is isomorphic to the trivial presheaf. Let $V \subset U$ be two connected open subsets of X. The nonsingular parts $U - \Sigma$ and $V - \Sigma$ are non-empty since $X - \Sigma$ is dense in X. By lemma 3, $U - \Sigma$ and $V - \Sigma$ are connected. Thus $\Gamma(U - \Sigma, \mathbb{R}^m_{X-\Sigma}) = \mathbb{R}^m$, $\Gamma(V - \Sigma, \mathbb{R}^m_{X-\Sigma}) = \mathbb{R}^m$, and the diagram

$$\begin{array}{ccc} \Gamma(U - \Sigma, \mathbb{R}^m_{X-\Sigma}) & = & \mathbb{R}^m \\ {\scriptstyle \text{restr}}\downarrow & & \downarrow{\scriptstyle 1_{\mathbb{R}^m}} \\ \Gamma(V - \Sigma, \mathbb{R}^m_{X-\Sigma}) & = & \mathbb{R}^m \end{array} \tag{7}$$

commutes. By definition of the pushforward, we have a commutative diagram

$$\begin{array}{ccc} \Gamma(U, i_*\mathbb{R}^m_{X-\Sigma}) & = & \Gamma(U-\Sigma, \mathbb{R}^m_{X-\Sigma}) \\ {\scriptstyle \text{restr}}\downarrow & & \downarrow{\scriptstyle \text{restr}} \\ \Gamma(V, i_*\mathbb{R}^m_{X-\Sigma}) & = & \Gamma(V-\Sigma, \mathbb{R}^m_{X-\Sigma}) \end{array} \tag{8}$$

Concatenating (7) and (8), we see that $i_*\mathbb{R}^m_{X-\Sigma}$ is the constant sheaf with stalk $\mathbb{R}^m$ on X. □

On normal spaces, strong transversality of local systems characterizes those systems that extend as local systems over the whole space:

Proposition 3. *Let X^n be normal. A Poincaré local system $\mathcal{S}$ on $X - \Sigma$ is strongly transverse to Σ if and only if it extends as a Poincaré local system over all of X. Such an extension is unique.*

Proof. Suppose $(\mathcal{S}, \phi)$ is a given strongly transverse Poincaré local system of rank m on $X - \Sigma$, we will construct its extension to X. By $\mathcal{X}_i$ denote the i-dimensional elements of $\mathcal{X}$. Let $Z \in \mathcal{X}$ be an open stratum. As X is normal, $Lk(Z)$ is connected and itself normal. By lemma 3, $Lk(Z) - \Sigma$ is connected. By assumption

$$\nu : \Pi_1(Lk(Z) - \Sigma) \longrightarrow \Pi_1(X - \Sigma) \xrightarrow{\mu(\mathcal{S})} \mathfrak{Vect}_m$$

is trivial. Now of course $\nu = \mu(\mathcal{S}|_{Lk(Z)-\Sigma})$, hence in particular the monodromy representation $\mu(\mathcal{S}|_{Lk(Z)-\Sigma})_z$ of $\pi_1(Lk(Z) - \Sigma, z)$ is trivial (for any $z \in Lk(Z) - \Sigma$) and the restriction

$$\mathcal{S}|_{Lk(Z)-\Sigma} \cong \mathbb{R}^m_{Lk(Z)-\Sigma} \tag{9}$$

is constant. Set $U_k = X - X_{n-k}$ and $i_k : U_k \hookrightarrow U_{k+1}$ be the inclusion. We will produce the extension by induction on the codimension k of strata. Let

$$\mathcal{S}_2 = \mathcal{S}, \phi_2 = \phi$$

on $U_2 = X - \Sigma$. Then $\mathcal{S}_2$ is locally constant on U_2 and $\mathcal{S}_2|_{Lk(Z_{n-2})}$ is constant, for every $Z_{n-2} \in \mathcal{X}_{n-2}$, using (9) above ($Lk(Z_{n-2}) = Lk(Z_{n-2}) - \Sigma$). Assume inductively that $(\mathcal{S}_k, \phi_k)$ is a Poincaré local system on U_k extending all previous extensions, that is, $(\mathcal{S}_k, \phi_k)|_{U_{k-1}} \cong (\mathcal{S}_{k-1}, \phi_{k-1}), \ldots, (\mathcal{S}_k, \phi_k)|_{U_2} \cong (\mathcal{S}_2, \phi_2)$, and such that

$$\mathcal{S}_k|_{Lk(Z_{n-k})} \cong \mathbb{R}^m_{Lk(Z_{n-k})} \tag{10}$$

is constant, all $Z_{n-k} \in \mathcal{X}_{n-k}$ (note $Lk(Z_{n-k}) \subset U_k$). Put

$$\mathcal{S}_{k+1} = i_{k*}\mathcal{S}_k, \phi_{k+1} = i_{k*}\phi_k.$$

The key issue is to prove that $\mathcal{S}_{k+1}$ is locally constant on U_{k+1} and that

$$\mathcal{S}_{k+1}|_{Lk(Z_{n-k-1})}$$

is constant for every $Z_{n-k-1} \in \mathcal{X}_{n-k-1}$. We show locally constant first. Let $Z_{n-k} \in \mathcal{X}_{n-k}$. It is enough to find, for each $x \in Z_{n-k}$, an open neighborhood $V_x \subset U_{k+1}$ such that $\mathcal{S}_{k+1}|_{V_x} \cong \mathbb{R}^m_{V_x}$. Let V_x be a distinguished neighborhood of x, i.e. there is a homeomorphism

$$\psi : V_x \xrightarrow{\cong} \mathbb{R}^{n-k} \times c^\circ Lk(Z_{n-k})$$

($c^\circ Lk(Z_{n-k}) = [0,1) \times Lk(Z_{n-k})/(0,x) \sim (0,y)$ is the open cone on $Lk(Z_{n-k})$) and a commutative diagram

$$\begin{tikzcd}
U_k \cap V_x \arrow[r, "\cong", "\psi|"'] \arrow[d, "i_k|"'] & \mathbb{R}^{n-k} \times (0,1) \times Lk(Z_{n-k}) \arrow[d, "1\times\iota"] \\
V_x \arrow[r, "\cong", "\psi"'] & \mathbb{R}^{n-k} \times c^\circ Lk(Z_{n-k})
\end{tikzcd}$$

(ι is the inclusion of the punctured cone into the cone). Applying lemma 1 to the diagram of open inclusions

$$\begin{tikzcd}
U_k \arrow[d, "i_k"'] & U_k \cap V_x \arrow[l, "j|"] \arrow[d, "i_k|"] \\
U_{k+1} & V_x \arrow[l, "j"]
\end{tikzcd}$$

we obtain $j^* i_{k*} \mathcal{S}_k \cong i_k|_* j|^* \mathcal{S}_k$. As $\psi|$ is a homeomorphism and $\mathcal{S}_k$ is locally constant, the direct image $\psi|_* j|^* \mathcal{S}_k$ is locally constant. Thus

$$\psi|_* j|^* \mathcal{S}_k \cong \mathbb{R}^{n-k} \times (0,1) \times \mathcal{S}_k|_{Lk(Z_{n-k})}$$

According to our induction hypothesis (10), $\mathcal{S}_k|_{Lk(Z_{n-k})}$ is constant. So $\psi|_* j|^* \mathcal{S}_k$ is constant on $\mathbb{R}^{n-k} \times (0,1) \times Lk(Z_{n-k})$ and consequently, using that $Lk(Z_{n-k})$ is connected, $(1 \times \iota)_* \psi|_* j|^* \mathcal{S}_k$ is constant on $\mathbb{R}^{n-k} \times c^\circ Lk(Z_{n-k})$ (at this point normality is crucial, for if $U_k \cap V_x$ had several components, then the stalks over $\mathbb{R}^{n-k} \times \{c\}$ of the pushforward under $1 \times \iota$ would be a direct sum of copies of $\mathbb{R}^m$, one summand for each connected component). Then

$$\psi_*^{-1}(1 \times \iota)_* \psi|_* j|^* \mathcal{S}_k \cong \mathbb{R}^m_{V_x}$$

and

$$\begin{aligned}\mathcal{S}_{k+1}|_{V_x} &= (i_{k*}\mathcal{S}_k)|_{V_x} = j^* i_{k*} \mathcal{S}_k \cong i_k|_* j|^* \mathcal{S}_k \cong \psi_*^{-1} \psi_* i_k|_* j|^* \mathcal{S}_k \\ &\cong \psi_*^{-1}(1 \times \iota)_* \psi|_* j|^* \mathcal{S}_k \cong \mathbb{R}^m_{V_x}.\end{aligned}$$

Next, we prove that $\mathcal{S}_{k+1}|_{Lk(Z_{n-k-1})}$ is constant, where $Z_{n-k-1} \in \mathcal{X}_{n-k-1}$. Let i be the inclusion $i : X - \Sigma = U_2 \hookrightarrow U_{k+1}$, that is, $i = i_k i_{k-1} \cdots i_2$. The inclusion $j : Lk(Z_{n-k-1}) \hookrightarrow U_{k+1}$ is normally nonsingular with trivial normal bundle, so that lemma 2 applies to $\mathcal{S}_2$ and the diagram

$$\begin{tikzcd} X - \Sigma \arrow[d, "i"'] & (X - \Sigma) \cap Lk(Z_{n-k-1}) = Lk(Z_{n-k-1}) - \Sigma \arrow[l, "j|"'] \arrow[d, "i|"] \\ U_{k+1} & Lk(Z_{n-k-1}) \arrow[l, "j"'] \end{tikzcd}$$

yielding

$$j^* i_* \mathcal{S}_2 \cong i|_* j|^* \mathcal{S}_2.$$

By (9), $j|^* \mathcal{S}_2 = \mathcal{S}|_{Lk(Z_{n-k-1})-\Sigma} \cong \mathbb{R}^m_{Lk(Z_{n-k-1})-\Sigma}$. Therefore,

$$\begin{aligned} \mathcal{S}_{k+1}|_{Lk(Z_{n-k-1})} &= j^* i_* \mathcal{S}_2 \cong i|_* j|^* \mathcal{S}_2 \cong i|_* \mathbb{R}^m_{Lk(Z_{n-k-1})-\Sigma} \\ &\cong \mathbb{R}^m_{Lk(Z_{n-k-1})}, \end{aligned}$$

where the last isomorphism is provided by lemma 4. This finishes the induction step, the sought extension is $(i_* \mathcal{S}, i_* \phi)$ with $i : X - \Sigma \hookrightarrow X$.

For the converse direction, suppose $(\mathcal{S}, \phi)$ is a Poincaré local system on X and $Z \in \mathcal{X}$. A point $z \in Z$ has a distinguished neighborhood homeomorphic to $\mathbb{R}^l \times c^\circ Lk(Z)$. Since the cone $c^\circ Lk(Z)$ is contractible, the restriction $\mathcal{S}|_{c^\circ Lk(Z)}$ is constant. In particular, $\mathcal{S}|_{Lk(Z)-\Sigma}$ is constant. Therefore, the functor

$$\begin{tikzcd} \Pi_1(Lk(Z) - \Sigma) \arrow[r] \arrow[dr, "\mu(\mathcal{S}|_{Lk(Z)-\Sigma})"'] & \Pi_1(X - \Sigma) \arrow[d, "\mu(\mathcal{S}|_{X-\Sigma})"] \\ & \mathfrak{Vect}_m \end{tikzcd}$$

is trivial, and $(\mathcal{S}|_{X-\Sigma}, \phi|_{X-\Sigma})$ is strongly transverse to Σ.

As for uniqueness, given $(\mathcal{S}, \phi)$ on X, we will show $\mathcal{S} \cong i_*(\mathcal{S}|_{X-\Sigma})$, $i : X - \Sigma \hookrightarrow X$. Let $x \in \Sigma$ and $U_x \subset X$ a small connected open neighborhood of x. Note that $U_x - \Sigma \neq \varnothing$, since $X - \Sigma$ is dense in X, and $U_x - \Sigma$ is connected by lemma 3. Thus, using that $\mathcal{S}$ is locally constant, restriction of sections

$$\Gamma(U_x, \mathcal{S}) \xrightarrow{\simeq} \Gamma(U_x - \Sigma, \mathcal{S}) = \Gamma(U_x, i_* \mathcal{S}|_{X-\Sigma})$$

is an isomorphism. □

Remark 1. The normality assumption is not necessary for the "if"-direction.

The following examples show that the normality assumption can not be omitted in the "only if"-direction and in the uniqueness statement.

Example 1. (We suppress mentioning pairings ϕ.) Let X^3 be the pseudomanifold $X = S^1 \times S^1 \times \mathbb{R}/\sim$, where $(x_1, y_1, z_1) \sim (x_2, y_2, z_2)$ iff $x_1 = x_2$ and $z_1 = z_2 = 0$. We denote the image of (x, y, z) under the collapse $S^1 \times S^1 \times \mathbb{R} \to X$ by $(x, [y], z)$. X is stratified as $X \supset \Sigma \supset \varnothing$, with singular set $\Sigma = S^1 \times [S^1] \times \{0\}$. Note that X is not normal, the link of Σ at the point $(x, [y], 0)$ is the disjoint union $\{x\} \times S^1 \times \{-1\} \sqcup \{x\} \times S^1 \times \{+1\}$. We define a local system $\mathcal{S}$ on $X - \Sigma$ as follows: Over $S^1 \times S^1 \times (-\infty, 0)$, let $\mathcal{S}$ be the constant sheaf with stalk $\mathbb{R}$. Let $\mathcal{M}$ denote the Möbius sheaf with stalk $\mathbb{R}$ over a circle. Over $S^1 \times S^1 \times (0, +\infty)$, let $\mathcal{S}$ be $\pi_1^* \mathcal{M}$, $\pi_1 : S^1 \times S^1 \times (0, +\infty) \to S^1$ projection to the first coordinate. Now as $\mathcal{S}_{\{x\} \times S^1 \times \{-1\} \sqcup \{x\} \times S^1 \times \{+1\}}$ is constant for all $x \in S^1$, $\mathcal{S}$ is strongly transverse to Σ. However, an extension $\bar{\mathcal{S}}$ to all of X does clearly not exist, as the restriction of $\bar{\mathcal{S}}$ to say $S^1 \times \{[y]\} \times \mathbb{R}$ would have to be of the form $(\bar{\mathcal{S}}|_{S^1 \times \{[y]\} \times \{0\}}) \times \mathbb{R}$, which is impossible.

Example 2. Consider the projection to the first coordinate $\pi_1 : S^1 \times S^1 \to S^1$ and fix $p \in S^1$. Let X^2 be the singular space obtained from the torus by collapsing $\pi_1^{-1}(p) = \{p\} \times S^1$ to a point. The projection induces a map $f : X \to S^1$ so that

$$\begin{array}{ccc} S^1 \times S^1 & \xrightarrow{\text{coll}} & X \\ & {\scriptstyle \pi_1} \searrow \quad \swarrow {\scriptstyle f} & \\ & S^1 & \end{array}$$

commutes. The local system $\bar{\mathcal{S}} = f^* \mathcal{M}$, with $\mathcal{M}$ as in the previous example, is nontrivial on X as it has nontrivial monodromy around the generator of $\pi_1(X)$. On the other hand, the restriction away from the singularity, $\bar{\mathcal{S}}|_{X-\Sigma}$, is trivial, and so can also be extended trivially into Σ. This shows that if X is not normal, then uniqueness of extensions fails as well.

Proposition 3 allows us to state yet another useful characterization of strongly transverse local systems:

Corollary 1. *Let X^n be normal. A Poincaré local system $\mathcal{S}$ on $X - \Sigma$ is strongly transverse to Σ if and only if its monodromy functor $\mu(\mathcal{S}) : \Pi_1(X - \Sigma) \to \mathfrak{Vect}_m$ factors (up to isomorphism of functors) through $\Pi_1(X)$:*

$$\begin{array}{ccc} \Pi_1(X - \Sigma) & \xrightarrow{incl_*} & \Pi_1(X) \\ {\scriptstyle \mu(\mathcal{S})} & \searrow & \downarrow \\ & & \mathfrak{Vect}_m \end{array}$$

Proof. Suppose $(\mathcal{S}, \phi)$ is strongly transverse. By proposition 3, there exists a unique extension to a Poincaré local system $(\bar{\mathcal{S}}, \bar{\phi})$ on X, which has an associated monodromy functor $\mu(\bar{\mathcal{S}}) : \Pi_1(X) \to \mathfrak{Vect}_m$. Since $\bar{\mathcal{S}}|_{X-\Sigma} = \mathcal{S}$, we have $\mu(\mathcal{S}) \cong \mu(\bar{\mathcal{S}}) \circ \text{incl}_*$.

Conversely, suppose $\mu(\mathcal{S})$ factors as $\mu(\mathcal{S}) \cong \nu \circ \mathrm{incl}_*$, $\nu : \Pi_1(X) \to \mathfrak{Vect}_m$. Then ν determines a Poincaré local system $(\bar{\mathcal{S}}, \bar{\phi})$ on X as described in section 4.2, such that $\mu(\bar{\mathcal{S}}) \cong \nu$. Hence $\mu(\mathcal{S}) \cong \mu(\bar{\mathcal{S}}) \circ \mathrm{incl}_*$ and $(\bar{\mathcal{S}}, \bar{\phi})$ is an extension of $(\mathcal{S}, \phi)$. By proposition 3, $\mathcal{S}$ is strongly transverse to Σ. □

Remark 2. The normality assumption is not necessary for the "if"-direction.

Corollary 2. *Let X^n be normal. A Poincaré local system $(\mathcal{S}, \phi)$ on $X^n - \Sigma$ strongly transverse to Σ has a K-theory signature*

$$[\mathcal{S}]_K \in \begin{cases} KO(X), & \text{if } n \equiv 0(4) \\ KU(X), & \text{if } n \equiv 2(4) \end{cases}$$

Proof. By proposition 3, $(\mathcal{S}, \phi)$ has a unique extension to a Poincaré local system $(\bar{\mathcal{S}}, \bar{\phi})$ on X. We now proceed as in [Mey72]. Let $\mathcal{S}^c$ denote the flat vector bundle associated to the locally constant sheaf $\bar{\mathcal{S}}$, that is

$$\mathcal{S}^c|_{X_0} = \widetilde{X_0} \times_\pi \mathbb{R}^m$$

over a path component X_0 of X, where $\mathbb{R}^m$ is given the usual topology, $\pi = \pi_1(X_0)$, and π acts on $\mathbb{R}^m$ by means of the monodromy $\mu(\bar{\mathcal{S}})$ of $\bar{\mathcal{S}}$. A suitable choice of Euclidean metric on $\mathcal{S}^c$ induces (using $\bar{\phi}$) a vector bundle automorphism

$$A : \mathcal{S}^c \longrightarrow \mathcal{S}^c$$

such that $A^2 = 1$ (if $\bar{\phi}$ is symmetric) or $A^2 = -1$ (if $\bar{\phi}$ is anti-symmetric). Thus in the case $n \equiv 0(4)$, $\mathcal{S}^c$ decomposes as a direct sum of vector bundles

$$\mathcal{S}^c = \mathcal{S}_+ \oplus \mathcal{S}_-$$

corresponding to the ± 1-eigenspaces of A. Put

$$[\mathcal{S}]_K = [\mathcal{S}_+] - [\mathcal{S}_-] \in KO(X).$$

In the case $n \equiv 2(4)$, A defines a complex structure on $\mathcal{S}^c$ and we obtain the complex vector bundle $\mathcal{S}_{\mathbb{C}}$ and its conjugate bundle $\mathcal{S}^*_{\mathbb{C}}$; we put

$$[\mathcal{S}]_K = [\mathcal{S}^*_{\mathbb{C}}] - [\mathcal{S}_{\mathbb{C}}] \in KU(X).$$

□

4.3. Remarks on flatness and characteristic classes

We continue to use the notation of the proof of corollary 2. The bundle automorphism A does in general *not* correspond to an automorphism of sheaves $A : \bar{\mathcal{S}} \to \bar{\mathcal{S}}$, and the bundles $\mathcal{S}_\pm, \mathcal{S}_\mathbb{C}$ are in general *not* flat. Let us discuss the anti-symmetric case. Suppose $(\bar{\mathcal{S}}, \bar{\phi})$ is a Poincaré local system for which $\mathcal{S}_\mathbb{C}$ and $\mathcal{S}^*_\mathbb{C}$ turn out to be flat, so that the classifying map $X \longrightarrow BGL(r;\mathbb{C})$ for $\mathcal{S}_\mathbb{C}$ factors (up to homotopy) as

$$\begin{array}{ccc} B\pi & \xrightarrow{B\mu} & BGL(r;\mathbb{C}) \\ & \nwarrow \quad \nearrow & \\ & X & \end{array}$$

where $B\mu$ is induced by a homomorphism $\mu : \pi \longrightarrow GL(r;\mathbb{C})$ and $X \longrightarrow B\pi$ is the classifying map of the universal cover of X. Now Kamber and Tondeur [KT68] prove that the rational Chern classes in positive degree of flat $GL(r,\mathbb{C})$-bundles are trivial. Hence

$$\widetilde{\mathrm{ch}}[\mathcal{S}]_K = \widetilde{\mathrm{ch}}(\mathcal{S}_\mathbb{C}) - \widetilde{\mathrm{ch}}(\mathcal{S}^*_\mathbb{C}) = 0,$$

and consequently also $\sigma(X;\mathcal{S}) = 0$, by the twisted signature formula (theorem 1). However, in general of course $\sigma(X;\mathcal{S}) \neq 0$ (cf. section 8).

The vanishing results of Kamber-Tondeur yield the following simplifications for the Chern character of the K-theory signature of a Poincaré local system:

Proposition 4. *Let $(\mathcal{S},\phi)$ be a Poincaré local system. If ϕ is symmetric, then*

$$\mathrm{ch}[\mathcal{S}]_K = 2\,\mathrm{ch}(\mathcal{S}_+) - \mathrm{rk}\,\mathcal{S},$$

and if ϕ is anti-symmetric, then

$$\mathrm{ch}[\mathcal{S}]_K = \mathrm{rk}\,\mathcal{S} - 2\,\mathrm{ch}(\mathcal{S}_\mathbb{C}).$$

Proof. We discuss the symmetric case first. The Chern character of a real bundle ξ is defined to be the Chern character $\mathrm{ch}(\xi\otimes\mathbb{C})$ of the complexification of ξ. For $\mathcal{S}^c$, the flat vector bundle associated to $\mathcal{S}$, the complexification $\mathcal{S}^c\otimes\mathbb{C}$ is a flat $GL(p+q;\mathbb{C})$-bundle and by [KT68] has trivial rational Chern classes in positive degrees. Thus

$$\mathrm{rk}\,\mathcal{S} = \mathrm{ch}(\mathcal{S}^c) = \mathrm{ch}(\mathcal{S}_+\oplus\mathcal{S}_-) = \mathrm{ch}(\mathcal{S}_+) + \mathrm{ch}(\mathcal{S}_-),$$

and

$$\mathrm{ch}[\mathcal{S}]_K = \mathrm{ch}(\mathcal{S}_+) - \mathrm{ch}(\mathcal{S}_-) = 2\,\mathrm{ch}(\mathcal{S}_+) - \mathrm{rk}\,\mathcal{S}.$$

Now the anti-symmetric case: The complex bundle $\mathcal{S}_\mathbb{C}$ has $\mathcal{S}^c$ as its underlying real bundle. Therefore,

$$\mathcal{S}^c\otimes\mathbb{C} \cong \mathcal{S}_\mathbb{C}\oplus\mathcal{S}^*_\mathbb{C}.$$

Again, as $\mathcal{S}^c \otimes \mathbb{C}$ is flat,

$$\operatorname{rk}\mathcal{S} = \operatorname{ch}(\mathcal{S}^c \otimes \mathbb{C}) = \operatorname{ch}(\mathcal{S}_{\mathbb{C}} \oplus \mathcal{S}_{\mathbb{C}}^*) = \operatorname{ch}(\mathcal{S}_{\mathbb{C}}) + \operatorname{ch}(\mathcal{S}_{\mathbb{C}}^*),$$

which implies

$$\operatorname{ch}[\mathcal{S}]_K = \operatorname{ch}(\mathcal{S}_{\mathbb{C}}^*) - \operatorname{ch}(\mathcal{S}_{\mathbb{C}}) = \operatorname{rk}\mathcal{S} - 2\operatorname{ch}(\mathcal{S}_{\mathbb{C}}).$$

□

5. The twisted signature formula

Let $\epsilon_* : H_0(X;\mathbb{Q}) \to \mathbb{Q}$ denote the augmentation homomorphism.

Theorem 1. *Let X^n be a closed oriented Whitney stratified normal Witt space of even dimension with singular set Σ, and let $(\mathcal{S}, \phi)$ be a Poincaré local system on $X - \Sigma$, strongly transverse to Σ. Then*

$$\sigma(X;\mathcal{S}) = \epsilon_*(\widetilde{\operatorname{ch}}([\mathcal{S}]_K) \cap L(X)) \in \mathbb{Z}$$

where $L(X) \in H_{2*}(X;\mathbb{Q})$ is the total L-class of X.

Proof. First note that $\sigma(X;\mathcal{S})$ is indeed defined: By proposition 3, $\mathcal{S}$ extends as a Poincaré local system over all of X. Thus if Lk is the link of an odd-codimensional stratum of X, then the restriction $\mathcal{S}|_{Lk-\Sigma}$ is a constant sheaf (see also the proof of proposition 3), and

$$IH_k^{\bar{m}}(Lk;\mathcal{S}) = 0,$$

since X is a Witt space ($\dim Lk = 2k$). Therefore (as pointed out in section 4.1)

$$\phi : \mathcal{D}\mathcal{S}[-n] \xrightarrow{\simeq} \mathcal{S}$$

extends to a self-duality isomorphism

$$\bar{\phi} : \mathcal{D}\mathbf{IC}^\bullet_{\bar{m}}(X;\mathcal{S})[n] \xrightarrow{\simeq} \mathbf{IC}^\bullet_{\bar{m}}(X;\mathcal{S})$$

and the twisted signature $\sigma(X;\mathcal{S})$ is defined.

Next, we show that the expression

$$\epsilon_*(\widetilde{\operatorname{ch}}([\mathcal{S}]_K) \cap L(X)) \tag{11}$$

is a cobordism invariant of $(X^n, \mathcal{S}, \phi)$ for globally defined Poincaré local systems $(\mathcal{S}, \phi)$. Let $(Y^{n+1}, \partial Y)$ be a compact oriented Witt space with boundary ∂Y, singular set Σ_Y, and $(\mathcal{T}, \psi)$ a Poincaré local system on all of Y. By naturality,

$$\widetilde{\operatorname{ch}}([\mathcal{T}|_{\partial Y}]_K) = j^*\widetilde{\operatorname{ch}}([\mathcal{T}]_K) \in H^{2*}(\partial Y;\mathbb{Q})$$

with $j : \partial Y \hookrightarrow Y$ the inclusion. Thus,

$$\begin{aligned}\epsilon_*(\widetilde{\mathrm{ch}}([\mathcal{T}|_{\partial Y}]_K) \cap L(\partial Y)) &= \langle \widetilde{\mathrm{ch}}([\mathcal{T}|_{\partial Y}]_K), L(\partial Y)\rangle \\ &= \langle j^*\widetilde{\mathrm{ch}}([\mathcal{T}]_K), L(\partial Y)\rangle \\ &= \langle \widetilde{\mathrm{ch}}([\mathcal{T}]_K), j_*L(\partial Y)\rangle.\end{aligned}$$

Now $j_*L(\partial Y) = 0$ by proposition 1 and we have

$$\epsilon_*(\widetilde{\mathrm{ch}}([\mathcal{T}|_{\partial Y}]_K) \cap L(\partial Y)) = 0,$$

proving (11) to be cobordism invariant.

The twisted signature $\sigma(X; \mathcal{S})$ is a cobordism invariant as well: We prove this using the bordism group Ω_*^{SD} as introduced in [Ban02, chapter 4]. Let $(Y^{n+1}, \partial Y)$, Σ_Y be as above and $(\mathcal{T}, \psi)$ be a Poincaré local system on $Y - \Sigma_Y$, strongly transverse to Σ_Y. Consider the diagram of inclusions

$$\begin{tikzcd} \partial Y - \Sigma_Y \arrow[r, "j|"] \arrow[d] & Y - \Sigma_Y \arrow[d] & (\mathrm{int}\, Y) - \Sigma_Y \arrow[l, "i|"'] \arrow[d] \\ \partial Y \arrow[r, "j"] & Y & \mathrm{int}\, Y \arrow[l, "i"'] \end{tikzcd}$$

The restriction

$$j|^*\psi : j|^*\mathcal{T} \times j|^*\mathcal{T} \longrightarrow \mathbb{R}_{\partial Y - \Sigma_Y}$$

induces a self-duality isomorphism in the derived category $D^b(\partial Y)$

$$\bar{\psi}_\partial : \mathcal{D}\mathbf{IC}^\bullet_{\bar{m}}(\partial Y; j|^*\mathcal{T})[n] \xrightarrow{\simeq} \mathbf{IC}^\bullet_{\bar{m}}(\partial Y; j|^*\mathcal{T})$$

as $\mathcal{T}$ being strongly transverse to Σ_Y implies that $\mathcal{T}$ extends as a Poincaré local system over Y (proposition 3), so is constant on links of odd-codimensional strata, and

$$\sigma(\partial Y; \mathcal{T}|_{\partial Y - \Sigma_Y}) = \sigma(\mathbf{IC}^\bullet_{\bar{m}}(\partial Y; j|^*\mathcal{T}), \bar{\psi}_\partial).$$

The triple $(\partial Y, \mathbf{IC}^\bullet_{\bar{m}}(\partial Y; j|^*\mathcal{T}), \bar{\psi}_\partial)$ defines an element in Ω_n^{SD}. The restriction to the interior

$$i|^*\psi : i|^*\mathcal{T} \times i|^*\mathcal{T} \longrightarrow \mathbb{R}_{\mathrm{int}\, Y - \Sigma_Y}$$

induces a self-duality isomorphism in $D^b(\mathrm{int}\, Y)$

$$\bar{\psi}_0 : \mathcal{D}\mathbf{IC}^\bullet_{\bar{m}}(\mathrm{int}\, Y; i|^*\mathcal{T})[n+1] \xrightarrow{\simeq} \mathbf{IC}^\bullet_{\bar{m}}(\mathrm{int}\, Y; i|^*\mathcal{T}).$$

In the terminology of [Ban02], the triple $(Y^{n+1}, \mathbf{IC}^\bullet_{\bar{m}}(\mathrm{int}\, Y; i|^*\mathcal{T}), \bar{\psi}_0)$ is an *admissible cobordism.* The boundary of such an admissible cobordism is defined to be

$$\partial(Y^{n+1}, \mathbf{IC}^\bullet_{\bar{m}}(\mathrm{int}\, Y; i|^*\mathcal{T}), \bar{\psi}_0) = (\partial Y, j^! Ri_! \mathbf{IC}^\bullet_{\bar{m}}(\mathrm{int}\, Y; i|^*\mathcal{T}), \partial\bar{\psi}_0),$$

where $\partial\bar{\psi}_0$ is induced by $j^* Ri_*(\bar{\psi}_0)$ under the canonical identification

$$j^* Ri_* \mathbf{A}^\bullet \xrightarrow{\simeq} j^! Ri_! \mathbf{A}^\bullet[1],$$

for any $\mathbf{A}^\bullet \in D^b(\text{int}\, Y)$, [Ban02, lemma 4.1]. Now by [Ban02, lemma 4.4],

$$\partial(Y^{n+1}, \mathbf{IC}^\bullet_{\bar{m}}(\text{int}\, Y; i|^*\mathcal{T}), \bar{\psi}_0) = (\partial Y, \mathbf{IC}^\bullet_{\bar{m}}(\partial Y; j|^*\mathcal{T}), \bar{\psi}_\partial)$$

Thus [Ban02, corollary 4.1] implies

$$\sigma(\partial Y; \mathcal{T}|_{\partial Y - \Sigma_Y}) = \sigma(\mathbf{IC}^\bullet_{\bar{m}}(\partial Y; j|^*\mathcal{T}), \bar{\psi}_\partial) = 0.$$

We return to the given X^n, $(\mathcal{S}, \phi)$ on $X - \Sigma$, strongly transverse to Σ. Consider the identity map

$$[X \xrightarrow{1} X] \otimes 1 \in \Omega_n^{Witt}(X) \otimes \mathbb{Z}[\tfrac{1}{2}]$$

By proposition 2,

$$\Omega_*^{SO}(X) \otimes \mathbb{Z}[\tfrac{1}{2}] \longrightarrow \Omega_*^{Witt}(X) \otimes \mathbb{Z}[\tfrac{1}{2}]$$

is onto.

Hence there exists a smooth oriented manifold M^n, a continuous map $f : M \to X$ and $r, s \in \mathbb{Z}$ such that

$$[M \xrightarrow{f} X] \otimes \frac{r}{2^s} = [X \xrightarrow{1} X] \otimes 1 \in \Omega_n^{Witt}(X) \otimes \mathbb{Z}[\tfrac{1}{2}]$$

and we have

$$r[M \xrightarrow{f} X] = 2^s[X \xrightarrow{1} X] \in \Omega_n^{Witt}(X).$$

Let $(\bar{\mathcal{S}}, \bar{\phi})$ denote the extension of $(\mathcal{S}, \phi)$ to X, proposition 3. On M, we consider the Poincaré local system $(f^*\bar{\mathcal{S}}, f^*\bar{\phi})$. Then

$$\begin{aligned} 2^s\sigma(X; \mathcal{S}) &= r\sigma(M; f^*\bar{\mathcal{S}}) && \text{(cobordism invariance)} \\ &= r\epsilon_*(\widetilde{\text{ch}}([f^*\bar{\mathcal{S}}]_K) \cap L(M)) && \text{(by Atiyah/Meyer)} \\ &= 2^s\epsilon_*(\widetilde{\text{ch}}([\mathcal{S}]_K) \cap L(X)) && \text{(cobordism invariance)} \end{aligned}$$

$\square$

6. The twisted L-class formula

Let X^n be a compact oriented stratified pseudomanifold and let

$$j : Y^m \hookrightarrow X^n$$

be a normally nonsingular inclusion of an oriented stratified pseudomanifold Y^m. Consider an open neighborhood $E \subset X$ of Y, the total space of an $\mathbb{R}^{n-m}$- vector bundle over Y, and put $E_0 = E - Y$, the total space with the zero-section removed. Let $u \in H^{n-m}(E, E_0)$ denote the Thom class. If $\pi : E \to Y$ denotes the projection, then the composition

$$H_k(X) \xrightarrow{i_*} H_k(X, X-Y) \underset{\cong}{\xleftarrow{e_*}} H_k(E, E_0) \underset{\cong}{\xrightarrow{u\cap -}} H_{k-n+m}(E) \underset{\cong}{\xrightarrow{\pi_*}} H_{k-n+m}(Y)$$

defines a map

$$j^! : H_k(X) \longrightarrow H_{k-n+m}(Y).$$

We recall the existence and uniqueness result on L-classes of self-dual complexes of sheaves (alluded to in section 2):

Theorem 2 ([CS91]). *Let $\boldsymbol{S}^\bullet \in D^b(X)$ be a self-dual complex of sheaves. There exist unique classes $L_k(\boldsymbol{S}^\bullet) \in H_k(X; \mathbb{Q})$ with the following properties:*

(i) $\epsilon_* L_0(\boldsymbol{S}^\bullet) = \sigma(\boldsymbol{S}^\bullet)$

(ii) If $j : Y^m \hookrightarrow X^n$ is a normally nonsingular inclusion with trivial normal bundle, then

$$L_{k-n+m}(j^! \boldsymbol{S}^\bullet) = j^! L_k(\boldsymbol{S}^\bullet)$$

Now let X be a closed Witt space with singular set Σ, and $(\mathcal{S}, \phi)$ a Poincaré local system on $X - \Sigma$ such that a self-dual extension $(\mathbf{IC}^\bullet_{\bar{m}}(X; \mathcal{S}), \bar{\phi})$ exists.

Definition 4. *The* twisted L-classes

$$L_k(X; \mathcal{S}) \in H_k(X; \mathbb{Q})$$

of X with coefficients in $\mathcal{S}$ *are the L-classes of the self-dual sheaf $\boldsymbol{S}^\bullet = \boldsymbol{IC}^\bullet_{\bar{m}}(X; \mathcal{S})$ as provided by theorem 2:*

$$L_k(X; \mathcal{S}) = L_k(\boldsymbol{S}^\bullet).$$

The proof of our twisted L-class formula requires the following multiplicative property of the Thom map $j^!$:

Lemma 5. *If $x \in H^p(X)$, $y \in H_{p+k}(X)$, then*

$$j^!(x \cap y) = (-1)^{p(n-m)} j^* x \cap j^! y.$$

Proof. With $e : (E, E_0) \to (X, X - Y)$ the inclusion, put $\xi = e^* x \in H^p(E)$. Let $\eta \in H_{p+k}(E, E_0)$ be the unique element such that $e_* \eta = i_* y$. Let $\iota : Y \to E$ be the inclusion as the zero section so that $\pi^* \iota^* \xi = \xi$. We have

$$\begin{aligned}
\pi_*(u \cap (\xi \cap \eta)) &= \pi_*((u \cup \xi) \cap \eta) \\
&= \pi_*(((-1)^{p(n-m)} \xi \cup u) \cap \eta) \\
&= (-1)^{p(n-m)} \pi_*(\xi \cap (u \cap \eta)) \\
&= (-1)^{p(n-m)} \iota^* \xi \cap \pi_*(u \cap \eta) \\
&= (-1)^{p(n-m)} j^* x \cap \pi_*(u \cap \eta).
\end{aligned}$$

Now $\pi_*(u \cap \eta) = j^! y$ as $e_* \eta = i_* y$. Also

$$e_*(\xi \cap \eta) = e_*(e^* x \cap \eta) = x \cap e_* \eta = x \cap i_* y = i_*(x \cap y),$$

so that $\pi_*(u \cap (\xi \cap \eta)) = j^!(x \cap y)$. □

Theorem 3. *Let X^n be a closed oriented Whitney stratified normal Witt space with singular set Σ, and let $(\mathcal{S}, \phi)$ be a Poincaré local system on $X - \Sigma$, strongly transverse to Σ. Then*

$$L(X; \mathcal{S}) = \widetilde{\mathrm{ch}}([\mathcal{S}]_K) \cap L(X).$$

Proof. For $\mathbf{S}^\bullet = \mathbf{IC}^\bullet_{\bar{m}}(X; \mathcal{S})$ define a set of homology classes by

$$L_k^\cap(\mathbf{S}^\bullet) = (\widetilde{\mathrm{ch}}[\mathcal{S}]_K \cap L(X))_k.$$

We shall verify that the $L_k^\cap$ satisfy (i) and (ii) of theorem 2. Firstly,

$$\epsilon_* L_0^\cap(\mathbf{S}^\bullet) = \epsilon_*(\widetilde{\mathrm{ch}}[\mathcal{S}]_K \cap L(X)) = \sigma(X; \mathcal{S}) = \sigma(\mathbf{S}^\bullet)$$

by the twisted signature formula, theorem 1. Now let $j : Y^m \hookrightarrow X^n$ be a normally nonsingular inclusion with trivial normal bundle. Then

$$j^! \mathbf{S}^\bullet = j^! \mathbf{IC}^\bullet_{\bar{m}}(X; \mathcal{S}) \cong \mathbf{IC}^\bullet_{\bar{m}}(Y; \mathcal{S}|_Y)$$

and

$$\begin{aligned}
L_{k-n+m}^\cap(j^! \mathbf{S}^\bullet) &= L_{k-n+m}^\cap(\mathbf{IC}^\bullet_{\bar{m}}(Y; \mathcal{S}|_Y)) \\
&= (\widetilde{\mathrm{ch}}[\mathcal{S}|_Y]_K \cap L(Y))_{k-n+m} \\
&= \sum_{p \geq 0} (\widetilde{\mathrm{ch}}[\mathcal{S}|_Y]_K)_{2p} \cap L_{2p+k-n+m}(Y) \\
&= \sum_{p \geq 0} j^*(\widetilde{\mathrm{ch}}[\mathcal{S}]_K)_{2p} \cap j^! L_{2p+k}(X) \\
&= \sum_{p \geq 0} j^!((\widetilde{\mathrm{ch}}[\mathcal{S}]_K)_{2p} \cap L_{2p+k}(X)) \\
&= j^!(\widetilde{\mathrm{ch}}[\mathcal{S}]_K \cap L(X))_k \\
&= j^! L_k^\cap(\mathbf{S}^\bullet),
\end{aligned}$$

confirming (ii) for $L^{\cap}$. The uniqueness statement of theorem 2 implies

$$L(X;\mathcal{S}) = L(\mathbf{S}^\bullet) = L^{\cap}(\mathbf{S}^\bullet).$$

□

7. Applications

7.1. Supernormal Spaces

Recall the

Definition 5. *X^n is* supernormal*, if for any components Z, Z' of open strata with* $\dim Z' > \dim Z \leq n-2$, *the link $Lk(Z) \cap Z'$ is simply connected.*

Theorem 1 implies

Corollary 3. *If X^n is supernormal, then for any Poincaré local system $(\mathcal{S}, \phi)$ on $X - \Sigma$*

$$\sigma(X;\mathcal{S}) = \epsilon_*(\widetilde{\mathrm{ch}}([\mathcal{S}]_K) \cap L(X)).$$

Proof. Let $Z \in \mathcal{X}$ be a pure stratum and let X_0 be the component of X which contains $Lk(Z)$. By lemma 3, $X_0 - \Sigma$ is connected. Thus by supernormality $Lk(Z) \cap (X_0 - \Sigma) = Lk(Z) - \Sigma$ is simply connected. It follows that

$$\Pi_1(Lk(Z) - \Sigma) \longrightarrow \Pi_1(X - \Sigma) \xrightarrow{\mu(\mathcal{S})} \mathfrak{Vect}_m$$

is isomorphic to the trivial functor and $(\mathcal{S}, \phi)$ is strongly transverse to Σ. Now apply theorem 1. □

Remark 3. To obtain the conclusion of the corollary, less than supernormality is actually needed. Indeed it is sufficient to require that X be normal and that the image of $\pi_1(Lk(Z) - \Sigma)$ in $\pi_1(X - \Sigma)$ vanish for all $Z \in \mathcal{X}$.

7.2. Stratified Maps

A synthesis of our characteristic class formula (theorem 1) and the Cappell-Shaneson signature formula (3), section 1.3, yields the following result. We use the notation introduced in section 1.3.

Theorem 4. *Let $f : Y^m \longrightarrow X^n$ be a stratified map of oriented compact Whitney stratified spaces with only even-codimensional strata, X normal and $m - n$ even. Assume that for all components Z of strata of X, the Poincaré local system $\mathcal{S}_f^Z$ is strongly transverse to the singularities of $\overline{Z}$. Then*

$$\sigma(Y) = \epsilon_*(\widetilde{\mathrm{ch}}([\mathcal{S}_f^{X-\Sigma}]_K) \cap L(X)) + \sum_{Z \in \mathcal{X}} \epsilon_*(\widetilde{\mathrm{ch}}([\mathcal{S}_f^Z]_K) \cap L(\overline{Z})). \tag{12}$$

We formulate the corresponding corollary for supernormal targets of stratified maps.

Corollary 4. *Let $f : Y^m \longrightarrow X^n$ be a stratified map of oriented compact Whitney stratified spaces with only even-codimensional strata, X supernormal and $m-n$ even. Then*

$$\sigma(Y) = \epsilon_*(\widetilde{\mathrm{ch}}([\mathcal{S}_f^{X-\Sigma}]_K) \cap L(X)) + \sum_{Z \in \mathcal{X}} \epsilon_*(\widetilde{\mathrm{ch}}([\mathcal{S}_f^Z]_K) \cap L(\overline{Z})).$$

Proof. By its very definition, supernormality is a property that is inherited by all $\overline{Z}$, $Z \in \mathcal{X}$. □

As a special case of theorem 4, let us discuss a locally trivial fiber bundle $F \to Y \xrightarrow{f} X$. Rather than deriving the result from theorem 4, we adapt part of the proof of theorem 1. Comparing formula (13) below to the general formula (12), it is interesting to observe that for fiber bundles, the singular contributions of (12) vanish.

Theorem 5. *Let X^n, Y^m be oriented compact Whitney stratified Witt spaces of even dimension. If $f : Y \to X$ is a fiber bundle projection with Witt space fiber F^{2d}, then*

$$\sigma(Y) = \epsilon_*(\widetilde{\mathrm{ch}}([\mathcal{S}_f^X]_K) \cap L(X)), \tag{13}$$

where the Poincaré local system $\mathcal{S}_f^X$ has stalks $IH_d^{\bar{m}}(F_x)$, $x \in X$.

Proof. Let G be the structure group of the fiber bundle f and $EG \to BG$ the universal principal G-bundle. Let $g : X \to BG$ be the classifying map for the principal G-bundle associated to f, so that $Y \cong g^*(EG \times_G F)$. Consider $[X \xrightarrow{g} BG] \otimes 1 \in \Omega_n^{Witt}(BG) \otimes \mathbb{Z}[\frac{1}{2}]$. By proposition 2, there exists a smooth oriented manifold M^n, $h : M \to BG$ and $r, s \in \mathbb{Z}$ such that

$$r[M \xrightarrow{h} BG] = 2^s[X \xrightarrow{g} BG] \in \Omega_n^{Witt}(BG).$$

Then, as in the proof of theorem 1,

$$\begin{aligned} 2^s\sigma(Y) &= 2^s\sigma(g^*(EG \times_G F)) \\ &= r\sigma(h^*(EG \times_G F)) \\ &= r\epsilon_*(\widetilde{\mathrm{ch}}([\mathcal{S}_\pi^M]_K) \cap L(M)) \text{ (by Atiyah)} \\ &= 2^s\epsilon_*(\widetilde{\mathrm{ch}}([\mathcal{S}_f^X]_K) \cap L(X)) \end{aligned}$$

where $\pi : h^*(EG \times_G F) \to M$ is the projection. □

Our next example is concerned with a map to a manifold which is a bundle except over a set of codimension at least 3, where the fiber is allowed to change. We will be able to calculate the top term.

Proposition 5. *Let M^n be an oriented compact manifold and Y^m an oriented compact Whitney stratified space with only even-codimensional strata, $m-n$ even. If $f : Y^m \to M^n$ is a stratified map with respect to a Whitney stratification of M of the form $M \supset \Sigma = X_{n-4} \supset X_{n-6} \supset \ldots \supset X_0 \supset \varnothing$ (only even-codimensional strata), then*

$$\sigma(Y) = \epsilon_*(\widetilde{\mathrm{ch}}([\mathcal{S}_f^{M-\Sigma}]_K) \cap L(M)) + \sum_{Z \in \mathcal{X}} \sigma(\overline{Z}; \mathcal{S}_f^Z).$$

Proof. We have to show that $\mathcal{S}_f^{M-\Sigma}$ is strongly transverse to Σ. Since Σ is of codimension ≥ 3 in M,

$$\Pi_1(M-\Sigma) \xrightarrow{\simeq} \Pi_1(M)$$

is an equivalence of categories, by general position in M. Thus the monodromy functor $\mu(\mathcal{S}_f^{M-\Sigma})$ factors, up to isomorphism of functors, as

$$\begin{tikzcd} \Pi_1(M-\Sigma) \arrow[r, "\simeq"] \arrow[dr, "\mu(\mathcal{S}_f^{M-\Sigma})"'] & \Pi_1(M) \arrow[d] \\ & \mathfrak{Vect} \end{tikzcd}$$

and $\mathcal{S}_f^{M-\Sigma}$ is strongly transverse by corollary 1 (and remark 2). □

8. An example

We shall discuss the example of a space X having one isolated singularity. The space will be obtained by collapsing a submanifold of a nonsingular space. It will come equipped with a local coefficient system $\mathcal{S}$ on the top stratum which is constant on the link of the singularity.

Let M^m and N^n be closed, connected, oriented, even-dimensional manifolds and let $\mathcal{T}$ be a Poincaré local system on M. (Below we will consider in detail the case $M = \Sigma_2$, a surface of genus 2, and $N = \mathbb{P}^2$, complex projective space.) Define $Y = M \times N$. If $s \in M$ is the base-point, let

$$X = Y/(\{s\} \times N)$$

be the pseudomanifold obtained from Y by collapsing $\{s\} \times N$ to a point. The space X has a singular stratum $X_0 = f(\{s\} \times N)$ containing a single point, where $f : Y \to X$ denotes the collapse map. With $D^m \subset M$ a small disc about s, the link of X_0 in X is given by

$$Lk(X_0) = \partial(D^m \times N) = S^{m-1} \times N.$$

Next, we equip the top stratum $X - X_0$ with a Poincaré local system $\mathcal{S}$ which will be strongly transverse to X_0. On Y, we consider the pull-back $\pi_1^* \mathcal{T}$

under the first factor projection $\pi_1 : Y \to M$. With $f|$ denoting the restriction $f| : f^{-1}(X - X_0) \to X - X_0$ (a homeomorphism), set

$$\mathcal{S} = f|_*(\pi_1^* \mathcal{T})|_{f^{-1}(X-X_0)}.$$

The goal is to calculate $L(X; \mathcal{S})$ and in particular $\sigma(X; \mathcal{S})$.

By the L-class version of formula (3) in section 1.3 applied to the stratified map $f : Y \to X$,

$$f_* L_k(Y) = L_k(X) + j_* L_k(X_0; -).$$

Here, j is the inclusion $j : X_0 \hookrightarrow X$ and $L_k(X_0; -)$ is a potential contribution of the singularity which vanishes for $k > 0$ as it lives in $H_k(X_0) = 0$. For $k = 0$, (3) asserts that the difference

$$\sigma(Y) - \sigma(X)$$

is given by a term near the singularity, $\sigma(E)$, where

$$E = (D^m \times N) \cup_{S^{m-1} \times N} c(S^{m-1} \times N)$$

(or use Novikov additivity to see this). It is rather clear that $\sigma(E) = 0$, however we would like to point out that this follows from the stronger fact that a certain class of singular spaces (which includes E), whose signature frequently arises as a potential contribution from singularities in problems involving stratified maps, actually has no middle dimensional intersection homology:

Lemma 6. *Let A^{2a-1} and B^{2b} be closed Witt spaces. Then the middle dimensional intersection homology of the pseudomanifold*

$$E^{2(a+b)} = cA \times B \cup_{A \times B} c(A \times B)$$

vanishes:

$$IH_{a+b}^{\bar{m}}(E) = 0.$$

Proof. Throughout the proof, let $IH_*^c(-)$ denote middle-perversity intersection homology with compactly supported chains. If X is of dimension $2n - 1$, then

$$IH_i^c(c^\circ X) = \begin{cases} IH_i^c(X), & i < n \\ 0, & i \geq n, \end{cases} \tag{14}$$

where $c^\circ X$ is the open cone on X. Consider the Mayer-Vietoris sequence

$$\to IH_i^c(A \times B) \xrightarrow{\alpha_i} IH_i^c(c^\circ A \times B) \oplus IH_i^c(c^\circ(A \times B)) \xrightarrow{\beta_i} IH_i^c(E) \xrightarrow{\partial_i}$$

For $i = a + b$, (14) implies $IH^c_{a+b}(c^\circ(A \times B)) = 0$. The Künneth formula holds for Witt spaces and middle perversity intersection homology. Thus

$$IH^c_{a+b}(c^\circ A \times B) = \bigoplus_{\substack{i+j=a+b \\ i<a}} IH^c_i(A) \otimes IH^c_j(B),$$

using (14) again. We conclude that α_{a+b} is surjective and $\beta_{a+b} = 0$. On the other hand, (14) shows that $IH^c_{a+b-1}(c^\circ(A \times B)) = IH^c_{a+b-1}(A \times B)$. Therefore the second component of α_{a+b-1} is injective, thus α_{a+b-1} is injective and $\partial_{a+b} = 0$. $\square$

The lemma applies to the present discussion with $A = S^{m-1}$ so that $cA = D^m$. Consequently,

$$f_* L(Y) = L(X).$$

It is straightforward to check that $\widetilde{\mathrm{ch}}[\mathcal{R}]_K$ is natural in the Poincaré local system $\mathcal{R}$, that is, if $g : A \to B$ is continuous and $\mathcal{R}$ is a Poincaré local system over B, then $\widetilde{\mathrm{ch}}[g^*\mathcal{R}]_K = g^*\widetilde{\mathrm{ch}}[\mathcal{R}]_K$.

Using the commutative diagram

$$\begin{tikzcd} D^m \times N \arrow[r] \arrow[d, "\pi_1|"'] & Y \arrow[d, "\pi_1"] \\ D^m \arrow[r] & M \end{tikzcd}$$

(the horizontal arrows are inclusions), we have

$$(\pi_1^*\mathcal{T})|_{D^m \times N} \cong \pi_1|^*(\mathcal{T}|_{D^m}) \cong \pi_1|^*(\mathbb{R}^r_{D^m}) = \mathbb{R}^r_{D^m \times N}.$$

In particular, the restriction

$$\mathcal{S}|_{Lk(X_0)} \cong \mathbb{R}^r_{S^{m-1} \times N}$$

is constant, that is, $\mathcal{S}$ is strongly transverse to X_0. By proposition 3, $\mathcal{S}$ extends uniquely to a Poincaré local system $\bar{\mathcal{S}}$ on all of X (here this is of course $p^*\mathcal{T}$, where $Y \xrightarrow{f} X \xrightarrow{p} M$ factors π_1) and

$$f^*\bar{\mathcal{S}} \cong \pi_1^*\mathcal{T}.$$

By theorem 1,

$$\begin{aligned} \sigma(X; \mathcal{S}) &= \langle \widetilde{\mathrm{ch}}[\bar{\mathcal{S}}]_K, L(X)\rangle = \langle \widetilde{\mathrm{ch}}[\bar{\mathcal{S}}]_K, f_*L(Y)\rangle \\ &= \langle f^*\widetilde{\mathrm{ch}}[\bar{\mathcal{S}}]_K, L(Y)\rangle = \langle \widetilde{\mathrm{ch}}[f^*\bar{\mathcal{S}}]_K, L(Y)\rangle \\ &= \langle \widetilde{\mathrm{ch}}[\pi_1^*\mathcal{T}]_K, L(Y)\rangle = \langle \pi_1^*\widetilde{\mathrm{ch}}[\mathcal{T}]_K, L(Y)\rangle \\ &= \langle \widetilde{\mathrm{ch}}[\mathcal{T}]_K, \pi_{1*}L(Y)\rangle = \langle \widetilde{\mathrm{ch}}[\mathcal{T}]_K, \sigma(N)L(M)\rangle \\ &= \sigma(N)\langle \widetilde{\mathrm{ch}}[\mathcal{T}]_K, L(M)\rangle. \end{aligned}$$

The homology L-class satisfies

$$L_{m+n-4k}(M \times N) = \sum_{p+q=k} L_{m-4p}(M) \times L_{n-4q}(N).$$

By theorem 3,

$$\begin{aligned} L(X; \mathcal{S}) &= \widetilde{\mathrm{ch}}[\bar{\mathcal{S}}]_K \cap L(X) \\ &= \widetilde{\mathrm{ch}}[\bar{\mathcal{S}}]_K \cap f_* L(Y) \\ &= f_*(\pi_1^* \widetilde{\mathrm{ch}}[\mathcal{T}]_K \cap L(Y)) \\ &= f_*((\widetilde{\mathrm{ch}}[\mathcal{T}]_K \times 1) \cap (L(M) \times L(N))) \\ &= f_*((\widetilde{\mathrm{ch}}[\mathcal{T}]_K \cap L(M)) \times L(N)). \end{aligned}$$

Alternatively, one can compute $\sigma(X; \mathcal{S})$ using the resolution $f : Y \to X$ and formula (3). One observes that $\sigma(Y; \pi_1^* \mathcal{T}) = \sigma(N)\sigma(M; \mathcal{T})$; (3) asserts that the difference

$$\sigma(Y; \pi_1^* \mathcal{T}) - \sigma(X; \mathcal{S})$$

is given by a term near the singularity,

$$\sigma(E; \pi_1^* \mathcal{T})$$

(or use Novikov additivity to see this). Now

$$(\pi_1^* \mathcal{T})|_{D^m \times N} \cong \mathbb{R}^r_{D^m \times N}$$

is constant, so

$$\sigma(E; \pi_1^* \mathcal{T}) = \sigma(E)\sigma(\mathbb{R}^r_E) = 0.$$

Consequently, $\sigma(X; \mathcal{S}) = \sigma(N)\sigma(M; \mathcal{T})$.

We specialize to the example of a 6-dimensional space X, taking $M = \Sigma_2$, a closed oriented surface of genus 2, and $N = \mathbb{P}^2$. The total L-class of X will be calculated and applying theorem 1 we shall see that

$$\sigma(X; \mathcal{S}) = 4. \tag{15}$$

We give a sample construction of a local system based on an example of [Mey72].

Define a Poincaré local system $\mathcal{T}$ on Σ_2 by the representation

$$\mu : \pi_1 \Sigma_2 \longrightarrow Sp(2; \mathbb{R}),$$

$$\mu(a_1) = \begin{pmatrix} -5 & 1 \\ -\frac{27}{2} & \frac{5}{2} \end{pmatrix}, \ \mu(a_2) = \begin{pmatrix} -4 & -1 \\ 33 & 8 \end{pmatrix}, \ \mu(b_1) = \begin{pmatrix} -4 & 1 \\ -33 & 8 \end{pmatrix}, \ \mu(b_2) = \begin{pmatrix} -5 & -1 \\ \frac{27}{2} & \frac{5}{2} \end{pmatrix},$$

where $\pi_1 \Sigma_2 = \langle a_1, a_2, b_1, b_2 | [a_1, b_1][a_2, b_2] = 1 \rangle$. If $H^2(\Sigma_2) = \mathbb{Q}\langle \gamma \rangle$ with $\langle \gamma, [\Sigma_2] \rangle = 1$, then $\mathcal{T}$ has the property that the first Chern class

$$c_1(\mathcal{T}_{\mathbb{C}}) = -\gamma,$$

where $[\mathcal{T}]_K = [\mathcal{T}^*_{\mathbb{C}}] - [\mathcal{T}_{\mathbb{C}}] \in KU(\Sigma_2)$, [Mey72].

Y^6, X^6, f, π_1, and $\mathcal{S}$ are constructed as in the general discussion above. All (co)homology will be taken with rational coefficients. We denote the cohomology ring of $\mathbb{P}^2$ by $H^*(\mathbb{P}^2) = \mathbb{Q}[\alpha]/(\alpha^3 = 0)$, $\alpha \in H^2(\mathbb{P}^2)$. We have

$$\begin{aligned} H^0(Y) &= \mathbb{Q}\langle 1 \times 1 \rangle, \\ H^2(Y) &= \mathbb{Q}\langle 1 \times \alpha, \gamma \times 1 \rangle, \\ H^4(Y) &= \mathbb{Q}\langle 1 \times \alpha^2, \gamma \times \alpha \rangle, \\ H^6(Y) &= \mathbb{Q}\langle \gamma \times \alpha^2 \rangle, \end{aligned}$$

and

$$\begin{aligned} H^0(X) &= \mathbb{Q}\langle 1 \times 1 \rangle, \\ H^2(X) &= \mathbb{Q}\langle \gamma \times 1 \rangle, \\ H^4(X) &= \mathbb{Q}\langle \gamma \times \alpha \rangle, \\ H^6(X) &= \mathbb{Q}\langle \gamma \times \alpha^2 \rangle. \end{aligned}$$

The Hirzebruch L-class of $\mathbb{P}^2$ is

$$\begin{aligned} \ell_0(\mathbb{P}^2) &= 1 \in H^0(\mathbb{P}^2), \\ \ell_1(\mathbb{P}^2) &= \alpha^2 \in H^4(\mathbb{P}^2). \end{aligned}$$

Thus, using multiplicativity of the L-class and $\ell_1(\Sigma_2) = 0$,

$$\ell_1(Y) = 1 \times \alpha^2.$$

Taking Poincaré duals, we obtain the homology L-classes of Y:

$$\begin{aligned} L(Y) &= \ell(Y) \cap [Y] \\ &= (1 \times 1 + 1 \times \alpha^2) \cap [\Sigma_2 \times \mathbb{P}^2] \\ &= [\Sigma_2] \times [\mathbb{P}^2] + (1 \cap [\Sigma_2]) \times (\alpha^2 \cap [\mathbb{P}^2]) \\ &= [\Sigma_2] \times [\mathbb{P}^2] + [\Sigma_2] \times [*] \end{aligned}$$

(where $[*] \in H_0(\mathbb{P}^2)$ is the homology class of a point), i.e.

$$\begin{aligned} L_2(Y) &= [\Sigma_2] \times [*] \in H_2(Y), \\ L_6(Y) &= [\Sigma_2] \times [\mathbb{P}^2] \in H_6(Y). \end{aligned}$$

Therefore,

$$\begin{aligned} L_2(X) &= f_* L_2(Y) = [\Sigma_2] \times [*] \in H_2(X), \\ L_6(X) &= f_* L_6(Y) = [\Sigma_2] \times [\mathbb{P}^2] \in H_6(X). \end{aligned}$$

We continue by calculating $\widetilde{\mathrm{ch}}[\mathcal{T}]_K$ on the surface. If ξ is any complex line bundle on Σ_2, then

$$\widetilde{\mathrm{ch}}(\xi) = e^{2c_1(\xi)} = 1 + 2c_1(\xi).$$

Hence

$$\begin{aligned}\widetilde{\mathrm{ch}}[\mathcal{T}]_K &= \widetilde{\mathrm{ch}}([\mathcal{T}^*_{\mathbb{C}}] - [\mathcal{T}_{\mathbb{C}}])\\ &= 1 + 2c_1(\mathcal{T}^*_{\mathbb{C}}) - 1 - 2c_1(\mathcal{T}_{\mathbb{C}})\\ &= -2c_1(\mathcal{T}_{\mathbb{C}}) - 2c_1(\mathcal{T}_{\mathbb{C}})\\ &= -4c_1(\mathcal{T}_{\mathbb{C}})\\ &= 4\gamma\end{aligned}$$

and

$$\widetilde{\mathrm{ch}}[\pi_1^*\mathcal{T}]_K = \pi_1^*(4\gamma) = 4(\gamma \times 1).$$

As a side-remark, this shows that if $\mathcal{R}$ is any Poincaré local system on a surface Σ_g given by a representation $\pi_1(\Sigma_g) \to Sp(2;\mathbb{R})$, then the twisted signature $\sigma(\Sigma_g;\mathcal{R})$ is divisible by 4. In fact, if $F \to E \to \Sigma_g$ is a surface bundle, then $\sigma(E)$ is divisible by 4 (and is actually 0 for $g \leq 1$ or $\mathrm{genus}(F) \leq 2$) according to [Mey73].

By theorem 1,

$$\begin{aligned}\sigma(X;\mathcal{S}) &= \langle \widetilde{\mathrm{ch}}[\bar{\mathcal{S}}]_K, L(X)\rangle\\ &= \langle 4(\gamma \times 1), [\Sigma_2] \times [*] + [\Sigma_2] \times [\mathbb{P}^2]\rangle\\ &= 4\langle \gamma, \Sigma_2\rangle\\ &= 4.\end{aligned}$$

The zero dimensional component $L_0(X;\mathcal{S}) = \sigma(X;\mathcal{S})[*] \times [*]$ is in fact the only non-zero component of the total twisted L-class $L(X;\mathcal{S})$: By theorem 3,

$$\begin{aligned}L(X;\mathcal{S}) &= \widetilde{\mathrm{ch}}[\bar{\mathcal{S}}]_K \cap L(X)\\ &= f_*(4(\gamma \times 1) \cap ([\Sigma_2] \times [*] + [\Sigma_2] \times [\mathbb{P}^2]))\\ &= f_*(4[*] \times [*] + 4(\gamma \cap [\Sigma_2]) \times (1 \cap [\mathbb{P}^2]))\\ &= 4[*] \times [*] + 4f_*([*] \times [\mathbb{P}^2])\\ &= 4[*] \times [*].\end{aligned}$$

References

[Ati69] Atiyah, M.F.: The signature of fibre-bundles. Global Analysis; Papers in honor of K. Kodaira (D.C. Spencer and S. Iyanayaga, eds.), Princeton Univ. Press, Princeton, 1969, pp. 73–84

[Ban02] Banagl, M.: Extending intersection homology type invariants to non-Witt spaces. Memoirs Amer. Math. Soc. **160**(760), 1–83 (2002)

[CHS57] Chern, S.S., Hirzebruch, F., Serre, J.-P.: The index of a fibered manifold. Proc. Amer. Math. Soc. **8**, 587–596 (1957)

[CS91] Cappell, S.E., Shaneson, J.L.: Stratifiable maps and topological invariants. J. Amer. Math. Soc. **4**, 521–551 (1991)

[CSW91] Cappell, S.E., Shaneson, J.L., Weinberger, S.: Classes topologiques caractéristiques pour les actions de groupes sur les espaces singuliers. C.R. Acad. Sci. Paris Sér. I Math. **313**, 293–295 (1991)

[GM80] Goresky, M., MacPherson, R.D.: Intersection homology theory. Topology **19**, 135–162 (1980)

[HBJ92] Hirzebruch, F., Berger, Th., Jung, R.: Manifolds and modular forms. Aspects of Math., vol. E20, Vieweg, 1992

[Hir69] Hirzebruch, F.: The signature of ramified coverings. Collected Math. Papers in Honor of Kodaira, Tokyo University Press, Tokyo, 1969, pp. 253–265

[Kod67] Kodaira, K.: A certain type of irregular algebraic surfaces. J. Anal. Math. **19**, 207–215 (1967)

[KT68] Kamber, F., Tondeur, Ph.: Flat manifolds. Lecture Notes in Math., no. 67, Springer Verlag, Berlin-New York, 1968

[Mey72] Meyer, W.: Die Signatur von lokalen Koeffizientensystemen und Faserbündeln. Bonner Math. Schriften **53** (1972)

[Mey73] Meyer, W.: Die Signatur von Flächenbündeln. Math. Ann. **201**, 239–264 (1973)

[Sie83] Siegel, P.H.: Witt spaces: A geometric cycle theory for KO-homology at odd primes. Amer. J. Math. **105**, 1067–1105 (1983)

[Sul70] Sullivan, D.: Geometric topology notes, part I. MIT, Cambridge, MA, 1970